SLOW VISCOUS MOTION OF A SOLID PARTICLE IN A SPHERICAL CAVITY

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Outline

1) Addressed problem and assumptions

2) Key issues and available literature

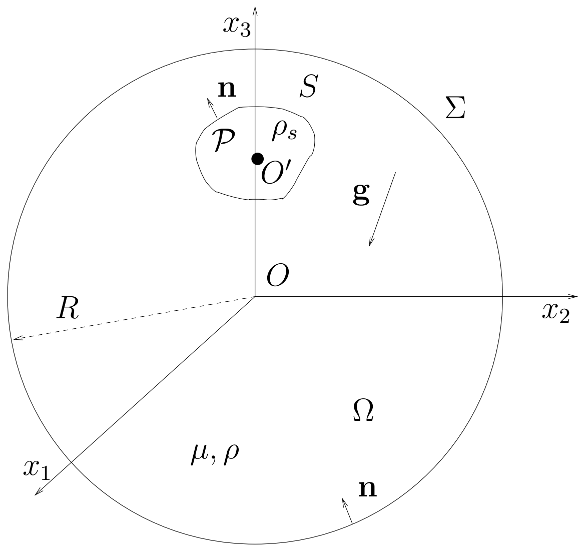
3) Boundary approach and suitable Green tensor

4) Numerical implementation and comparisons

5) Numerical results for a non-spherical particle

6) Concluding remarks

Addressed problem



- A Newtonian liquid (ρ, μ). Applied uniform gravity field $\mathbf{g}$
- The liquid is confined by a solid and motionless cavity Σ with attached Cartesian coordinates (O, x_1, x_2, x_3)
- A solid arbitrary-shaped particle $\mathcal{P}$ with center of mass O', uniform density ρ_s and smooth surface S with $\mathbf{n}$ the unit outward normal
- The particle translates at $\mathbf{U}$ (velocity of O') and rotates at $\mathbf{W}$

Basic issues

Experienced surface traction $\mathbf{f}$ on S?
Resulting hydrodynamic force $\mathbf{F}$ and torque $\mathbf{\Gamma}$ (about O') on $\mathcal{P}$?

Assumptions and governing equations

- The particle and its rigid-body motion $(\mathbf{U}, \mathbf{W})$ have length and velocity scales a and V.
- Assuming that $\mathrm{Re} = \rho V a/\mu \ll 1$ one neglects inertia effects and obtains a quasi-steady flow $(\mathbf{u}, p+\rho\mathbf{g}.\mathbf{x})$ in the liquid domain Ω

Creeping steady flow

$$\mu\nabla^2\mathbf{u} = \nabla p \ \text{ and } \ \nabla.\mathbf{u} = 0 \text{ in } \Omega,$$

$$\mathbf{u} = \mathbf{0} \ \text{ on } \ \Sigma,$$

$$\mathbf{u} = \mathbf{U} + \mathbf{W} \wedge \mathbf{x}' \text{ on } S \text{ with } \mathbf{x}' = \mathbf{O}'\mathbf{M}$$

Introducing the stress tensor $\boldsymbol{\sigma}$ such that $\sigma_{ij} = -p\delta_{ij} + \mu(u_{i,j} + u_{j,i})$, one looks at

$$\mathbf{f} = \boldsymbol{\sigma}.\mathbf{n} \ \text{ on } \ S,$$

$$\mathbf{F} = \int_S \mathbf{f}dS, \ \boldsymbol{\Gamma} = \int_S \mathbf{x}' \wedge \mathbf{f}dS$$

Two basic Problems

- Problem 1: $(\mathbf{U}, \mathbf{W})$ prescribed. Evaluation of $\mathbf{F}$ and $\boldsymbol{\Gamma}$?
- Problem 2: freely-suspended particle $\mathcal{P}$ with volume $\mathcal{V}$. Obtain $(\mathbf{U}, \mathbf{W})$ by enforcing

$$\mathbf{F} = (\rho - \rho_s)\mathcal{V}\mathbf{g}, \quad \boldsymbol{\Gamma} = \mathbf{0}$$

Auxiliary Stokes flows and key surface tractions

- $(\mathbf{u}_t^{(i)}, p_t^{(i)})$ and $(\mathbf{u}_r^{(i)}, p_r^{(i)})$ for $i = 1, 2, 3$. Stokes flows with

$$\mathbf{u}_t^{(i)} = \mathbf{u}_r^{(i)} = \mathbf{0} \text{ on } \Sigma, \mathbf{u}_t^{(i)} = \mathbf{e}_i \text{ and } \mathbf{u}_t^{(i)} = \mathbf{e}_i \wedge \mathbf{x}' \text{ on } S$$

- Resulting surface tractions $\mathbf{f}_t^{(i)}$ and $\mathbf{f}_r^{(i)}$ on S

Use for Problem 1

$$\mathbf{F} = -\mu\{\mathbf{A}_t.\mathbf{U} + \mathbf{B}_t.\mathbf{W}\}, \quad \mathbf{\Gamma} = -\mu\{\mathbf{A}_r.\mathbf{U} + \mathbf{B}_r.\mathbf{W}\}$$

$$-\mu A_L^{i,j} = \int_S \mathbf{f}_L^{(i)}.\mathbf{e}_j dS, \quad -\mu B_L^{i,j} = \int_S (\mathbf{x}' \wedge \mathbf{f}_L^{(i)}).\mathbf{e}_j dS$$

Use for Problem 2

The rigid-body migration $(\mathbf{U}, \mathbf{W})$ is obtained by solving

$$\mu\{\mathbf{A}_t.\mathbf{U} + \mathbf{B}_t.\mathbf{W}\} = (\rho_s - \rho)\mathcal{V}\mathbf{g}$$

$$\mu\{\mathbf{A}_r.\mathbf{U} + \mathbf{B}_r.\mathbf{W}\} = \mathbf{0}$$

- Well-posed linear system
- Unique solution $(\mathbf{U}, \mathbf{W})$

Available literature?

• Restricted to a spherical particle!

• Case of a translating sphere located at the cavity center
- Cunningham (1910), Williams (1915)
by obtaining the stream function (exact solution)

• Case of a sphere not located at the cavity center
- Use of bipolar coordinates (well adapted to the fluid domain geometry)
- Jeffery (1915), Stimson & Jeffery (1926), O'Neill & Majumdar (1970a, 1970b)
- recently: accurate calculations by Jones (2008)

• Merits
- very accurate solution (if carefully implemented)
- able to deal with small sphere-cavity gaps!
- provides very nice benchmatk tests for other methods to be developed

• Drawbacks
- cumbersome approach (tricky analytical manipulations)
- provides the net force $\mathbf{F}$ and torque $\mathbf{\Gamma}$ but still uneasy to calculate the surface tractions $\mathbf{f}_t^{(i)}$ and $\mathbf{f}_r^{(i)}$ on S
- not possible to cope with one non-spherical particle or with several particles!

Quite different boundary approach

Green tensors

- $\mathbf{y}$ source point or pole in the entire domain $\mathcal{D} = \Omega \cup \mathcal{P}$
- $\mathbf{x}$ observation point. For $j = 1, 2, 3$ one introduces a Stokes flows $(\mathbf{v}^{(j)}, p^{(j)})$,

$$\mu \nabla^2 \mathbf{v}^{(j)} = \nabla p^{(j)} - \delta_{3d}(\mathbf{x} - \mathbf{y})\mathbf{e}_j, \ \nabla . \mathbf{v}^{(j)} = 0 \ \text{ in } \ \mathcal{D}$$

- Resulting Green tensor $\mathbf{G}$ with Cartesian components

$$G_{kj}(\mathbf{x}, \mathbf{y}) = \mathbf{v}^{(j)}(\mathbf{x}, \mathbf{y}).\mathbf{e}_k$$

Remark, examples

- A Green tensor: not unique (no prescribed boundary conditions)
- Widely-employed free-space Green tensor $\mathbf{G}^{\infty}$ such that

$$8\pi\mu G^{\infty}_{kj}(\mathbf{x}, \mathbf{y}) = \frac{\delta_{kj}}{|\mathbf{x} - \mathbf{y}|} + \frac{[(\mathbf{x} - \mathbf{y}).\mathbf{e}_j][(\mathbf{x} - \mathbf{y}).\mathbf{e}_k]}{|\mathbf{x} - \mathbf{y}|^3}$$

- Specific Green tensor $\mathbf{G}^c$ for the given cavity Σ :

$$G^{c}_{jk}(\mathbf{x}, \mathbf{y}) = 0 \quad \text{for } \mathbf{x} \text{ on } \Sigma$$

Relevant integral representations and boundary-integral equations

- One looks at $\mathbf{f} = f_k \mathbf{e}_k$ on S for $\mathbf{u} = \mathbf{U} + \mathbf{\Omega} \wedge \mathbf{x}'$ on S

- Due to this velocity boundary condition, one gets a single-layer integral representation

$$[\mathbf{u}.\mathbf{e}_j](\mathbf{x}) = -\int_{S \cup \Sigma} f_k(\mathbf{y}) G_{kj}(\mathbf{y}, \mathbf{x}) dS(\mathbf{y}) \text{ for } \mathbf{x} \text{ in } \Omega \cup S;\ j = 1, 2, 3.$$

(Here $\mathbf{x}$ is the pole)

- Associated Fredholm boundary-integral equation of the first kind

$$[\mathbf{U} + \mathbf{\Omega} \wedge \mathbf{x}'].\mathbf{e}_j = -\int_{S \cup \Sigma} f_k(\mathbf{y}) G_{kj}(\mathbf{y}, \mathbf{x}) dS(\mathbf{y}) \text{ for } \mathbf{x} \text{ on } S;\ j = 1, 2, 3.$$

(solution unique up to $c\mathbf{n}$ with c constant)

- Valid for any Green tensor $\mathbf{G}$!

- Because $G^c_{jk}(\mathbf{y}, \mathbf{x}) = 0$ for $\mathbf{y}$ on Σ, one replaces $S \cup \Sigma$ with S in the above integrals!

- Additional general property: $G^c_{jk}(\mathbf{x}, \mathbf{y}) = G^c_{kj}(\mathbf{y}, \mathbf{x})$ under the condition $G^c_{jk}(\mathbf{x}, \mathbf{y}) = 0$ on Σ

Green tensor G^c for the spherical cavity

Obtained (in a different form not suitable for numerics) by Oseen 1927!

- Pole $\mathbf{y}$ and obervation point $\mathbf{x}$.

$$\mathbf{y}' = \frac{R^2\mathbf{y}}{|\mathbf{y}|^2},\ \mathbf{t} = \frac{\mathbf{y}}{|\mathbf{y}|},\ \mathbf{a} = \mathbf{x} - (\mathbf{x}.\mathbf{t})\mathbf{t},\ \mathbf{h} = \frac{|\mathbf{y}|}{R}(\mathbf{x} - \mathbf{y}'),\ h = |\mathbf{h}|$$

$$G^c_{jk}(\mathbf{x},\mathbf{y}) = G^\infty_{jk}(\mathbf{x},\mathbf{y}) - \frac{\delta_{jk}}{h} - \frac{(\mathbf{x}.\mathbf{e}_j)(\mathbf{x}.\mathbf{e}_k)}{h^3} + \frac{(\mathbf{t}.\mathbf{e}_j)(\mathbf{t}.\mathbf{e}_k)}{h}[\frac{|\mathbf{x}|^2}{h^2} - 1]$$
$$-[\frac{2|\mathbf{y}|\mathbf{t}.\mathbf{x}}{h^3}](\mathbf{t}.\mathbf{e}_j)(\mathbf{t}.\mathbf{e}_k) + |\mathbf{y}|[\frac{(\mathbf{t}.\mathbf{e}_j)(\mathbf{x}.\mathbf{e}_k) + (\mathbf{t}.\mathbf{e}_k)(\mathbf{x}.\mathbf{e}_j)}{h^3}]$$
$$-\frac{[|\mathbf{x}|^2 - R^2][|\mathbf{y}|^2 - R^2]}{2}\{\frac{\delta_{jk}}{R^3h^3} - \frac{3}{R^2}[\frac{(\mathbf{h}.\mathbf{e}_j)(\mathbf{h}.\mathbf{e}_k)}{h^5}]$$
$$-2\frac{\mathbf{t}.\mathbf{e}_k}{R^2}[\frac{\mathbf{t}.\mathbf{e}_j}{h^3} - \frac{3(\mathbf{h}.\mathbf{e}_j)(\mathbf{h}.\mathbf{t})}{h^5}] + \frac{3E}{R^4h}[\delta_{jk} - (\mathbf{t}.\mathbf{e}_k)(\mathbf{t}.\mathbf{e}_j)]$$
$$+\frac{3\mathbf{a}.\mathbf{e}_k}{R}[-\frac{E}{R^3h}\{\frac{|\mathbf{y}|\mathbf{h}.\mathbf{e}_j}{Rh^2} + \frac{2\mathbf{a}.\mathbf{e}_j}{|\mathbf{a}|^2}\} + \frac{\mathbf{E}.\mathbf{e}_j}{R^4h^2[|\mathbf{x}|^+_-(\mathbf{x}.\mathbf{t}))]} + \mathbf{a}.\mathbf{e}_j[\frac{(2R^2)^+_-|\mathbf{y}||\mathbf{x}|}{R^4h^2|\mathbf{a}|^2}]]\}$$

$$E = \{|\mathbf{x}|^+_-\frac{2R^2\mathbf{x}.\mathbf{t}}{R^2 + Rh^-_+|\mathbf{x}||\mathbf{y}|}\}/\{|\mathbf{x}|^+_-\mathbf{x}.\mathbf{t}\}, \mathbf{E} =^-_+ |\mathbf{y}|\mathbf{x} + [|\mathbf{y}||\mathbf{x}|^+_-(1^-_+2)R^2]\mathbf{t}^+_-2[\frac{2R^2|\mathbf{y}|\mathbf{x} + [R^3h^-_+R^2|\mathbf{y}||\mathbf{x}|]\mathbf{t}}{R^2 + Rh^-_+|\mathbf{y}||\mathbf{x}|}]$$

with upperscripts or subscripts for $\mathbf{x}.\mathbf{t} \geq 0$ or $\mathbf{x}.\mathbf{t} < 0$, respectively

Numerical strategy

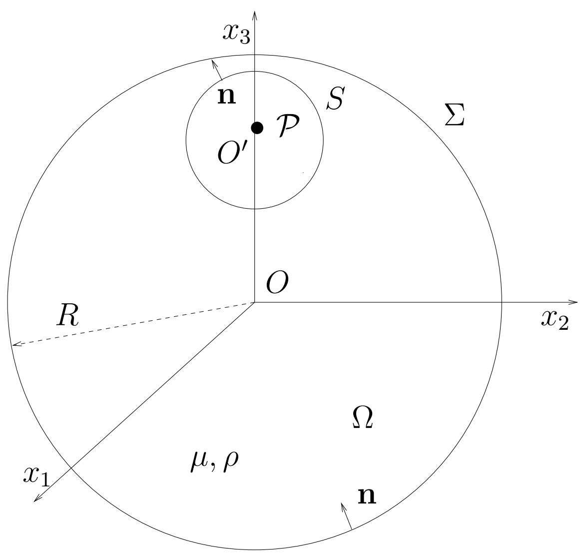
- Isoparametric triangular curvilinear Boundary Elements on S and, if needed, on the cavity Σ
- Discretize each boundary-integral equation. This requires to accurately deal with the case of a source $\mathbf{x}$ on a boundary element (a refined treatment is needed with the use of local polar coordinates)
- Solve each resulting linear systems $AX = Y$ by Gaussian elimination
- The use of $\mathbf{G}^c$ permits one to solely mesh the particle's surface (worth for a large cavity)

Benchmarks are needed!

- As seen before, $\mathbf{G}^c$ is available for a spherical cavity
- Comparisons with both analytical and numerical results for a spherical particle (previously-mentioned literature)
- Sphere located or not located at the cavity center

Case of a spherical particle

Adopted notations



- A spherical cavity with center O and radius R
- A spherical particle with radius a and center O'

$$\mathbf{OO'} = d\mathbf{e}_3 \quad \text{and } 0 \leq d < R - a$$

- $R - (d + a)$ is the sphere-cavity gap
- Normalized sphere-cavity gap $\eta = (R - d - a)/a$

Numerical comparisons for a sphere located at the cavity center

- Here $O = O'$ and $d = 0$. Sphere with radius $a < R$ translating at the velocity $\mathbf{e}_i$.

$$\mathbf{F} = -6\pi\mu a c(a/R)\mathbf{e}_i, \qquad \mathbf{\Gamma} = \mathbf{0}$$

- Analytical formula for the occurring dimensionless resistance coefficient c

$$c(\beta) = \frac{1-\beta^5}{1 - \frac{9\beta}{4} + \frac{5\beta^3}{2} - \frac{9\beta^5}{4} + \beta^6}, \qquad \beta = a/R < 1.$$

- A $N-$*node* mesh on the sphere and, if needed, 1058 nodal points on the cavity Σ

Two computed values of the above coefficient c

- c_s : using the Green $\mathbf{G}^\infty$ and putting Stokeslets on both S and Σ
- c_c : using the Green tensor $\mathbf{G}^c$ and Stokeslets on S
- Notation: $\Delta c_l = |c_l/c - 1|$

A translating sphere
located at the cavity center

N	R/a	c_s	Δc_s	c_c	Δc_c
74	1.1	3258.137	1.00613	2097.155	0.29128
242	1.1	2124.983	0.30842	1949.547	0.01030
1058	1.1	1777.331	0.09436	1676.260	0.00353
exact	1.1	1624.089	0	1624.089	0
74	2.	7.223525	0.00968	7.218993	0.01030
242	2.	7.289179	0.00068	7.284937	0.00126
1058	2.	7.297493	0.00046	7.293273	0.00012
exact	2.	7.294118	0	7.294118	0
74	5.	1.749799	0.00344	1.749640	0.00353
242	5	1.755232	0.00035	1.755073	0.00044
1058	5.	1.755937	0.00005	1.755777	0.00004
exact	5.	1.755845	0	1.755845	0

Computed quantities $c_s, \Delta c_s, c_c$ and Δc_c
versus the number N of collocation points on S

Arbitrarily-located sphere

- Here $\mathbf{OO'} = d\mathbf{e}_3$ with $0 \leq d < R - a$.

For symmetry reasons one confines the attention to four cases.

- (i) A sphere translating at the velocity $\mathbf{e_1}$: $\mathbf{F} = -6\pi\mu a c_1 \mathbf{e}_1$ and $\mathbf{\Gamma} = 8\pi\mu a^2 s \mathbf{e}_2$
- (ii) A sphere translating at the velocity $\mathbf{e_3}$: $\mathbf{F} = -6\pi\mu a c_3 \mathbf{e}_3$ and $\mathbf{\Gamma} = \mathbf{0}$
- (iii) A sphere rotating at the velocity $\mathbf{e_1}$: $\mathbf{F} = -8\pi\mu a^2 s \mathbf{e}_2$ and $\mathbf{\Gamma} = -8\pi\mu a^3 t_1 \mathbf{e}_1$
- (iv) A sphere rotating at the velocity $\mathbf{e_3}$: $\mathbf{F} = \mathbf{0}$ and $\mathbf{\Gamma} = -8\pi\mu a^3 t_3 \mathbf{e}_3$

Comparisons for the computed coefficients c_1, c_3, t_1, t_3 and s

- Accurate computations obtained elsewhere by using the bipolar coordinates (Jones 2008, here labelled Jones in each reported table)
- $R = 4a$ and two values of the normalized gap $\eta = (R - d - a)/a$ are selected: $\eta = 0.5$ and $\eta = 0.1$ (small sphere-cavity gap).
- 4098 nodal points are put on the cavity Σ when using the Green tensor $\mathbf{G}^\infty$

Comparisons for a sphere not located at the cavity center with $\eta = (R - d - a)/a = 0.5$

N	Method	c_1	c_3	t_1	t_3	s
74	$\mathbf{G}^\infty$	2.6330	4.6730	1.1640	1.0789	0.11870
74	$\mathbf{G}^c$	2.6327	4.6714	1.1639	1.0789	0.11861
242	$\mathbf{G}^\infty$	2.6473	4.7107	1.1639	1.0755	0.11927
242	$\mathbf{G}^c$	2.6471	4.7090	1.1639	1.0755	0.11920
1058	$\mathbf{G}^\infty$	2.6488	4.7144	1.1639	1.0755	0.11938
1058	$\mathbf{G}^c$	2.6486	4.7127	1.1639	1.0755	0.11932
Jones	Bipolar	2.6487	4.7131	1.1639	1.0755	0.11933

Comparisons for a sphere not located at the cavity center with $\eta = (R - d - a)/a = 0.1$

N	Method	c_1	c_3	t_1	t_3	s
74	$\mathbf{G}^{\infty}$	3.9016	15.552	1.6065	1.1960	0.20206
74	$\mathbf{G}^{c}$	3.9009	15.413	1.6052	1.1960	0.20138
242	$\mathbf{G}^{\infty}$	3.9273	18.886	1.6145	1.1939	0.19108
242	$\mathbf{G}^{c}$	3.9237	18.636	1.6134	1.1938	0.19001
1058	$\mathbf{G}^{\infty}$	3.9159	18.832	1.6171	1.1945	0.18494
1058	$\mathbf{G}^{c}$	3.9121	18.711	1.6160	1.1945	0.18353
Jones	Bipolar	3.9121	18.674	1.6163	1.1945	0.18344

Numerical results for a non-spherical particle

- Ellipsoid with semi-axis (a_1, a_2, a_3) and surface admitting the equation

$$(x_1/a_1)^2 + (x_2/a_2)^2 + ([x_3 - d]/a_3)^2 = 1$$

- Ellipsoid-cavity normalized separation parameter λ with

$$0 < \lambda = d/a_3 < (R - a_3)/a_3$$

- 8 friction coefficients c_i, t_i, s_1 and s_2 such that

$$\begin{aligned}
&\mathbf{A}_T^{(i)} = 6\pi\mu a_3 c_i \mathbf{e}_i,\ \mathbf{B}_R^{(i)} = 8\pi\mu a_3^3 t_i \mathbf{e}_i,\\
&\mathbf{B}_T^{(1)} = -8\pi\mu a_3^2 s_1 \mathbf{e}_2,\ \mathbf{B}_T^{(2)} = 8\pi\mu a_3^2 s_2 \mathbf{e}_1,\ \mathbf{B}_T^{(3)} = \mathbf{0},\\
&\mathbf{A}_R^{(1)} = 8\pi\mu a_3^2 s_2 \mathbf{e}_2,\ \mathbf{A}_R^{(2)} = -8\pi\mu a_3^2 s_1 \mathbf{e}_1,\ \mathbf{A}_R^{(3)} = \mathbf{0}
\end{aligned}$$

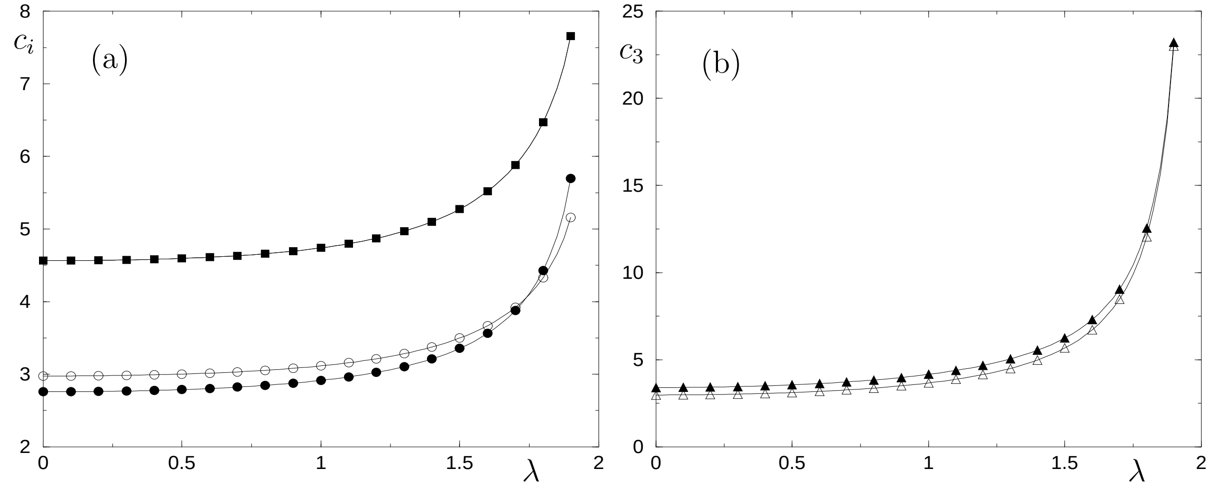
Comparisons for two selected ellipsoids

- A sphere with radius a_3 (clear symbols)
- The ellipsoid $a_1 = 5a_3/3, a_2 = 0.6a_3$

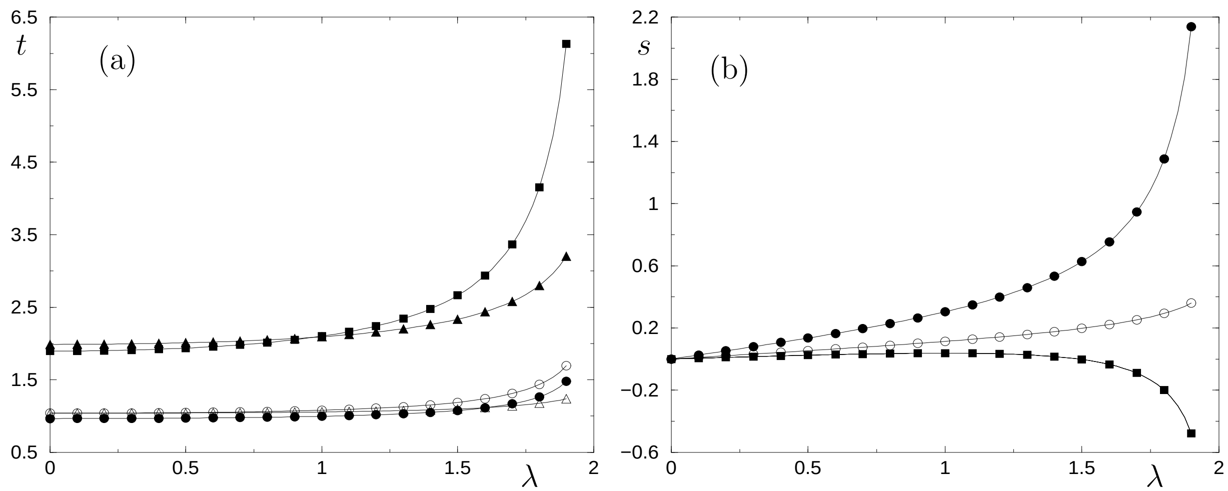
having the same volume as the sphere (filled symbols)

Friction coefficients

Normalized coefficients c_i for the sphere (clear symbols) and the ellipsoid (filled symbols).



(a) Coefficients c_1 (circles) and c_2 (squares). (b) Coefficients c_3 (triangles)



(a) Coefficients t_1 (circles), t_2 (squares) and t_3 (triangles). (b) Coefficients s_1 (circles) and s_2 (squares)

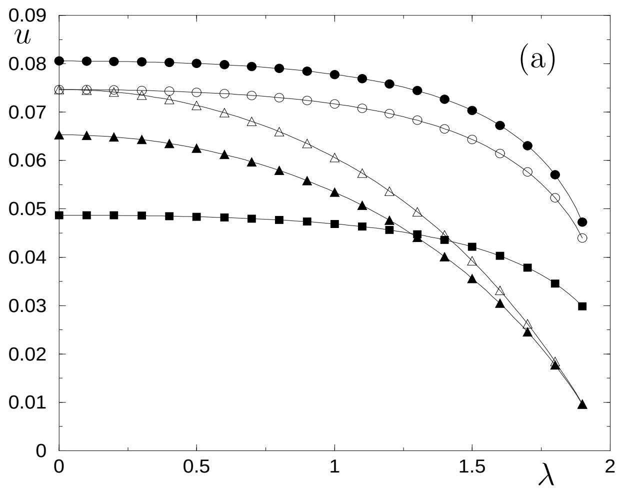
Settling normalized translational and angular velocities

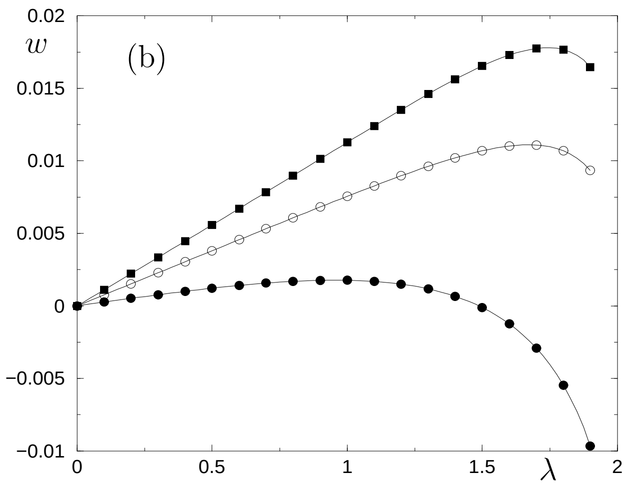
Setting $U'_s = (\rho_s - \rho)a^2 g/\mu$ one gets

$$(i)\ \text{If } \mathbf{g} = g\mathbf{e}_1 : \ \mathbf{U} = U'_s u_1 \mathbf{e}_1,\ \mathbf{W} = aU'_s w_2 \mathbf{e}_2$$

$$(ii)\ \text{If } \mathbf{g} = g\mathbf{e}_2 : \ \mathbf{U} = U'_s u_2 \mathbf{e}_2,\ \mathbf{W} = -aU'_s w_1 \mathbf{e}_1$$

$$(iii)\ \text{If } \mathbf{g} = g\mathbf{e}_3 : \ \mathbf{U} = U'_s u_3 \mathbf{e}_3,\ \mathbf{W} = \mathbf{0}$$





Normalized velocities for the sphere (clear symbols) and the ellipsoid (filled symbols).
(a) Translational velocities u_1 (circles), u_2 (squares) and u_3 (triangles).
(b) Angular velocities w_1 (circles) and w_2 (squares)

Concluding remarks

• A new approach based on a boundary-integral formulation

• Valid for arbitrarily-shaped particles!

• Easy implementation and nicely retrieves for a spherical particle results obtained elsewhere using a quite different (bipolar coordinates) approach

• Two tested approaches resorting to the free-space Green tensor and the Green tensor complying with the no-slip condition on the motionless spherical cavity

• The second one makes it possible to solely mesh the particle surface and offers more accurate results

• Numerical results reveal that a particle behaviour is slightly sensitive to its shape

• In future: cope with the challenging case of a collection of particles!