LINEAR STABILITY OF THE LAMINAR FLOW IN THE CHANNEL WITH TRANSVERSELY CORRUGATED WALLS

PART II: TRANSIENT GROWTH OF DISTURBANCES

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WHY IS THE NORMAL MODE ANALYSIS INSUFFICIENT ?

1. The NMA is relevant only for the long-time (asymptotic) behavior of the disturbance field.

2. If large transient amplification of disturbances is possible then small perturbations of the flow may quickly rise up to the magnitude which is sufficient to trigger nonlinear effects. Then the flow will probably evolve to a new (laminar or chaotic) state, even though the Reynolds number Re_L is subcritical and all normal modes of the undisturbed motion are nominally stable.

SO WHAT ARE WE GOING TO DO?

1. We will calculate the form of the "most dangerous" (or optimal) disturbances, i.e. such that give rise to the largest possible amplification.

2. We will carry out the parametric study of the transient growth phenomenon.

3. We will look at the structures in the disturbance velocity field accompanying the transient growth process.

TRANSIENT GROWTH IN LINEAR DYNAMICAL SYSTEMS (I)

Linear ODE system:
$$\tfrac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} \ , \ \ \mathbf{x}(0) = \mathbf{x}_0 \ ,$$

Solution:
$$\mathbf{x}(t) = \exp(\mathbf{A}t)\,\mathbf{x}_0 \quad , \quad \exp(\mathbf{A}t) = \sum_{k-0}^{\infty} \tfrac{1}{k!}\mathbf{A}^k t^k \ .$$

Assume A is diagonalizable
$$A = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1} \ , \quad \mathbf{V} = [\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\,...\,|\, \mathbf{v}_N] \quad , \quad \boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1,..,\lambda_N\}$$

Then
$$\mathbf{x}(t) = \mathbf{V}\exp(\boldsymbol{\Lambda}t)\mathbf{V}^{-1}\mathbf{x}_0 = \mathbf{V}\exp(\boldsymbol{\Lambda}t)\,\mathbf{y}_0 \quad , \quad \exp(\boldsymbol{\Lambda}t) = \mathrm{diag}\{e^{\lambda_1 t}, e^{\lambda_2 t},.., e^{\lambda_{N_A} t}\}.$$

The magnitude (energy norm) of the solution
$$e(t) \equiv \left\|x(t)\right\|_E^2 = \left\langle \mathbf{x}(t), \mathbf{E}\mathbf{x}(t)\right\rangle = \left\langle \mathbf{y}_0, \mathbf{M}(t)\mathbf{y}_0\right\rangle$$

where
$$\mathbf{M}(t) = \exp(\boldsymbol{\Lambda}^H t)\underbrace{\mathbf{V}^H\mathbf{E}\mathbf{V}}_{\mathbf{M}_0}\exp(\boldsymbol{\Lambda}t) = \exp(\boldsymbol{\Lambda}^H t)\mathbf{M}_0 \exp(\boldsymbol{\Lambda}t)$$

TRANSIENT GROWTH IN LINEAR DYNAMICAL SYSTEMS (2)

$$\mathrm{Re}(\lambda_k) < 0\,,\;\; k = 1,2,...,N \quad\rightarrow\quad \lim_{t\to 0} x(t) = 0 \quad\rightarrow\quad x(t) \equiv 0 \;\;\text{is asymptotically stable}$$

THE „MOST DANGEROUS" (OR OPTIMAL) INITIAL CONDITIONS

For a given t = τ find such y_0 that $e(0) = \langle \mathbf{y}_0, \mathbf{M}_0 \mathbf{y}_0 \rangle = 1$ and $e(\tau) = \langle \mathbf{y}_0, \mathbf{M}(\tau)\,\mathbf{y}_0 \rangle$ is the largest possible.

Solution of the constrained optimization problem

Extended functional
$$\boldsymbol{F}(\mathbf{y}_0) = \langle \mathbf{y}_0, \mathbf{M}(\tau)\mathbf{y}_0 \rangle - \varepsilon \big[\langle \mathbf{y}_0, \mathbf{M}_0 \mathbf{y}_0 \rangle - 1 \big]$$

The variation of F
$$\delta \boldsymbol{F}(\mathbf{y}_0, \mathbf{w}) = 2\,\mathrm{Re}\langle \mathbf{w}, [\mathbf{M}(\tau) - \mu \mathbf{M}_0]\mathbf{y}_0 \rangle$$

δF vanishes for all vectors **w** iff
$$\mathbf{M}(\tau)\,\mathbf{y}_0 = \varepsilon \mathbf{M}_0\,\mathbf{y}_0$$

The optimal I.C. $\mathbf{x}_0^m = \mathbf{V}\mathbf{y}_0^m$, where $\mathbf{y}_0^m$ is the eigenvector with the largest eigenvalue ε_m. Moreover

$$\sup_{\mathbf{x}_0 : \langle \mathbf{x}_0, \mathbf{E}\mathbf{x}_0 \rangle = 1} e(\tau) = \langle \mathbf{y}_0^m, \mathbf{M}(\tau)\,\mathbf{y}_0^m \rangle = \varepsilon_m$$

TRANSIENT GROWTH IN LINEAR DYNAMICAL SYSTEMS (3)

ANALYSIS OF THE ,,TOY PROBLEM''

REAL EIGENVALUES $0 > \lambda_2 > \lambda_1$ **AND EIGENVECTORS** $\mathbf{v}_1$ **AND** $\mathbf{v}_2$ **ARE GIVEN.**

General solution
$$\mathbf{x}(t) = C_1 \mathbf{v}_1 e^{\lambda_1 t} + C_2 \mathbf{v}_2 e^{\lambda_2 t}$$

If $x(0) = x_0$ then
$$C_1 = \frac{\langle \mathbf{x}_0, \mathbf{v}_1 \rangle - p\langle \mathbf{x}_0, \mathbf{v}_2 \rangle}{1 - p^2} \quad , \quad C_2 = \frac{\langle \mathbf{x}_0, \mathbf{v}_2 \rangle - p\langle \mathbf{x}_0, \mathbf{v}_1 \rangle}{1 - p^2} \quad ,$$

where $p = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ is the inner product of the eigenvectors.

Assume $\mathbf{E} = \mathbf{I}$. Then the norm is defined as
$$e(t) \equiv \|\mathbf{x}(t)\|_2^2 = \langle \mathbf{x}(t), \mathbf{x}(t) \rangle$$

The matrices $\mathbf{M}_0$ and $\mathbf{M}(\tau)$:
$$M_0 = \begin{bmatrix} 1 & p \\ p & 1 \end{bmatrix} \quad , \quad M(\tau) = \begin{bmatrix} e^{2\lambda_1 \tau} & p\,e^{(\lambda_1 + \lambda_2)\tau} \\ p\,e^{(\lambda_1 + \lambda_2)\tau} & e^{2\lambda_2 \tau} \end{bmatrix}.$$

TRANSIENT GROWTH IN LINEAR DYNAMICAL SYSTEMS (4)

ANALYSIS OF THE ,,TOY PROBLEM'' (CONTINUED …)

Let's fix t = τ. The largest root of the characteristic polynomial $\det\left[\mathbf{M}(\tau) - \mu\,\mathbf{M}_0\right] = 0$ is

$$\mu_m = \frac{e^{2\lambda_1\tau} - 2p^2 e^{(\lambda_1+\lambda_2)\tau} + e^{2\lambda_2\tau} + (e^{\lambda_2\tau} - e^{\lambda_1\tau})\sqrt{(e^{\lambda_2\tau} - e^{\lambda_1\tau})^2 - 4p^2 e^{(\lambda_1+\lambda_2)\tau}}}{2(1-p^2)}$$

The optimal initial condition $\mathbf{y}_0^m$ can be determined:

$$\frac{y_{0,1}^m}{y_{0,2}^m} = -p\,\frac{e^{(\lambda_1+\lambda_2)\tau} - \mu_m}{e^{2\lambda_1\tau} - \mu_m} \quad , \qquad \left(y_{0,1}^m\right)^2 + 2p\,y_{0,1}^m y_{0,2}^m + \left(y_{0,2}^m\right)^2 = 1 \quad \text{(normalization)}.$$

TRANSIENT GROWTH IN LINEAR DYNAMICAL SYSTEMS (5)

ANALYSIS OF THE ,,TOY PROBLEM'' (CONTINUED …)

CASE 1: The eigenvectors are orthogonal

For **p = 0** eigenvectors are **orthogonal**, the matrix A is **symmetric** (hence, the operator exp(At) is **normal**) and $\mu_m = e^{2\lambda_2 \tau}$. We also have $y_{0,1}^m = 0$ and $y_{0,2}^m = 1$.

If eigenvectors are orthogonal then e(t) diminishes monotonically – no transient growth occurs!

CASE 2: Multiply regular eigenvalue

Assume $\lambda_1 = \lambda_2 = \lambda < 0$ and two linearly independent eigenvectors exist. Then

$$\mu_m = e^{2\lambda \tau},$$

while the optimal IC is not uniquely determined.

The norm e(t) decays monotonically to zero and no transient growth is observed.

TRANSIENT GROWTH IN LINEAR DYNAMICAL SYSTEMS (6)

ANALYSIS OF THE ,,TOY PROBLEM" (CONTINUED …)

NONORTHOGONALITY OF THE EIGENSOLUTIONS OR DEGENERATED EIGENVALUES IS NECESSARY FOR THE TRANSIENT GROWTH TO APPEAR.

Example: $\lambda_1 = -1,\ \lambda_2 = -0.01$.
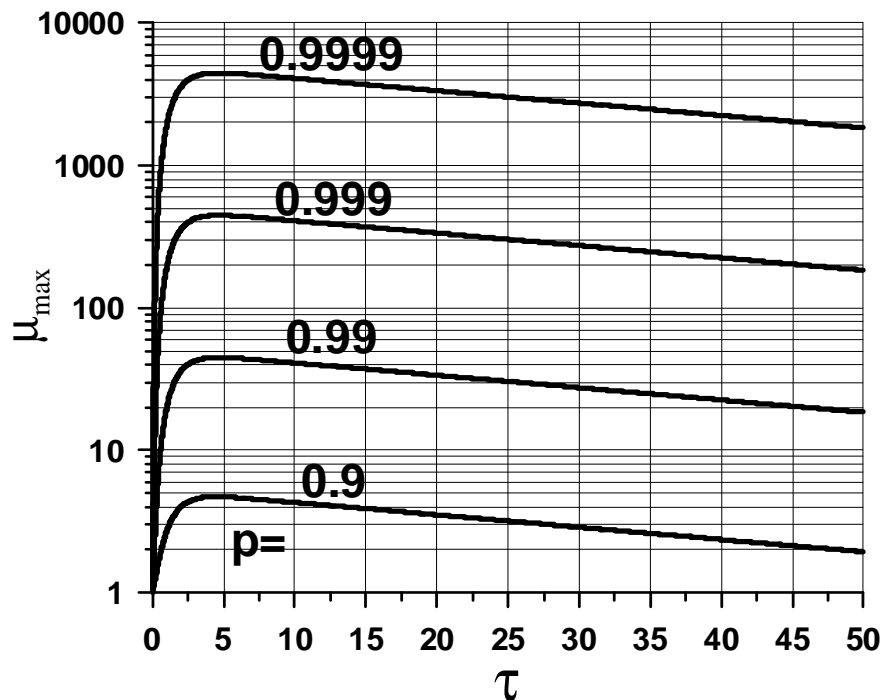
p	μ_{opt}	τ_{opt}	y_1^{opt}	y_2^{opt}
0.9	4.7204	4.435	2.0598	-2.2941
0.99	44.894	4.630	7.0165	-7.0888
0.999	446.757	4.650	22.3435	-22.3663
0.9999	4465.395	4.650	70.7053	-70.7125

For $p = 1 - \varepsilon$ the following asymptotic scaling holds

$$\mu_{opt} \cong O(\varepsilon^{-1})\ ,\quad |\, y_j^{opt}\,| \cong O(\varepsilon^{-1/2}).$$

The optimal time τ_{opt} converges to the constant depending on the largest eigenvalue λ_2.

TRANSIENT GROWTH IN LINEAR DYNAMICAL SYSTEMS (7)

ANALYSIS OF THE „TOY PROBLEM'' (CONTINUED …)

The mechanism of the transient growth
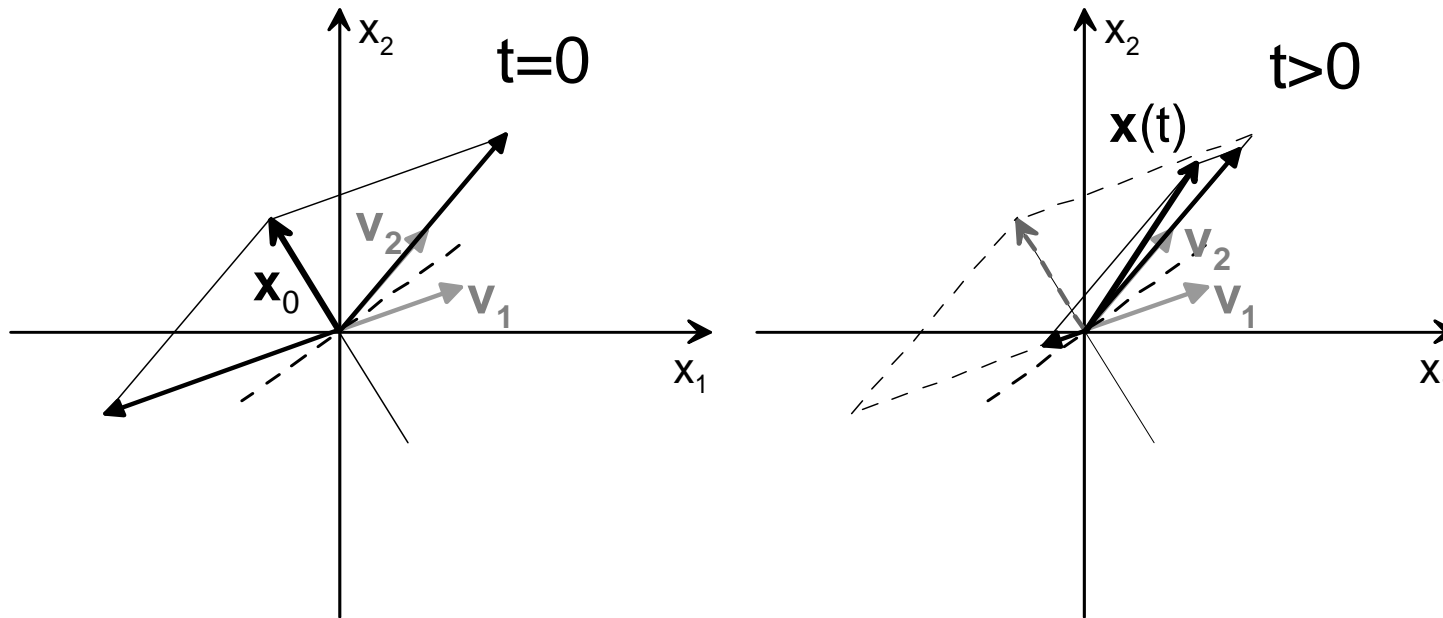
$$e(0) = 1 \qquad\qquad e(t) > e(0)$$

The growth is possible only if $|p| > \dfrac{2\sqrt{\lambda_1 \lambda_2}}{|\lambda_1 + \lambda_2|}$!

TRANSIENT ENERGY GROWTH IN THE POISEUILLE FLOW (1)

Disturbance velocity field:
$$\mathbf{v}(t, x, y, z) = [g_u, g_v, g_w](t, y) \exp[i(\delta x + \beta z)] + C.C.$$

Initial/boundary value problem (linear theory) is formulated as follows:

OS Eq.
$$\{\partial_t (\partial_{yy} - k^2) + i\beta[W_0(\partial_{yy} - k^2) - D^2 W_0] - \tfrac{1}{Re}(\partial_{yy} - k^2)^2\} g_v = 0,$$

Sq Eq.
$$[\partial_t + i\beta W_0 - \tfrac{1}{Re}(\partial_{yy} - k^2)]\theta = i\,\delta D W_0\, g_v, \qquad \leftarrow \textbf{\textcolor{red}{forcing term !}}$$

Bound.C.
$$g_v(t, \pm 1) = \partial_y g_v(t, \pm 1) = \theta(t, \pm 1) = 0,$$

Init. C.
$$g_v(0, y) = g_v^0(y) \;,\; \theta(0, y) = \theta^0(y) \,,$$

where
$$W_0(y) = 1 - y^2 \;,\; k = \sqrt{\delta^2 + \beta^2} \,.$$

TRANSIENT ENERGY GROWTH IN THE POISEUILLE FLOW (2)

The following special case in very instructive …

Assume that $\beta = 0$. Then all OS and Sq modes are attenuated and not moving, i.e. all eigenvalues are purely imaginary and have negative real parts. Consider the initial conditions:

$$g_v(0,y) = \hat{G}_v(y) \text{ - selected O-S mode (stable!)} \quad , \quad \theta(0,y) \equiv 0$$

The solution of the initial/boundary value problem can be written as

$$g_v(t,y) = \hat{G}_v(y)\exp(-\hat{\zeta}^{OS}t) \ ,$$

$$\theta(t,y) = i\,\delta\sum_{(j)}[\int_{-1}^{1} DW_0(y)\hat{G}_v(y)\Xi_j^*(y)\,dy]\frac{\exp(-\hat{\zeta}^{OS}t)-\exp(-\zeta_j^{Sq}t)}{\hat{\zeta}^{OS}-\zeta_j^{Sq}}\,\Xi_j(y),$$

where:

$$\hat{\sigma}^{OS} = -i\,\hat{\zeta}^{OS} \text{ - purely imaginary eigenvalue of the selected O-S mode, } \hat{\zeta}^{OS} > 0,$$

$$\sigma_j^{Sq} = -i\,\zeta_j^{Sq} \text{ - purely imaginary eigenvalue of all Sq modes, } \zeta_j^{Sq} > 0.$$

TRANSIENT ENERGY GROWTH IN THE POISEUILLE FLOW (3)

Special case continued …

For short times we can expand in the power series and get

$$\theta(t,y) = i\delta t \sum_{(j)} \left[\int_{-1}^{1} DW_0(y)\hat{G}_v(y)\Xi_j^*(y)dy \right] \Xi_j(y) + O(t^2) \approx i\delta DW_0(y)g_v(0,y)t + O(t^2)$$

For $\beta = 0$ we have $\theta(t,y) = -i\delta g_w(t,y)$ hence

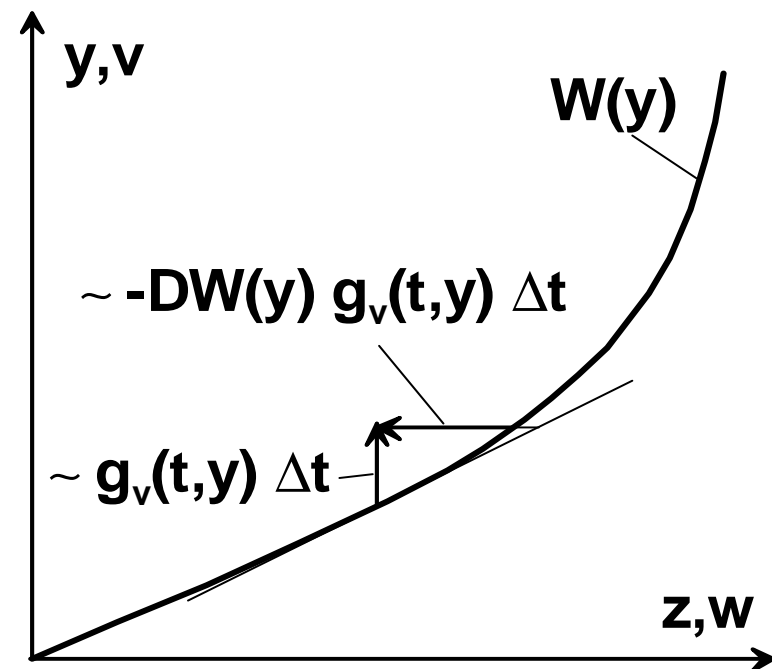$$g_w(t,y) \approx \underbrace{-DW_0(y)g_v(0,y)t}_{\text{TRANSIENT LINEAR GROWTH}} + O(t^2).$$

For a short time interval we have

$$g_w(t+\Delta t,y) \approx -DW_0(y)g_v(t,y)\Delta t,$$

which is interpreted physically as the

LIFT-UP EFFECT

TRANSIENT ENERGY GROWTH IN THE POISEUILLE FLOW (4)

The disturbance energy norm $\quad \varepsilon(t) = \frac{1}{2k^2} \int_{-1}^{1} \{ |\partial_y g_v|^2 + k^2 |g_v|^2 + |\theta|^2 \} dy$

Let's fix $t = \tau$ and find $\left\{ g_v^M(t,y), \theta^M(t,y) \right\}$ such that $\varepsilon(0) = 1$ and $\varepsilon(\tau)$ is the largest possible. The maximal disturbance energy depends on τ, i.e. $\varepsilon = \varepsilon_M(\tau)$. If Re < Re$_L$ then $\varepsilon_M(\tau)$ attains the maximum ε_{opt} at $\tau = \tau_{opt}$. The corresponding initial conditions are called **the optimal disturbances**.
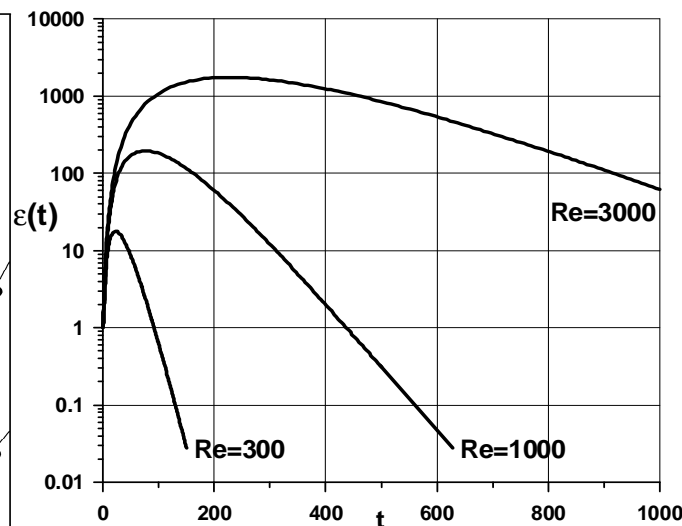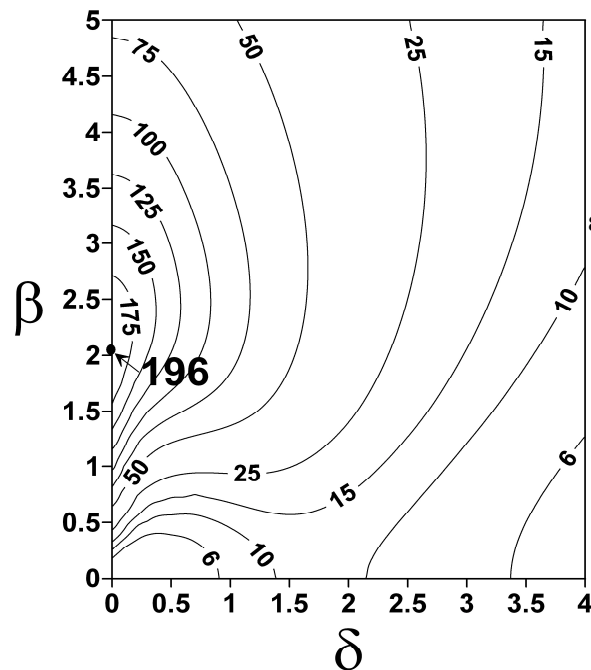
Left: the map of ε_{opt} as the function of δ and β for Re=1000.

Right: the energy of the disturbances evolving in time from the optimal IC, computed for different Reynolds numbers. The wave numbers are δ=0 and β=2.04 (optimal for all values of Re).

$$\varepsilon_{opt} \approx 1.96 \cdot 10^{-4} \, \mathrm{Re}^2 \quad , \quad \tau_{opt} \approx 0.076 \cdot \mathrm{Re}$$

TRANSIENT ENERGY GROWTH IN THE POISEUILLE FLOW (5)

Kinematical structure of the optimal disturbances
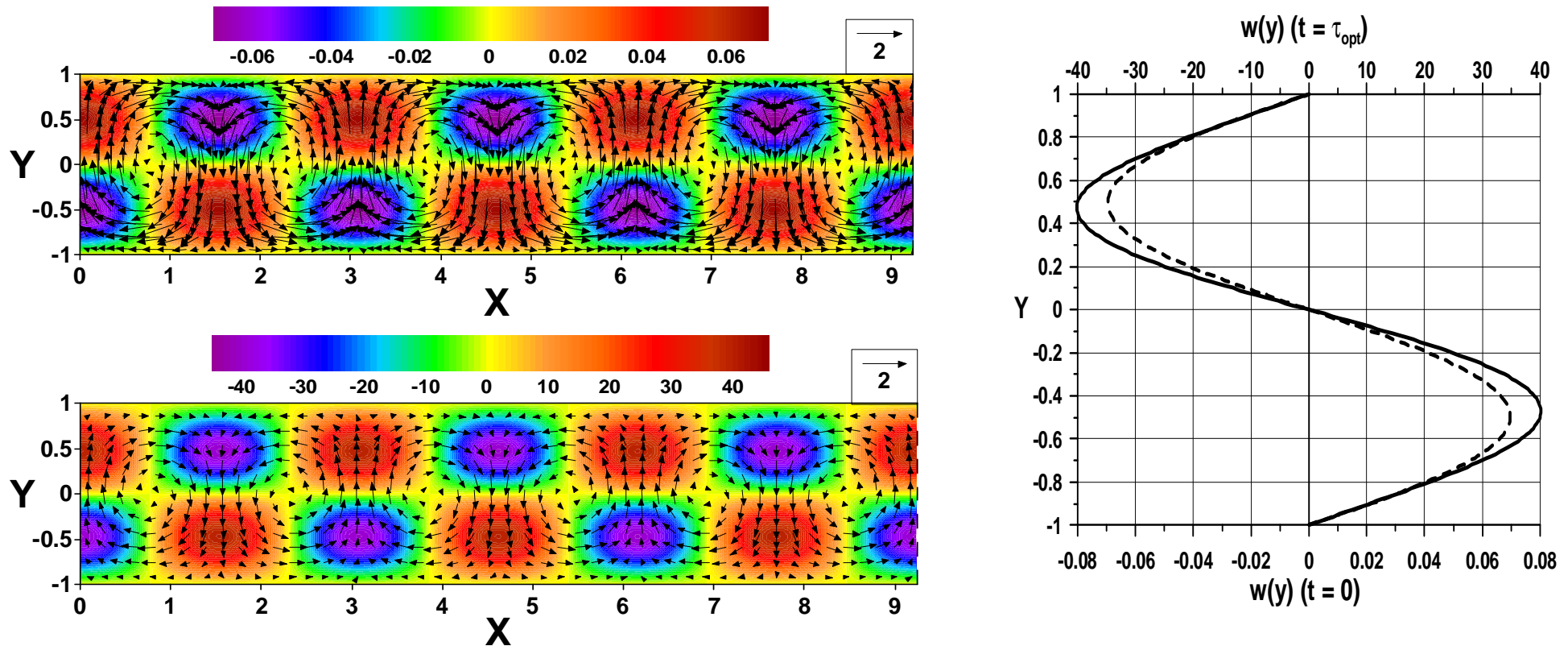
Left: the spanwise structure of the velocity field at the time t=0 (top) and t=τ_{opt}=76 (bottom), computed for Re=1000, δ=0 and β=2.04. The corresponding energy growth factor is 196.

Right: The velocity profile of the "streamwise streak" at the time t=0 and t=τ_{opt}=76. The streamwise velocity is amplified about 600 times (!). All parameters like for the contour maps.

TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (1)

General remarks:

(1) Each eigenmode in the wavy channel originates from a certain eigensolution of the Poiseuille flow with the wave vector $\kappa_m = [\kappa_x, 0, \kappa_z] = [\delta_* + m\alpha, 0, \beta]$, m = -M$_S$,..,0,..,M$_S$. We seek initial disturbances in the form of the linear combination of various spanwise Fourier harmonics.

(2) Theoretically (i.e. with M$_S$ approaching infinity) the ensemble of all eigenmodes is periodic with the Floquet number δ_*, and the period is equal to the wave number α. Thus, it is actually sufficient to consider δ_* in the range $[0, \alpha)$ or $[-\alpha/2, \alpha/2)$.

(3) Numerical calculations show that the largest transient growth is observed for the disturbances which are streamwise independent ($\beta = 0$), exactly like in the case of the Poiseuille flow.

(4) If $\beta = \delta_* = 0$ then the simplified description of the disturbance dynamics must be slightly changed. In such case the 0-th Fourier modes of the vertical velocity and vorticity components are zero. Thus, they cannot be used to evaluate the 0-th Fourier modes of the streamwise and spanwise components of the velocity – the latter are included into governing equations as the explicit unknowns. Also, in order to obtain well-posed problem two additional constrains must be imposed on the disturbance field. Here we assume that the mean values of the streamwise and spanwise pressure gradients remain unchanged.
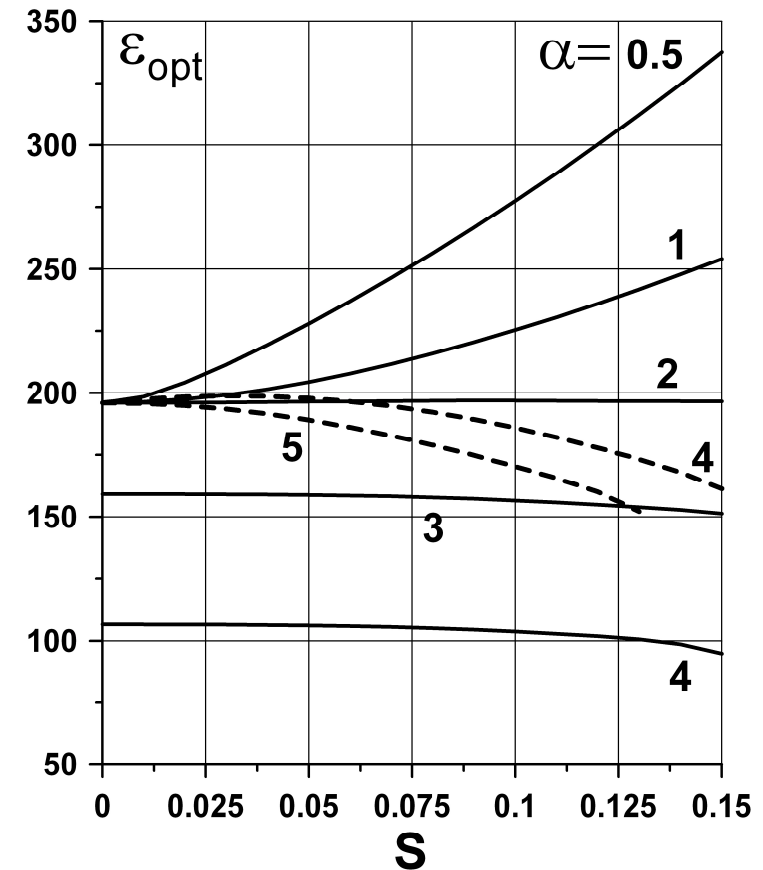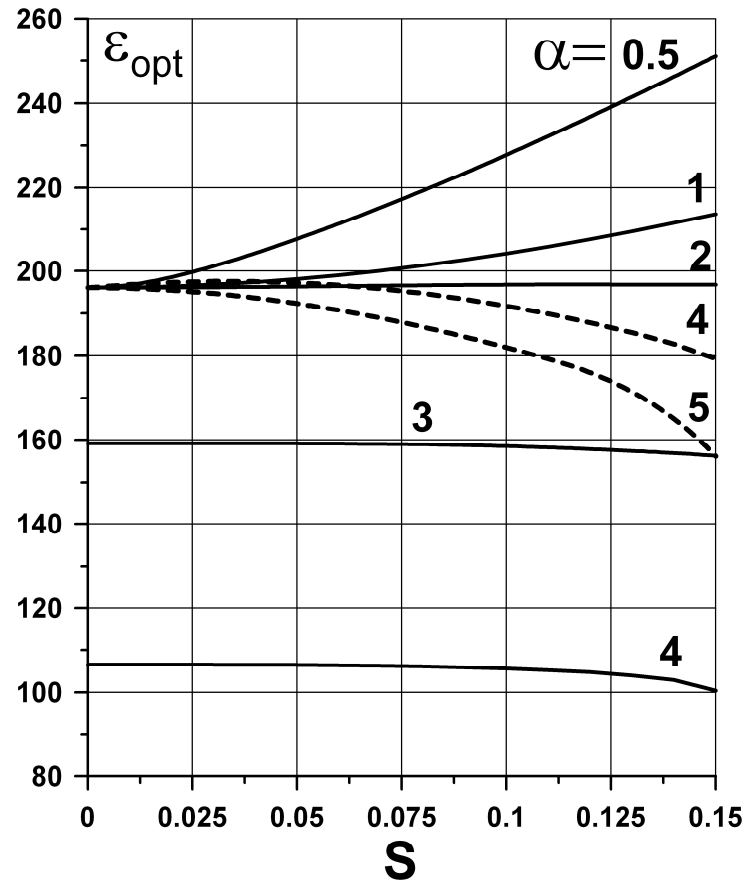
TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (2)
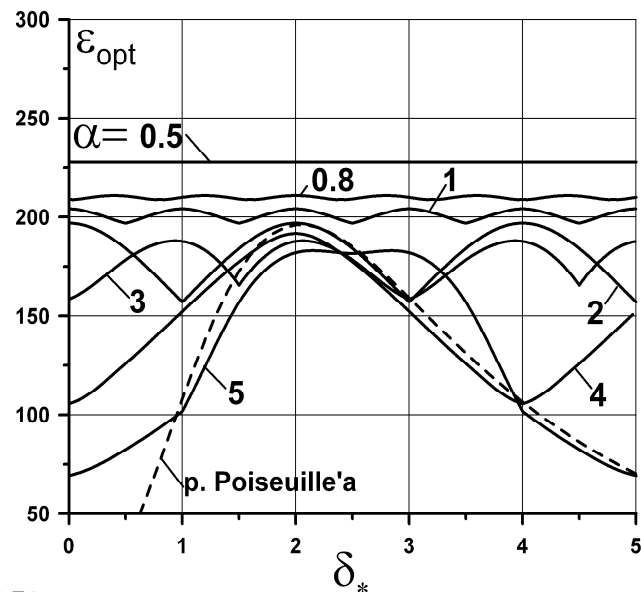
Optimal energy amplification (OEA) as the function of the Floquet parameter δ_*, computed for different values of the wall wave number α (Re = 1000, S=0.1). Dashed line corresponds to the referential flow with the same Reynolds number and the spanwise wane number equal $\delta = \delta_*$.
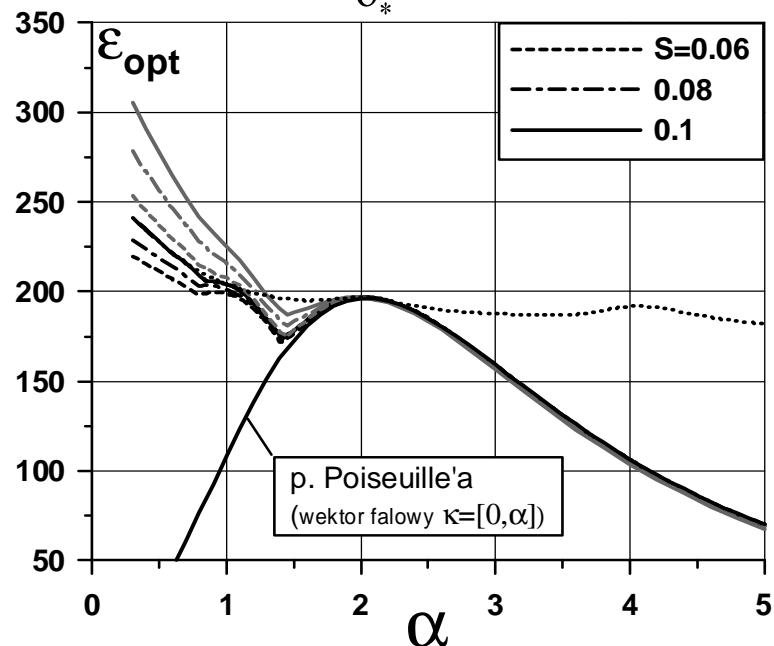
OEA as the function of the wall wave number α computed for different amplitudes of the corrugation of the bottom (black lines) and both walls (symmetric case, gray lines). Reynolds number Re=1000, the Floquet parameter $\delta_* = 0$. The dot line shows the value of the optimal amplification obtained with the Floquet parameter adjusted in such a way that the Fourier mode with the spanwise wave number equal 2 is always present.
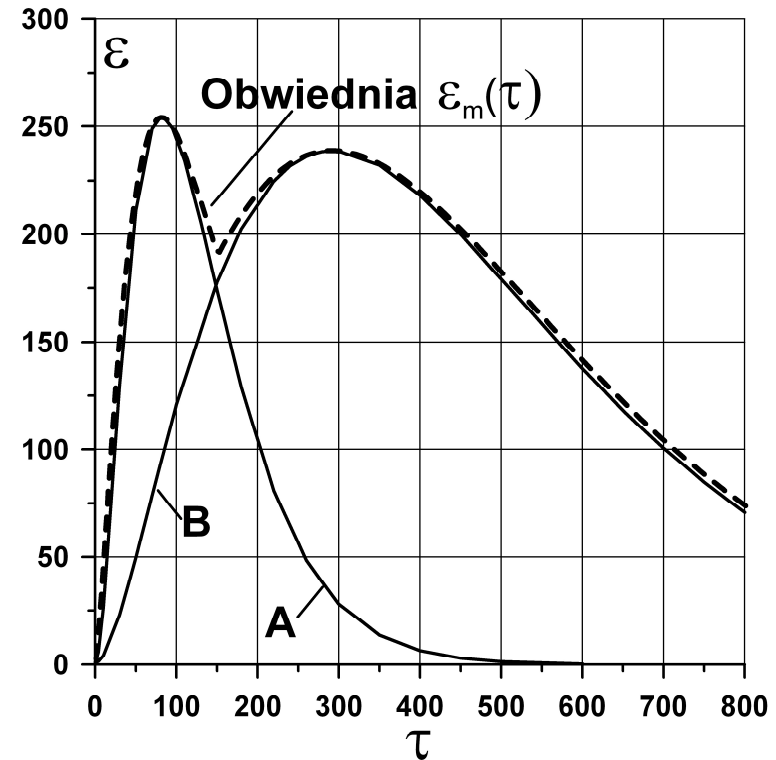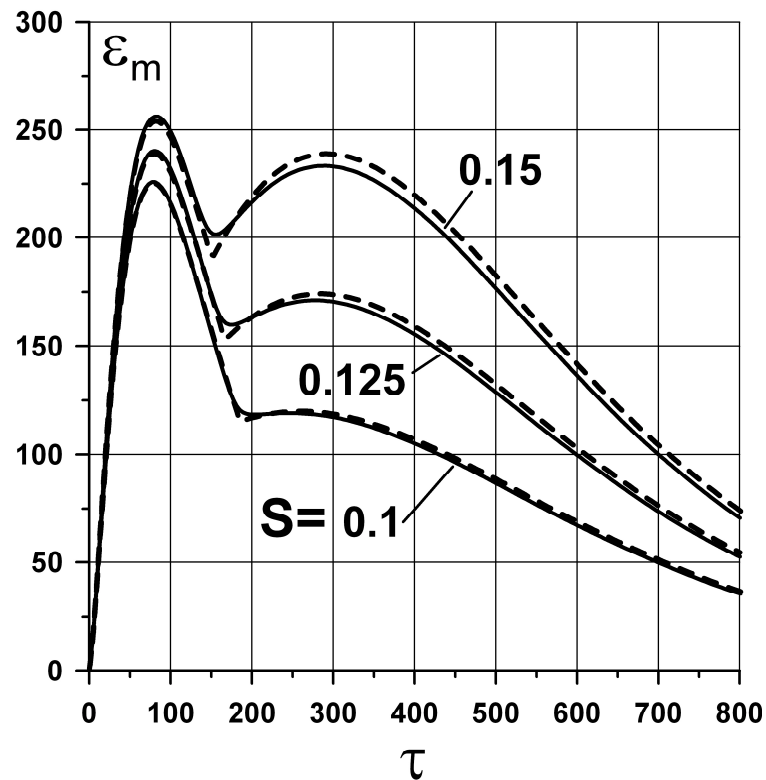
TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (3)

OEA as the function of the amplitude S computed for 1-sided (left) and 2-sided symmetric (right) wall corrugation with different geometric period. The Reynolds number Re=1000. The Floquet parameter δ_* is either 0 (continuous lines) or 2 (dashed lines).
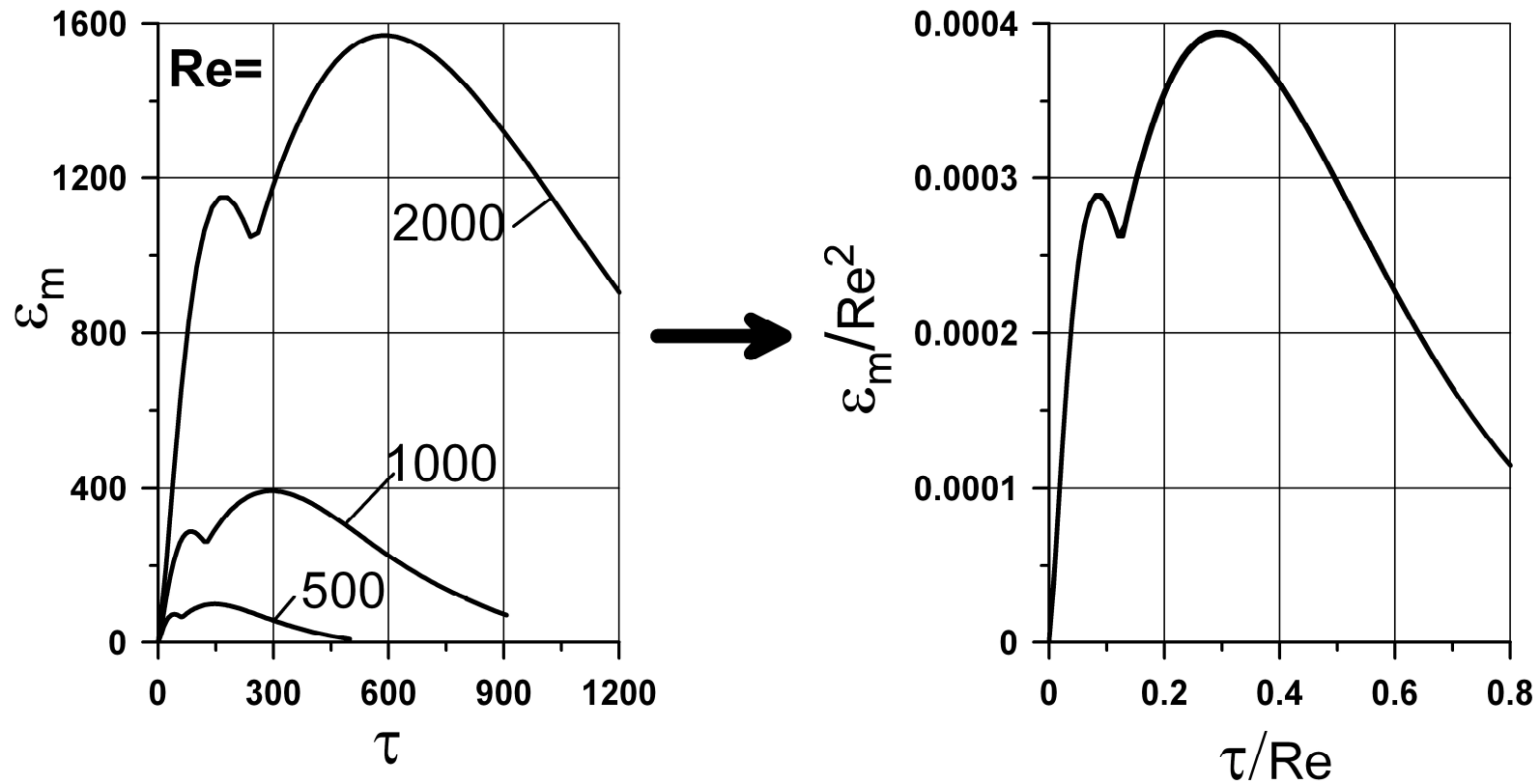
TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (4)

Left: the maximal energy amplification as the function of the time instant τ computed for Re=1000, α=1, δ_*=0 and different values of the amplitude S (2-sided symmetric wavy walls)

Right: time histories of the disturbance energy computed for the initial conditions corresponding to local maxima of the envelope function $\varepsilon = \varepsilon_M(\tau)$.
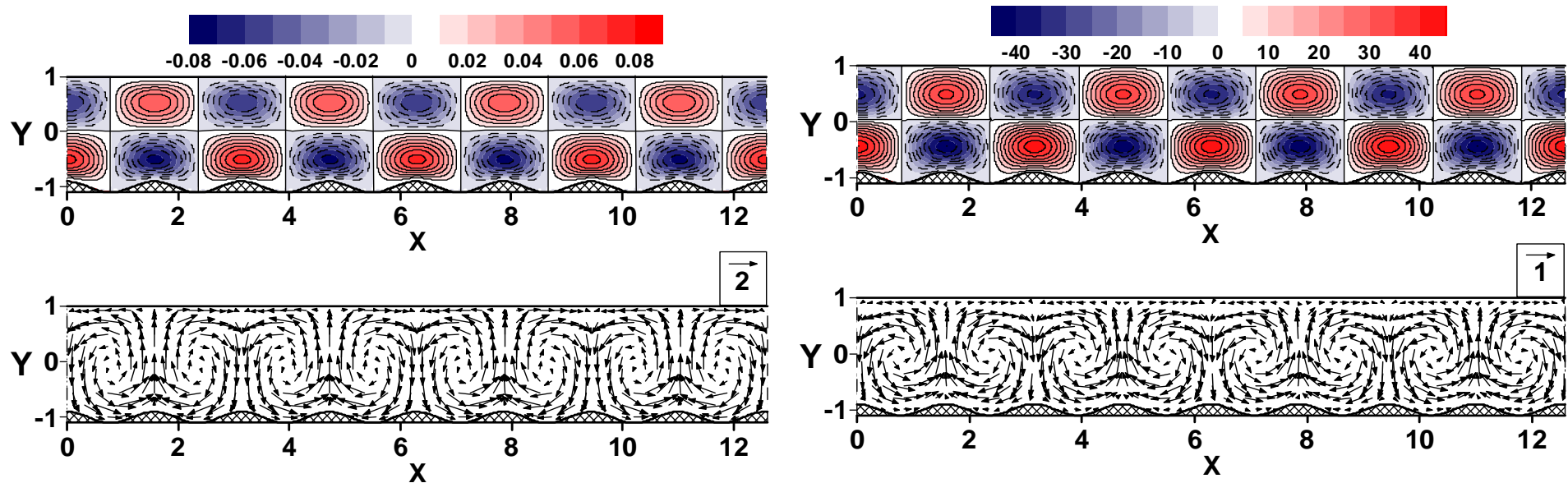
TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (5)

The universal scaling of the maximal energy growth envelope function $\varepsilon = \varepsilon_M(\tau)$ with respect to the Reynolds number (essentially the same as for the Poiseuille flow) (2-sided symmetric wall corrugation with S=0.2, α=1 and δ*=0).
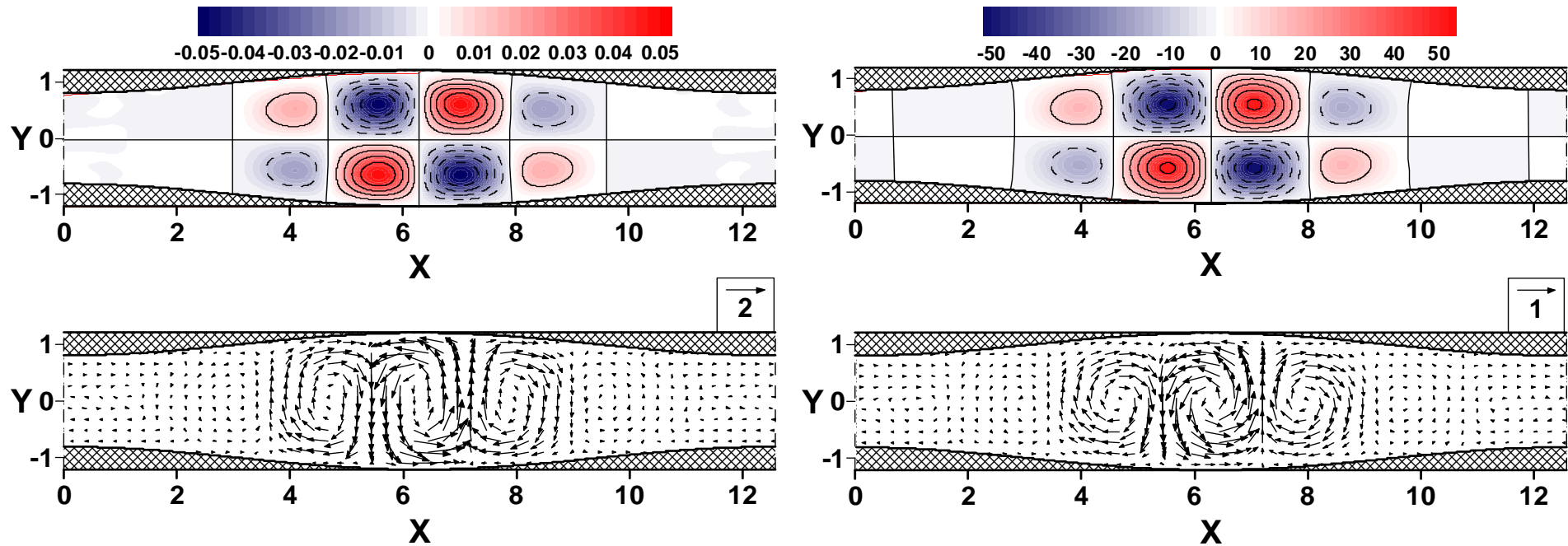
TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (6)

Flow structures for the first maximum (lift-up effect) …

Spanwise structure of the velocity field corresponding to the optimal initial disturbance in the wavy channel with S=0.1 and α=4, computed for Re=1000, δ∗=2 and β=0. The spanwise period of the velocity field is 2 times larger that the wall period. **Left:** optimal initial condition (t=0). **Right:** the state obtained for the optimal time t≈75. The optimal amplification of disturbance energy is 192. The streamwise component of the disturbance velocity is amplified about 600 times.

TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (7)

Flow structures for the first maximum (lift-up effect) continued …

As above but this time the wall wave number is $\alpha=0.5$ and the amplitude $S=0.2$. The energy amplification attains the optimal value of 408 at the time instant $t \approx 96$. The streamwise velocity is amplified about 1000 times (!)

TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (8)

## Flow structures for the second maximum …
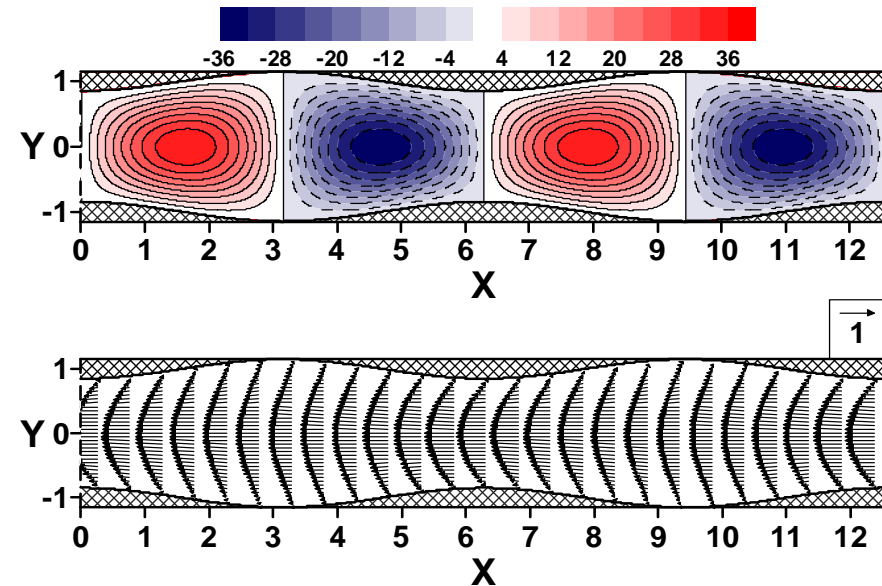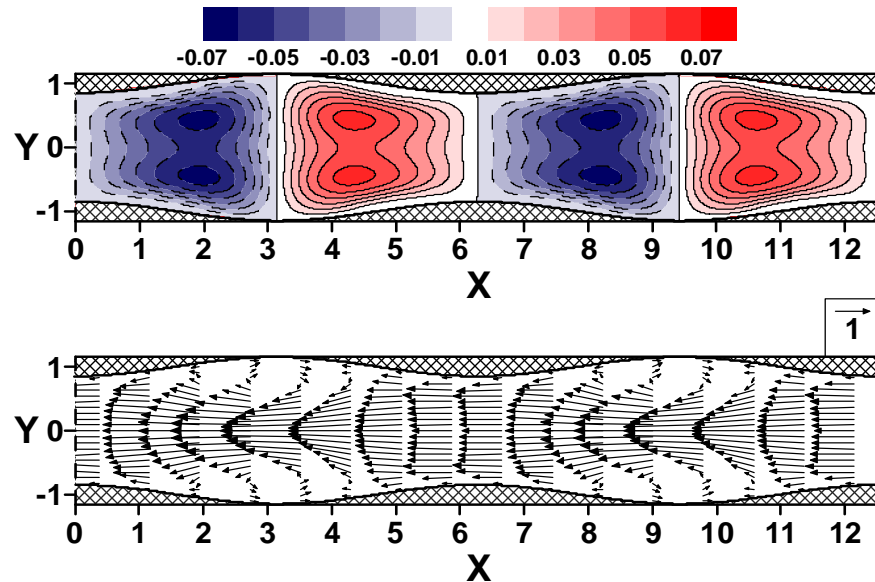
Spanwise structure of the disturbance velocity field calculated for S=0.15, α=1, Re=1000 and δ⁎=β=0.

Left: optimal initial condition (t=0). Note the lack of the streamwise vortices, very small vertical velocity component and dominating cross-flow with nonzero volumetric rate.

Right: the state obtained for the optimal time t≈292. The optimal amplification of disturbance energy is about 239. The strong streamwise streak structure emerges but quite different then before. The cross-flow still exists but is weakly attenuated. The streamwise component of the disturbance velocity is amplified about 550 times.

TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (9)

Two mechanism of the transient energy growth
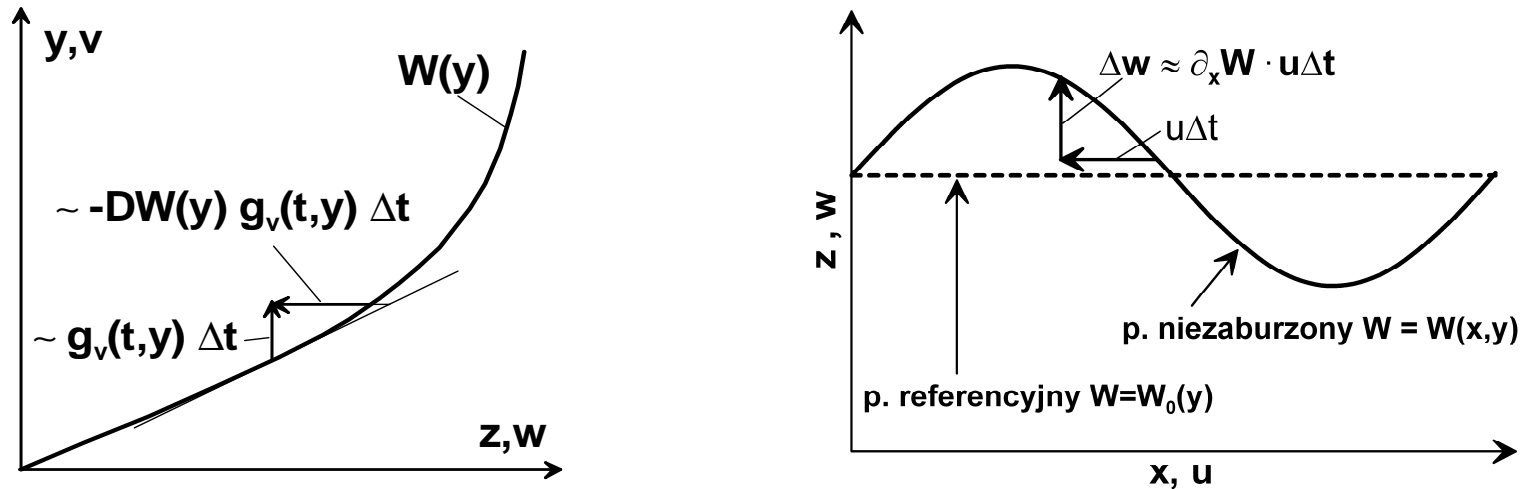
LEFT: The **lift-up mechanism**: weakly attenuated spanwise-periodic array of streamwise-oriented vortices generate vertical disturbances which are transformed into the streamwise streaks with the large amplitude. This mechanism of energy growth is common for all 2D parallel or nearly parallel flows (Poiseuille, Couette, Blasius, Falkner-Skan, etc.).

RIGHT: The **push-aside mechanism**: weakly attenuated cross-flow disturbances interact with the spanwise-modulated velocity of the basic flow and generate strong streamwise streaks, having however a different structure then those created by the lift-up effect.

From the mathematical viewpoint, transient growth of disturbances appears due to strong nonorthogonality of a few least attenuated normal modes (the evolutionary linear operator is nonnormal).

SUMMARY

1. Large transient growth of the disturbance energy is possible providing that Fourier modes with the spanwise wave number equal or close to 2 are present in the disturbance field. Such situation is natural for the wall corrugation with the large period where "most active" Fourier modes will always appear as superhamonics.

2. Two "modes" of the transient growth may appear: the short-time mode and long-time mode. The first one is driven by the lift-up mechanism and characteristic time of the extreme energy amplification is less that 100. The long-time mode (appears for larger amplitudes S and $\alpha \approx 1$) is driven by the push-aside effect, which is not present in the referential flow. The characteristic time scale of the push-aside effect is roughly 3 times larger that for the lift-up effect.

3. The transient velocity structures generated by lift-up and push-aside mechanisms differ in the form of the streamwise streaks and in the character of the transversal motion. The lift-up effect is connected to the presence of the slowly decaying vortices occupying the whole volume of fluid (good for mixing!). However, in the flow structures generated by the push-aside effect the disturbed motion is predominantly horizontal (the vertical velocity is very small), thus mixing in such disturbance field may be less effective.

4. The streamwise streaks are known to be very unstable structures – later development of secondary instabilities should occur and lead to even more complicated 3D velocity field with enhanced mixing properties. Interesting effects may also expected due to the secondary instability of the cross-flow in the disturbance velocity field generated by the push-aside effect. Much further research is required in this respect, including high-resolution DNS.