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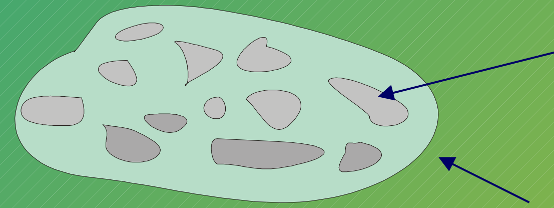
THE BEST ESTIMATIONS OF STIELTJES FUNCTIONS AND TEIR APPLICATIONS IN MECHANICS OF INHOMOGENEOUS MEDIA

Abstrtact

Starting from the truncated power expansions at real points and infinity it has been established in a unified and coherent form the two methods of estimation of a Stieltjes function called S- and T- multipoint continued fraction methods. As practical applications the bounds on effective transport coefficients of two –phase media with periodical microstructure has been calculated.

1. Subject of investigations
2. Approximation of a Stieltjes function
3. Estimation of a Stieltjes function
4. Relations for inclusion regions
5. Inequalities for Padé approximants
6. Exchangeable power series
7. *S*- multipoint continued fraction method
8. *T*- multipoint continued fraction method
9. Comparison with earlier results
10. Conclusion

1. Subject of investigations



$$z=\left(\frac{\lambda_2}{\lambda_1}-1\right)$$

Two-phase composite

$$Q(z)=\int_0^1 \frac{d\gamma(u)}{1+zu},\ d\gamma(u)\ge 0,\ z\in \mathbf{C}\setminus[-\infty,-1],\ Q(-1)\le 1.$$

Conductivity coefficient

Approximation of conductivity coefficient

$$Q(z)=\sum_{i=0}^{p_j-1} c_{ij}(z-x_j)^i+O((z-x_j)^{p_j}),\ \ x_j\in \mathbf{R},\quad j=1,2,\dots,N,$$

Expansions at x_j

$$zQ(z)=\sum_{i=0}^{p_\infty-1} c_{i\infty}\left(\frac{1}{z}\right)^i+O\left(\frac{1}{z}\right)^{p_\infty}$$

Expansions at infinity

$$Q(z)=c_{i(N+1)}+O(z+1)=Q(-1)+O(z+1),\ Q(-1)\le 1.$$

Expansion at -1

2. Approximation of Stieltjes functions

Two-phase composite

$$f_1(z) = \int_0^{1/\rho} \frac{d\gamma(u)}{1+zu},\ d\gamma(u) \ge 0,\ z \in \mathbf{C} \setminus [-\infty, -\rho],\quad f_1(\zeta) \le \eta.$$

Stieltjes function

Approximation of Stieltjes functions

$$f_1(z) = \sum_{i=0}^{p_j - 1} c_{ij}(z - x_j)^i + O((z - x_j)^{p_j}),\ \ x_j \in \mathbf{R},\quad j = 1, 2, \ldots, N.$$

Expansion at x_j

$$z f_1(z) = \sum_{i=0}^{p_\infty - 1} c_{i\infty}\left(\frac{1}{z}\right)^i + O\left(\frac{1}{z}\right)^{p_\infty}.$$

Expansions at infinity

$$f_1(z) = c_{i(N+1)} + O(z+1) = Q(\xi) + O(z - \xi),\ Q(\xi) \le \eta.$$

Expansion at -1

3. Estimations of Steltjes functions

$$f_1(z)\,{}^{p_1,p_2,\dots,p_N,1}_{x_1,x_2,\dots,x_N,\xi} = \left\{ f_1(z)^{p1}_{x_1}, f_1(z)^{p2}_{x_2}, \dots, f_1(z)^{pN}_{x_N}, f_1(z)^{p_\infty}_{\infty}, f_1(z)^{1}_{\xi} \right\},$$

$$f_1(z)^{p_j}_{x_j} = \sum_{i=0}^{p_j-1} c_{ij}(z-x_j)^i + O((z-x_j)^{p_j}),\ j=1,2,\dots,N$$

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$$f_1(z)^{p_\infty}_{\infty} = \sum_{i=0}^{p_\infty-1} c_{i\infty}\left(\frac{1}{z}\right)^i + O\left(\frac{1}{z}\right)^{p_\infty}, \quad f_1(z)^1_\xi = \eta + O(z-\xi).$$

Notations of truncated power expansions of $f_1(z)$

$$\Gamma\,{}^{p_1,\dots,p_N,p_\infty 1}_{x_1,\dots,x_N,\infty,\xi} = \left\{ f_1;\ f_1(z) = \int_0^\infty \frac{d\gamma_1(u)}{1+zu},\ f_1(z)\,{}^{p_1,\dots,p_N,p_\infty,1}_{x_1,\dots,x_N,\infty,\xi},\ f_1(\xi)=\eta,\ d\gamma_1(u)\ge 0 \right\}.$$

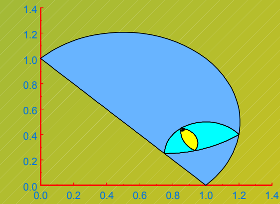
Set of all Stieltjes func-tions $f_1(z)$ satisfying in-put data

$$\Phi_{P+p_\infty,1}(z) = \Gamma\,{}^{p_1,p_2,\dots,p_N,p_{\infty,}1}_{x_1,x_2,\dots,x_N,\infty,\xi}(z),\ \ P=\sum_{j=1}^{N} p_j + 1,\ f_1(z)\in\Phi_{P+p_\infty,1}(z).$$

Inclusion region for $f_1(z)$

$$\phi_{P+p_\infty,1}(z) = \left\{ F_{P+p_\infty,1}(z,u);\ -1\le u\le 1 \right\}.$$

Complex boundaries for $f_1(z)$



$$F_{P+p_\infty,1}(z,u);\ -1\le u\le 1$$

Bounding function for $f_1(z)$

4. Relations for inclusion regions

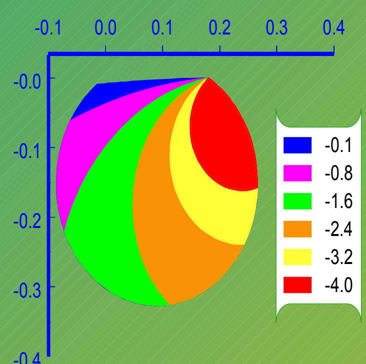
Let power expasion of Stieltjes function $f_1^{\xi\eta}(z)$ *be given*

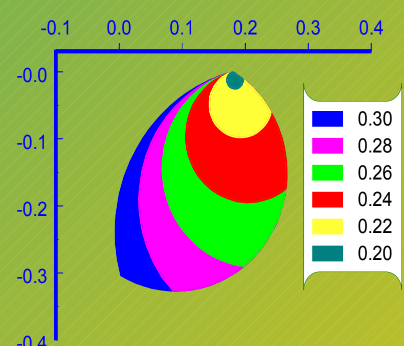
$$f_1^{\xi_1,\eta_1}(z) = f_1(z)_{\eta_1}\begin{matrix} p_1^I,\dots,p_N^I,p_\infty^I 1 \\ \\ x_1,\dots,x_N,\infty,\xi_1 \end{matrix}\quad , \qquad f_1^{\xi_2,\eta_2}(z) = f_1(z)_{\eta_2}\begin{matrix} p_1^{II},\dots,p_N^{II},\ p_\infty^{II},1 \\ \\ x_1,\dots,x_N,\ \infty,\ \xi_2 \end{matrix}$$

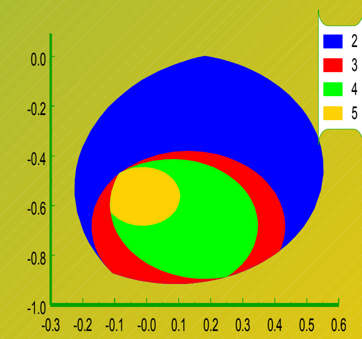
For $z \in \mathbf{C} \setminus [-\infty,-\rho],\ x_j \in \mathbf{R},\ j = 1,2,\dots,N$ *inclusion regions* $\Phi_{P,1}^{\xi,\eta}(z)$ *satisfy the following relations*

$$f_1^{\xi_1,\eta_1}(z) \in \Phi_{P_I,1}^{\xi_1,\eta_1}(z) \subset \Phi_{P_{II},1}^{\xi_2,\eta_2}(z),$$

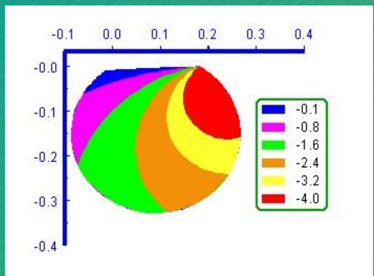
provided that $\xi_1 \le \xi_2\ ,\ \eta_1 \le \eta_2\ ,\ P_{II} \le P_I.$

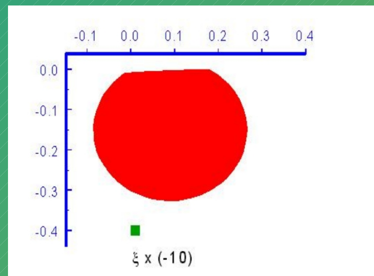


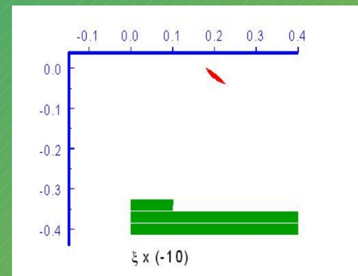




5. Graphic illustration for fundamental inclusion relations



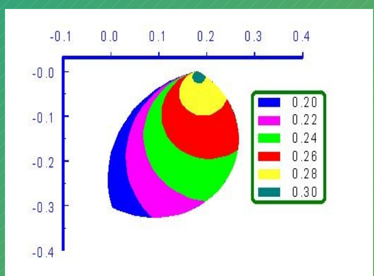


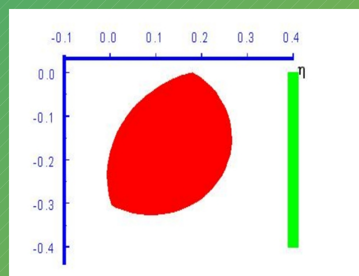


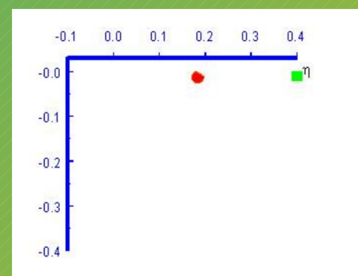
$$f_1(z) = \eta + O(z-\xi), \quad f_1(z) = g_1 + O(z)$$

$$F_{2,1}^{\xi,\eta}(z,u) = \frac{g_1}{1 + z\dfrac{(\eta - g_1)}{-\eta\xi}F_1(z-\xi,u)}$$

$$-4 \le \xi \le -0.1, \; \eta = 0.3, \; P = 2.$$



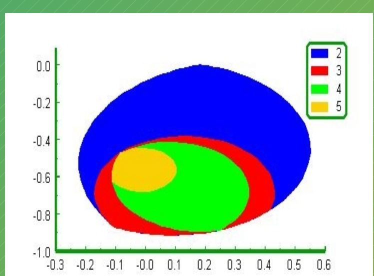


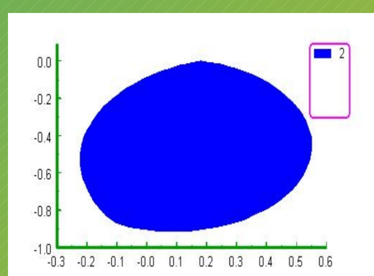


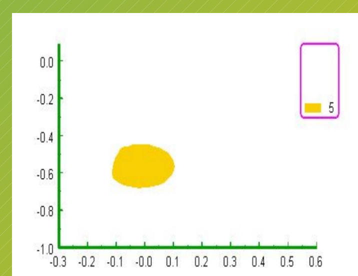
$$f_1(z) = \eta + O(z-\xi), \quad f_1(z) = g_1 + O(z)$$

$$F_{2,1}^{\xi,\eta}(z,u) = \frac{g_1}{1 + z\dfrac{(\eta - g_1)}{-\eta\xi}F_1(z-\xi,u)}.$$

$$\xi = -2, \quad 0.2 \le \eta \le 0.3, \; P = 2.$$







$$f_1(z) = \eta + O(z-\xi), \; f_1(z) = 0.187 + O(z-2),$$

$$f_1(z) = 0.115 + (z-5), \; f_1(z) = \frac{1}{z}\left(1 - 4.02\frac{1}{z} + O\left(\frac{1}{z}\right)^2\right)$$

$$\xi = -1, \; \eta = 0.4694, \; 2 \le P \le 5.$$

6. Basic inequalities for Padé approximants

Padé approksimants $F_{P+p_\infty,1}(x,0)$ ***and*** $F_{P+p_\infty,1}(x,-1)$ ***constructed for power series of Stieltjes***

$$f_1(x) = f_1(x) = f_1(z)\begin{array}{l} p_1^I,\ldots,p_N^I,p_\infty^I,1 \\ \\ x_1,\ldots,x_N,\infty,\xi \end{array}$$

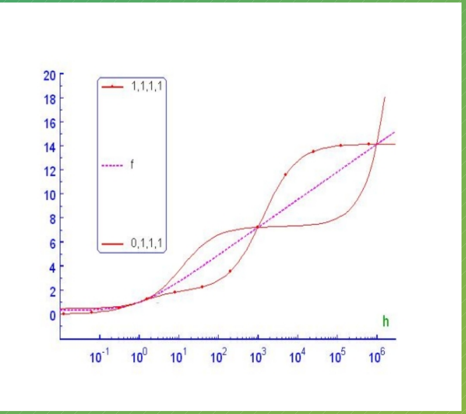
satisfy the following inequalities

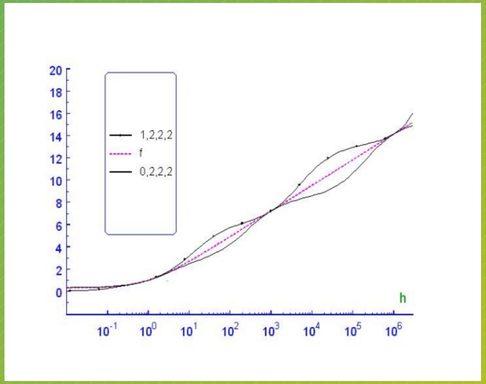
$$(-1)^{L_P(x)}F_{(P+p_\infty),1}(x,0) \le (-1)^{L_P(x)} f_1(x) \le (-1)^{L_P(x)}F_{(P+p_\infty),1}(x,-1),$$

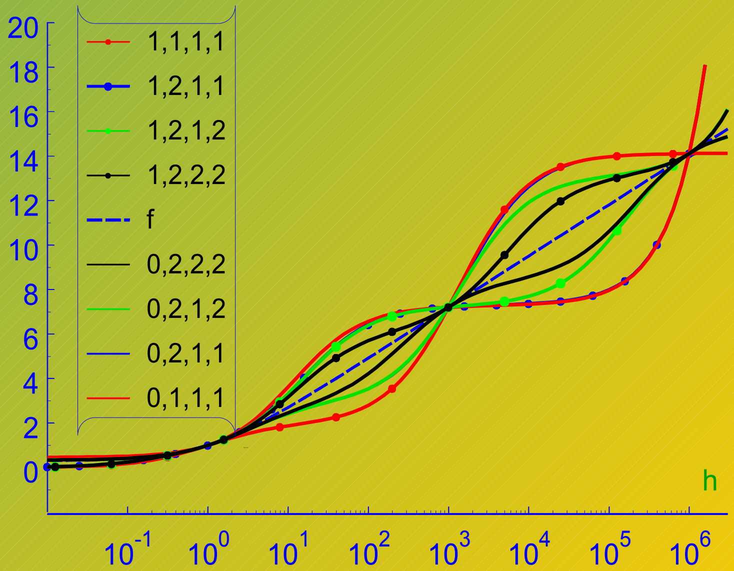
where step function

$$L_P(x) = L_P(x) = \sum_{j=1}^{N} p_j H(x-x_j)+1,\; P=\sum_{i=1}^{N} p_i + 1.$$

depends on input data only







7. Exchangeable power series

$$f_1(z)=\sum_{i=0}^{p_j-1} c_{ij}(z-x_j)^i+O((z-x_j)^{p_j}),\quad x_j\in\mathbf{R},\quad j=1,2,\dots,N,$$

$$zf_1(z)=\sum_{i=0}^{p_\infty-1} c_{i\infty}\left(\frac{1}{z}\right)^i+O\left(\frac{1}{z}\right)^{p_\infty}$$

$$f_1^{x_{N+1}}(z)=\frac{1}{z}\sum_{i=0}^{p_\infty-1} c_{i\infty}\left(\frac{1}{z}-\frac{1}{x_{N+1}}\right)^i+O\left(\left(\frac{1}{z}-\frac{1}{x_{N+1}}\right)^{p_\infty}\right).$$

$$f_1^{x_{N+1}}(z)=\sum_{i=0}^{p_{N+!}-1} c_{i(N+1)}(z-x_{N+1})^i+O((z-x_{N+1})^{p_{N+1}}).$$

$$f_1(z)=c_{i(N+1)}+O(z-\xi)=Q(\eta)+O(z-\xi),\ Q(\xi)\le\eta.$$

$$F_{0+3,1}(z,-1)=\frac{c_{0\infty}\frac{1}{z}}{1-\frac{c_{1\infty}}{c_{0\infty}}\frac{1}{z}}$$

Example

$$f_1(z)=\frac{1}{z}\left(c_{0\infty}+c_{1\infty}\left(\frac{1}{z}\right)+O\left(\left(\frac{1}{z}\right)^2\right)\right)$$

$$f_1^{x_{N+1}}(z)=\frac{c_{0\infty}}{x_{N+1}}-\frac{c_{0\infty}x_{N+1}+c_{1\infty}}{x_{N+1}^3}(z-x_{N+1})+O\left((z-x_{N+1})^2\right),$$

$$F_{0+3,1}^{x_{N+1}}(z,-1)=\frac{\frac{c_{0\infty}}{x_{N+1}}}{1+\frac{\frac{c_{0\infty}x_{N+1}+d_1(\infty)}{x_{N+1}^3}(z-x_{N+1})}{\frac{c_{0\infty}}{x_{N+1}}}}.$$

Block diagram of the S-multipoint continued fraction method

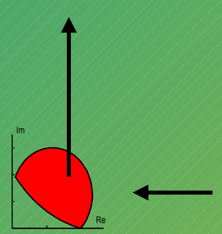
$$f_1(z) = \eta + O(x-\xi),\ f_1(z) = \sum_{i=0}^{p_j-1} c_{ij}(z\text{-}x_j)^i + O((z\text{-}x_j)^{p_j}),\quad \xi < min(x_j, j = 1,...N)$$

Multipoint S-transformations

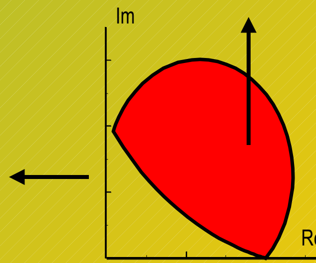
$$f_1(z) = \mathop{\bigvee}_{j=1}^{N} \mathop{\bigvee}_{i=P_{j-1}+1}^{p_j} \frac{g_i}{1+(z-x_j)} \times \frac{w_P \overline{f}_P(z)}{1}$$

$\Phi_{P,1}(z)$

$\Phi_{1,P}(z)$



$$F_{P,1}(z,u) = \mathop{\bigvee}_{j=1}^{N} \mathop{\bigvee}_{i=P_{j-1}+1}^{p_j} \frac{g_i}{1+(z-x_j)} \times \frac{w_P F_1(z-\xi, u)}{1}$$



$f_1(z) \in \Phi_{P,1}(z)$ $\qquad z \in C \setminus (-\infty, \xi)$ $\qquad f_P(z) \in \Phi_{1,P}(z)$

Examples

Initial Input Data

$$f_1(z)=1+O(z),\ f_1(z)=\frac{1}{7}+O(z-3)$$

Parametric Input Data

$$f_1(z)=1+O(z),\ f_1(z)=\frac{1}{7}+O(z-3),\ f_1(z)=\eta+O(z-\xi)$$

Parametric Estimation

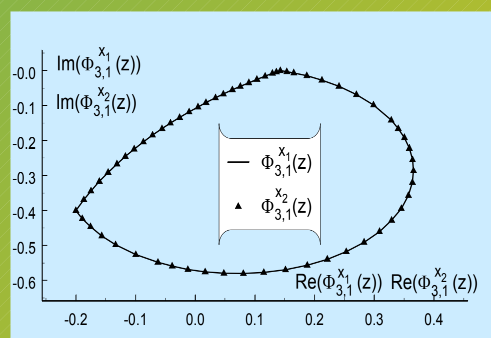
$$F^{\eta,\xi}_{3,1}(z,u)=\cfrac{1}{1+\cfrac{2z}{1-\cfrac{(\eta-2\eta\xi-1)(z-3)}{(\eta\xi-3\eta-\xi+3)}F_1(z,u)}}$$

Optimal Estimation

$$F_{3,1}(z,u)=\lim_{\xi\to 0}\lim_{\eta\to\infty}\cfrac{1}{1+\cfrac{2z}{1-\cfrac{(\eta-2\eta\xi-1)(z-3)}{(\eta\xi-3\eta-\xi+3)}F_1(z,u)}}$$

Final Result

$$F_{3,1}(z,u)=\cfrac{1}{1+\cfrac{2z}{1+\cfrac{(z-3)}{3}F_1(z,u)}}$$



Initial Input Data

$$f_1(z)=\frac{1}{7}+O(z-3),\ f_1(z)=1+O(z)$$

Parametric Input Data

$$f_1(z)=\frac{1}{7}+O(z-3),\ f_1(z)=1+O(z),\ f_1(z)=\eta+O(z-\xi)$$

Parametric Estimation

$$F^{\eta,\xi}_{3,1}(z,u)=\frac{1}{7}\cfrac{1}{1+\cfrac{2}{7}\cfrac{z-3}{1-\cfrac{(\eta-2\eta\xi-1)z}{\xi(7\eta-1)}F_1(z,u)}}$$

Optimal Estimation

$$F_{3,1}(z,u)=\lim_{\xi\to 0}\lim_{\eta\to\infty}\frac{1}{7}\cfrac{1}{1+\cfrac{2}{7}\cfrac{z-3}{1-\cfrac{(\eta-2\eta\xi-1)z}{\xi(7\eta-1)}F_1(z,u)}}$$

Final Result

$$F_{3,1}(z,\tau)=\frac{1}{7}\cfrac{1}{1+\cfrac{2}{7}(z-3)F_1(z,\tau)}$$

Block diagram of the T-multipoint continued fraction method

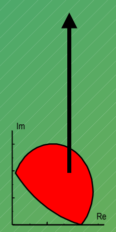
$$f_1(z)=\sum_{i=0}^{p_j-1} c_{ij}(z-x_j)^i+O((z-x_j)^{p_j}),\ zf_1(z)=\sum_{i=0}^{p_j-1} c_{i\infty}\left(\frac{1}{z}\right)^i+O\left(\frac{1}{z}\right)^{p_\infty},$$

$$f_1(z)=\eta+O(x-\xi),\quad \xi<min(x_j,\ j=1,\dots N)$$

Multipoint T-transformations

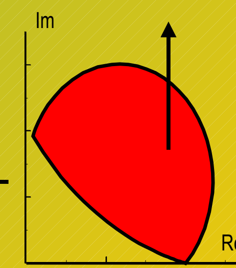
$$f_1(z)=\mathop{\mathrm{V}}_{j=1}^{N}\mathop{\mathrm{V}}_{i=P_{j-1}+1}^{p_j}\frac{g_i}{1+(z-x_j)e_i+(z-x_j)}\times\frac{w_P\bar{f}_P(z)}{1}$$

$\Phi_{P+p_\infty,1}(z)$



$$F_{P+p_\infty,1}(z,u)=\mathop{\mathrm{V}}_{j=1}^{N}\mathop{\mathrm{V}}_{i=P_{j-1}+1}^{p_j}\frac{g_i}{1+(z-x_j)e_i+(z-x_j)}\times\frac{w_P F_1(z-\xi,u)}{1}$$

$\Phi_{1,P}(z)$



$f_1(z)\in\Phi_{P+p_\infty,1}(z)$

$z\in C\backslash(-\infty,\ \xi)$

$f_P(z)\in\Phi_{1,P}(z)$

Numerical example

$$f_1(z) = \frac{1}{z}\left(1 + \frac{2.5}{z}\ln\frac{12+z}{20+5z}\right)$$

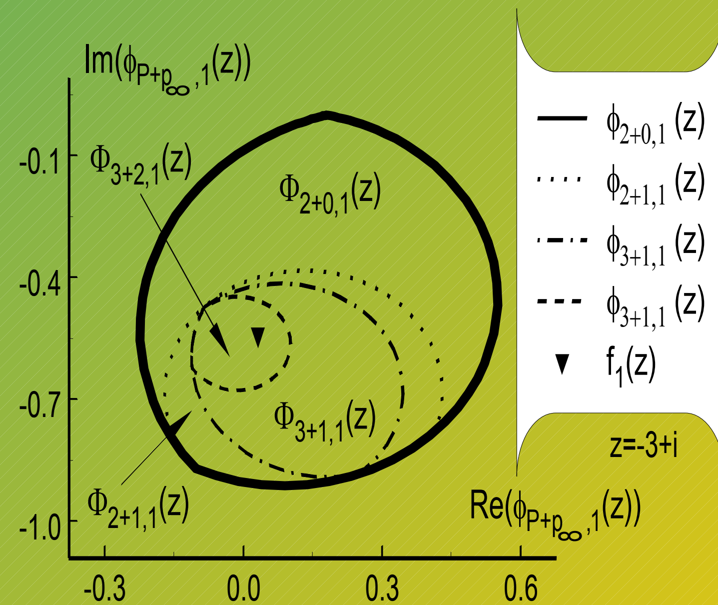
$$f_1(z)_{-1}^{+1} = 0.4694 + O(z+1), \quad f_1(z)_2^1 = 0.18073 + O(z-2),$$

$$f_1(z)_5^1 = 0.11527 + O(z-5), \quad f_1(z)_\infty^2 = \frac{1}{z}\left(1 - 4.0236\frac{1}{z} + O\left(\frac{1}{z}\right)^2\right).$$

$$F_{3+2,1}(z,u) = \cfrac{g_1}{1 + (z-2)e_2 + \cfrac{(z-2)g_2}{1 + (z-5)e_3 + (z-5)F_{1,3}(z,u,e_3)}},$$

$$F_{1,3}(z,u,e_3) = W_3(e_3)F_1(z+1,u),$$

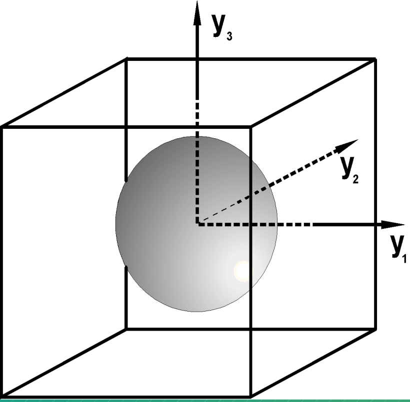
$$g_1 = 0.180733, \; e_2 = 0.18073, \; g_2 = 0.0086, \; e_3 = 0.0967, \; W_3 = 0.0111.$$



Comparison with the results obtained earlier

$$F_{P+p_\infty,1}(z,u)=\mathop{V}_{j=1}^{N}\ \mathop{V}_{i=P_{j-1}+1}^{p_j}\frac{g_i}{1+(z-x_j)e_i+(z-x_j)}\times\frac{w_P F_1(z-\xi,u)}{1}$$

$F_{P,1}(z,u)=\mathop{V}_{j=1}^{N}\ \mathop{V}_{i=P_{j-1}+1}^{P_j}\frac{g_i}{1+(z-x_j)}\times\frac{F_{1,p}(z,u)}{1}$	S. Tokarzewski, Continued fraction approach to the bounds on transport coefficients of two phase media,.IFTR Reports 4 (2005)
$F_{P,1}(z,u)=\mathop{V}_{j=1}^{1}\ \mathop{V}_{i=P_{j-1}+1}^{P_j}\frac{g_i}{1+(z-x_j)}\times\frac{F_{1,p}(z,u)}{1}$	G. Baker, Jr, Essentials of Padé Approximants ,Academic Press, 1975 , Chapter 17, Section A G. Baker, Jr, P. Graves-Morris, Padé Approximants ,Cambridge Press, 1996 , Chapter 5
$F_{P,1}(z,u)=\mathop{V}_{j=1}^{N}\ \mathop{V}_{i=j}^{j}\frac{g_i}{1+(z-x_j)}\times\frac{F_{1,p}(z,u)}{1}$	G. Baker, Jr, Essentials of Padé Approximants, Academic Press, 1975, Chapter 17, Section B



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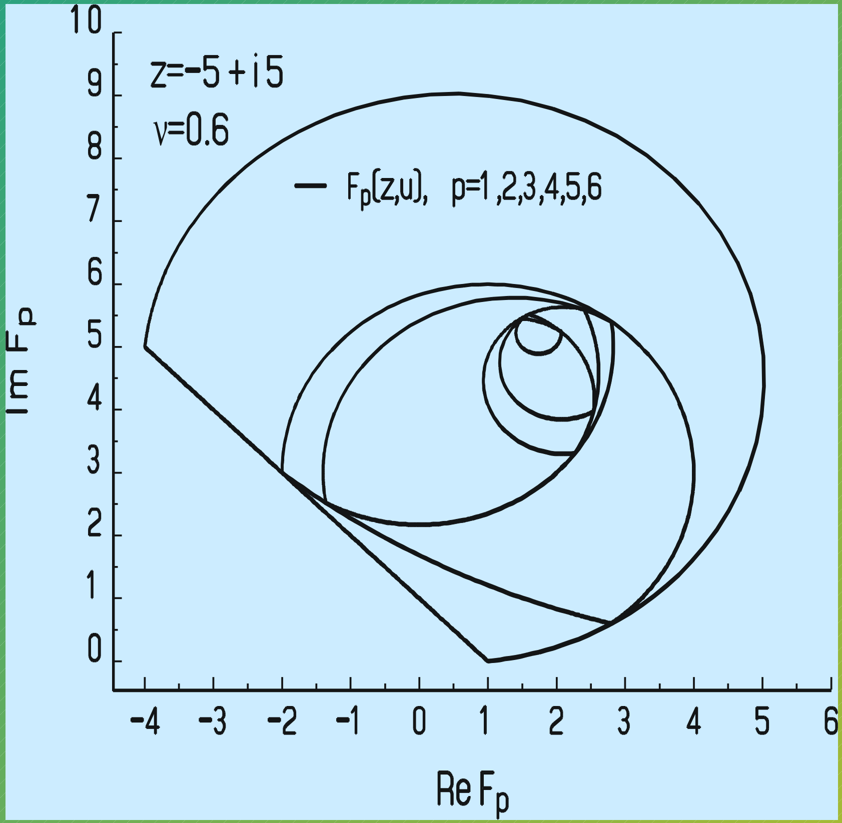
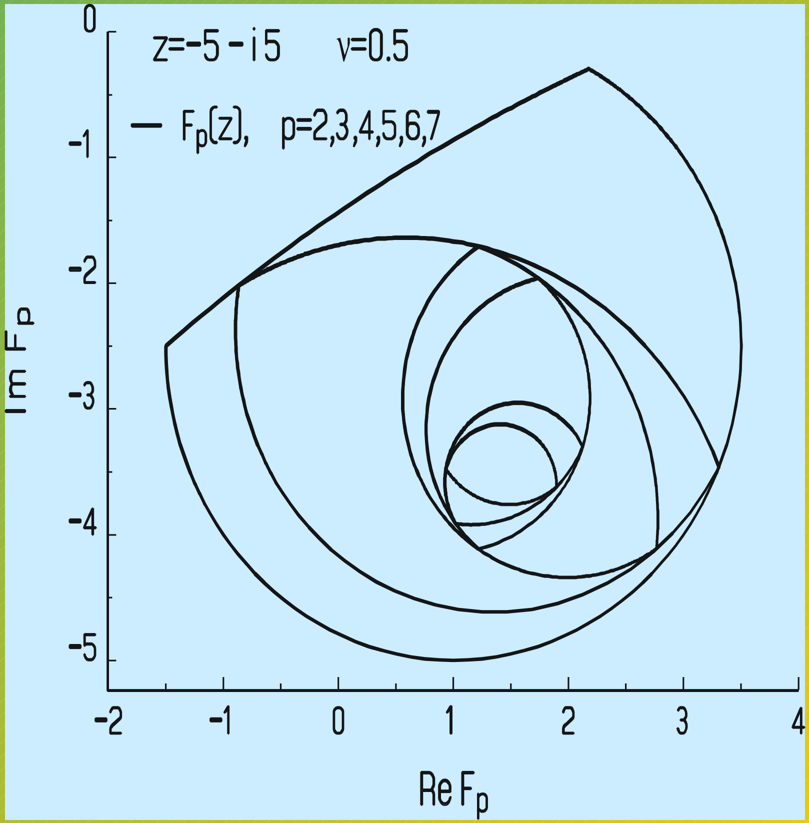
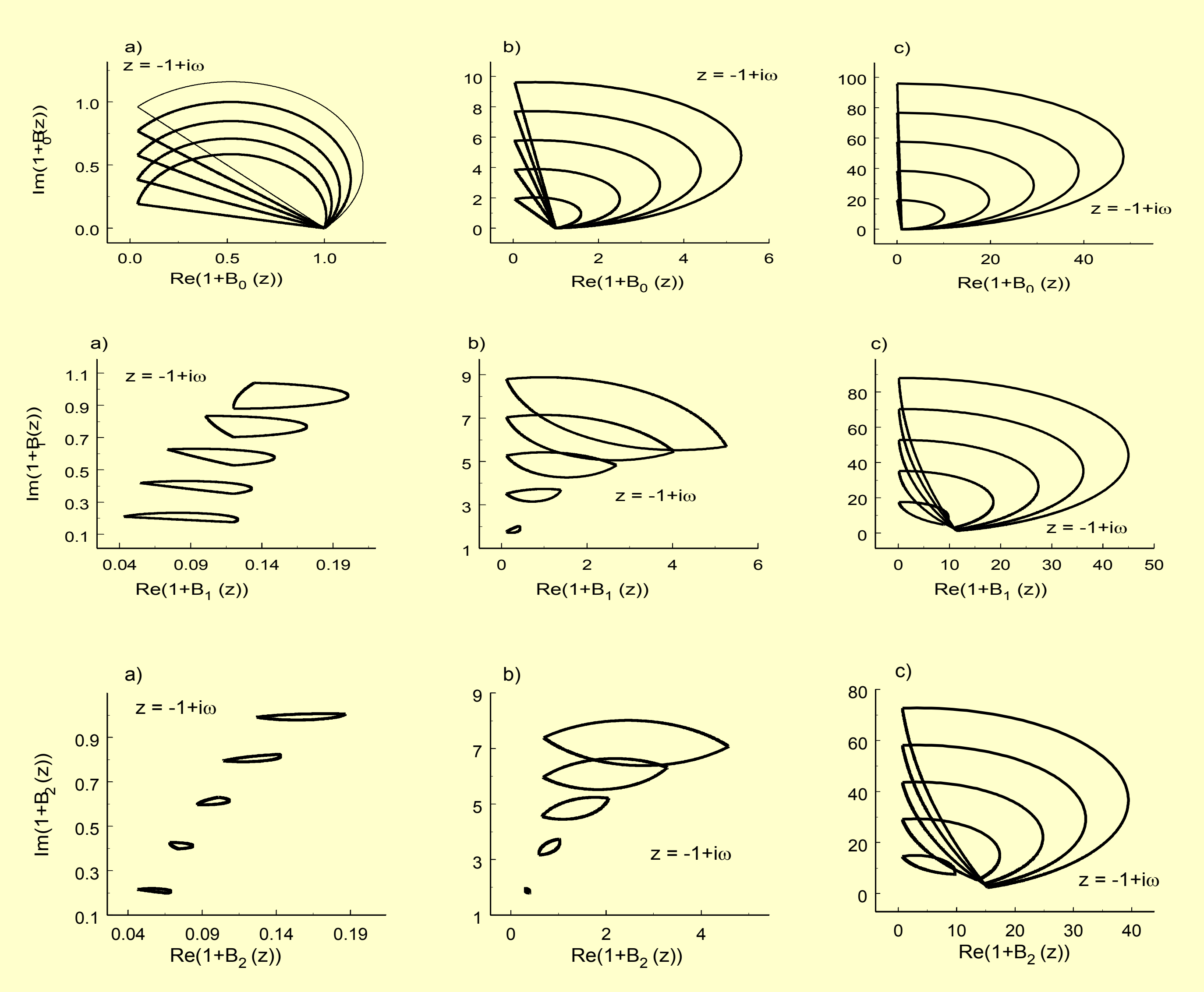
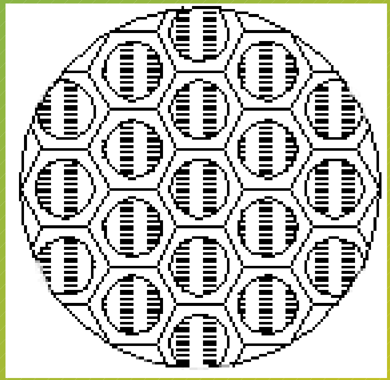




S. Tokarzewski, J.J. Telega, M. Pindor, J. Gilewicz, *A note on total bounds on complex transport moduli of parametric two-phase media*, ZAMP, **54,** 713-726, 2003.





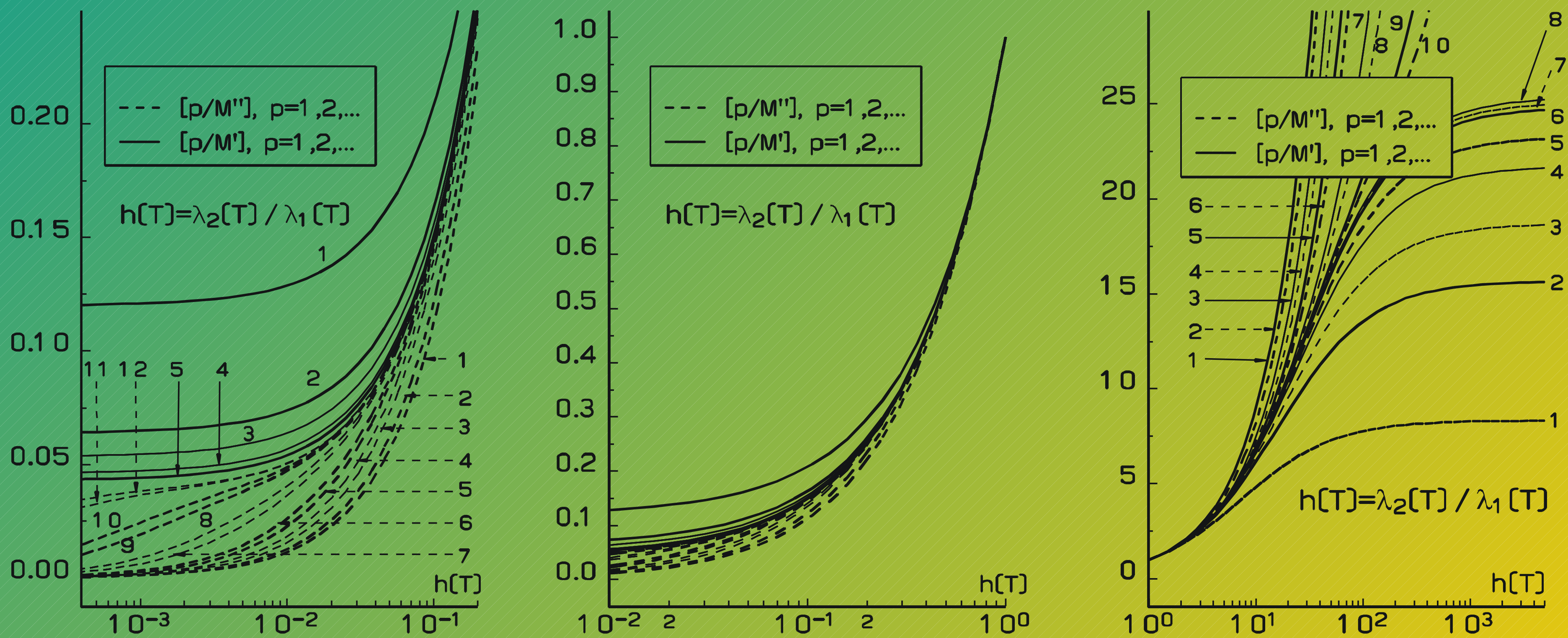
Parametric bounds:

ω=0.2, 0.4, 0.6, 0.8, 1;

ω=2, 4, 6, 8, 10;

ω=20, 40, 60, 80, 100

A. Gałka, J.J.Telega, S.Tokarzewski, Heat Equation with Temperature-Dependent Conductivity Coefficients and Macroscopic Properties of Microheterogeneous Media, Mathematical and Computer Modelling. **33,** 927-942, 1997.



Sequences of Padé approximants forming universal bounds on the effective conductivity of hexagonal array of cylinders with volume fraction ϕ =0.88

Telega, J., Tokarzewski, S., and Gałka, A., Modelling torsional properties of human bones by multipoint Padé approximants, In Numerical Analysis and Its Applications (Berlin 2001), L. Vulkov, J. Waśniewski, and P. Yalamov, Eds., Springer-Verlag, pp. 741-748.

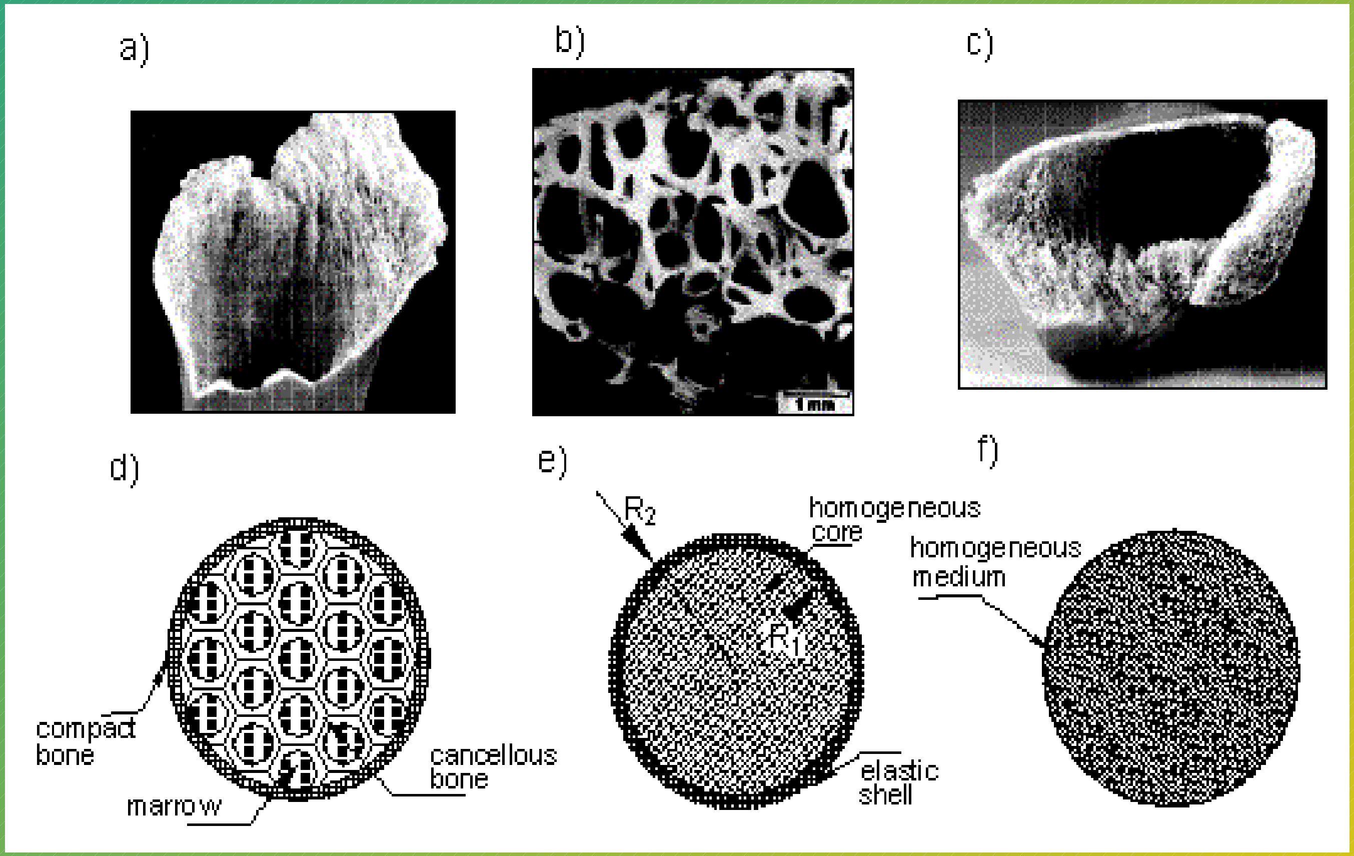
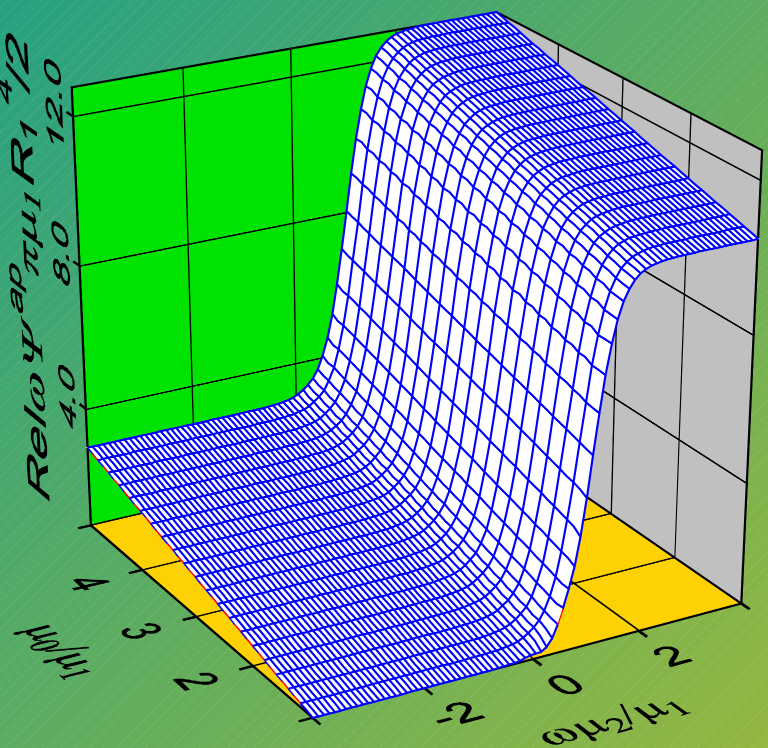
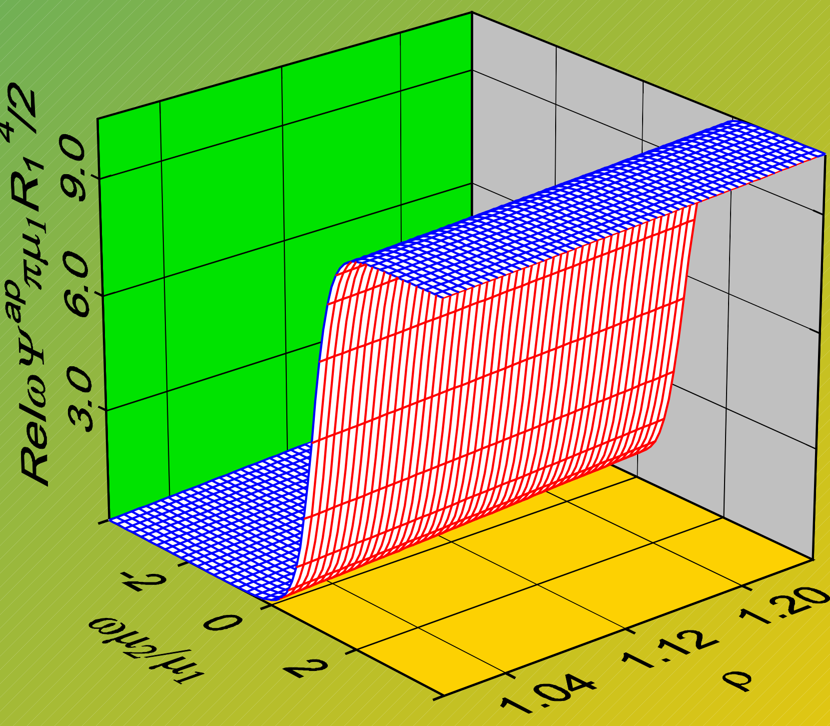


Fig.1. Microstructure of a cancellous bone- a,b,c; Three steps of a process of modelling of human bone- d,e,f

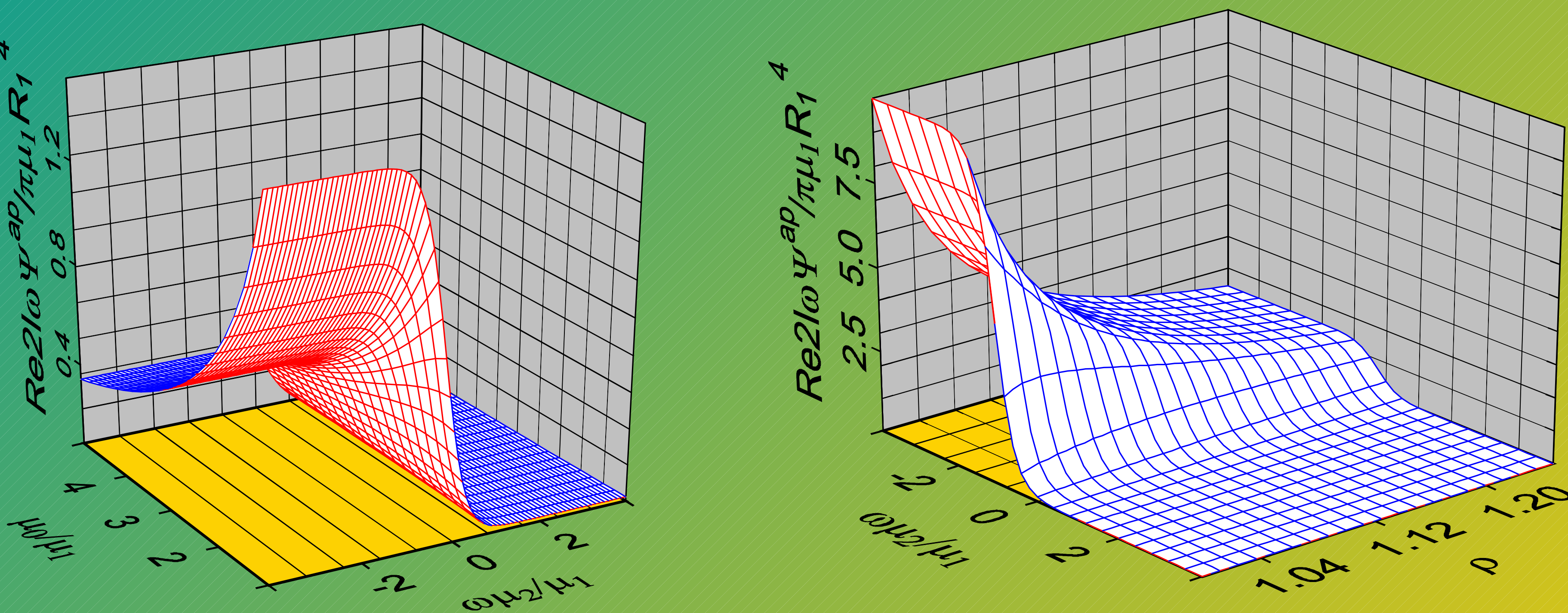
Effective torsion modulus





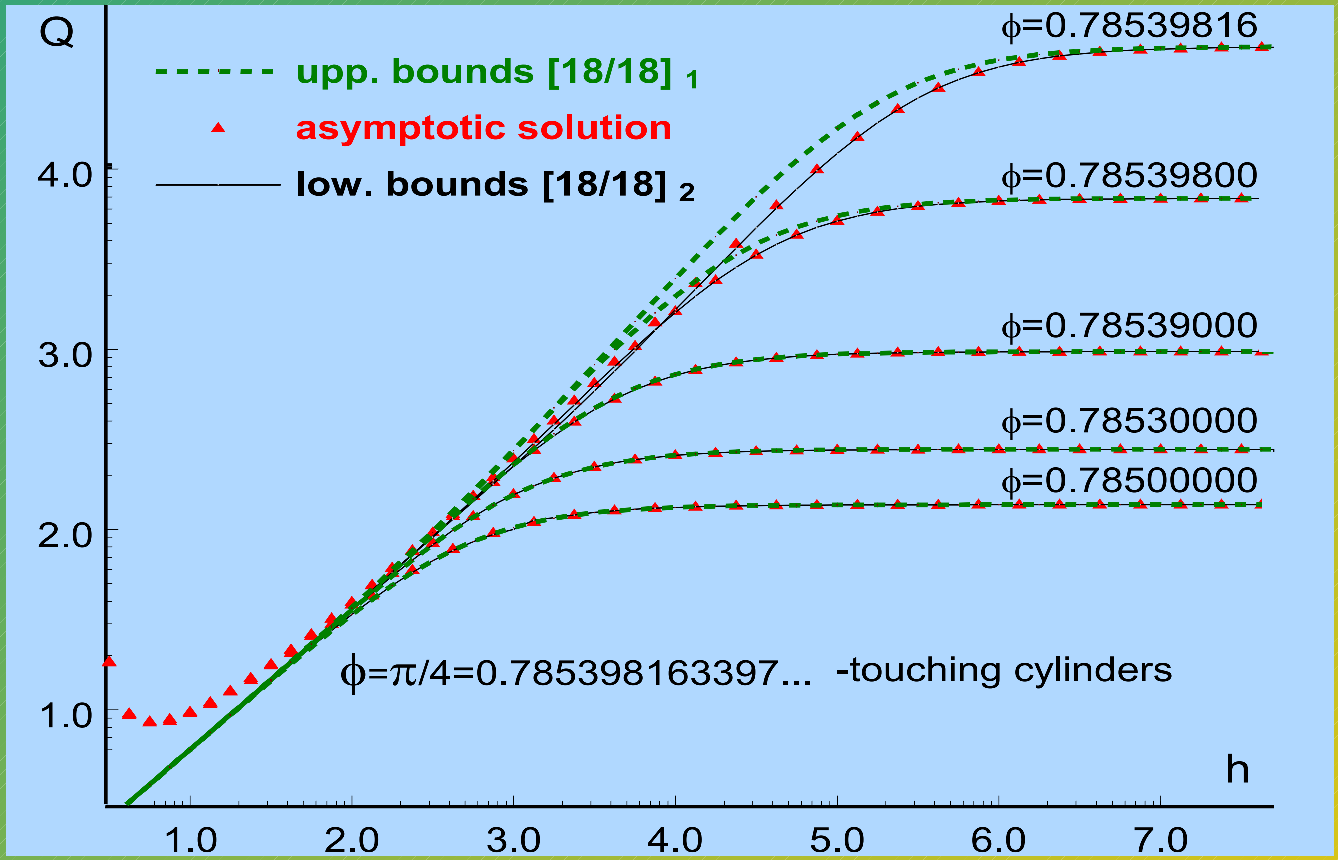
Hydraulic stiffening of a model of human bone

Effective torsional compliance



Hydraulic stffening of a model of human bone

S.Tokarzewski, J.J.Telega, Bounds on efffective moduli by analytical continuation of the Stieltjes function expanded at zero and infinity, Z. angew. Math. Phys. **48,** 1-20, 1997.



Upper and lower bounds of effective conductivity of square array of cylinders

Conclusions

1. For the first time in literature it is incorporated to estimates of Stieltjes function the truncated power expansions available at infinity.
2. S- i T- transformation methods are reccurent ones. Hence they do not reqired to solve the sets of complicated equations.
3. Estimates of Stieltjes functions obtained by means of S- and T- multipoint continued fraction methods are the best. It means that it is not possible to improve them starting from given input data.
4. As an examples of applications several numerical computations have been carried out. The optimal bounds are evaluated on the effective transport coefficients of two phase media such as dielectric constants of the arrays of spheres, thermal conductivities of regular lattices of cylinders and rigidities of a porous bars modelling a macroscopical behaviour of a human bone.