LINEAR STABILITY OF THE LAMINAR FLOW IN THE CHANNEL WITH TRANSVERSELY CORRUGATED WALLS

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MOTIVATION

1. Need for developing novel efficient methods to improve the heat/mass transport and mixing in the laminar flows, possibly introducing only small amount of additional hydraulic resistance. Potential applications include such devices as:

- Oxygenators, dialisators, small bioreactors,
- Compact heat exchangers, cooling systems for microelectronic systems,
- Devices and technologies using micro- and nanoflows (vide T. Kowalewski's talk during 17-th PCFM in Bełchatów).

2. Assessment of the influence of the surface roughness on the flow stability and transition process, formulation of the rational criteria of the "hydraulic smoothness", investigation of the possibilities of affecting the flow field by small modifications of the wall geometry (passive flow control), etc. (vide J.M. Floryan's talk during the 17-th PCFM in Bełchatów).

Research up to now:

- Most works deal with the longitudinal corrugation or large-amplitude waviness.
- Transversal corrugation - some DNS and experimental investigation of the friction drag reduction in turbulent boundary layers by means of surface riblets.
- There exist several theoretical and experimental studies of the effect of the short-wave transversal corrugation (modeling the presence of the riblets) on the stability of the Tollmien-Schlichting waves in the channel and boundary layer flows (Ehrenstein 1996, Luchini 1995, Grek et al. 1995, 1996).

THE SCOPE OF THE CURRENT RESEARCH

1. Development of the spectrally accurate numerical method for the basic (undisturbed) flow in the channel with transversely wavy walls.

2. Derivation of the linear equations governing the evolution of small perturbations in time and space.

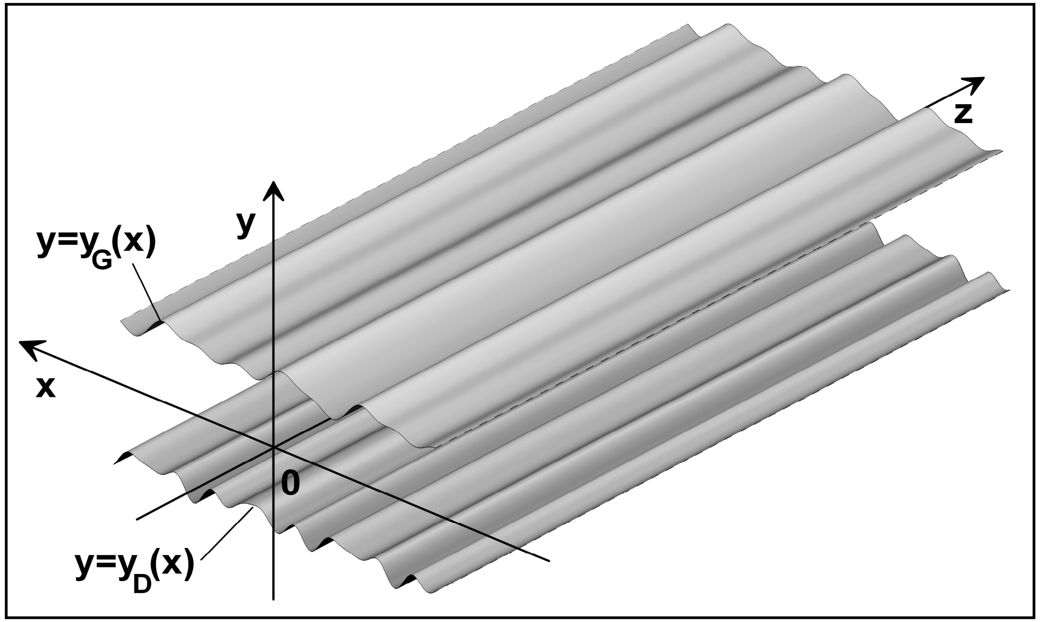
3. Design of the spectral tau-Galerkin method for these equations including the immersed-boundary treatment of the boundary conditions (to avoid domain transformations).

4. The analysis of the asymptotic stability by investigating the properties of the normal modes: determination of the least attenuated or mostly amplified mode and the numerical study of its parametric variation.

5. Investigation of the short-term dynamic behavior of the small perturbations: determination of the initial perturbations which give rise to the largest possible transient energy growth (optimal initial conditions), numerical study of the parametric variations and the corresponding flow structures, identification of mechanisms of the transient growth phenomenon.

LAMINAR FLOW IN THE TRANSVERSELY CORRUGATED CHANNEL (1)

The flow domain:



$$y = y_D(x) = \sum_{n=-\infty}^{\infty} A_n^D\, e^{in\alpha x}, \qquad A_{-n}^D = \left(A_n^D\right)^*,$$

$$y = y_G(x) = \sum_{n=-\infty}^{\infty} A_n^G\, e^{in\alpha x}, \qquad A_{-n}^G = \left(A_n^G\right)^*,$$

$$A_0^D = -1, \;\; A_0^G = 1.$$

Corrugation period: $\lambda_X = 2\pi/\alpha$.

Referential domain: the region between two parallel planes $y = -1$ and $y = 1$.

Referential flow: the incompressible viscous flow in the referential domain driven in the z-direction by the constant pressure gradient k_0 and/or the upper wall movement with the velocity w_G.

The RF's velocity field: $\mathbf{v}_0 = [0,0,w_0(y)]$, $w_0(y) = \frac{\mathrm{Re}\, k_0}{2}[y^2 - 1] + \frac{1}{2} w_G (y+1)$

The Poiseuille flow: $w_G = 0$, $k_0 = -2/\mathrm{Re}$ $\rightarrow$ $w_0(y) = 1 - y^2$.

LAMINAR FLOW IN THE TRANSVERSELY CORRUGATED CHANNEL (2)

Incompressible viscous flow in the wavy channel

Velocity field (x-periodic): $\mathbf{v}_B = [0,0,W(x,y)] = [0,0,w_0] + [0,0,W'(x,y)]$

BVP for W':
$$\begin{cases} \Delta W' = 0 \ \text{in} \ \Omega = \{(x,y) : x \in (0,\lambda_X)\,,\, y_D(x) < y < y_G(x)\}\,, \\ w_0(y_D(x)) + W'(x, y_D(x)) = 0 \ , \ w_0(y_G(x)) + W'(x, y_G(x)) = 0\,. \end{cases}$$

Semi-numerical solution (immerse-boundary approach)

Extended domain
$$\Omega_{ext} = \left\{(x,y) : x \in R,\, y \in [Y_{min}, Y_{max}]\right\},$$
$$Y_{min} = \min_{x\in[0,\lambda_X]} y_D(x)\ ,\ Y_{max} = \max_{x\in[0,\lambda_X]} y_G(x).$$

General solution
$$W'(x,y) = C_0 + S_0 y + \sum_{n\neq 0}[C_n \cosh(n\alpha y) + S_n \sinh(n\alpha y)]\, e^{in\alpha x}\ ,$$

where the coefficients $\{C_n\}$ and $\{S_n\}$ should be chosen so that the B.C. are satisfied.

LAMINAR FLOW IN THE TRANSVERSELY CORRUGATED CHANNEL (3)

Semi-numerical solution (immerse-boundary approach) continued …

Velocity distribution at the bottom wall $y = Y_D(x)$

$$\varepsilon_D(x) \equiv W(x, y_D(x)) =$$
$$= w_0(y_D(x)) + C_0 + S_0 y_D(x) + \sum_{n\neq 0} [C_n \cosh(n\alpha y_D(x)) + S_n \sinh(n\alpha y_D(x))]\, e^{in\alpha x}$$

The following Fourier expansions will be used: $w_0(y_D(x)) = \sum_{m=-\infty}^{\infty} B_m^D e^{im\alpha x}$,

$$\zeta_m^D(x) \equiv \cosh(m\alpha y_D(x)) = \sum_{n=-\infty}^{\infty} G_{m,n}^D e^{im\alpha x} \quad , \quad \gamma_m^D(x) \equiv \sinh(m\alpha y_D(x)) = \sum_{n=-\infty}^{\infty} H_{m,n}^D e^{im\alpha x} .$$

Then $\varepsilon_D(x) = \sum_{m=-\infty}^{\infty} E_m^D e^{im\alpha x} = C_0 + \sum_{m=-\infty}^{\infty} (B_m^D + S_0 A_m^D + \sum_{n\neq 0} C_n G_{n,m-n}^D + \sum_{n\neq 0} S_n H_{n,m-n}^D)\, e^{im\alpha x}$.

We have obtained the explicit form of the Fourier coefficients for the bottom wall distribution of the velocity error. The coefficients for the x-periodic function $\varepsilon_G(x)$ are calculated in the similar way.

LAMINAR FLOW IN THE TRANSVERSELY CORRUGATED CHANNEL (4)

The boundary conditions at both wall are enforced by setting:

$$E_m^D = 0 \quad , \quad E_m^G = 0 \, , \quad m = 0, \pm 1, \pm 2, \ldots$$

After truncation the finite dimensional linear algebraic system for $\{C_m\}$ and $\{S_m\}$ is obtained.

Solution in the sense of the least squares

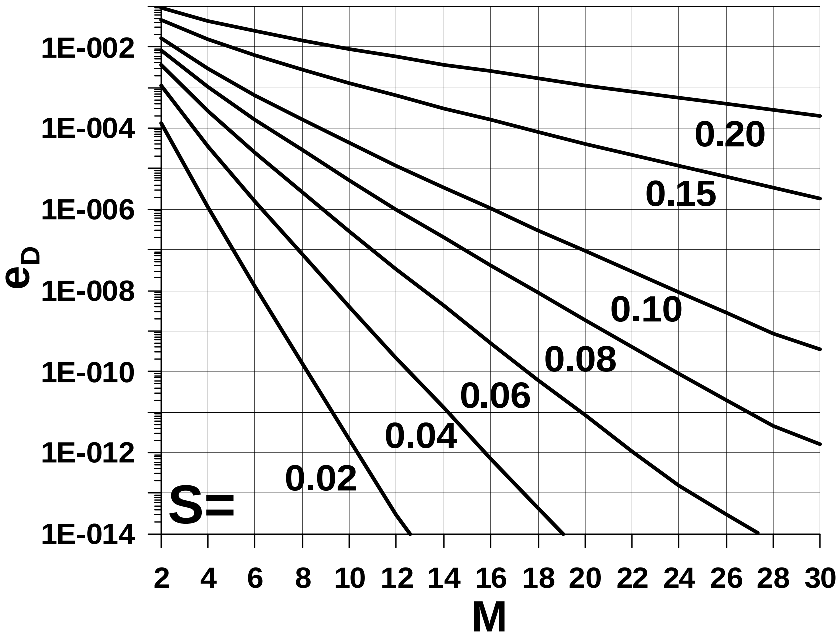
- The Fourier coefficients of the boundary error for $|m| \leq M$ are to be cancelled.
- The velocity field is approximated by the truncated Fourier expansion containing the modes for $|n| \leq N$, where $N < M$.
- Overdetermined linear system $\mathbf{P\,c} = \mathbf{r}$ is solved as the least squares problem, i.e. the vector **c** is computed from $\mathbf{P}^{\mathrm{H}}\mathbf{P\,c} = \mathbf{P}^{\mathrm{H}}\mathbf{r}$. Using the QR decomposition of the matrix macierzy **P**, the problem is transformed to the upper-triangular system $\mathbf{R\,c} = \mathbf{Q}^{\mathrm{T}}\mathbf{r}$.

LAMINAR FLOW IN THE TRANSVERSELY CORRUGATED CHANNEL (5)

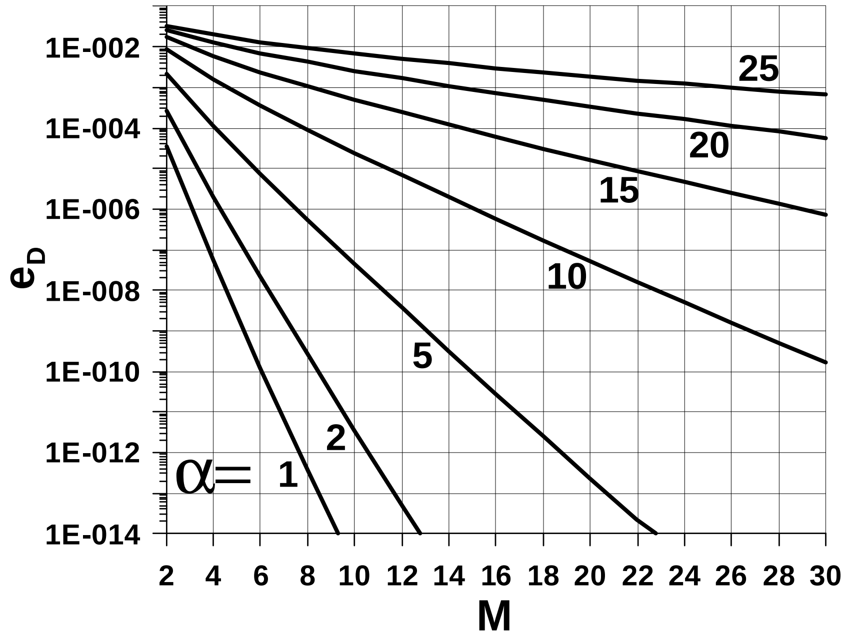
Numerical tests

The test case: $y_D(x) = -1 + S\cos\alpha x, \quad y_G(x) \equiv 1$ (flat wall).

The norm of the boundary error: $e_{D,G} = \sup\limits_{x\in[0,\lambda_X]} \left| W(x, y_{D,G}(x)) \right|.$

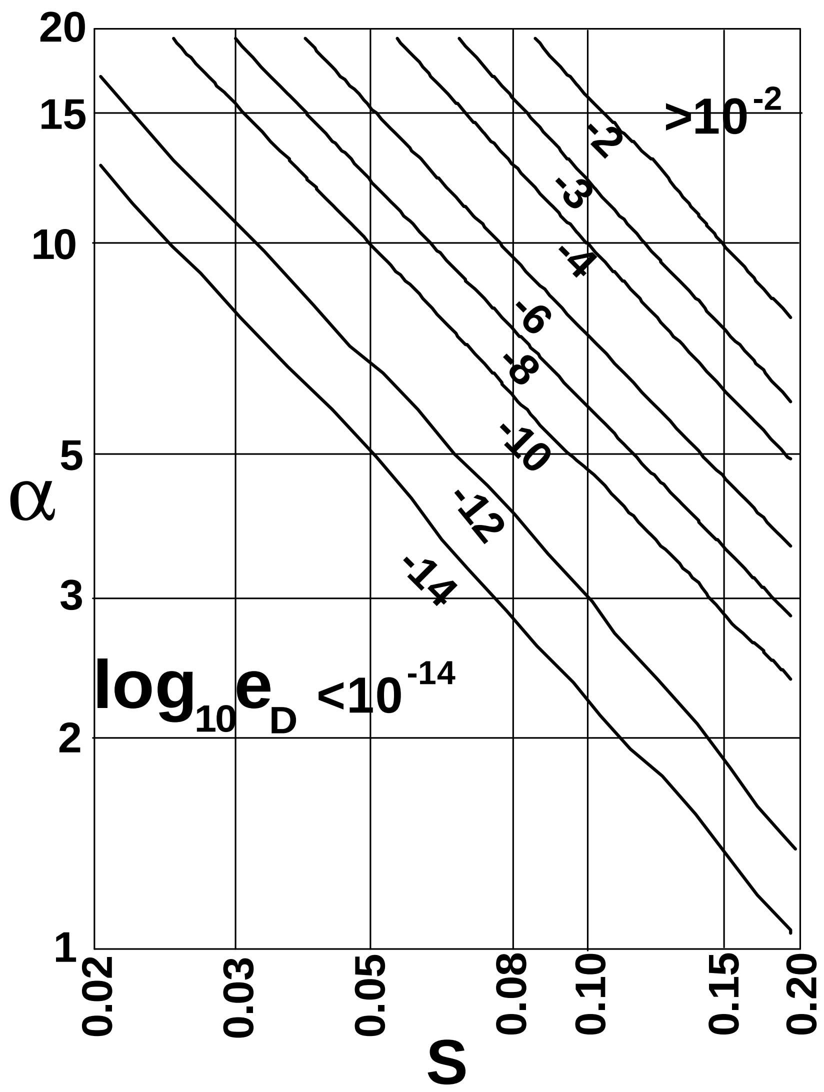


$\alpha = 5$



$S = 0.05$

LAMINAR FLOW IN THE TRANSVERSELY CORRUGATED CHANNEL (6)



Norm e_D as the function of the amplitude S and the wall shape periodicity (wave number α)

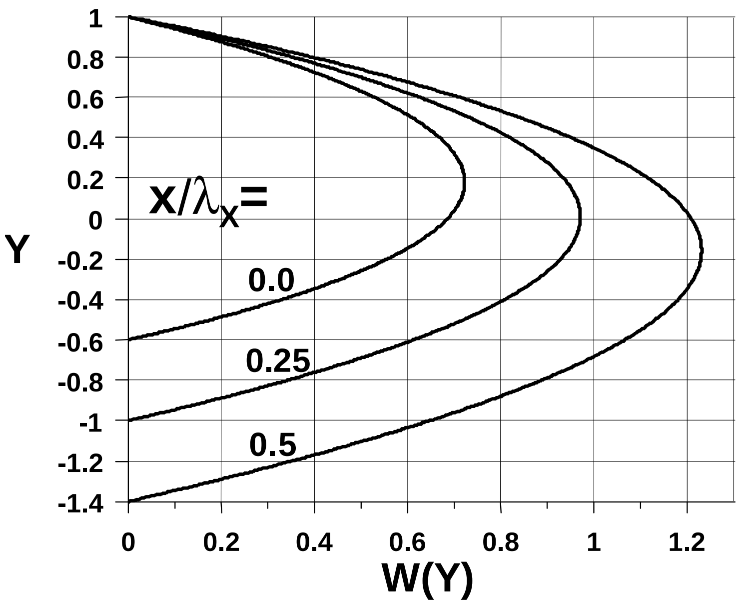
With the logarithmic scaling of both S and α the izolines of $\log_{10}e_D$ are nearly parallel straight lines with the slope coefficient close to –1. We conclude that the izolines of the norm e_D as basically the same as the izolines of the product $S \cdot \alpha$, which can be interpreted as follows: the magnitude of the e_D is dependent on the ratio between the amplitude S and the geometric period $\lambda_X = 2\pi/\alpha$ of the wall shape.

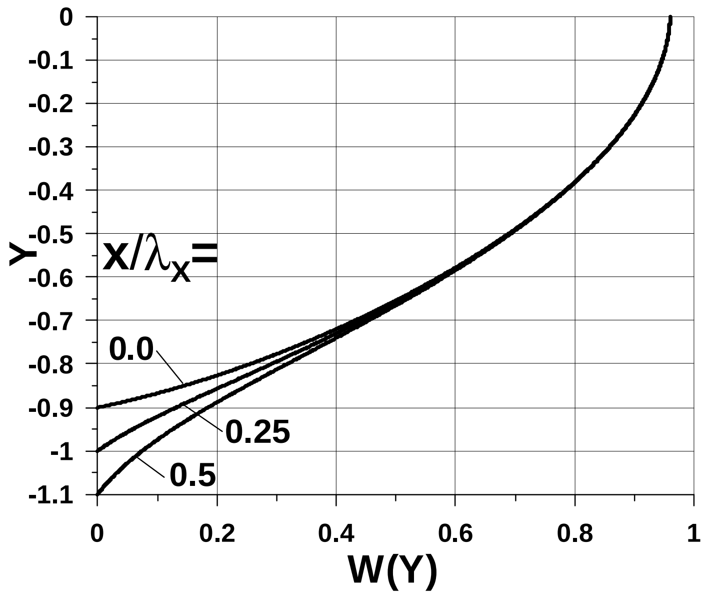
LAMINAR FLOW IN THE TRANSVERSELY CORRUGATED CHANNEL (7)

Velocity profiles of the basic flow

$$y_D(x) = -1 + 0.4 \cdot \cos(x)$$

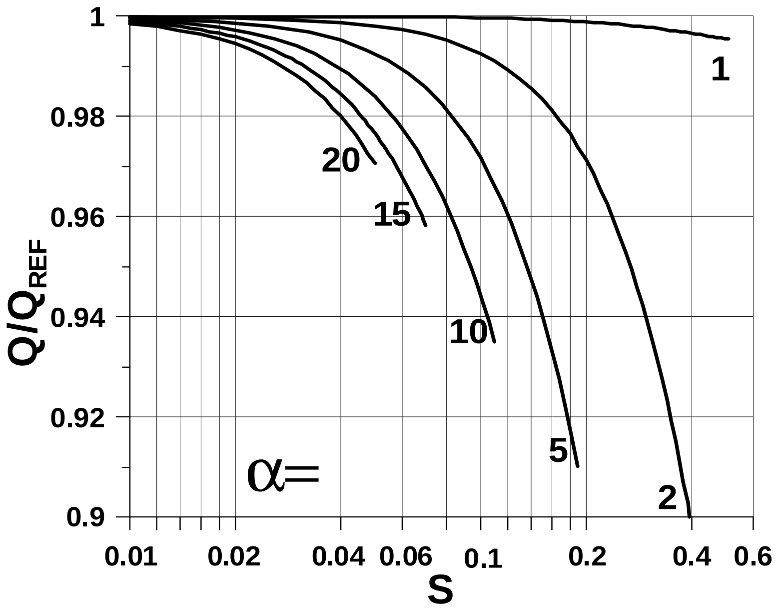
$$y_D(x) = -1 + 0.1 \cdot \cos(10\,x).$$

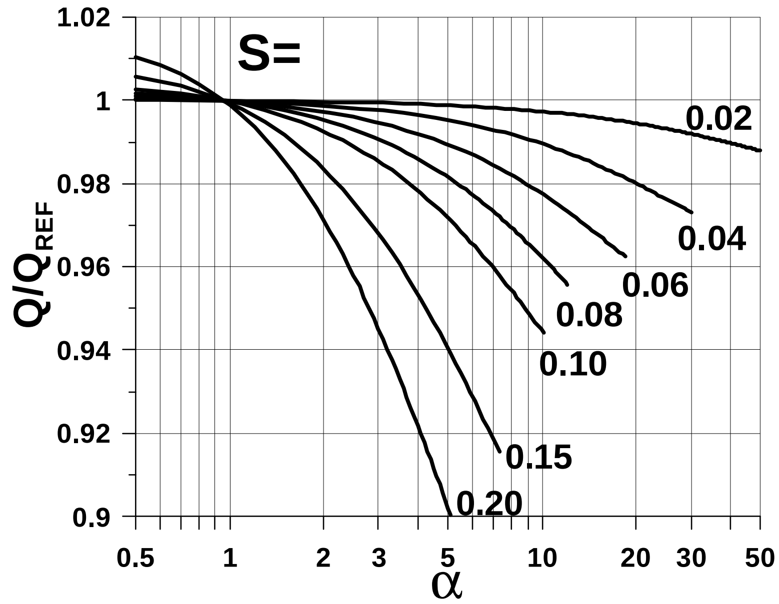




LAMINAR FLOW IN THE TRANSVERSELY CORRUGATED CHANNEL (8)

Corrugation-induced flow resistance





The ratio of the volumetric flow rate in the wavy channel Q to the reference value Q_{REF}=4/3 plotted as the function of S and α. The streamwise pressure gradients in the wavy and reference channels are the same.

DYNAMICS OF SMALL DISTURBANCES – LINEAR THEORY (1)

Derivation of the governing equations

Basic (undisturbed) flow: $\mathbf{v}_B = \mathbf{v}_B(x,y,z) \quad , \quad p_B = p_B(x,y,z)$

Disturbance velocity and pressure: $\mathbf{v}' = \mathbf{v}'(t,x,y,z) \quad , \quad p' = p'(t,x,y,z)$

Disturbed flow: $\mathbf{v} = \mathbf{v}_B + \mathbf{v}' \quad , \quad p = p_B + p'$

Linearized Navier-Stokes and the continuity equations

$$\partial_t \mathbf{v}' + \left(\mathbf{v}_B \cdot \nabla\right)\mathbf{v}' + \left(\mathbf{v}' \cdot \nabla\right)\mathbf{v}_B = -\nabla p' + \tfrac{1}{Re}\Delta \mathbf{v}' \ , \qquad \nabla \cdot \mathbf{v}' = 0$$

Our basic flow is $\mathbf{v}_B = [\,0,0,W(x,y)\,]$, $p_B = p_B(z)$, thus

$$\begin{cases} \partial_t u + W\,\partial_z u = -\partial_x p' + \tfrac{1}{Re}\Delta u \ , \\ \partial_t v + W\,\partial_z v = -\partial_y p' + \tfrac{1}{Re}\Delta v \ , \\ \partial_t w + W\,\partial_z w + u\,\partial_x W + v\,\partial_y W = -\partial_z p' + \tfrac{1}{Re}\Delta w \ , \\ \partial_x u + \partial_y v + \partial_z w = 0 \, . \end{cases}$$

DYNAMICS OF SMALL DISTURBANCES – LINEAR THEORY (2)

Basic flow velocity field: $W(x,y) = \sum_{n=-\infty}^{\infty} F_W^n(y)\, e^{in\alpha x}\,,\;\; F_W^{-n} = (F_W^n)^*$,

$$F_W^0(y) = w_0(y) + W_0(y) \quad , \quad F_W^m = W_n \quad , \quad n \neq 0\,,$$

General form of the disturbance velocity (Floquet) can be written as follows

$$[u,v,w]\,(t,x,y,z) = e^{i(\delta_* x + \beta z)}\,[G_u, G_v, G_w](t,x,y) + \text{c.c.} \qquad ,$$

where:

- G_u, G_v and G_w – complex x-periodic functions (period $\lambda_X = 2\pi/\alpha$),
- β - streamwise wave number (real)
- δ_* - Floquet parameter (real). The fractional part of δ_*/α determines the character of disturbance variation with respect to the space variable x (spanwise direction): if this number is rational the variation is periodic, otherwise it is quasi-periodic. In particular, when δ_* is equal to some integer multiplicity of α (which is in fact equivalent to $\delta_* = 0$), the period of the spanwise variation of the disturbance field is equal to the geometric period of the wall corrugation λ_x.

DYNAMICS OF SMALL DISTURBANCES – LINEAR THEORY (3)

The disturbance velocity field can be re-written in the following form

$$[u,v,w](t,x,y,z)=\sum_{m=-\infty}^{\infty}[g_u^m,g_v^m,g_w^m](t,y)\,e^{i(t_m x+\beta z)}+c.c.\;,\quad t_m=\delta_*+m\alpha.$$

Vertical (y) vorticity component: $\eta=\sum_{m=-\infty}^{\infty}\theta^m(t,y)\,e^{i(t_m x+\beta z)}+C.C.\;,\quad \theta^m=i\,(\beta g_u^m-t_m g_w^m).$

Continuity equation: $\partial_x u+\partial_y v+\partial_z w=0 \quad\Rightarrow\quad it_m g_u^m+\partial_y g_v^m+i\beta g_w^m=0$ (m=...,-1,0,...) .

Hence: $g_u^m=i\,(t_m\partial_y g_v^m-\beta\theta^m)\big/k_m^2\,,\; g_w^m=i\,(\beta\partial_y g_v^m+t_m\theta^m)\big/k_m^2\,,$ where $k_m^2=t_m^2+\beta^2.$

Simplified description is possible $\quad \left\{g_u^m,g_v^m,g_w^m\right\}_{m=-\infty}^{\infty}\Rightarrow\left\{g_v^m,\theta_v^m\right\}_{m=-\infty}^{\infty}$

DYNAMICS OF SMALL DISTURBANCES – LINEAR THEORY (4)

... which leads to the following set of the governing equations (m=...,-1,0,1,...):

$$\partial_t L^m g_v^m + S^m g_v^m + \sum_{n>0}\left(H_V^{m,n} g_v^{m-n} + \hat{H}_V^{m,n} g_v^{m+n} + H_\theta^{m,n}\theta^{m-n} + \hat{H}_\theta^{m,n}\theta^{m+n}\right) = 0,$$

$$\partial_t \theta^m + Q^m\theta^m - i t_m D F_W^0 g_v^m + \sum_{n>0}\left(E_V^{m,n} g_v^{m-n} + \hat{E}_V^{m,n} g_v^{m+n} + E_\theta^{m,n}\theta^{m-n} + \hat{E}_\theta^{m.n}\theta^{m+n}\right) = 0 \ ,$$

where

$$L^m = \partial_{yy} - k_m^2 \quad , \quad S^m = i\beta\,[F_W^0\, L^m - D^2 F_W^0] - \tfrac{1}{Re}(L^m)^2 \quad , \quad Q^m = i\beta F_W^0 - \tfrac{1}{Re} L^m \,,$$

$$H_V^{m,n} = i\beta\,[\tfrac{\beta^2 + t_{m-n} t_{m+n}}{k_{m-n}^2} F_W^n\,\partial_{yy} - (D^2 + k_m^2) F_W^n + \tfrac{2n\alpha t_{m-n}}{k_{m-n}^2} DF_W^n\,\partial_y\,] \quad ,$$

$$\hat{H}_V^{m,n} = i\beta\,[\tfrac{\beta^2 + t_{m-n} t_{m+n}}{k_{m+n}^2} (F_W^n)^*\,\partial_{yy} - (D^2 + k_m^2)(F_W^n)^* - \tfrac{2n\alpha t_{m+n}}{k_{m+n}^2} (DF_W^n)^*\,\partial_y\,] \;,$$

$$H_\theta^{m,n} = -i\,\tfrac{2n\alpha\beta^2}{k_{m-n}^2}\left[F_W^n\,\partial_y + DF_W^n\right] \qquad , \qquad \hat{H}_\theta^{m,n} = i\,\tfrac{2n\alpha\beta^2}{k_{m+n}^2}\left[(F_W^n)^*\,\partial_y + (DF_W^n)^*\right],$$

$$E_V^{m,n} = i\,\tfrac{n\alpha}{k_{m-n}^2}(\beta^2 + t_m t_{m-n}) F_W^n\;\partial_y - i t_m DF_W^n \qquad , \qquad E_\theta^{m,n} = i\beta\,\tfrac{\beta^2 + t_m t_{m-2n}}{k_{m-n}^2} F_W^n \;,$$

$$\hat{E}_V^{m,n} = -i\,\tfrac{n\alpha}{k_{m+n}^2}(\beta^2 + t_m t_{m+n})(F_W^n)^*\;\partial_y - i t_m (DF_W^n)^* \qquad , \qquad \hat{E}_\theta^{m,n} = i\beta\,\tfrac{\beta^2 + t_m t_{m+2n}}{k_{m+n}^2} (F_W^n)^* \;.$$

Homogeneous B.C.: $\quad [u,v,w](t,x,y_D(x),z) = [u,v,w](t,x,y_G(x),z) = \mathbf{0}.$

SPECTRAL DISCRETIZATION OF THE GOVERNING EQUATIONS

Chebyshev approximation of the amplitude functions (in the y-interval implied by the extension of the computational domain to Ω_{ext}):

$$g_v^n(t,y) \approx \sum_{k=0}^{K_V} \Gamma_k^n(t)\, T_k(y) \quad , \quad \theta^n(t,y) \approx \sum_{k=0}^{K_\theta} \Theta_k^n(t)\, T_k(y).$$

The residua of all 4th order PDEs are made orthogonal (with respect to the Chebyshev inner product) to the subspace spanned by the polynomials $T_0, T_1, ..., T_{K_V - 4}$. Similarly, the residua of all 2nd order PDEs are made orthogonal to the subspace spanned by $T_0, T_1, ..., T_{K_\theta - 2}$. After the representation of the disturbance velocity field is restricted to the Fourier harmonics from the range $-M_S,...,0,...,M_S$, the following linear ODE system is obtained

$$\boldsymbol{M} \tfrac{d}{dt} \mathbf{z} + \boldsymbol{L}\, \mathbf{z} = \mathbf{0}$$

where
$$\mathbf{z} = \left[\left\{ \Gamma_0^n, \Gamma_1^n, .., \Gamma_{K_V - 1}^n, \Gamma_{K_V}^n ; \Theta_0^n, \Theta_1^n, .., \Theta_{K_\theta - 1}^n, \Theta_{K_\theta}^n \right\}, n = -M_S, .., 0, .. M_S \right]^T.$$

The dimension of this system is $M_G = (2M_S + 1)\cdot(K_V + K_\theta - 4)$, so it is not yet closed! Additional equation must arrive to account for the boundary conditions …

NUMERICAL TREATMENT OF THE BOUNDARY CONDITIONS

For convenience, consider the velocity component u. At the bottom wall $y = y_D(x)$ we have

$$u(t,x,y_D(x),z) \approx e^{i(\delta_* x+\beta z)} \sum_{n=-M_S}^{M_S} \left[\sum_{m=-M_S}^{M_S} i\left(\frac{t_m}{k_m^2} \sum_{k=1}^{K_V} \left(\zeta_{n-m}^k\right)_D \Gamma_k^m(t) - \frac{\beta}{k_m^2} \sum_{k=0}^{K_\theta} \left(\xi_{n-m}^k\right)_D \Theta_k^m(t) \right) \right] e^{in\alpha x},$$

where $(\xi_n^k)_D$ and $(\zeta_n^k)_D$ denote the Fourier coefficients for the composite function $T_k \circ y_D$ and $DT_k \circ y_D$, respectively. We demand that all coefficients of the above expansion for $|n| \le M_S$ vanish identically. Proceeding the same way for the remaining velocity components and the upper wall, we arrive at the following $6 \cdot (2M_S + 1)$ linear algebraic equations (n = -M_S,...,0,...,M_S)

$$\sum_{m=-M_S}^{M_S} \left(\frac{t_m}{k_m^2} \sum_{k=1}^{K_V} \left(\zeta_{n-m}^k\right)_{D,G} \Gamma_k^m - \frac{\beta}{k_m^2} \sum_{k=0}^{K_\theta} \left(\xi_{n-m}^k\right)_{D,G} \Theta_k^m \right) = 0 \quad , \quad \sum_{m=-M_S}^{M_S} \left(\sum_{k=0}^{K_V} \left(\xi_{n-m}^k\right)_{D,G} \Gamma_k^m \right) = 0$$

$$\sum_{m=-M_S}^{M_S} \left(\frac{\beta}{k_m^2} \sum_{k=1}^{K_V} \left(\zeta_{n-m}^k\right)_{D,G} \Gamma_k^m + \frac{t_m}{k_m^2} \sum_{k=0}^{K_\theta} \left(\xi_{n-m}^k\right)_{D,G} \Theta_k^m \right) = 0$$

Summarizing: The approximate description of the dynamics of small perturbations is provided by the following linear differential/algebraic system

$$\boldsymbol{M} \frac{d}{dt} \mathbf{z} + \boldsymbol{L} \mathbf{z} = \mathbf{0} \quad , \quad \boldsymbol{B} \mathbf{z} = \mathbf{0} \; .$$

THE NORMAL MODES ANALYSIS (1)

The method of the normal modes

The governing equations admit solutions in the special form called a normal mode:

$$[g_u^m, g_v^m, g_w^m](t,y) = [G_u^m, G_v^m, G_w^m](y)\exp(-i\sigma t) \quad , \qquad \sigma = \sigma_R + i\sigma_I \text{ - complex frequency.}$$

The disturbance velocity field of the NM

$$[u,v,w](t,x,y,z) = \sum_{m=-\infty}^{\infty}[G_u^m, G_v^m, G_w^m](y)\, e^{i(t_m x+\beta z-\sigma t)} + C.C.,$$

Vertical vorticity component of the NM:

$$\eta(t,x,y,z) = \sum_{m=-\infty}^{\infty}\Xi^m(y)\, e^{i(t_m x+\beta z-\sigma t)} + C.C., \qquad \theta^m(t,y) = \Xi^m(y)\exp(-i\sigma t).$$

- Sufficient condition for the instability of the basic flow: at least one unstable NM exists, i.e. for at least one NM the imaginary part σ_I of the corresponding complex frequency is positive.
- The critical Reynolds number Re_L (of the linear theory) is the largest Re value such that for all NMs are at least neutrally stable (all σ_I-s are nonpositive).

THE NORMAL MODES ANALYSIS (2)

In the finite dimensional approximation the NM are defined as $\mathbf{z}(t) = \boldsymbol{\varsigma} e^{-i\sigma t}$. After this form is inserted into the linear DAE system, the following generalized eigenvalue problem is obtained

$$\mathbf{P}\boldsymbol{\zeta} = i\sigma\mathbf{Q}\boldsymbol{\zeta} \; ,$$

where $\mathbf{P} = \begin{bmatrix} \boldsymbol{L} \\ \boldsymbol{B} \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} \boldsymbol{M} \\ \mathbf{0} \end{bmatrix}$.

The above problem is complex and nonsymmetric, with the dimension $M_G=(2M_S+1)\cdot(K_V+K_\theta+2)$. Typical parameters for the current research are: M_S from 8 to 12, K_v and K_θ from 50 to 60. This gives the dimension M_G ranging from 1700 to 3000.

Numerical methods used in this study:

- Most often: the method of inverse iterations used as the tool for the parametric continuation (or tracing) of selected NMs with respect to various parameters like wave numbers, corrugation amplitude and the Reynolds number.
- Occasionally: the evaluation of the full spectra by means of the QZ method (LAPACK) performed in order to initiate parametric continuations or to check whether the mode being traced is indeed the most amplified (or the least attenuated) one.

THE NORMAL MODES ANALYSIS (3)

Digression: normal modes for the referential Poiseuille flow

The Orr-Sommerfeld/Squire eigenvalue problem for the parallel flow with $\mathbf{V} = [0,0,w_0(y)]$):

- OS Eq.: $\left\{(D^2 - k^2)^2 - i\,Re\left[(\beta w_0 - \sigma)(D^2 - k^2) - \beta D^2 w_0\right]\right\} G_v = 0$,
- Sq Eq.: $[\,D^2 - k^2 - i\,Re(\beta w_0 - \sigma)\,]\Theta = i\,Re\,\delta\, D w_0\, G_v$,
- B.C.: $G_v(-1) = G_v(1) = 0$, $DG_v(-1) = DG_v(1) = 0$, $\Theta(-1) = \Theta(1) = 0$.

where $G_V(y)$, $\Theta(y)$ - the complex functions of the spatial variable y describing the distribution of the disturbance amplitude of the vertical component of the disturbance velocity and vorticity, β, δ - streamwise and spanwise wave numbers and $k = \sqrt{\beta^2 + \delta^2}$ is the length of the wave vector $\boldsymbol{\kappa} = [\kappa_z, \kappa_x] = [\beta, \delta]$.

THE NORMAL MODES ANALYSIS (4)

Digression continued ...

Two families of the normal modes for the parallel flows:

- Orr-Sommerfeld modes: the function G_V is the eigensolution of the Orr-Sommerfeld Eq., while the function Θ is the solution of the Squire Eq. with the nonzero right-hand side. The streamwise and spanwise components of the disturbance velocity field are uniquely implied by the form of G_V and Θ. If $\delta = 0$ (lack of spanwise variations) then the velocity field of the OS mode is two-dimensional (the spanwise component is zero). **Some OS modes can become unstable for sufficiently large Reynolds numbers, but the first mode which looses stability (i.e. for the lowest Re values) is always the fundamental 2D mode (Squire Theorem).**

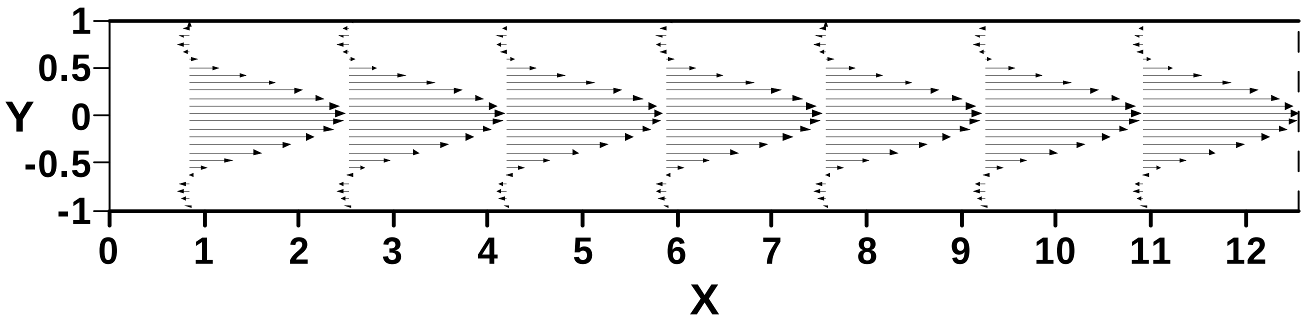
- Squire modes: the function Θ is the eigensolution of the homogeneous Sq Eq., while the function G_V vanishes identically. Hence, the disturbance velocity of the Sq mode is purely horizontal - it contains the components only in the streamwise and spanwise directions. If $\delta = 0$ then the velocity field of such mode contains only spanwise component which is y- and z-dependent. If $\beta = 0$ then the velocity of the Sq mode contains only streamwise component which is y- and x-dependent. In such case, the real part of the corresponding eigenvalue is zero meaning that the Squire mode describes the stationary (i.e. not traveling) disturbances. **In the Poiseuille flow all Sq modes are always stable**.

THE NORMAL MODES ANALYSIS (5)

Ehrenstein (1996): the short-wave (α ≈ 12) transversal corrugation (modeling the presence of the surface riblets) with the amplitude equal to 5% of the channel height lowers the critical Reynolds number to the value of 2600. The unstable mode has been recognized as the 2-D fundamental OS mode (mode OS_1) subject to the 3-D modification evoked by the wall corrugation. Further reduction of Re_L for larger amplitudes is certainly possible but at the cost of an excessive hydraulic resistance.

QUESTION: Is there any other normal mode which can be destabilized by the long-wave (α ≈ 1) transversal corrugation at least as effectively as the OS_1 mode ?

ANSWER IS POSITIVE! – this mode is the least attenuated (or fundamental) Squire mode Sq_1 with the wave vector $\boldsymbol{\kappa} = [\beta, 0]$. Its velocity contains only the spanwise component thus we will refer to this mode as the **fundamental transversal mode**. It describes the disturbances in the form of the traveling wave. The velocity of the wave measured with respect to the channel walls is slightly smaller that 1.

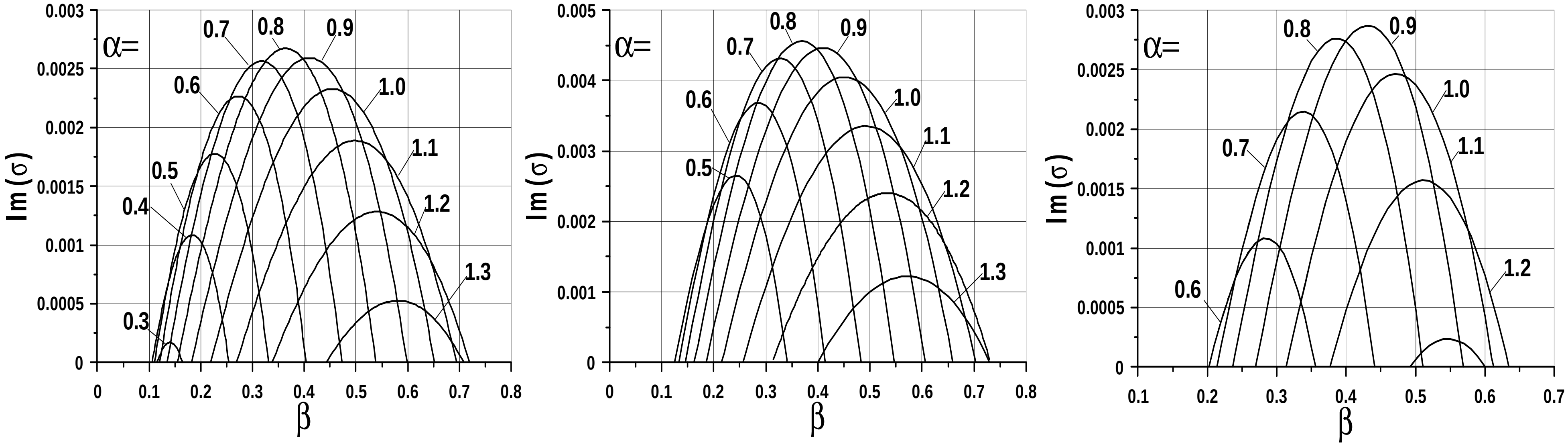


The profile of the spanwise velocity of the original mode Sq_1 (Poiseuille flow) computed for Re=250 and the wave vector $\boldsymbol{\kappa} \equiv [\beta, \delta] = [0.5, 0]$.

THE NORMAL MODES ANALYSIS (6)

Consider again our test geometry: $y_D(x) = -1 + S\cos\alpha x$, $y_G(x) \equiv 1$ (flat wall).

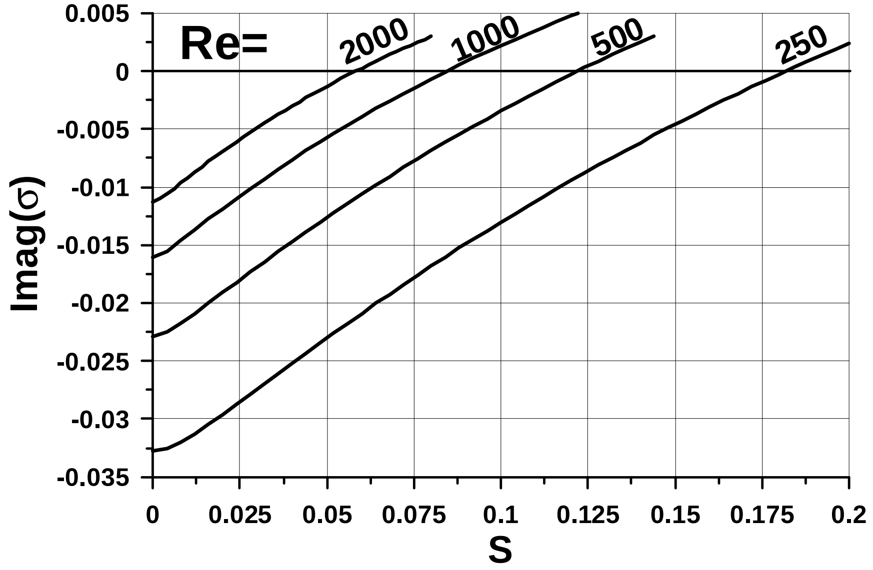
The amplification rate σ_I=Im(σ) of the mode Sq_1 as the function of the wave number α and β

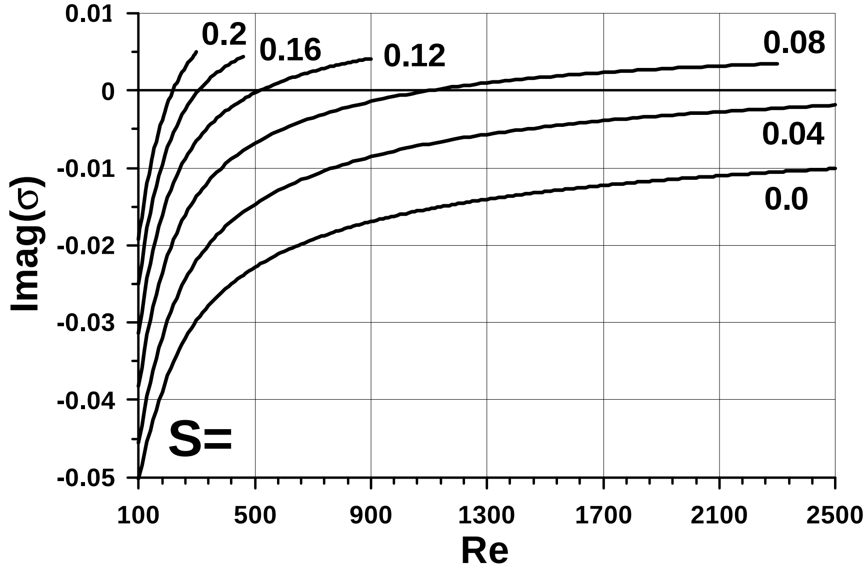


(a) S=0.1, Re=1000 (b) S=0.15, Re=500 (c) S=0.2, Re=250

THE NORMAL MODES ANALYSIS (7)

The amplification rate σ_I of the mode Sq_1 as the function the amplitude of the wall corrugation S and the Reynolds number Re (α = 1 and β = 0.5)

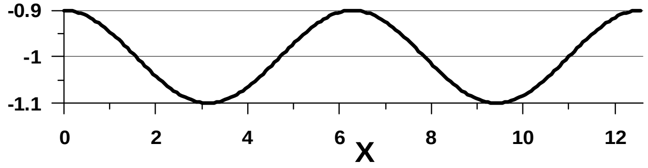




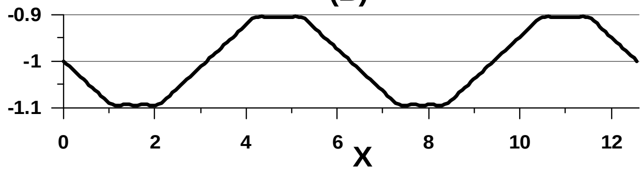
THE NORMAL MODES ANALYSIS (8)

Wall corrugation with more complicated shape…

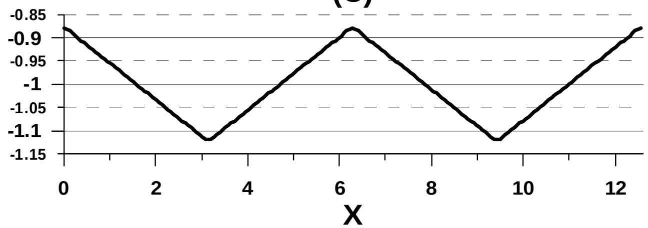
(A)



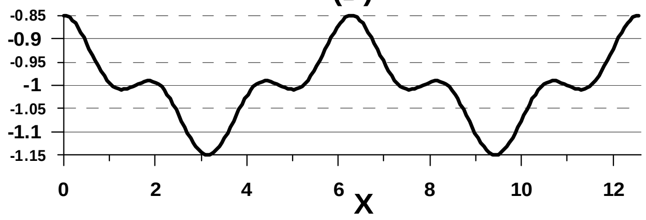
(B)



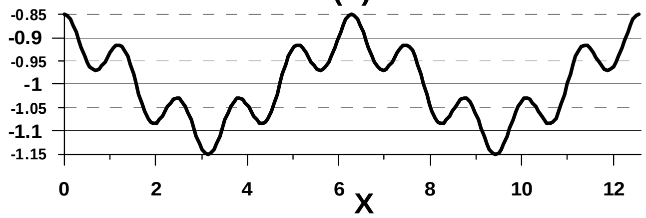
(C)



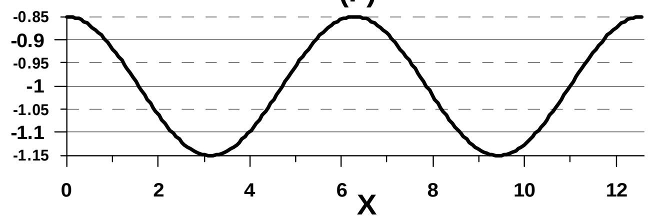
(D)



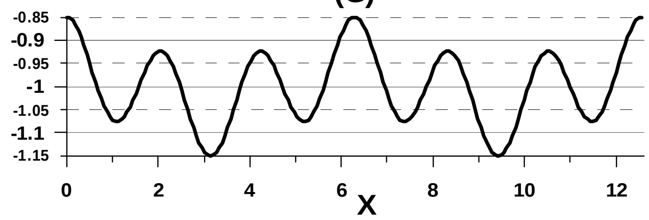
(E)



(F)

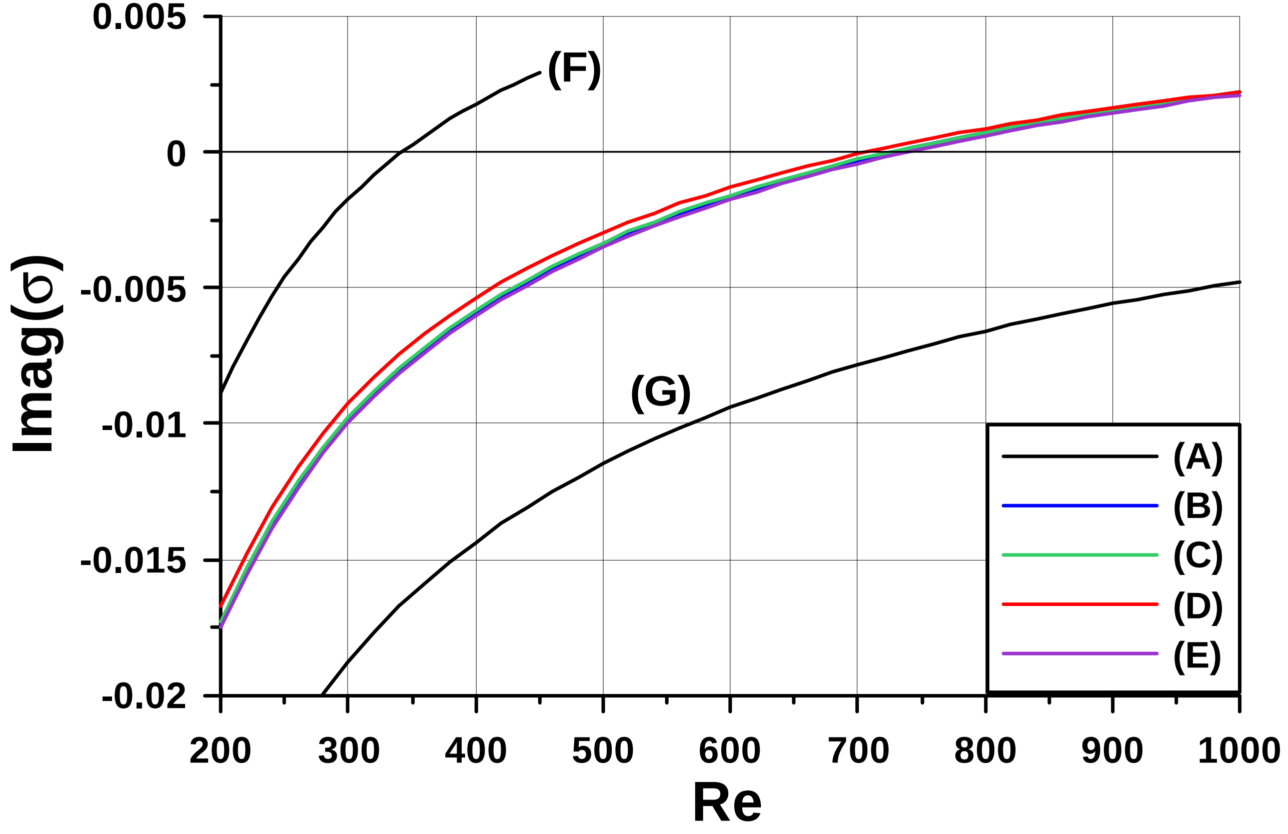


(G)



THE NORMAL MODES ANALYSIS (9)

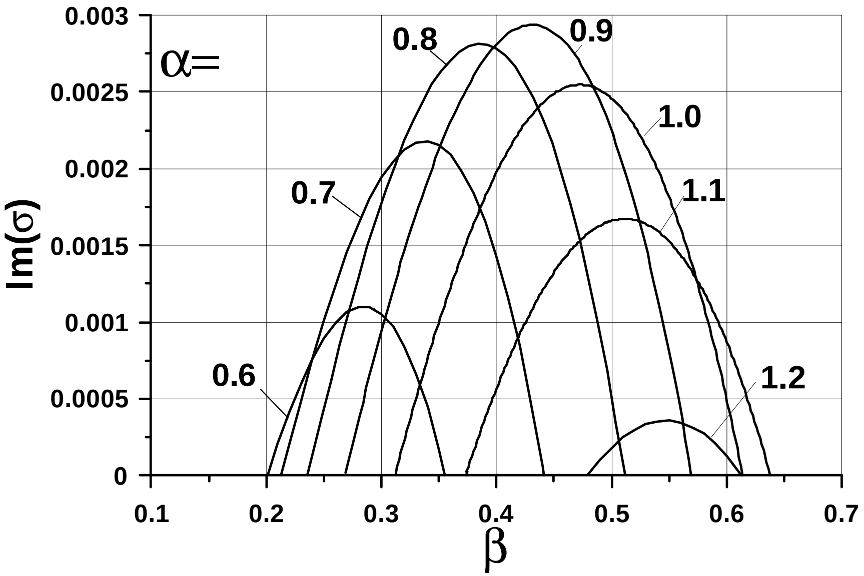
The amplification rate σ_I plotted versus Reynolds number Re for different wall shapes (α=1, β=0.5)

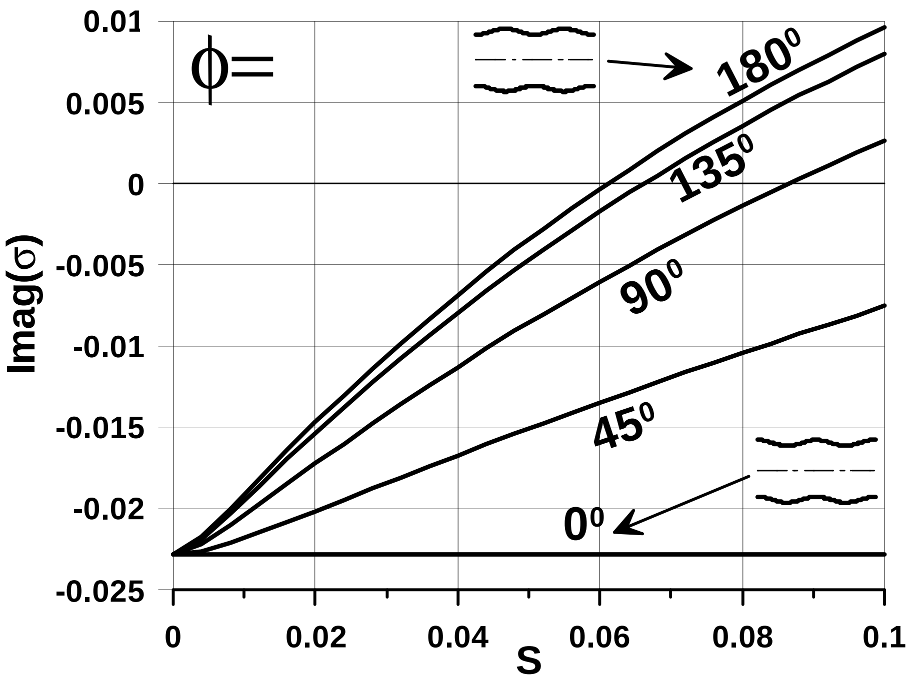


THE NORMAL MODES ANALYSIS (10)

Flow in the channel with two wavy walls

$$y_D(x) = -1 + S\cos(\alpha x) \ , \ y_G(x) = 1 + S\cos(\alpha x + \phi)$$



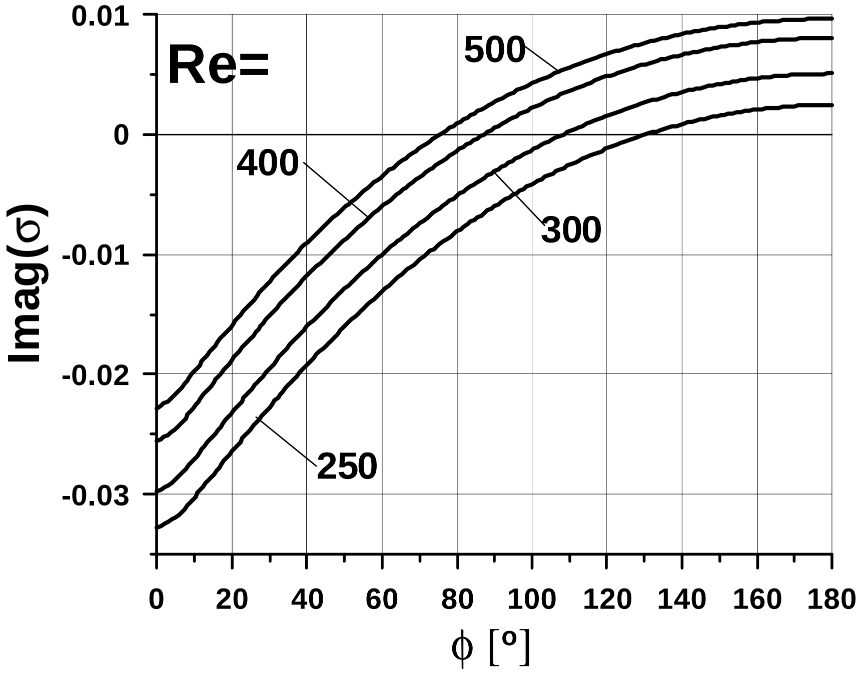


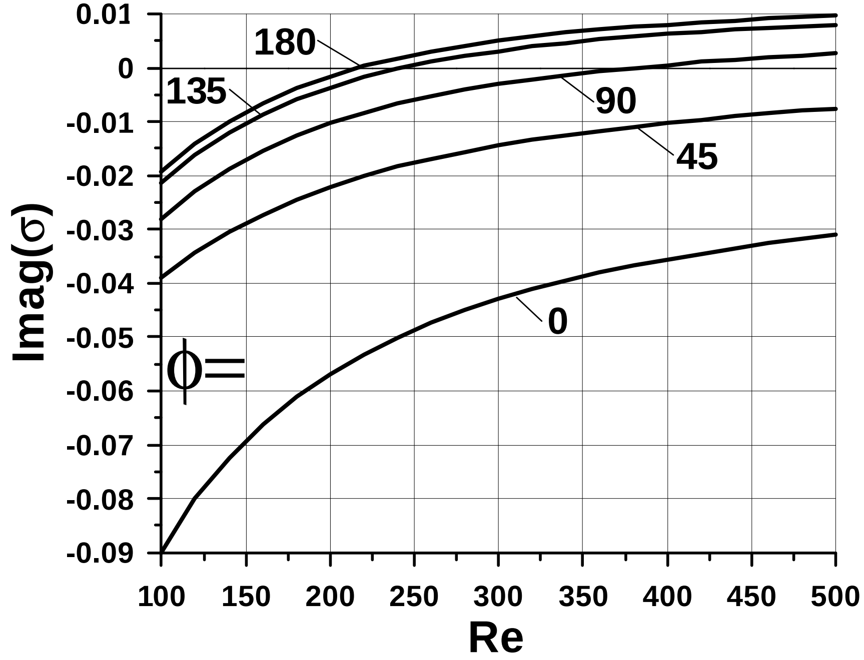
Re=250, S=0.1, φ=180^0 (symmetry)

Re=500, α=1, β=0.5

THE NORMAL MODES ANALYSIS (11)

The effect of the phase shift ϕ (continued)

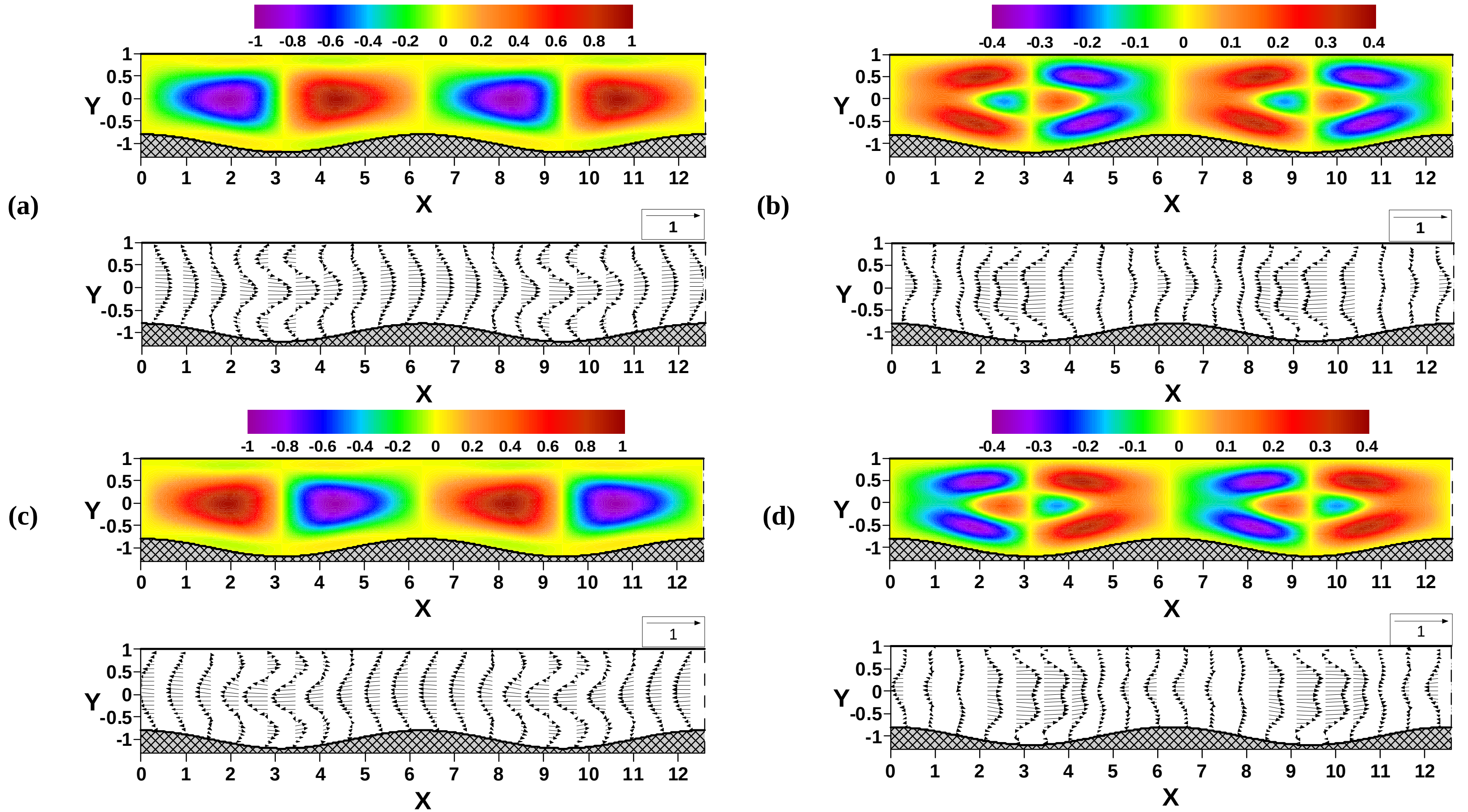




$S=0.1, \alpha=1, \beta=0.5$

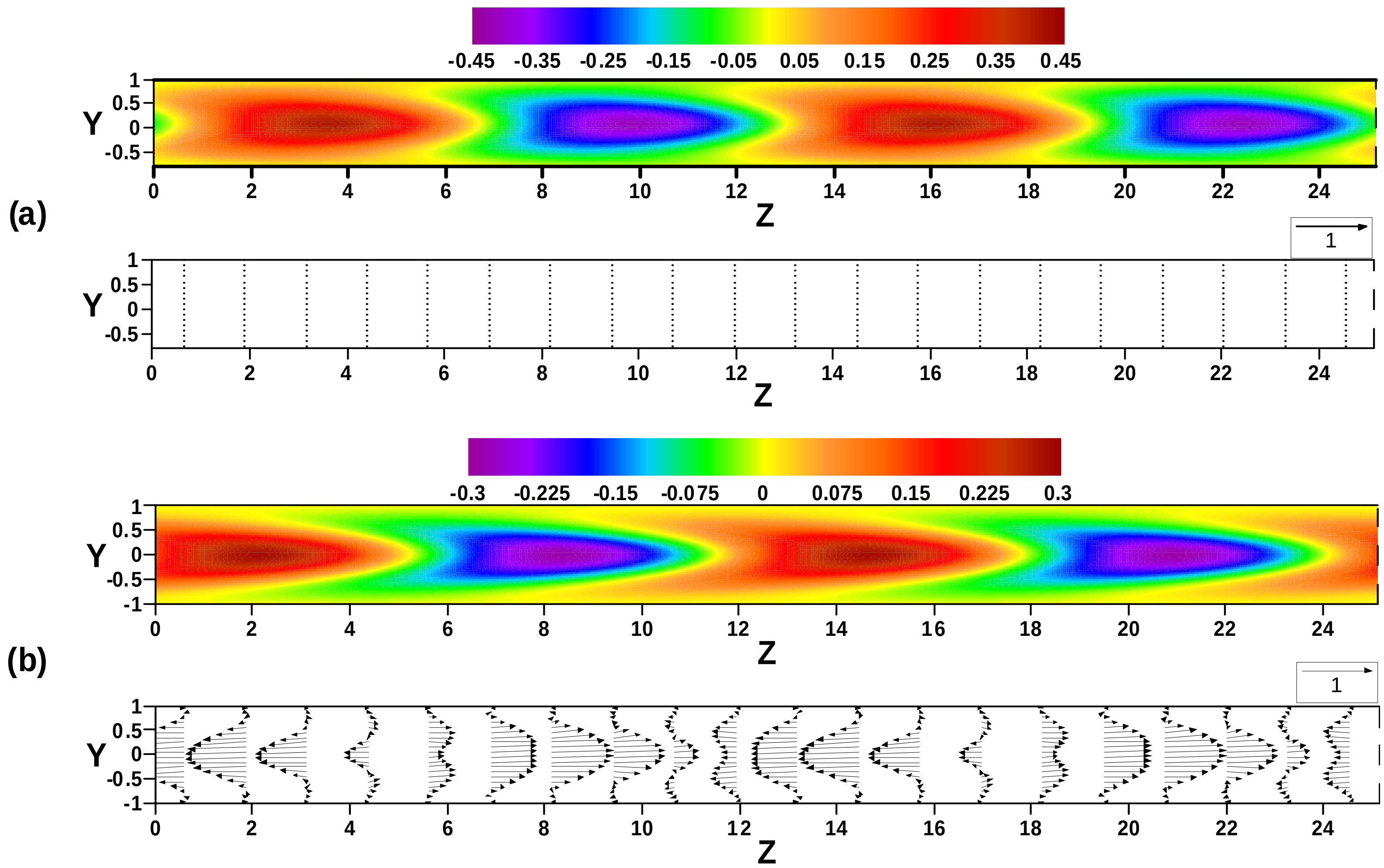
THE NORMAL MODES ANALYSIS (12)

Structure of the disturbance velocity field for the unstable mode Sq_1



The structure of the disturbance velocity field of the mode Sq_1 calculated at various spanwise sections: (a) z=0, (b) $z=0.25\lambda_Z$ (c) $z=0.5\lambda_Z$ and (d) $z=0.75\lambda_Z$. The wall parameters S=0.2 and α=1, wave numbers β=0.5 and δ_*=0, the Reynolds number Re=250.

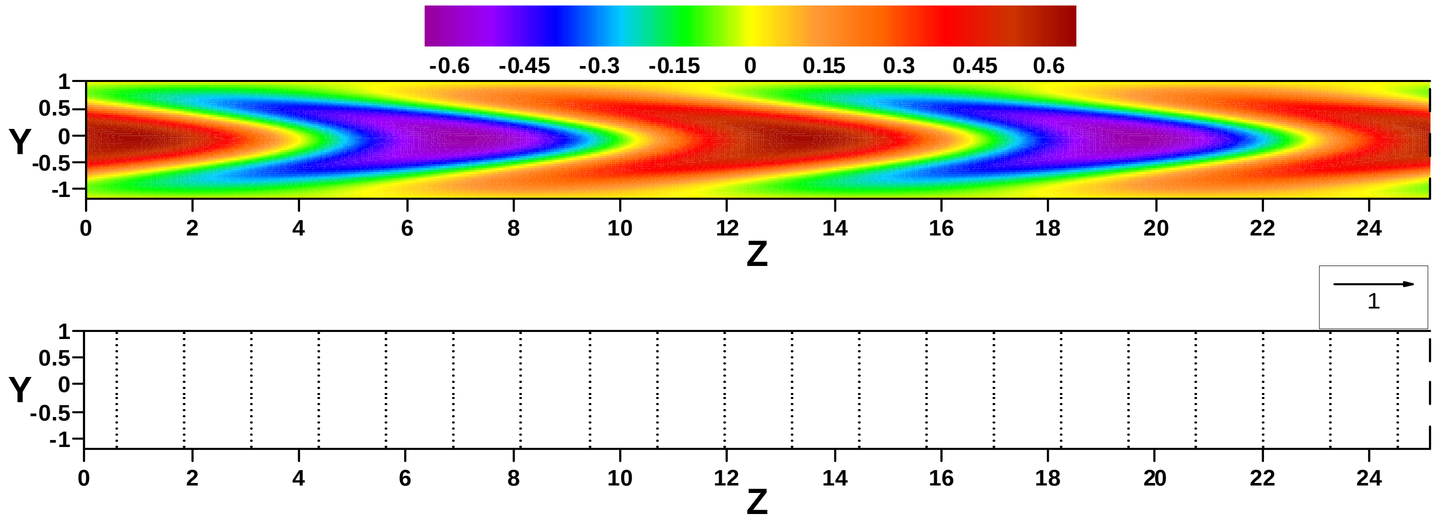
THE NORMAL MODES ANALYSIS (13)



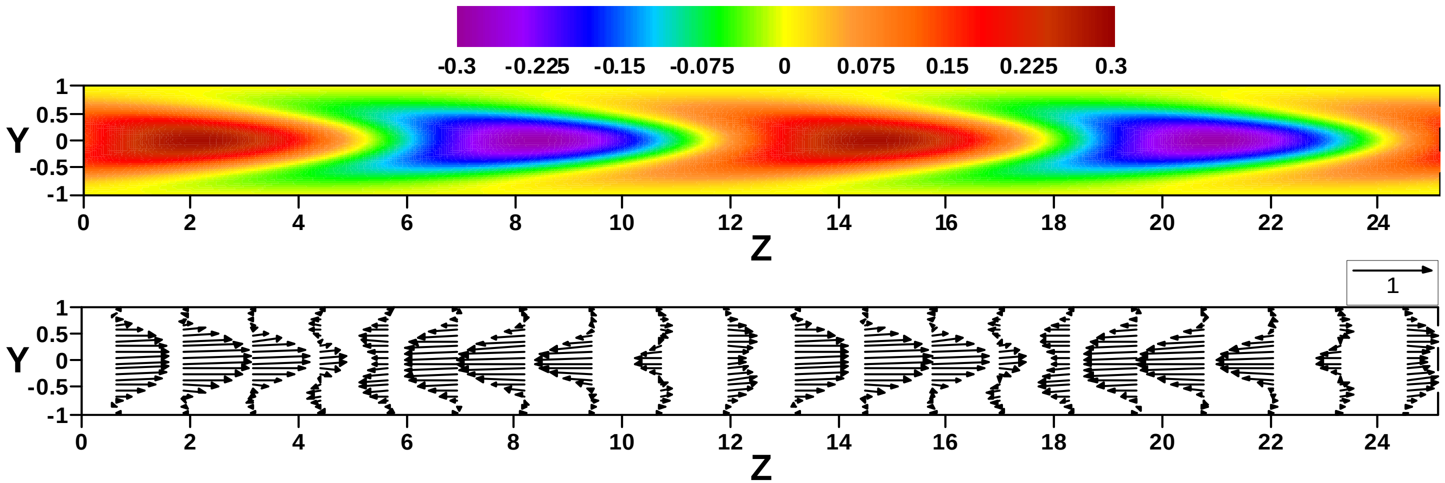
Structure of the velocity field of the mode Sq_1 calculated at different streamwise sections: (a) x=0, (b) $x=0.25\lambda_X$. ($S=0.2$, $\alpha=1$; $\beta=0.5$, $\delta_*=0$; $Re=250$)

THE NORMAL MODES ANALYSIS (14)

(c)

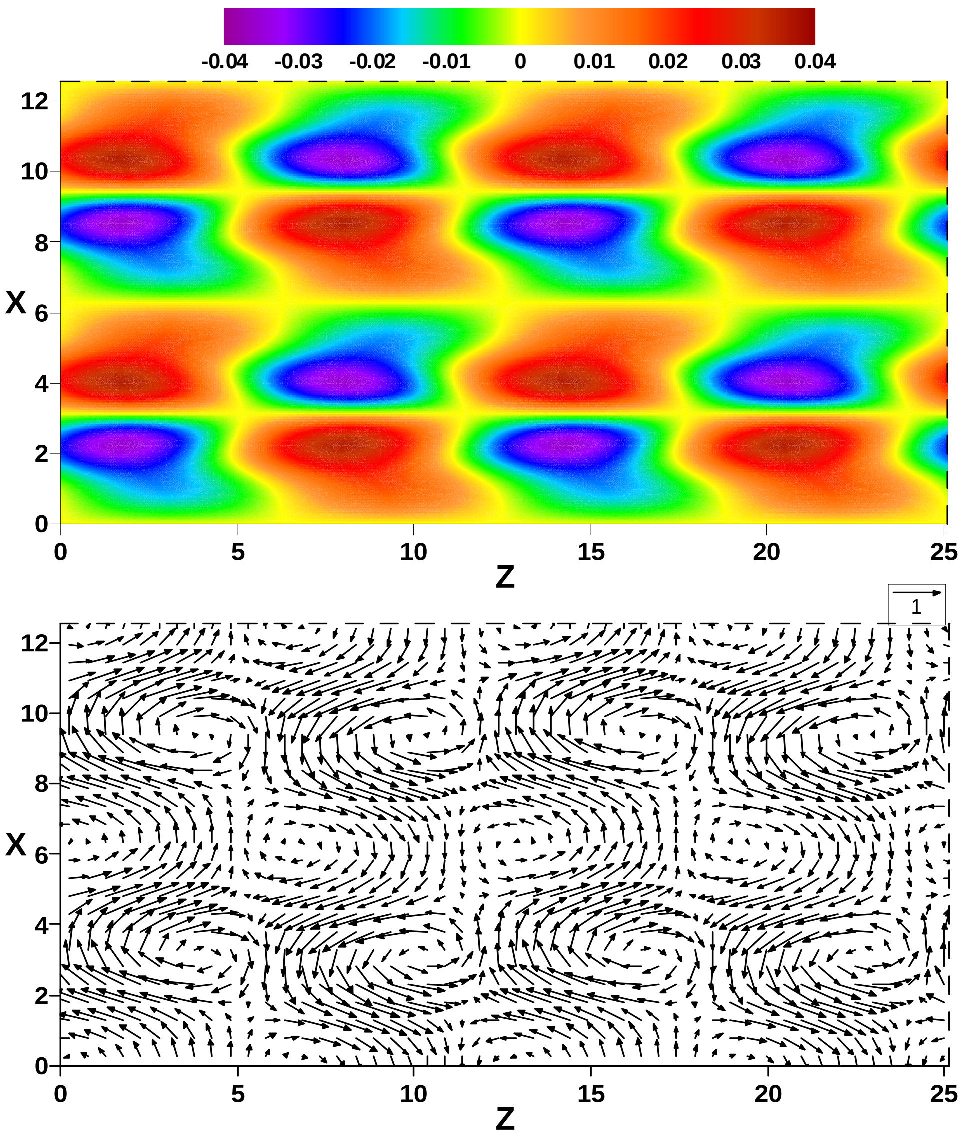


(d)



Structure of the velocity field of the mode Sq_1 calculated at different streamwise sections (continued): (c) $x=0.5\lambda_X$, (d) $x=0.75\lambda_X$. ($S=0.2$, $\alpha=1$; $\beta=0.5$, $\delta_*=0$; $Re=250$)

THE NORMAL MODES ANALYSIS (15)



Structure of the velocity field of the unstable mod Sq_1 computed in the horizontal plane y=0. (S=0.2, α=1; β=0.5, δ_*=0; Re=250)

THE SUMMARY OF THE NORMAL MODES ANALYSIS

1. Transversal wall corrugation with $\alpha \approx 0.9$ (the corrugation wave length λ_X is about 3.5 times larger than the average wall distance) is the most efficient in destabilizing the fundamental Squire mode with $\delta_* = 0$ and $\beta \approx \alpha/2 \approx 0.4$. The computational results show drastic reduction of the critical Reynolds number. For instance, for the one-sided corrugation with the amplitude S=0.2 it can be as low as 200. For both-sided symmetric waviness with S=0.4 and α=1.13 the value of Re_L is approximately equal to 58.6 (!!!).

2. Analysis of different wall shapes leads to the conclusion that the destabilization effect depends on the spectral structure of the corrugation rather than its total magnitude. Radical reduction of the critical Reynolds number is obtained when Fourier harmonics from the certain range of wave numbers are present.

3. In the case of both-sided corrugation the destabilization depends strongly on the phase shift. The strongest effect is achieved for the sinuous (or varicose) geometry, while it practically disappears for the serpentine (or snake-like) geometry. It is directly connected to the amount of the spanwise modulation of the basic velocity profile near the central plane of the channel: larger modulation means less stable flow.

4. The velocity field of the unstable Squire mode has mostly horizontal components, the vertical velocity is by one order of magnitude smaller. The space-periodic patterns of inclined vortices are observed.