Institute of Fundamental Technological Research PAS
Department of Theory of Continuous Media
Laboratory of Analytical Mechanics and Field Theory

NONLINEAR MODELS OF COLLECTIVE AND INTERNAL DEGREES OF FREEDOM IN MECHANICS AND FIELD THEORY. SYMMETRY PROBLEMS.

by
Vasyl Kovalchuk

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Prof. dr hab. Jan Jerzy Sławianowski

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The White Rabbit put on his spectacles.
'Where shall I begin, please your Majesty?' he asked.
'Begin at the beginning,' the King said gravely,
'and go on till you come to the end: then stop.'
(Lewis Carroll *Alice in Wonderland*)

To my beloved wife, relatives, and friends,
who are making my life better.

Contents

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Nomenclature

Repeated indices are generally summed, unless otherwise indicated (Einstein convention of summation).

Capital Latin indices A, B, C, and so on generally run over the n material coordinate labels, usually taken as $1, 2, \ldots, n$.

Latin indices i, j, k, and so on from the middle of the Latin alphabet generally run over the n spatial coordinate labels, usually taken as $1, 2, \ldots, n$.

Greek indices μ, ν, and so on from the middle of the Greek alphabet generally run over the $(n+1)$ space-time coordinate labels $0, 1, \ldots, n$.

Latin indices r, s, and so on from the second part of the Latin alphabet generally run over the infinite number of coordinate labels, usually taken as $0, 1, \ldots, \infty$.

Greek indices ρ, σ, and so on from the second part of the Greek alphabet generally run over the infinite number of coordinate labels, usually taken as $1, \ldots, \infty$.

The tensor $\varepsilon^{\mu\nu\sigma}$ is defined as the totally antisymmetric quantity with $\varepsilon^{123} = +1$.

A hat over any quantity indicates the co-moving representation of that quantity defined on the material space.

A dot over any quantity denotes the time-derivative of that quantity.

Quantum operators are generally indicated by letters in boldface.

Dirac matrices γ_μ are defined so that

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}.$$

Also, $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ and $\beta = i\gamma^0 = \gamma_4$.

The step function $\theta(s)$ has the value $+1$ for $s > 0$ and 0 for $s < 0$.

The complex conjugate, transpose, and Hermitian adjoint of a matrix or vector A are denoted A^*, A^T, and $A^+ = A^{*T}$, respectively. A Dirac conjugation of bispinors is defined by the following expression:

$$\overline{\Psi} = \Psi^+ \gamma^0.$$

Units are usually used with $\hbar$ and the speed of light taken to be unity. Throughout $-e$ is the rationalized charge of the electron, so that the constant of the fine structure is given as follows:

$$\alpha = e^2/4\pi = 1/137.$$

Notation

All notation is defined where first introduced. Most symbols have only local meaning and so are not listed here. The following symbols are usually assigned the indicated significance:

$\mathcal{M}$	the differential manifold of physical space or space-time
$T_x\mathcal{M}$	its tangent space at the point $x \in \mathcal{M}$
$F\mathcal{M}$	the principle fibre bundle of linear frames in $\mathcal{M}$
Q	the configuration space of the considered problem
$q^1, \ldots, q^n$	the generalized coordinates
$p_1, \ldots, p_n$	the canonical momenta conjugated to the generalized coordinates
$\mathcal{M}, \mathcal{N}$	the physical and material spaces
V, U	the linear spaces of translations which are corresponding to the physical and material spaces $\mathcal{M}$ and $\mathcal{N}$
$\mathrm{Af}(\mathcal{N}, \mathcal{M})$	the manifold of all affine mappings of $\mathcal{N}$ into $\mathcal{M}$ (including non-invertible ones)
$\mathrm{AfI}(\mathcal{N}, \mathcal{M})$	the manifold of affine isomorphisms of $\mathcal{N}$ onto $\mathcal{M}$
$\mathrm{L}(U, V)$	the manifold of all linear mappings of U into V
$\mathrm{LI}(U, V)$	the manifold of linear isomorphisms of U onto the V
$\mathrm{L}(U) = \mathrm{L}(U, U)$	the endomorphisms
$\mathcal{F}(V)$	the manifold of linear frames in V
$\widehat{\eta}$	the metric tensor in the material space
g	the metric tensor in the physical space
a	the Lagrangian radius-vector (material variables)
$x(t, a)$	the Eulerian radius-vector (physical variables)
$\varphi : a \mapsto x(t, a)$	the mapping from the material space to the physical one
${}^{E}v(t, x)^i$	the Eulerian velocity field

${}^{L}v(a)^{i}$	the Lagrangian velocity field
ρ_0	the Lagrangian density of mass
Φ	the density of forces per unit mass
$\mathcal{F} = \rho_0 \Phi$	the Lagrangian density of forces per unit non-deformed volume
f	the Eulerian density of forces per unit deformed volume
H^r	the complete system of real-valued functions
$q^i{}_r$	the coefficient of the expansion of the configurations $x(t, a)$ with respect to the mode functions H^r
M^{ri}	the H^r-moments of the distribution of linear momentum
N^{ri}	the H^r-moments of the distribution of forces within the body
Q^{rs}	the quadrupole moments of the Lagrangian mass distribution (H-collective coefficients of inertia)
$\widehat{J}^{AB}$	the co-moving components of the tensor of inertia
M	the total mass of the body
T	the kinetic energy
$\mathcal{P}$	the power of forces
P^i	the total linear momentum of the body
F^i	the total force acting on the body
$\Omega, \widehat{\Omega}$	the affine quasi-velocity in the laboratory and co-moving reference frames
$\Sigma, \widehat{\Sigma}$	the canonical momenta conjugated to the affine quasi-velocity in the laboratory and co-moving representations
$K, \widehat{K}$	the kinematical affine hypermomentum in the laboratory and co-moving reference frames
ω	the angular velocity of the affine motion
d	the deformation rate
$\widehat{G}, \widehat{E}$	the Green and Lagrange deformation tensors of the material space
C, e	the Cauchy and Euler deformation tensors of the physical space
$\mathcal{K}_a$	the deformation invariants
$C(p)$	the p-order Casimir invariants on $\mathrm{GL}(n, \mathbb{R})$
L	the angular momentum of the centre of mass of the body with respect to a fixed origin

S	the internal angular momentum (spin)
$\widehat{V}$	the vorticity (Dyson)
$\|S\|, \|\widehat{V}\|$	the magnitudes of the spin and vorticity
$\mathrm{Diff}(\mathcal{M})$	the group of all diffeomorphisms of $\mathcal{M}$
$\mathrm{SDiff}(\mathbb{R}^n)$	the group of volume-preserving diffeomorphisms of $\mathbb{R}^n$ onto itself
$\mathrm{GAf}(n, \mathbb{R})$	the n-dimensional affine group
$\mathrm{L}(n, \mathbb{R})$	the group of all real $n \times n$ matrices
$\mathrm{GL}(n, \mathbb{R})$	the group of invertible real $n \times n$ matrices
$\mathrm{GL}^+(n, \mathbb{R})$	the group of invertible real $n \times n$ matrices with positive determinants
$\mathrm{GL}(n, \mathbb{C})$	the group of invertible complex $n \times n$ matrices
$\mathrm{SL}(n, \mathbb{R})$	the group of invertible real $n \times n$ matrices with determinants equal to $+1$
$\mathrm{SL}(n, \mathbb{R})'$	the covering group consisting of all trace-less matrices
$\mathrm{SO}(n, \mathbb{R})$	the special orthogonal group over reals
$\mathrm{SO}(n, \mathbb{R})'$	the covering group consisting of all antisymmetric matrices
$\mathrm{Symm}(n, \mathbb{R})$	the symmetric matrices
$\mathrm{Symm}^+(n, \mathbb{R})$	the symmetric and positively-definite matrices
$\mathrm{Pr}(n, \mathbb{R})$	the projective group
(x_1, x_2, x_3, x_4)	the cross-ratio of any four points placed on the same line
$\mathrm{U}(n)$	the unitary group
$\mathrm{U}(n)'$	the covering group consisting of all anti-hermitian matrices, i.e., $A^+ = -A$
$\overline{\mathrm{GL}(n, \mathbb{R})}$	the covering group of the real linear group
$\mathrm{Spin}(n)$	the covering group of the orthogonal group $\mathrm{SO}(n, \mathbb{R})$
$\mathrm{SU}(2)$	the covering group of the orthogonal group $\mathrm{SO}(3, \mathbb{R})$
$\mathrm{Co}(1, 3)$	the conformal group preserving the metric up to a functional multiplier
$\mathrm{U}(2, 2)$	the covering group of the conformal group
$L^2(\mu)$	the Hilbert space
$\langle \cdot, \cdot \rangle$	the scalar product of two functions
$g[\cdot, \cdot]$	the scalar product of two vector-valued functions
$[A, B]$	the commutator of $A, B \in \mathrm{L}(U)$
$\mathbb{I}$, $\mathbb{I}_n$	the identity $n \times n$ matrix

μ	the positive volume measure representing the Lagrangian mass distribution
μ_φ	the positive volume measure representing the Eulerian mass distribution
α	the Haar measure on the affine group $\mathrm{GAf}(n, \mathbb{R})$
λ	the Haar measure on $\mathrm{GL}^+(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$
l	the Lebesgue measure on the linear group $\mathrm{L}(n, \mathbb{R})$
Ψ	the wave function
$\mathbf{H}, \mathbf{T}$	the quantum Hamiltonian and kinetic energy operators
$\mathcal{D}^s_{mm'}$	the matrix elements of unitary irreducible representations
σ_{ab}	the Pauli matrices
G	the Lie group
$G' = T_e G$	the Lie algebra of the Lie group G
TG	the tangent bundle of the Lie group G
T^*G	the cotangent bundle of the Lie group G
Ad_g	the adjoint transformation
Ad^*_g	the adjoint of Ad_g
$\mathrm{An}G'$	the set of functionals vanishing on the Lie group G'
$L_k(g)$, $R_k(g)$	the left and right regular translations on the group G
Σ, $\widehat{\Sigma}$	the Hamiltonian generators of the groups of left and right regular translations
$\{\cdot, \cdot\}$	the Poisson bracket
$\Gamma_{\mu\nu}$	the Riemannian structure on the group G
$\Delta(\Gamma)$	the Laplace-Beltrami operator based on the $\Gamma_{\mu\nu}$
$\mathcal{L}$	the Lagrangian density
$u^{s,m}_{\vec{p}}, v^{r,m}_{\vec{p}}$	the Dirac bispinors, i.e., the amplitudes of plane harmonic waves with positive and negative frequencies
ω^s	the Dirac 3-spinor
$\mathbf{a}^+, \mathbf{a}$	the creation and annihilation operators
$D_{\mathrm{ret}}(\vec{r}, t; \vec{r}_0, t_0)$	the retarding Green function
I_1	the modified Bessel function of the first kind
K_1	the modified Bessel function of the second kind
$H^{(1)}_1, H^{(2)}_1$	the Hankel functions, i.e., the Bessel functions of the third kind
$T(\mathbf{A}(t), \mathbf{B}(t'))$	the chronological multiplication of two operators

Introduction

Typical physical theories are based on groups preserving bilinear or sesquilinear forms. These forms are fixed and belong to the absolute, non-dynamical sector of the theory. Typical examples are Euclidean and pseudo-Euclidean scalar products in space or space-time. In field theory and quantum physics one deals with unitary and pseudo-unitary groups preserving Hermitian scalar products in target spaces of field multiplets. In tetrad models of gravitation the internal space is again endowed with Minkowskian geometry. It is something that differs from "external" geometry of special relativity; internal Lorentz group which rules it must not be mixed up with the external group of special relativity.

External and internal metrics are machines contracting tensor indices and enabling us to build scalars and scalar densities necessary for constructing Lagrangians. The resulting models are linear or weakly (perturbatively) nonlinear, in particular, scalars and densities quadratic in dynamical variables do exist. But it is clear that the affine (Tales) geometry is mathematically more primary. Does a hypothetical affine physics with metrics appearing as byproducts exist? It was the old idea of Ne'eman, Hehl, Sardanashvilly, and others [60, 59, 65] in $\mathrm{GL}(4,\mathbb{R})$-gauge approaches to gravitation. One of them is an alternative tetrad model of gravitation based on the global group $\mathrm{GL}(4,\mathbb{R})$ of internal symmetries [157, 159, 164]. Do realistic classical and quantum models ruled by affine groups exist? In typical theories the affine group occurs but in a rather different context. In nuclear matter theory it is used as a "non-invariance group". It rules kinematics but does not preserve dynamics. Instead of this, its generators satisfy some system of commutation relations with Hamiltonian, and essential information concerning the energy levels may be obtained on the basis of some ladder procedure. There also exist classical and quantum models of an affinely-rigid body with kinematics ruled by the affine group but with dynamics violating this symmetry and compatible at most with the Euclidean subgroup. Without potential, on the purely geodetic level, such models are non-physical because they predict the unlimited contraction or expansion of the body. They are also deprived of the aesthetic quality of invariant geodetic systems on Lie groups [3]; the possibility of rigorous analytical solution is also lost. Obviously, with appropriate potential terms, such models are applicable in a wide range of physical phenomena like nuclear and molecular dynamics, macroscopic elasticity, molecular crystals, micro-structured continua, and even astrophysical objects [13, 41]. Nevertheless, at least from the academic point of view, it is an interesting idea to replace such models by affinely invariant geodetic models, metrically-rigid bodies by affinely-rigid ones, and spin by affine spin (hypermomentum) generating affine transformation centred at the centre of mass [144, 124, 171, 59]. It is interesting that replacing (pseudo)Euclidean and (pseudo)unitary symmetries by the affine one, one obtains essential (non-perturbative) nonlinearity, e.g., the generalized Born-Infeld-type nonlinearity in field theory [162].

The model of an affinely-rigid body is widely used in continuum mechanics and in various issues of dynamics of many-body systems. This is a special case of collective degrees of freedom and moments methods in many-body problems, mechanics of continuum media, and field theory, which is connected with classical discretization procedures (e.g., like Rietz, Galerkin, etc.) and finite-elements method. It is known that quite often problems with infinite (even non-denumerable) degrees of freedom can be effectively described in good approximation with the help of finite-dimensional dynamical systems. When we pass on from systems described by partial differential equations to the model based on ordinary differential equations that is always a great simplification in the sense of qualitative analysis, as well as some approximate calculations and numerical techniques.

Methods based on collective models and moments procedures are efficient when the described phenomena have so strong non-local character that only few parameters which depend on one-body variables on the equal basis are necessary for its satisfactory description. These non-local "collective" quantities can be approximately separated from the rest and satisfy as functions of time in good approximation autonomous system of ordinary differential equations. It takes place, for instance, when the length of the elastic waves excited in the body is comparable with its size. The very appearance of collective modes is closely connected with the geometry of the physical space and any other spaces which occur in the kinematical description of the problem. As a general rule, the configuration space of collective degrees of freedom is some Lie group (connected with the afore-mentioned geometrical structures) or the homogeneous space of Lie group. The relationship between the geometry and the structure of physical theories on the fundamental level causes that these are, as a rule, groups of automorphisms of various space structures. A typical example is the usual rigid body and the corresponding dynamical systems on the orthogonal group that are prototypes of all other generalizations. The extreme example of the system with group-theoretical background based on the rigid body concept is hydrodynamics of ideal incompressible fluid (see, e.g., [3]). There is a similar formulation for the elastic theory. We have there, of course, no discretization because of essentially infinite-dimensional language but some collective effective modes, that describe a microscopic system in a simpler way than a discrete statistical aggregate of many particles, can be introduced. The heuristical methods based on the analogy with finite-dimensional groups allow us to guess or postulate some kinds of solutions and then verify them by direct substitution into the equations of motion. While there is not yet any effective and well-formulated theory of infinite-dimensional "Lie groups", any descriptions of situations which lie between a rigid body and a continuum medium are even more valuable. The simplest model of this kind is an affinely-rigid body, i.e., the body that is rigid in the sense of affine geometry. Thus, during any admissible motion all affine relations between material points, i.e., the set of material straight lines, their parallelism, and all mutual ratios of segments placed on the parallel lines, are conserved. Then the configuration space of collective modes is identical to the affine group or, more precisely, its homogeneous space with trivial isotropy groups. There are other related but more general deformation modes based on the projective or conformal groups.

Such "rigid" deformations appear not only in macroscopic elastic bodies when excited waves have lengths comparable with linear dimensions of the body, but also in some astrophysical problems, e.g., virial coefficients of Chandrasekhar [23], the theory of shape of the Earth, vibrations and rotations of stars, etc. The dynamics of microscopic objects like molecules and nuclei of atoms can also use this description of collective modes, e.g., the molecule of phosphor P_4 has only affine degrees of freedom. The dynamics of molecules and various supra-molecular structures is connected to the theory of media with structure, like molecular crystals, which in the continuous limit give us the continuum media with structure, e.g., micromorphic (generalized micropolar) or micromorphic of higher order. One of the first applications of affine degrees of freedom as a model of internal modes was A. C. Eringen's micromorphic theory [40, 41, 42, 43]. Then affine models become a subject in itself but nowadays the consideration of dynamical systems describing the behaviour of the affinely-rigid body is highly motivated by needs of the theory of structured media. Physically interesting are not only the three-dimensional cases but also two- and one-dimensional ones, e.g., models of layers and strings with microstructure. From the point of view of pure analytical mechanics these models are strongly interrelated with the theory of integrable (and superintegrable) systems and the problems of symmetry and degeneration.

Microscopic applications (including those on the molecular and supra-molecular level) demand a quantum description of our models. The simplest solution is then the Schrödinger quantization in the sense of wave functions on the relevant group (affine, linear, orthogonal, projective, and so on) or their homogeneous space. The quasi-classical approximation can be also useful in some applications in mechanics of large molecules, fullerens, etc., where in the description of the same object the quantum and classical phenomena take place simultaneously. This may be relevant to some fundamental physical problems like the decoherence or quantum measurement.

As for more practical applications of the model of the affinely-rigid body we can mention the finite elements method and numerical techniques. For instance, in the very popular version of the finite elements method the body is subjected to the triangulation on the "small" elements, e.g., cubes (symplices), which deform in approximation homogeneously. In this way the continuum is described as finite aggregate of mutually interacting affinely-rigid bodies. This enables the use of some reliable methods combining analytical and numerical techniques [113, 114].

As it has been already mentioned, the model of the affinely-rigid body first appeared in the theory of micromorphic media and was derived by A. C. Eringen and his school [40, 41, 42, 43]. As a theory based on the differential geometry, Hamiltonian, and quantum mechanics it was developed in various aspects by professor J. J. Sławianowski and his co-workers [45, 46, 50, 51, 52, 53, 54, 90, 91, 92, 93, 124, 129, 131, 132, 133, 134, 135, 136, 138, 139, 141, 143, 144, 151, 160, 165, 166, 167, 168, 169, 174, 175, 176, 177, 178, 189]. The technical term "pseudo-rigid body" is also used [25, 26, 27, 28, 29, 30, 31, 33, 34, 79, 98, 99, 103, 180, 181]. A sophisticated mathematical analysis based on the qualitative theory of dynamical systems and symplectic geometry was made among others by M. Roberts and C. Wulff [44, 109, 205]. These problems have been the subject of research of many leading scientific centres in the field of applied mechanics (mechanical engineering) as well as mathematical foundations connected with a theory of dynamical systems.

Approbation of results

The great deal of original results presented in this PhD thesis have been already published [74, 170, 171, 172, 173, 174, 175, 176]. Thus,

- Although two earlier written articles [12, 73] are not directly connected with the subject of the thesis, nevertheless they may be considered as an instructive example of application of the moments method described in Section 1.2. In the article [12] relativistic corrections for a resonant interaction of two atoms are calculated. It is shown that the leading correction is inversely proportional to the interatomic distance. In the article [73] the influence of the relativistic attraction forces on large distances between neutral atoms upon thermodynamical characteristics, in particular upon the value of the second virial coefficient, is investigated. These forces are described by means of Casimir-Polder formula. Analytical expression of the second virial coefficient within the high-temperature approximation is obtained. For atomic hydrogen and inert gas the constant of the relativistic interaction is evaluated by means of Slater-Kirkwood approximation formula, in which the constant of the relativistic interaction is expressed by the constant of the dipole-dipole interaction.

- Chapter 2 and Section 3.1 are structurally based on the article [175] where models of collective and internal degrees of freedom with not only kinematics but also dynamics invariant under the action of the affine group and its subgroups are discussed. The relationship with the dynamics of integrable one-dimensional lattices is shown. It appears that the affinely-invariant geodetic models can encode the dynamics of something like elastic vibrations.

- Section 2.1 is based on the article [171] where (pseudo-)Riemannian metrics on the affine group are discussed. A special stress is laid on metric structures invariant under left or right regular translations by elements of the total affine group or some of its geometrically distinguished subgroups. Also some non-geodetic problems in corresponding Riemannian spaces are discussed.

- Section 2.3 is based on the article [172] where the concept of an n-dimensional projectively-rigid body is introduced, and its connection to the concept of an $(n+1)$-dimensional incompressible affinely-rigid body is analyzed. The equations of geodetic motion for such a projectively-rigid body are obtained. As an instructive example, the special case of $n = 1$ is investigated.

- Chapter 3 (except Section 3.1) is structurally based on the article [176] where the quantized version of the classical description of collective and internal affine modes is discussed. The Schrödinger quantization is performed and, as a consequence, the quantized problem is effectively reduced from n^2 to n degrees of freedom. Some possible applications in nuclear physics and other quantum many-body problems are suggested. Also the possibility of half-integer angular momentum in composed systems of spinless particles is discussed.

- Section 3.2 is based on the article [173] where the classical and quantum dynamical systems on Lie groups and their homogeneous spaces are described. The special stress is laid on the dynamics of deformable bodies and the mutual coupling between rotations and deformations. Deformative modes are discretized, i.e., it is assumed that the relevant degrees of freedom are controlled by a finite number of parameters. Mainly the situation when the effective configuration space is identical with affine group (affinely-rigid bodies) is considered. The special attention is paid to the left- and right-invariant geodetic systems, i.e., when there is no potential term and the metric tensor underlying the kinetic energy form is invariant under left and/or right regular translations on this group. The dynamics of elastic vibrations may be encoded in this way in the very form of kinetic energy. Although special attention is paid to invariant geodetic systems, the potential case is also taken into account.

- Section 4.2 is based on the article [170] where Klein-Gordon-Dirac equation is discussed, i.e., a linear differential equation with constant coefficients obtained by superposing Dirac and d'Alembert operators. A general solution for Klein-Gordon-Dirac equation as a superposition of two Dirac plane harmonic waves with different masses is obtained. The multiplication rules for Dirac bispinors with different masses are found. Lagrange formalism is applied to receive the energy-momentum tensor and 4-current. It appears, in particular, that the scalar product is a superposition of Klein-Gordon and Dirac scalar products. The primary approach to the canonical formalism is suggested. The limit cases of equal masses and one mass equal to zero are calculated.

- Section 4.3 is based on the article [74] where Green function for Klein-Gordon-Dirac equation is obtained. The case with the dominating Klein-Gordon term is considered. There seems to be formal analogy between this problem and a certain problem for a 4-dimensional particle moving in an external field. The explicit relationships between the wave functions, Green functions, and initial conditions are established with the help of the T-exponent formalism.

The so-far-obtained results were also presented in the form of posters and talks at the various international symposia, conferences, meetings, and workshops:

- XXXIII Symposium on Mathematical Physics, Toruń, Poland, June 5–9, 2001. Poster title: *Klein-Gordon-Dirac equation: physical justification and quantization attempts.*

- XX Workshop on Geometric Methods in Physics, Białowieża, Poland, July 1–7, 2001. Poster title: *Some investigations of the quantum interpretation of Klein-Gordon-Dirac field.*

- XXXIV Symposium on Mathematical Physics, Toruń, Poland, June 14–18, 2002. Poster title: *Invariant geodetic problems on the affine group and related Hamiltonian systems.*

- XXI Workshop on Geometric Methods in Physics, Białowieża, Poland, June 30 – July 6, 2002. Poster title: *Geodetic problems and related Hamiltonian systems.*

- XVII International Conference for Physics Students, Budapest, Hungary, August 21–28, 2002. Talk title: *n-dimensional left and right invariant problems on the affine group.*

- V International Conference "Symmetry in Nonlinear Mathematical Physics", Kyiv, Ukraine, June 23–29, 2003. Poster title: *Affine models of collective and internal degrees of freedom. Problems of dynamical affine invariance.*

- XXII Workshop on Geometric Methods in Physics, Białowieża, Poland, June 29 – July 5, 2003. Poster title: *Problems of affine invariance in dynamics of collective and internal degrees of freedom.*

- XVIII International Conference for Physics Students, Odense, Denmark, August 7–13, 2003. Poster title: *Problems of dynamical affine invariance.*

- Second Junior European Meeting "Control Theory and Stabilization", Turin, Italy, December 3–5, 2003. Talk title: *Hamiltonian systems on linear and projective groups and their control.*

- XXXVI Symposium on Mathematical Physics, Toruń, Poland, June 9–12, 2004. Poster title: *Quantum and classical invariant geodetic problems on the projective group* $Pr(n, \mathbb{R})$.

Chapter 1

Collective and internal degrees of freedom in analytical mechanics

1.1 The origin of collective and internal modes

Let us consider a complex system, e.g., multi-bodies one, with an infinite (even non-denumerable) number of degrees of freedom or with a finite but rather large number of degrees of freedom. Then let us suppose that degrees of freedom of such a system are hierarchically ordered in such a way that a relatively small part of them is approximately decoupled from the remaining ones and ruled by approximately autonomous dynamics. This hierarchy of degrees of freedom usually appears due to some peculiarities of intermolecular forces and quite often has to do with geometry of the physical space or some other spaces relevant for the problem. The leading parameters deciding about the main dynamical features of the system are usually referred to as *collective modes*. The rules of the collective dynamics are either derived from the micro-model or somehow guessed on the basis of certain natural symmetry demands.

In other words, we say that a "large" system of material points has collective modes when [174, 175]:

- there exists a "small" number of parameters $q^1, \ldots, q^n$ that are dynamically relevant, i.e., they satisfy an approximately autonomous system of evolution equations and these corresponding evolution equations describe behaviour of the system,

- for our purposes the kinematical information about the system encoded in those parameters is sufficient,

- and (very important!) $q^1, \ldots, q^n$ are non-local in the sense that all degrees of freedom of individual particles, i.e., positions and velocities, enter the q^i-variables on essentially equal footing, with the same strength, order of magnitude, so to speak.

On this non-local character of relevant modes is based the idea of various moment approaches (like traditional methods developed by Rietz, Tshebyshev, Galerkin, and others), virial coefficients, and so on [177].

More rigorously, as a mathematical model of collective degrees of freedom we can realize some quotient manifolds of the multi-particle configuration (state) space Q or their submanifolds (e.g., representatives of cosets).

Such a model may be successfully used for describing collective degrees of freedom of extended bodies. But we can also try to deal with some essentially point-like dimensionless objects [167]. This becomes unavoidable, for instance, if the physical space $\mathcal{M}$ is a general differential manifold, no longer a flat space. Then there are no extended in-any-way-rigid bodies, only infinitesimal ones are well-defined. Roughly speaking, the extended bodies shrink and, finally, they become injected into a tangent space $T_x\mathcal{M}$.

So, *internal modes* of objects which are essentially non-extended, or which are so small that details of their spatial structure are hidden, are described in such a way that their configuration spaces Q are fibre bundles over the space or space-time manifold $\mathcal{M}$. The base points describe their spatial localization, whereas the fibre points are internal variables.

Usually the typical models of collective and internal degrees of freedom have to do with Lie groups or their homogeneous spaces. Then, due to the analyticity of Lie groups, quite often some rigorous or at least approximate solutions based on functional series and special functions may be found. This concerns both classical and quantum levels.

The most natural and intuitive origin of collective modes is based on the mechanism of constraints and the d'Alembert principle. Collective motion is then "large", whereas non-collective one is "small" and merely reduced to some vibrations about the appropriate constraints submanifold. The collective kinetic energy, i.e., the dynamical metric element, is obtained from the restriction of the total one to the constraints surface (the first fundamental quadratic form). This corresponds to the classical relationship between the kinetic energy and inertia [3, 18, 19, 185]. In this case, as a rule, the collective kinetic energy is invariant under a proper subgroup of the group underlying geometry of the constraints submanifold.

But there is also another mechanism when the hidden non-collective motion is just "large", and the emerging collective modes have to do with the averaged behaviour of these hidden modes, i.e., with the time dependence of some relatively slowly-varying mean values. Then it is quite natural to expect that the collective Lagrangian will be based on a kinetic energy whose underlying dynamical metric tensor will be non-interpretable in terms of the restriction of the usual multi-particle metric tensor to the constraints manifold. Similarly, equations of motion do not need to be derivable from the usual d'Alembert principle based on the original spatial metric. Therefore, the relationship between the kinetic energy and inertia may become rather non-classical and, to some extent, exotic in comparison with the usual requirements (cf., e.g., the discussion by Capriz and Trimarco in [18, 19, 187]).

In such situations the only reasonable procedure is to postulate the kinetic term of the Lagrangian on the basis of some natural and physically justified remises. Let us mention two examples from two completely opposite scales of physical phenomena, namely, the atomic nuclei and vibrating-rotating stars (by the way, the neutron stars are, in a sense, exotic and gigantic nuclei with $Z = 0$ and enormous A). There are also such objects as kinetic bodies or various non-standard microstructure elements like gas bubbles, voids, and defects in solids [18, 19], though bubbles and voids can be hardly treated as constrained pieces of a substance or systems of material points.

1.2 Collective modes and moments approach

Let us consider an arbitrary classical system of material points (discrete or continuous) without retardation or memory [177] and call it the body. For the discrete case at least $(n+1)$ material points in the n-dimensional space should be provided. Let $(\mathcal{M}, V, \rightarrow)$ be an affine space, where $\mathcal{M}$ is a physical space in which our body is placed and V is a linear space of translations (free vectors) in $\mathcal{M}$. We may also introduce the metric tensor $g \in V^* \otimes V^*$ that makes this affine space a Euclidean one $(\mathcal{M}, V, \rightarrow, g)$. Let us suppose that we have labelled every material point of such a body in some way (e.g., we may choose those labels as initial positions of all points at the time instant t_0). Then let $(\mathcal{N}, U, \rightarrow)$ be an affine space, where $\mathcal{N}$ is the material space of such labels and U is corresponding linear space of translations in $\mathcal{N}$. Similarly, we may also introduce the metric tensor $\eta \in U^* \otimes U^*$ that makes this affine space a Euclidean one $(\mathcal{N}, U, \rightarrow, \eta)$. So, the position of the a-th material point at the time instant t will be denoted by $x(t, a)$ ($x \in \mathcal{M}$, $a \in \mathcal{N}$). Newton's equations of motion may be written as follows:

$$\frac{\partial^2 x}{\partial t^2}(t, a) = \varPhi \left[x(t, \cdot), \frac{\partial x}{\partial t}(t, \cdot); t, a \right], \tag{1.1}$$

where $\varPhi$ is a density of forces per unit mass. This system of ordinary differential equations is labelled by the "index" a, which may have a finite, denumerable, or continuous range. The functional dependence of the density $\varPhi$ on the evolution $(t, a) \rightarrow x(t, a)$ is underlined by using the square brackets.

1.2.1 Finite system of material points

If the system is finite, then it is customary to write a as a capital subscript $A = \overline{1, N}$. Then, the Newton equations of motion for an N-particle system can be written in the following form:

$$m_A \frac{d^2 x_A}{dt^2}(t) = F_A \left(x_1(t), \ldots, x_N(t); \frac{dx_1}{dt}(t), \ldots, \frac{dx_N}{dt}(t); t \right), \tag{1.2}$$

where m_A denotes the mass of the A-th material point, and $F_A = m_A \varPhi_A$ is the force affecting this point.

1.2.2 Continuous body

If the system is continuous, then the label a becomes a Lagrangian radius-vector (material variables). In the special case of simple elastic bodies free of external interactions (dynamically homogeneous physical space), the density $\varPhi$ is a local algebraic function of the first derivative of the Eulerian radius-vector x (physical variables) with respect to the Lagrangian radius-vector a, i.e.,

$$\varPhi \left[x(t, \cdot), \frac{\partial x}{\partial t}(t, \cdot); t, a \right] = \varPhi \left(\nabla_a x(t, \cdot); t, a \right).$$

Then the system (1.1) becomes a partial differential equation for the time-dependent vector field x:

$$\frac{\partial^2 x}{\partial t^2}(t, a) = \varPhi \left(\nabla_a x(t, \cdot); t, a \right).$$

If the body is materially homogeneous, then there is no dependence of Φ on the last argument a.

Equations of motion for a continuous body can be written as follows:

$$\rho_0(a)\frac{\partial^2 x}{\partial t^2}(t,a) = \mathcal{F}\left[x(t,\cdot), \frac{\partial x}{\partial t}(t,\cdot); t, a\right], \tag{1.3}$$

where ρ_0 denotes the Lagrangian density of mass, and $\mathcal{F} = \rho_0\Phi$ is the Lagrangian density of forces per unit non-deformed volume.

We can also rewrite the equation (1.3) in Eulerian terms:

$$\rho(t,x)\frac{Dv}{Dt}(t,x) = f\left[a(t,\cdot), v(t,\cdot); t, x\right], \tag{1.4}$$

where $a(t,\cdot)$ denotes the inverse mapping of $x(t,\cdot)$, i.e., $x(t,a(t,u)) = u$ for any t and u, $v(t,\cdot)$ denotes the Eulerian velocity field:

$$v^i(t,x) = -\frac{\partial x^i}{\partial a^j}\left(t, a(t,x)\right)\frac{\partial a^j}{\partial t}(t,x),$$

D/Dt is the substantial derivative:

$$\frac{Dv^i}{Dt} = \frac{\partial v^i}{\partial t} + v^j\frac{\partial v^i}{\partial x^j},$$

and f is the Eulerian density of forces per unit deformed volume (f does not depend on $a(t,\cdot)$ in the special case of fluids).

To have a closed system of equations, the continuity equation,

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0,$$

and (if we consider the full thermo-mechanical theory) an appropriate equation describing dynamics of the temperature field should be added to the equation (1.4).

1.2.3 Unique treatment of discrete and continuous systems

In a wide range of problems, including certain general considerations, there is no need to distinguish between continuous and discrete cases. Nevertheless, it must be stressed that there is a deep qualitative gap between both cases, and one must be very careful when considering them on the same footing. Fortunately, what is written below is neutral with respect to such delicate and troublesome problems.

Thus, we can use the general form of equations of motion (1.1). Let us represent the Lagrangian mass distribution in the material by a positive measure μ. If $\varphi : a \mapsto$

$x(t,a)$ is a mapping from the material space to the physical one, then the Euler mass distribution in a given configuration φ is represented by the measure μ_φ obtained from μ by the φ-transport:

$$\int (h \circ \varphi)(a) d\mu(a) = \int h(x) d\mu_\varphi(x)$$

for any function h on the physical space.

The material measure μ gives rise to the scalar product of any two functions on the set of material points, i.e.,

$$\langle f, h \rangle := \int \overline{f(a)} h(a) d\mu(a).$$

Let $L^2(\mu)$ denote the Hilbert space of functions which are square-integrable in the μ-sense, i.e., such ones that $\langle h, h \rangle < \infty$. Similarly, for the vector-valued material functions, i.e., functions assigning spatial vectors to material points, we define the scalar product as follows:

$$g[v, w] := g_{ij} \left\langle v^i, w^j \right\rangle = g_{ij} \int v^i(a) w^j(a) d\mu(a),$$

where g denotes the spatial metric tensor.

Kinetic energy and power of forces are given by the following functionals:

$$T\left[\frac{\partial x}{\partial t}(t,\cdot)\right] = \frac{1}{2}\, g\left[\frac{\partial x}{\partial t}(t,\cdot), \frac{\partial x}{\partial t}(t,\cdot)\right], \tag{1.5}$$

$$\mathcal{P}\left[x(t,\cdot), \frac{\partial x}{\partial t}(t,\cdot)\right] = g\left[\frac{\partial x}{\partial t}(t,\cdot), \varPhi\left[x(t,\cdot), \frac{\partial x}{\partial t}(t,\cdot); t, \cdot\right]\right]. \tag{1.6}$$

1.2.4 Moments techniques

Due to the separability of the Hilbert space $L^2(\mu)$ we can choose a complete system of real-valued functions H^r, $r = \overline{0,\infty}$. For certain reasons it is convenient to include here a constant function, e.g., $H^0 = 1$, and denote the remaining functions H^r by H^ρ, $\rho = \overline{1,\infty}$.

Let us calculate the moments of equations of motion (1.1) with respect to this complete system of functions H^r. The resulting equations have the form of balance laws:

$$\frac{dM^{ri}}{dt} = N^{ri}, \tag{1.7}$$

where $r = \overline{0,\infty}$, $i = \overline{1,3}$, and the quantities

$$M^{ri} = \left\langle H^r, \frac{\partial x^i}{\partial t} \right\rangle, \qquad N^{ri} = \left\langle H^r, \varPhi^i \right\rangle$$

are H^r-moments of the distribution of linear momentum and the distribution of forces within the body, respectively. They provide a global, collective representation of those distributions. It is clear that M^{0i} and N^{0i} are equal, respectively, to the total linear momentum of the body and the total force acting on it, i.e., $M^{0i} = P^i$ and $N^{0i} = F^i$.

In the case of continuous media, the system of moments equations (1.7) provide some kind of infinite discretization, because the system of equations (1.1) labelled by the continuous variable a is replaced by a countable system labelled by the discrete index r. However, in the case of solids, the dynamical moments N^{ri} are not functions of the kinematical moments M^{ri} alone, thus, the system (1.7) is not an effective dynamical system for M^{ri}.

On the other hand, the Lagrangian representation (1.1) is not effective for fluids. Thus, applying the moment techniques to fluids and to physical fields interacting with continua, one has to use the Eulerian form of equation of motion, i.e., (1.4). Instead of the material functions we should use then functions defined on the physical Euclidean space E^3. Projecting equations of motion (1.4) onto elements of an appropriately chosen complete system of spatial functions, we obtain a discrete system of equations. In certain problems of fluid dynamics such a system of equations of motion provides an effective tool for integration and qualitative analysis, e.g., calculating multipole moments of (1.4) with respect to Eulerian coordinates, we obtain the hierarchy of so-called virial equations used for a long time in hydrodynamical problems of astrophysics and in the theory of the shape of Earth. Nevertheless, even in the case of fluids the direct calculation of moments also does not lead automatically to a closed dynamical system.

1.2.5 Formal transition to collective modes

To transform the system of equations (1.7), at least formally, into a dynamical system, we has to introduce explicitly the H^r-moments of configurations, i.e., the quantities $\langle H^r, x^i \rangle$. Thus, we expand the configurations $x(t, \cdot)$ with respect to the mode functions H^r as follows:

$$x^i(t, a) = q^i{}_r(t) H^r(a). \tag{1.8}$$

If the system is finite or discrete, then it is simply a change of coordinates, i.e., the natural variables $x^i{}_A$ are replaced by generalized coordinates $q^i{}_r$, which depend in general on all radius-vectors x_A, i.e., q are collective coordinates parameterizing coherent multi-particle motions. Similarly, in the continuous case we could say that it is a change of representation, i.e., the one-particle coordinates $x^i(t,a)$ are replaced by the collective variables $q^i{}_r$. Then the quantity q is an amplitude of the intensity with which the collective mode H^r occurs in a given configuration x.

After the substitution of the expansion (1.8) into (1.7), we obtain a system of ordinary differential equations for expansion coefficients $q^i{}_r$, i.e.,

$$Q^{rs}\frac{d^2q^i{}_s}{dt^2} = N^{ri}\left(q, \frac{dq}{dt}, t\right), \tag{1.9}$$

where

$$Q^{rs} = \langle H^r, H^s\rangle = \int \overline{H^r(a)}H^s(a)d\mu(a)$$

are quadrupole moments of the Lagrangian mass distribution (they are H^r-collective coefficients of inertia).

The mode functions H^r do not need to be mutually orthogonal and normalized because there are numerous applications where non-orthonormal systems are more convenient. Thus, we do not assume that $Q^{rs} = \delta^{rs}$. It is also natural to use real functions H^r and real coefficients $q^i{}_s$ because if they were complex, then we would have to use additional conditions for coefficients $q^i{}_s$ to ensure the reality of Eulerian coordinates x^i.

In the equations (1.9) the motion of the centre of mass and the relative motion of constituents are mixed in a non-physical way. However, due to the fact that the system of mode functions H^r includes the constant function $H^0 = 1$, it is relatively easy to separate these two kinds of degrees of freedom. The radius-vector of the centre of mass can be expressed in the following form:

$$q^i = \frac{1}{M}Q^r q^i{}_r = q^i{}_0 + \frac{1}{M}Q^\rho q^i{}_\rho,$$

where $M = \int d\mu(a)$ is the total mass of the body, $Q^r = \int H^r(a)d\mu(a)$ is the dipole H^r-moment of the mass distribution, and the parameters $q^i{}_\rho$, $\rho = \overline{1,\infty}$, refer to the relative motion. Inertial properties of the relative motion with respect to the centre of mass reference frame (*internal motion*) are describe by the following coefficients:

$$Q^{\rho\sigma}_{\mathrm{int}} := Q^{\rho\sigma} - \frac{1}{M}Q^\rho Q^\sigma.$$

Remark: we implicitly use the assumption that all multipole moments Q are finite, e.g., the total mass M is finite. In statistical mechanics and general theory of dynamical systems are considered also infinite systems of particles with the infinite total mass. Then for such systems the above-mentioned scheme does not work. Moreover, the very splitting of motion into translational and internal parts breaks down because the centre of mass is no longer well-defined (except certain exceptional situations when the total linear momentum is finite).

Then the equations of motion (1.9) decouple into two system of balance laws, i.e.,

$$\frac{dp^i}{dt} = F^i, \qquad \frac{dM^{\rho i}_{\rm int}}{dt} = N^{\rho i}_{\rm int}, \tag{1.10}$$

where

$$p^i = M\frac{dq^i}{dt}, \qquad M^{\rho i}_{\rm int} = Q^{\rho\sigma}_{\rm int}\frac{dq^i{}_\sigma}{dt}, \qquad N^{\rho i}_{\rm int} = N^{\rho i} - \frac{1}{M}Q^\rho F^i.$$

Remark: there is a coupling between translational and internal dynamics because, in general, both F^i and $N^{\rho i}_{\rm int}$ depend on all arguments $(q^i, \dot{q}^i, q^i{}_\rho, \dot{q}^i{}_\rho, t)$. However, in the theory of multi-particle systems and in the continuum theory, we are usually interested in a pure relative (internal) motion when $F^i = 0$ and $N^{\rho i}_{\rm int}$ depend only on $(q^i{}_\rho, \dot{q}^i{}_\rho, t)$.

The kinetic energy (1.5) and power of forces (1.6) can be rewritten as follows:

$$T = \frac{1}{2}g_{ij}\frac{dq^i{}_r}{dt}\frac{dq^j{}_s}{dt}Q^{rs} = \frac{M}{2}g_{ij}\frac{dq^i}{dt}\frac{dq^j}{dt} + \frac{1}{2}g_{ij}\frac{dq^i{}_\rho}{dt}\frac{dq^j{}_\sigma}{dt}Q^{\rho\sigma}_{\rm int}, \tag{1.11}$$

$$\mathcal{P} = g_{ij}\frac{dq^i{}_r}{dt}N^{rj} = g_{ij}\frac{dq^i}{dt}F^j + g_{ij}\frac{dq^i{}_\rho}{dt}N^{\rho j}_{\rm int}. \tag{1.12}$$

So, if we deal with a continuous medium (the label a is a 3-dimensional radius-vector running over the material manifold), then the above-described procedure results in an infinite discretization, i.e., for any time instant t the continuous fields $x^i(a)$ describing configurations are replaced by infinite arrays of numbers $q^i{}_\rho$, $\rho = \overline{1,\infty}$. The choice of functions H^r, which represent the basic collective modes, depends on the physical nature of the considered problem (reasonable choices are suited to the structure of interactions). The collective character of these modes means that if we choose any r generalized coordinates $q^i{}_1, \ldots, q^i{}_r$, then the degrees of freedom corresponding to them are relatively autonomous, i.e., weakly affected by the degrees of freedom described by the rest of parameters $q^i{}_s$, $s > r$. In a wide range of problems it is convenient to choose and order the modes H^r in such a way that for increasing r the oscillations of functions H^r become faster (starting from the non-oscillating $H^0 = 1$). This corresponds to the *hierarchy* of increasing lengths of the excited waves. In many applications, like the dynamics of suspensions or macroscopic elastic problems, only a few long-wave modes are relevant for the considered phenomena and describes their qualitative features.

1.2.6 Holonomic constraints mechanism

If the system is continuous, then its equations of motion (1.1) are equivalent to an infinite system of ordinary differential equations (1.9). But such infinite systems are computationally non-effective and do not provide any real simplifications of the models. However, in situations when only a few approximately autonomous long-wave modes decide about the qualitative behaviour of the system, we can perform an effective finite discretization of (1.1) by imposing some *discretization constraints.*

Let us assume that from some theoretical considerations or from experimental data we know that in a given range of phenomena the motion of the system holds approximately in a finite-dimensional linear subspace Δ of $L^2(\mu)$, e.g., in the linear span of modes $H^0, \ldots, H^N$:

$$\Delta = \left\{ \sum_{r=0}^{N} c_r H^r \,\middle|\, c_r \in \mathbb{R} \right\}.$$

By "holds approximately" we mean that for any $\varepsilon > 0$ there exists an open range of initial conditions for which all trajectories do not leave the subspace Δ by the Hilbert distance larger than ε, uniformly all over the time axis. In other words, the subspace Δ has some attractive properties because of special organization of internal interactions, and for sufficiently small open domains of initial states the system performs small vibrations about the subspace Δ. This is a typical mechanism of *holonomic constraints* in mechanics. Performing orthogonal projections of those oscillating trajectories onto the subspace Δ, we obtain constrained trajectories as they are seen by any observers who do not notice oscillations or do not take them into account. Such constrained trajectories satisfy equations of motion following from *the d'Alembert principle.*

So, our constraints equations have the following form:

$$q^i{}_s = 0, \qquad s > N. \tag{1.13}$$

According to the d'Alembert principle, these equations of constraints should be substituted to the following modified equations of motion (with some *a priori* non-specified reaction forces):

$$\frac{\partial^2 x}{\partial t^2} = \Phi + \Phi_R,$$

where the power of reaction forces Φ_R is assumed to vanish on any virtual velocity field

$$\mathcal{V} = \sum_{r=0}^{N} \mathcal{V}_r H^r$$

compatible with the constraints equations (1.13), i.e.,

$$g\left[\mathcal{V}, \Phi_R\right] = 0.$$

Thus, we should also modify the equations (1.9) for the expansion coefficients ${q^i}_s$:

$$Q^{rs} \frac{d^2 {q^i}_s}{dt^2} = N^{ri} + N_R^{ri},$$

where $N_R^{ri} = \langle H^r, \Phi_R^i \rangle$ is the dynamical H^r-moment built of the reaction forces. Then we have the constraints conditions that

$$g_{ij} \sum_{r=0}^{N} N_R^{ri} {\mathcal{V}^j}_r = 0$$

for any system ${\mathcal{V}^j}_0, \ldots, {\mathcal{V}^j}_N$ of expansion coefficients for the virtual velocity field, which means that all moments of reaction forces with $r \leq N$ vanish, i.e., $N_R^{ri} = 0$ for $r = \overline{0, N}$. Therefore, the first $(N + 1)$-tuple of the moments equations is free of reaction forces, although the forces Φ_R themselves do not vanish.

All motions compatible with holonomic constraints (1.13) satisfy the subsystem of $(N + 1)$ first equations of motion (1.9) with algebraically substituted conditions (1.13), i.e.,

$$\left.\frac{dM^{ri}}{dt}\right|_{\Delta} = Q^{rs} \frac{d^2 {q^i}_s}{dt^2} = N^{ri}\left(q, \frac{dq}{dt}, t\right), \qquad r, s = \overline{0, N}.$$

All higher modes H^r, $r \geq N$, are then non-excited, and the above equations for $r \geq N$ are superfluous. However, they may be used for determining the reaction forces Φ_R.

After the separation of the translational and internal motions and denoting the indices restricted to the ranges $\overline{0, N}$ or $\overline{1, N}$ by the corresponding capital Latin or Greek letters, respectively, we obtain the following constrained equations of motion:

$$M \frac{d^2 q^i}{dt^2} = F^i, \qquad \frac{dM_{\text{int}}^{\Omega i}}{dt} = Q_{\text{int}}^{\Omega\Sigma} \frac{d^2 {q^i}_\Sigma}{dt^2} = N_{\text{int}}^{\Omega i}\left(q, \frac{dq}{dt}, t\right). \tag{1.14}$$

Remark: it is a generic situation that the matrices Q^{RS} and $Q^{\Omega\Sigma}$ are non-singular. This implies that the system of equations of motion (1.14) is regular, i.e., solvable with respect to the generalized accelerations $\ddot{q}^i{}_\Sigma$. Thus, (1.14) is an effective system of equations of motion for the Δ-constrained system and we obtained an effective discretization for continuous systems or computational simplifications for systems with a finite but rather large number of degrees of freedom.

Finally, the constrained kinetic energy expression is as follows:

$$T = \frac{1}{2} g_{ij} \frac{dq^i{}_R}{dt} \frac{dq^j{}_S}{dt} Q^{RS} = \frac{M}{2} g_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt} + \frac{1}{2} g_{ij} \frac{dq^i{}_\Omega}{dt} \frac{dq^j{}_\Sigma}{dt} Q^{\Omega\Sigma}_{\text{int}}.$$

1.2.7 Lagrangian multipoles and polynomial expansions

In a wide class of problems polynomial and trigonometric functions of material coordinates provide the most convenient and intuitive models of collective modes. These models are more than satisfactory because in any compact domain all sufficiently regular functions can be approximated by polynomials and Fourier series. At the same time both models are well-suited to the hierarchy of decreasing lengths of excited waves. The moments of fields with respect to homogeneous polynomials, first of all the moments of densities of extensive physical quantities, are known in physics (e.g., in electrostatics) as multipole moments and virial coefficients [23]. Usually a few lowest-order multipoles provide a satisfactory description of the spatial distribution of physical quantities within bounded domains.

The method of multipole moments is widely used in the continuum theory, e.g., in the mechanics of generalized media with internal degrees of freedom (*micromorphic continua*) [40, 41].

If we use homogeneous polynomials of Cartesian material coordinates $\{H^r\} = \left\{1, a^K, a^K a^L, a^K a^L a^M, \ldots\right\}$ as mode functions (in spite of their non-bounded character they belong to $L^2(\mu)$ because in realistic problems the measure μ is compactly-supported), then the expansion (1.8) becomes something very peculiar from the mathematical point of view, i.e., *the Taylor expansion* of analytical functions. This is an additional distinguishing feature of polynomial discretization and polynomial collective modes.

The moments M^{ri} and N^{ri} become the following tensorial objects (k-th order multipole moments of the material distribution of linear momentum and forces):

$$_k M^{A_1 \cdots A_k i} = \int a^{A_1} \cdots a^{A_k} \frac{\partial x^i}{\partial t}(t, a) d\mu(a), \tag{1.15}$$

$$ {}_kN^{A_1\cdots A_k i} \quad = \quad \int a^{A_1}\cdots a^{A_k}\,\Phi^i(t,a)d\mu(a). \tag{1.16}$$

They are mixed material-spatial quantities because the indices $(A_1,\ldots,A_k)$ refer to the material space and i is the usual spatial index, i.e., ${}_kM$, ${}_kN$ are partially Lagrangian and partially Eulerian objects.

The inertial moments Q^{rs} and Q^r are now given by Lagrangian multipoles of the mass distribution,

$$ {}_kQ^{A_1\cdots A_k} = \int a^{A_1}\cdots a^{A_k} d\mu(a), $$

which are completely symmetric tensors in the material space, and the internal parts $Q^{\rho\sigma}_{\rm int}$ are as follows:

$$ {}_{(\mu,\nu)}Q_{\rm int}^{A_1\cdots A_\mu B_1\cdots B_\nu} = {}_{(\mu+\nu)}Q^{A_1\cdots A_\mu B_1\cdots B_\nu} - \frac{1}{M}{}_\mu Q^{A_1\cdots A_\mu}{}_\nu Q^{B_1\cdots B_\nu}. $$

In particular, the dipole moment ${}_1Q^A = \int a^A d\mu(a) = Mq^A$ vanishes (and then the quadrupole moments ${}_2Q^{AB}$ and ${}_{(1,1)}Q^{AB}_{\rm int}$ are equal) if we choose the material coordinates a^K in such a way that the centre of mass in the reference frame is placed at zero, i.e., its radius-vector $q^A = 0$.

In the rigid-body mechanics we use very often the co-moving components of the tensor of inertia I^{AB}, which is algebraically equivalent to our quadrupole moment of the mass distribution, i.e.,

$$ I^{AB} = \eta^{AB}{}_2Q^{CD}\eta_{CD} - {}_2Q^{AB}, $$

where η is the material metric tensor.

The expansion of instantaneous configurations x through the collective generalized coordinates q for the polynomial functions H^r has the following form:

$$ x^i(t,a) = \sum_{k=0}^{\infty} {}_kq^i{}_{A_1\cdots A_k}(t)a^{A_1}\cdots a^{A_k}. \tag{1.17}$$

Obviously, the coefficients ${}_kq^i{}_{A_1\cdots A_k}$ are fully symmetric in material indices, thus, only those corresponding to $A_1 \leq \cdots \leq A_k$ are independent generalized coordinates.

Remark 1: in certain applications it is convenient to decrease the number of redundant variables and use the expansion expressed through spherical functions of angular variables, i.e.,

$$ x^i(t,a) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} q^i{}_{lm}(t)|a|^l Y^{lm}\left(\frac{a}{|a|}\right), \tag{1.18}$$

where $|a|$ is the length of the vector a, and for x to be a real quantity the coefficients ${q^i}_{lm}$ have to satisfy the following condition:

$$\overline{{q^i}_{lm}} = {q^i}_{l,-m}.$$

Remark 2: it should be stressed that, in general, ${}_0q^i$ do not coincide with coordinates of the spatial position of the centre of mass q^i, i.e.,

$$q^i - {}_0q^i = \frac{1}{M}\sum_{\varkappa=1}^{\infty} {}_\varkappa Q^{A_1\cdots A_\varkappa}\, {}_\varkappa {q^i}_{A_1\cdots A_\varkappa}$$

and

$$x^i(t,a) - q^i(t) = \sum_{\varkappa=1}^{\infty} {}_\varkappa {q^i}_{A_1\cdots A_\varkappa}(t)\left(a^{A_1}\cdots a^{A_\varkappa} - \frac{1}{M}{}_\varkappa Q^{A_1\cdots A_\varkappa}\right).$$

Using the polynomial expansion (1.17) the kinetic moments ${}_kM$ can be written in the following form:

$${}_kM^{A_1\cdots A_k i} = \sum_{n=0}^{\infty} {}_{(k+n)}Q^{A_1\cdots A_k B_1\cdots B_n}\frac{d\, {}_n{q^i}_{B_1\cdots B_n}}{dt}.$$

The translational and internal parts of them are as follows:

$${}_kM_{\mathrm{tr}}^{A_1\cdots A_k i} = {}_kQ^{A_1\cdots A_k}\frac{dq^i}{dt}, \tag{1.19}$$

$${}_kM_{\mathrm{int}}^{A_1\cdots A_k i} = \sum_{\nu=1}^{\infty} {}_{(k,\nu)}Q_{\mathrm{int}}^{A_1\cdots A_k B_1\cdots B_\nu}\frac{d\, {}_\nu{q^i}_{B_1\cdots B_\nu}}{dt}. \tag{1.20}$$

Similarly, for translational and internal parts of the dynamical moments ${}_kN$ we have

$${}_kN_{\mathrm{tr}}^{A_1\cdots A_k i} = \frac{1}{M}{}_kQ^{A_1\cdots A_k}F^i, \tag{1.21}$$

$${}_kN_{\mathrm{int}}^{A_1\cdots A_k i} = {}_kN^{A_1\cdots A_k i} - \frac{1}{M}{}_kQ^{A_1\cdots A_k}F^i. \tag{1.22}$$

Then the equations of motion (1.10) can be rewritten as follows:

$$M\frac{d^2q^i}{dt^2} = F^i, \qquad \frac{d\, {}_\varkappa M_{\mathrm{int}}^{A_1\cdots A_\varkappa i}}{dt} = {}_\varkappa N_{\mathrm{int}}^{A_1\cdots A_\varkappa i}, \qquad \varkappa = \overline{1,\infty}. \tag{1.23}$$

Finally, the kinetic energy term (1.11) has the following form:

$$T = \frac{M}{2}g_{ij}\frac{dq^i}{dt}\frac{dq^j}{dt} + \frac{1}{2}g_{ij}\sum_{\mu,\nu=1}^{\infty} {}_{(\mu,\nu)}Q_{\mathrm{int}}^{A_1\cdots A_\mu B_1\cdots B_\nu}\frac{d\, {}_\mu{q^i}_{A_1\cdots A_\mu}}{dt}\frac{d\, {}_\nu{q^j}_{B_1\cdots B_\nu}}{dt}.$$

The constraints equations (1.13) for the polynomial functions H^r mean that our configurations are described by polynomials of a given finite degree N, and then the infinite series in the expansion (1.17) has to be truncated at the N-th step. The effective system of equations of motion (1.23) with directly substituted constraints ${}_kq^i = 0$, $k > N$, has only N first balance equations in the internal sector.

1.2.8 Eulerian multipoles

The above derivation of collective dynamics is based on the multipoles in the material space, thus, it is automatically compatible with the d'Alembert principle. However, Lagrangian (material) multipoles are not very intuitive quantities and it seems more natural to work with Eulerian (spatial) multipoles, which have all tensorial indices in the physical space.

So, Eulerian multipole moments of linear momentum and forces are defined as follows:

$$
\begin{aligned}
{}_k m^{i_1 \cdots i_k j} &= \int x^{i_1}(t,a) \cdots x^{i_k}(t,a) \frac{\partial x^j}{\partial t}(t,a) d\mu(a), && (1.24)\\
{}_k n^{i_1 \cdots i_k j} &= \int x^{i_1}(t,a) \cdots x^{i_k}(t,a) \Phi^j(t,a) d\mu(a), && (1.25)
\end{aligned}
$$

where both terms are symmetric in the first k-tuples of indices. It is also possible to express Eulerian multipoles through Lagrangian ones and generalized collective coordinates, e.g., in the polynomial theory of degree N we have that

$$
\begin{aligned}
{}_k m^{i_1 \cdots i_k j} &= \sum_{l_1, \ldots, l_k = 0}^{N} {}_{(l_1 + \cdots + l_k)} M^{A_1 \cdots A_{l_1} \cdots Z_1 \cdots Z_{l_k} j} {}_{l_1} q^{i_1}{}_{A_1 \cdots A_{l_1}} \cdots {}_{l_k} q^{i_k}{}_{Z_1 \cdots Z_{l_k}},\\
{}_k n^{i_1 \cdots i_k j} &= \sum_{l_1, \ldots, l_k = 0}^{N} {}_{(l_1 + \cdots + l_k)} N^{A_1 \cdots A_{l_1} \cdots Z_1 \cdots Z_{l_k} j} {}_{l_1} q^{i_1}{}_{A_1 \cdots A_{l_1}} \cdots {}_{l_k} q^{i_k}{}_{Z_1 \cdots Z_{l_k}}
\end{aligned}
$$

(if we do not perform polynomial truncation and base on analytical functions, then we have infinite series above).

In other words, Eulerian multipoles of degree k depend linearly (with functional coefficients) on Lagrangian ones of degrees $0, \ldots, Nk$. Thus, the Eulerian multipoles of degrees $0, \ldots, N$ calculated for (1.4) do not lead to some effective equations of motion for the polynomially constrained body because on the right-hand side of these effective equations there occur also Lagrangian dynamical multipoles ${}_s N$, $s = \overline{(N+1), Nk}$, which are calculated for the total forces, i.e., both the given forces and the reactions maintaining constraints. However, d'Alembert principle implies only the vanishing of first $N+1$ reaction multipoles ${}_r N_R$, $r = \overline{0, N}$, but not the rest of them, i.e., ${}_s N_R$, $s = \overline{(N+1), Nk}$.

Nevertheless, it is instructive to write down explicitly the system of Eulerian moment equations. For non-constrained systems with configurations described by analytical functions we have that

$$\frac{d\,{}_k m}{dt} = {}_k n + \sum_{\varkappa=1}^{k} \sum_{l_1,\ldots,l_k,r=0}^{\infty} \left({}_{l_1}q \otimes \cdots \otimes \frac{d\,{}_{l_\varkappa}q}{dt} \otimes \cdots \otimes {}_{l_k}q \otimes \frac{d\,{}_r q}{dt} \right) {}_{(l_1+\cdots+l_k+r)}Q \quad (1.26)$$

or, equivalently,

$$\sum_{l_1,\ldots,l_k,r=0}^{\infty} \left({}_{l_1}q \otimes \cdots \otimes {}_{l_k}q \otimes \frac{d^2\,{}_r q}{dt^2} \right) {}_{(l_1+\cdots+l_k+r)}Q = {}_k n. \quad (1.27)$$

This balance law for ${}_k m$ does not reduce to any conservation principle even in the interaction-free case ${}_k n = 0$ because of the strongly nonlinear and purely kinematic term on the right-hand side of (1.26).
Remark: another important difference between Eulerian and Lagrangian moments is that, in contrast to ${}_k M$ and ${}_k N$, the multipoles ${}_k m$ and ${}_k n$ with $k > 1$ do not split into translational and internal parts. Moreover, they are very complicated superpositions of terms involving mutually mixed translational and internal coordinates.

1.2.9 Some exceptional properties of Eulerian dipoles

It is very important for physical applications that the Eulerian dipole moments

$$\begin{aligned} {}_1 m^{ij} &= \sum_{k=0}^{N} {}_k q^i{}_{A_1\cdots A_k}\, {}_k M^{A_1\cdots A_k j}, \\ {}_1 n^{ij} &= \sum_{k=0}^{N} {}_k q^i{}_{A_1\cdots A_k}\, {}_k N^{A_1\cdots A_k j} \end{aligned}$$

do not involve Lagrangian multipoles of any degree higher than N for the polynomially constrained situation. This implies that, in addition to the monopole equations describing the centre-of-mass motion, i.e.,

$$\frac{dp^i}{dt} = M \frac{d^2 q^i}{dt^2} = F^i,$$

the constrained forms of equations (1.26), (1.27) with $k = 1$ may be used as a subsystem of effective equations of motion, i.e.,

$$\frac{d\,{}_1 m}{dt} = {}_1 n + \sum_{r,s=0}^{N} \left(\frac{d_r q}{dt} \otimes \frac{d_s q}{dt} \right) {}_{(r+s)}Q \quad (1.28)$$

or, equivalently,

$$\sum_{r,s=0}^{N} \left({}_r q \otimes \frac{d^2\,{}_s q}{dt^2} \right) {}_{(r+s)}Q = {}_1 n.$$

Remark: the quantity ${}_1m$ is not a constant of motion even in the interaction-free case, and this non-conservation may be interpreted as a consequence of the parametric dependence of the kinetic energy (1.11) on the metric tensor, i.e., we can rewrite (1.28) as follows:

$$\frac{d}{dt}{}_1m^{ij} = {}_1n^{ij} + 2\frac{\partial T}{\partial g_{ij}}.$$

However, the skew-symmetric part of ${}_1m$ is conserved because the kinetic term preventing the conservation of ${}_1m$ itself is a symmetric tensor, i.e.,

$$\frac{d_1m^{[ij]}}{dt} = {}_1n^{[ij]}.$$

This is physically obvious because the quantity $J^{ij} := {}_1m^{[ij]}$ is nothing else but the total angular momentum of our system related to the origin of coordinates x^i and $D^{ij} := {}_1n^{[ij]}$ is the total moment of forces with respect to the same origin. To be more precise, in the three-dimensional case the angular momentum and the moment of forces are axial vectors J^i, D^i related to the skew-symmetric tensors J^{ij}, D^{ij} through the totally antisymmetric symbol ε^{ijk}, i.e.,

$$J^{ij} = \varepsilon^{ijk}J_k, \qquad D^{ij} = \varepsilon^{ijk}D_k.$$

The next distinguishing feature of the moments ${}_1m$ and ${}_1n$ among all other multipoles is their natural splitting into the translational and internal parts, i.e.,

$${}_1m^{ij}_{\rm tr} = q^ip^j = Mq^i\frac{dq^j}{dt}, \qquad {}_1m^{ij}_{\rm int} = \sum_{\varkappa=1}^{N} {}_{\varkappa}q^i{}_{A_1\cdots A_\varkappa}{}_{\varkappa}M^{A_1\cdots A_\varkappa j}_{\rm int},$$

$${}_1n^{ij}_{\rm tr} = q^iF^j, \qquad {}_1n^{ij}_{\rm int} = \sum_{\varkappa=1}^{N} {}_{\varkappa}q^i{}_{A_1\cdots A_\varkappa}{}_{\varkappa}N^{A_1\cdots A_\varkappa j}_{\rm int},$$

where the internal moments ${}_1m_{\rm int}$ and ${}_1n_{\rm int}$ are dipole moments with respect to the instantaneous position of the centre of mass, whereas the total moments ${}_1m$ and ${}_1n$ are related to a fixed origin in the physical space, i.e., the origin of coordinates x^i.

Remark: obviously, the same is true for the antisymmetric parts of dipole moments, i.e., they split into translational and internal parts:

$$J = L + S, \qquad D = D_{\rm tr} + D_{\rm int},$$

where $L^{ij} = {}_1m^{[ij]}_{\rm tr} = q^{[i}p^{j]}$ denotes the angular momentum of the centre of mass with respect to a fixed origin, $S^{ij} = {}_1m^{[ij]}_{\rm int}$ is the spin, i.e., the angular momentum in the centre of mass reference frame, and $D^{ij}_{\rm tr} = {}_1n^{[ij]}_{\rm tr} = q^{[i}F^{j]}$, $D^{ij}_{\rm int} = {}_1n^{[ij]}_{\rm int}$.

Thus, the balance law for the internal part of ${}_1m$ has the following form:

$$\frac{d\,{}_1m_{\rm int}}{dt} = {}_1n_{\rm int} + \sum_{\rho,\sigma=1}^{N}\left(\frac{d_\rho q}{dt}\otimes\frac{d_\sigma q}{dt}\right){}_{(\rho,\sigma)}Q_{\rm int} \qquad (1.29)$$

or, equivalently,

$$\sum_{\rho,\sigma=1}^{N}\left({}_\rho q\otimes\frac{d^2{}_\sigma q}{dt^2}\right){}_{(\rho,\sigma)}Q_{\rm int} = {}_1n_{\rm int}.$$

Remark 1: in many realistic problems the hierarchy of equations of motion is suited to the ordering of collective modes that corresponds to their decreasing relevance to the phenomena considered, e.g., we are often dealing with situation when the leading internal modes correspond to rigid rotations, finite homogeneous deformations, and higher-order polynomial excitations superposed over them, for instance, as small corrections. Then to single out the rotational dynamics it is convenient to split the equation (1.29) into the skew-symmetric and symmetric parts.

Remark 2: it is interesting that in the case of infinitesimal deformations and rotation-less motion the Eulerian equations of motion become effective and essentially coincide with Lagrangian equations because then the spatial and material multipoles differ by higher-order terms.

1.2.10 Variational dynamical models

Later on we will see that the exceptional properties of the Eulerian dipole moments have very profound geometrical and group-theoretical roots.

So, let us consider variational dynamical models with Lagrangians of the form $L = T - V(q)$, where T is the kinetic energy as above, and V is a potential energy depending only on generalized coordinates q. The corresponding Legendre transformations can be written as follows:

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = M\frac{dq^j}{dt}g_{ji}, \qquad {}_\varkappa p^{A_1\cdots A_\varkappa}{}_i = \frac{\partial L}{\partial {}_\varkappa \dot{q}^i{}_{A_1\cdots A_\varkappa}} = {}_\varkappa M^{A_1\cdots A_\varkappa j}_{\rm int} g_{ji}, \qquad (1.30)$$

where p_i, ${}_\varkappa p^{A_1\cdots A_\varkappa}{}_i$ are canonical momenta conjugated, respectively, to the generalized collective coordinates q^i, ${}_\varkappa q^i{}_{A_1\cdots A_\varkappa}$.

Remark: obviously, just as ${}_\varkappa q^i{}_{A_1\cdots A_\varkappa}$ the quantities ${}_\varkappa p^{A_1\cdots A_\varkappa}{}_i$ are totally symmetric in material indices $(A_1, \ldots, A_\varkappa)$, thus, the independent phase-space coordinates are only those q^i, ${}_\varkappa q^i{}_{A_1\cdots A_\varkappa}$, p_i, ${}_\varkappa p^{A_1\cdots A_\varkappa}{}_i$ with $i = \overline{1,3}$ and $A_1 \leq A_2 \leq \cdots \leq A_\varkappa$, $\varkappa = \overline{1,N}$. Nevertheless, in many applications it is more convenient to use the symmetric redundant representation, i.e., formally admit all possible orderings of indices.

An interesting fact is that the internal parts of material multipoles ${}_{\varkappa}M_{\rm int}^{A_1\cdots A_\varkappa j}$ are related to the canonical momenta ${}_{\varkappa}p^{A_1\cdots A_\varkappa}{}_i$ through the g-lowering of spatial indices. Thus, roughly speaking, the Lagrangian multipoles are Hamiltonian generators of translations in the configuration space of our problem, i.e.,

$$'q^i = q^i + a^i, \qquad {}'_{\varkappa}q^i{}_{A_1\cdots A_\varkappa} = {}_{\varkappa}q^i{}_{A_1\cdots A_\varkappa} + {}_{\varkappa}a^i{}_{A_1\cdots A_\varkappa}, \tag{1.31}$$

where a^i and ${}_{\varkappa}a^i{}_{A_1\cdots A_\varkappa}$ are understood as a constant vector and tensor.

In virtue of the transformation rules (1.30) the spatial dipole multipoles ${}_1m^{ij}$ can be expressed directly through the canonical variables q^i, ${}_{\varkappa}q^i{}_{A_1\cdots A_\varkappa}$, p_i, ${}_{\varkappa}p^{A_1\cdots A_\varkappa}{}_i$ as follows:

$${}_1m^i{}_j := \left({}_1m^{ih}_{\rm tr} + {}_1m^{ih}_{\rm int}\right) g_{hj} = q^i p_j + \sum_{\varkappa=1}^{N} {}_{\varkappa}q^i{}_{A_1\cdots A_\varkappa}\,{}_{\varkappa}p^{A_1\cdots A_\varkappa}{}_j. \tag{1.32}$$

It can be shown that the quantities on the right-hand side are Hamiltonian generators of the spatial linear group $\mathrm{GL}(3,\mathbb{R})$ that acts on the generalized collective variables and their corresponding canonical momenta as follows:

$$\begin{aligned} 'q^i &= U^i{}_j q^j, \qquad {}'_{\varkappa}q^i{}_{A_1\cdots A_\varkappa} = U^i{}_j\,{}_{\varkappa}q^j{}_{A_1\cdots A_\varkappa}, \\ 'p_i &= p_j \left(U^{-1}\right)^j{}_i, \qquad {}'_{\varkappa}p^{A_1\cdots A_\varkappa}{}_i = {}_{\varkappa}p^{A_1\cdots A_\varkappa}{}_j \left(U^{-1}\right)^j{}_i, \end{aligned} \tag{1.33}$$

where $U \in \mathrm{GL}(3,\mathbb{R})$. These transformation rules describe rigid rotations and homogeneous deformations of the body in the physical space.

Let us consider an infinitesimal transformations $U = \mathbb{I} + \varepsilon$, where $\mathbb{I}$ denotes the identity matrix, and ε is an arbitrary "small" matrix. Then for any function f of the phase-space variables we have the following Poisson-bracket expression for the increment of f under the infinitesimal transformations:

$$\delta f := f('q, {}'_{\varkappa}q, 'p, {}'_{\varkappa}p) - f(q, {}_{\varkappa}q, p, {}_{\varkappa}p) = \{f, {}_1m^i{}_j\}\varepsilon^j{}_i$$

(this formula is valid up to second-order terms in ε). Thus, in fact, the quantities ${}_1m^i{}_j$ are Hamiltonian generators of $\mathrm{GL}(3,\mathbb{R})$. Adding to them the total linear momentum p_i, that generates spatial translations $x^i \mapsto x^i + a^i$, we obtain the system of functions that generates the action of the spatial affine group $\mathrm{GAf}(3,\mathbb{R}) \simeq \mathrm{GL}(3,\mathbb{R}) \times_s \mathbb{R}^3$.

Thus, we have the following system of Poisson brackets corresponding to the structure constants of $\mathrm{GAf}(3,\mathbb{R})$:

$$\{{}_1m^i{}_j, {}_1m^k{}_l\} = {}_1m^k{}_j\delta^i{}_l - {}_1m^i{}_l\delta^k{}_j, \quad \{{}_1m^i{}_j, p_k\} = \delta^i{}_k p_j, \quad \{p_i, p_j\} = 0. \tag{1.34}$$

So, the crucial role of the monopole and dipole moments of linear momentum is based on the fact that they generate the spatial affine group. They are distinguished among all multipole moments by the very affine geometry of the physical space. Metaphorically speaking, this relationship between the affine geometry and the structure of internal interactions manifests itself through the balance laws for the monopole and dipole moments.

The spatial Euclidean group is generated by the total linear momentum p_i and the angular momentum $J^i{}_j = {}_1m^i{}_j - g^{il}{}_1m^k{}_l g_{kj}$. If there are no external forces, then for the usual non-polar continuum the balance of p_i and $J^i{}_j$ become the system of conservation laws for linear and angular momenta. The translational and internal parts of $J^i{}_j$ are, respectively, the orbital angular momentum $L^i{}_j$ that generates rotations of the centre of mass around the origin of spatial coordinates (without affecting the internal variables) and the spin $S^i{}_j$ that generates rotations around the centre of mass itself. The Poisson brackets for the internal parts ${}_1m_{\mathrm{int}}{}^i{}_j$, $S^i{}_j$ (just as for the translational ones ${}_1m_{\mathrm{tr}}{}^i{}_j$, $L^i{}_j$) have the same form as those for the total quantities ${}_1m^i{}_j$, $J^i{}_j$.

Transformations generated by higher-order multipoles are not geometrically interesting. Moreover, their Poisson brackets do not close to a Lie algebra, i.e., for $N > 1$ the system of Eulerian multipoles ${}_km$, $k = \overline{0,N}$, generates an infinite-dimensional Poisson-Lie algebra of phase-space functions.

Remark: besides of spatial transformations it is useful to consider also material ones that act on configurations through an appropriate action on the Lagrangian variables a. For instance, the transformations $(xU)(t,a) := x(t,Ua)$, $U \in \mathrm{GL}(3,\mathbb{R})$, describe the action of the material group $\mathrm{GL}(3,\mathbb{R})$, i.e., the group of material rotations and homogeneous deformations:

$$'q^i = q^i, \qquad {}'_{\varkappa}q^i{}_{A_1\cdots A_\varkappa} = {}_{\varkappa}q^i{}_{B_1\cdots B_\varkappa} U^{B_1}{}_{A_1}\cdots U^{B_\varkappa}{}_{A_\varkappa}. \tag{1.35}$$

1.3 Affinely-rigid body and additional constraints

In virtue of the above geometric discussion it is particularly interesting to consider such a system for which the polynomial discretization is truncated at the step of affine functions, i.e.,

$$x^i(t,a) = r^i(t) + \varphi^i{}_A(t)a^A \tag{1.36}$$

is an affine mapping from the material space $\mathcal{N}$ into the physical one $\mathcal{M}$, where $\varphi(t) \in \mathrm{LI}(U,V)$ is the linear part of this affine mapping (and also $\varphi^i{}_A$ is non-singular for any time instant t) and $\vec{r}(t)$ is the radius-vector of the centre of mass of our body in the physical space if the origin of material radius-vectors is placed at the reference position of the centre of mass. Here the quantities r^i and $\varphi^i{}_A$ simply denotes the generalized collective variables q^i and $q^i{}_A$.

Thus, at any fixed $t \in \mathbb{R}$ the configuration space Q of our problem is identical to the manifold of all affine isomorphisms from the material space $\mathcal{N}$ onto the physical one $\mathcal{M}$, i.e., $\mathrm{AfI}(\mathcal{N},\mathcal{M})$, which can be identified with the Cartesian product $\mathcal{M}\times \mathrm{LI}(U,V)$, where $\mathrm{LI}(U,V)$ is the manifold of all linear isomorphisms from the linear space U onto the linear space V. The first factor refers to *the translational motion*, i.e., to the motion of the centre of mass, and the linear part of affine mapping describes *the relative (internal) motion*.

The corresponding phase space may be identified with the manifold $P = \mathcal{M}\times \mathrm{LI}(U,V)\times V^*\times \mathrm{L}(V,U)$, where the factors V^* and $\mathrm{L}(V,U)$ refer to translational and internal canonical momenta in the sense of the obvious pairing between the canonical momenta $(p,\pi) \in V^*\times \mathrm{L}(V,U)$ and velocities $(\dot{r},\dot{\varphi}) \in \mathcal{M}\times \mathrm{LI}(U,V)$ understood as follows:

$$\langle (p,\pi)\,, (\dot{r},\dot{\varphi})\rangle := \langle p,\dot{r}\rangle + \mathrm{Tr}\,(\pi\dot{\varphi}) = p_i\dot{r}^i + \pi^A{}_i\dot{\varphi}^i{}_A.$$

In practical calculations it is often technically convenient, although it may be geometrically misleading, to identify both U and V with $\mathbb{R}^n$, then the configuration space Q becomes identical to the group space of the n-dimensional affine group $\mathrm{GAf}(n,\mathbb{R})$, i.e., to the homogeneous space of this group with trivial isotropy groups. This affine group may be identified with the semi-direct product $\mathrm{GL}(n,\mathbb{R}) \times_s \mathbb{R}^n$.
Remark: another natural model of Q is $\mathcal{M} \times \mathcal{F}(V)$, where $\mathcal{F}(V)$ denotes the manifold of all linear frames in V. By the way, as a model of internal degrees of freedom $\mathcal{F}(V)$ is essentially identical with $\mathrm{LI}(U,V)$ if we put $U = \mathbb{R}^n$ and use the natural isomorphism between linear mappings $\varphi \in \mathrm{LI}(\mathbb{R}^n,V)$ and co-moving frames $e \in \mathcal{F}(V)$ frozen into the body and attached at the centre of mass. Such a model of the configuration space Q has to be used if the body is infinitesimal and the relative motion is replaced by the dynamics of essentially internal degrees of freedom. Then $\mathbb{R}^n$ becomes the micro-material space of internal motion.

The considered system is called *an affinely-rigid body* because during any admissible motion all affine relations between its constituents are invariant, i.e., material straight lines remain straight lines, their parallelism is conserved, and all mutual ratios of segments placed on the same straight lines are constant. The conception of the affinely-rigid body is a generalization of the usual *metrically-rigid body*, i.e., such a body in which during any admissible motion all distances (metric relations) between its constituents are constant (see, e.g., [3]).

In the special case of continuous media the configuration space becomes the proper affine group $\mathrm{GAf}^{+}(n,\mathbb{R}) \simeq \mathrm{GL}^{+}(n,\mathbb{R}) \times_s \mathbb{R}^n$, i.e., the matrices $\varphi^i{}_A$ have positive determinants. For discrete systems the whole affine group $\mathrm{GAf}(n,\mathbb{R}) \simeq \mathrm{GL}(n,\mathbb{R}) \times_s \mathbb{R}^n$ is in principle admissible.

Among all polynomially-discretized models the affinely-rigid body is peculiar in that that its configuration space is a Lie group or, to be more precise, a group space. Transformations described by polynomials of a finite degree $N > 1$ do not form a group because their compositions result in raising the degree.

1.3.1 Traditional d'Alembert model

Let us now consider variational dynamical model based on the Lagrangian of the form $L = T - V(r,\varphi)$, where the potential term $V(r,\varphi)$ is independent on the generalized velocities, and T is the appropriate kinetic energy for the affinely-constrained body, i.e.,

$$\begin{aligned} T &= \frac{1}{2} g_{ij} \int \frac{\partial x^i}{\partial t}(t,a) \frac{\partial x^j}{\partial t}(t,a) d\mu(a) \\ &= T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{M}{2} g_{ij} \frac{dr^i}{dt}\frac{dr^j}{dt} + \frac{1}{2} g_{ij} \frac{d\varphi^i{}_A}{dt}\frac{d\varphi^j{}_B}{dt} \widehat{J}^{AB}, \end{aligned} \tag{1.37}$$

where $\widehat{J}^{AB}$ are the components of the quadrupole moment of the mass distribution, i.e.,

$$\widehat{J}^{AB} = {}_2Q^{AB} = {}_{(1,1)}Q^{AB}_{\mathrm{int}} = \int a^A a^B d\mu(a).$$

Remark: hereafter any quantity with all indices immersed in the material space $\mathcal{N}$ has a hat over it just like the quadrupole moment $\widehat{J}^{AB}$.

The Legendre transformations (1.30), i.e.,

$$p_i = \frac{\partial L}{\partial \dot{r}^i} = M g_{ij} \frac{dr^j}{dt}, \qquad \pi^A{}_i = \frac{\partial L}{\partial \dot{\varphi}^i{}_A} = \widehat{J}^{AB} \frac{d\varphi^j{}_B}{dt} g_{ji},$$

leads to the following kinetic term of the Hamiltonian:

$$\mathcal{T} = \mathcal{T}_{\rm tr} + \mathcal{T}_{\rm int} = \frac{1}{2M} g^{ij} p_i p_j + \frac{1}{2} g^{ij} \pi^A{}_i \pi^B{}_j \left(\widehat{J}^{-1}\right)_{AB}, \tag{1.38}$$

where, obviously, g^{ij} are components of the reciprocal contravariant metric of g.
Remark: this kinetic term T (and its underlying flat metric on Q) is invariant under Abelian additive translations in $Q = \mathcal{M} \times$ LI(U, V). Therefore, without the interaction term, i.e., for $L = T$, the Hamiltonian generators p_i, $\pi^A{}_i$ are constants of motion. However, such geodetic models for deformable bodies are physically non-interesting because they predict unlimited expansion, contraction, and passing through singular configurations with $\det(\varphi) = 0$. The latter, although non-acceptable in continuum mechanics, may be to some extent admissible in mechanics of discrete bodies. If we once decide that the internal configuration space is given by LI(U, V), then the above transformation group is only local. At the same time, even for purely geodetic systems there is no invariance under geometrically interesting affine groups of left or right affine regular translations in Q.

The equations of motion may be derived basing merely on the d'Alembert principle and its underlying spatial metric g in the physical space $\mathcal{M}$. The primary quantities of this approach are the monopole and dipole moments of the distributions of linear kinematical momentum and forces within the body.

Let $\mathcal{F}^i(t, r, x(t, a), dr/dt, (\partial x/\partial t)(t, a); a)$ denote the density of forces per unit mass. Then the Eulerian monopoles are simply the total quantities, i.e., the total kinematical momentum k^i (do not confuse it at this stage with the canonical one p_i) and the total force F^i affecting the centre of mass motion, i.e.,

$$\begin{aligned} k^i &= \int \frac{\partial x^i}{\partial t}(t, a) d\mu(a), \\ F^i &= \int \mathcal{F}^i \left(t, r, x(t, a), \frac{dr}{dt}, \frac{\partial x}{\partial t}(t, a); a\right) d\mu(a). \end{aligned}$$

The dipole moments with respect to the current position of the centre of mass are referred to as kinematical affine spin K^{ij} (do not confuse it at this stage with the canonical one $\Sigma^i{}_j = \varphi^i{}_A \pi^A{}_j$) and the affine moment of forces N^{ij}. They are given as follows:

$$\begin{aligned} K^{ij} &= \int \left(x^i(t, a) - r^i\right) \left(\frac{\partial x^j}{\partial t}(t, a) - \frac{dr^j}{dt}\right) d\mu(a), \\ N^{ij} &= \int \left(x^i(t, a) - r^i\right) \mathcal{F}^j \left(t, r, x(t, a), \frac{dr}{dt}, \frac{\partial x}{\partial t}(t, a); a\right) d\mu(a). \end{aligned}$$

If we assume that the motion is affine, the kinematical quantities k^i, K^{ij} can be written now as follows:

$$k^i = M\frac{dr^i}{dt}, \qquad K^{ij} = \varphi^i{}_A \frac{d\varphi^j{}_B}{dt}\widehat{J}^{AB},$$

and the equations of motion of the affinely-rigid body are equivalent to the balance laws of the first two Eulerian multipoles, i.e.,

$$\frac{dk^i}{dt} = F^i, \qquad \frac{dK^{ij}}{dt} = N^{ij} + \widehat{J}^{AB}\frac{d\varphi^i{}_A}{dt}\frac{d\varphi^j{}_B}{dt}, \tag{1.39}$$

or explicitly

$$M\frac{d^2r^i}{dt^2} = F^i, \qquad \varphi^i{}_A\frac{d^2\varphi^j{}_B}{dt^2}\widehat{J}^{AB} = N^{ij}. \tag{1.40}$$

Remark: the above derivation is quite general and valid for all kinds of forces, including non-potential and dissipative ones (then the contravariant forces F^i and hyperforce N^{ij} (affine dynamical moment) may depend on all possible arguments, i.e., t, r^i, $\varphi^i{}_A$, dr^i/dt, $d\varphi^i{}_A/dt$) because it relies only on the metric structure g in $\mathcal{M}$ and on the d'Alembert principle. Obviously, for potential models with Lagrangians $L = T - V(r,\varphi)$ they depend only on generalized coordinates and possibly on the time variable t itself, and then

$$F^i = -g^{ij}\frac{dV}{dr^j}, \qquad N^{ij} = -\varphi^i{}_A\frac{\partial V}{\partial \varphi^k{}_A}g^{kj}.$$

Similarly, one can easily show that

$$K^{ij} = \Sigma^i{}_m g^{mj}, \qquad k^i = g^{ij}p_j.$$

But these relationships become false when Lagrangian depends on velocities not only through the kinetic energy T but also through some generalized potential V, e.g., when magnetic or gyroscopic external forces are present.

Kinematical quantities k^i, K^{ij} are intuitive because of their direct operational interpretation in terms of positions and velocities. At the same time they are lowest-order multipoles (monopoles and dipoles) of the distribution of kinematical linear momentum within the body. On the other hand, their canonical counterparts p_i, $\Sigma^i{}_j$ have a very deep geometrical interpretation as Hamiltonian generators of fundamental transformation groups. Because of this, they are very often important constants of motions. In mechanics of affinely-rigid body, equations of motion are equivalent to the balance laws for p_i, $\Sigma^i{}_j$ or, in a sense equivalently, to the ones for k^i, K^{ij} because Lagrangians of non-dissipative models (or at least Lagrangians of non-dissipative background dynamics) establish some link between these concepts. Similarly, in rigid-body mechanics equations of motion are equivalent to the balance for p_i, $\Sigma^i{}_j - g^{ik}\Sigma^m{}_k g_{mj}$ or, equivalently, for k^i, S^{ij} (the usual spin).

Let us introduce some co-moving representations of the kinematical quantities like k^i, K^{ij} or F^i, N^{ij}, i.e., the components with respect to the reference frame affinely-frozen into the body,

$$\widehat{k}^A = \left(\varphi^{-1}\right)^A{}_i k^i, \qquad \widehat{F}^A = \left(\varphi^{-1}\right)^A{}_i F^i, \tag{1.41}$$

$$\widehat{K}^{AB} = \left(\varphi^{-1}\right)^A{}_i \left(\varphi^{-1}\right)^B{}_j K^{ij}, \qquad \widehat{K}^A{}_B = \left(\varphi^{-1}\right)^A{}_i K^i{}_j \varphi^j{}_B, \tag{1.42}$$

$$\widehat{N}^{AB} = \left(\varphi^{-1}\right)^A{}_i \left(\varphi^{-1}\right)^B{}_j N^{ij}, \qquad \widehat{N}^A{}_B = \left(\varphi^{-1}\right)^A{}_i N^i{}_j \varphi^j{}_B. \tag{1.43}$$

Remark: there is a delicate point touching on the distinction between mixed and contravariant Eulerian tensors, i.e., unlike the spatial dipole moments that are related through the spatial metric tensor g (in affine coordinates the metric components g_{ij} are constants, in particular, in orthonormal frames $g_{ij} = \delta_{ij}$), i.e.,

$$K^i{}_j = K^{ik} g_{kj}, \qquad N^i{}_j = N^{ik} g_{kj},$$

the material dipole moments are not interrelated through the constant reference metric $\widehat{\eta}_{AB}$ but through the Green deformation tensor $\widehat{G}_{AB} := g_{ij}\varphi^i{}_A \varphi^j{}_B$, i.e.,

$$\widehat{K}^A{}_B = \widehat{K}^{AC}\widehat{G}_{CB}, \qquad \widehat{N}^A{}_B = \widehat{N}^{AC}\widehat{G}_{CB}.$$

Only in non-deformed configurations, e.g., when $\widehat{G}_{AB} = \widehat{\eta}_{AB}$, mixed tensors are related to contravariant ones through linear expressions with constant coefficients $\widehat{\eta}_{AB}$. More generally, if some tensor objects in V are related to each other by the g-shifting of indices, then the corresponding co-moving objects in U are interrelated by the $\widehat{G}$-shifting. And conversely, if two tensors in U are interrelated by the $\widehat{\eta}$-shifting of indices, then their spatial counterparts in V are obtained from each other by the C-shifting, where $C \in V^* \otimes V^*$, i.e., $C_{ij} = \widehat{\eta}_{AB} \left(\varphi^{-1}\right)^A{}_i \left(\varphi^{-1}\right)^B{}_j$, is the Cauchy deformation tensor.

The quantities $\widehat{K}^A{}_B$ have a direct geometrical interpretation because they are Hamiltonian generators of the material transformations (1.35), i.e.,

$$\left('r^i, '\varphi^i{}_A, 'p_i, '\pi^A{}_i\right) = \left(r^i, \varphi^i{}_B U^B{}_A, p_i, \left(U^{-1}\right)^A{}_B \pi^B{}_i\right), \qquad U \in \mathrm{GL}(3, \mathbb{R}).$$

Thus, in addition to (1.34) we have the following system of Poisson brackets for spatial and material transformations:

$$\{\widehat{K}^A{}_B, \widehat{K}^C{}_D\} = \widehat{K}^A{}_D \delta^C{}_B - \widehat{K}^C{}_B \delta^A{}_D, \tag{1.44}$$

$$\{\widehat{K}^A{}_B, K^i{}_j\} = 0, \qquad \{\widehat{K}^A{}_B, p_j\} = 0$$

(the latter two vanish because spatial and material transformations mutually commute). In the special case of the affinely-rigid body the dipole moments ${K_{\rm tot}}^{i}{}_{j}$, $\widehat{K}^{A}{}_{B}$ can be written, on the basis of (1.32) and (1.42), in the following simple explicit forms:

$$
{K_{\rm tot}}^{i}{}_{j} = r^{i}k_{j} + K^{i}{}_{j} = r^{i}p_{j} + \varphi^{i}{}_{A}\pi^{A}{}_{j}, \qquad \widehat{K}^{A}{}_{B} = \pi^{A}{}_{i}\varphi^{i}{}_{B}.
$$

1.3.2 Additionally-constrained models

The equations of motion of the affinely-rigid body (1.39), (1.40) can be subject to some additional constraints on the basis of the d'Alembert principle. For instance,

- the equations of motion for the usual metrically-rigid body are given by the balance laws of linear momentum and the skew-symmetric part of the second equation in (1.40), i.e.,

$$
M\frac{d^{2}r^{i}}{dt^{2}} = F^{i}, \qquad \varphi^{[i}{}_{A}\frac{d^{2}\varphi^{j]}{}_{B}}{dt^{2}}\widehat{J}^{AB} = N^{[ij]}, \tag{1.45}
$$

- the equations of internal motion for an incompressible affinely-rigid body are given by the g-traceless part of the second equation in (1.40), i.e.,

$$
\varphi^{i}{}_{A}\frac{d^{2}\varphi^{j}{}_{B}}{dt^{2}}\widehat{J}^{AB} - \frac{1}{3}g_{kl}\varphi^{k}{}_{A}\frac{d^{2}\varphi^{l}{}_{B}}{dt^{2}}\widehat{J}^{AB}g^{ij} = N^{ij} - \frac{1}{3}g_{kl}N^{kl}g^{ij} \tag{1.46}
$$

  (the factor $1/3$ comes from the dimension of the physical space, i.e., in an n-dimensional space it would be replaced by $1/n$),

- if the only admissible modes of internal motion are dilatations, then we have the scalar equations that is the g-trace of the second equation in (1.40), i.e.,

$$
g_{ij}\varphi^{i}{}_{A}\frac{d^{2}\varphi^{j}{}_{B}}{dt^{2}}\widehat{J}^{AB} = g_{ij}N^{ij}, \tag{1.47}
$$

- if the body undergoes only rigid rotations and dilatations, then its dynamics is given by the system composed of equations (1.45) and (1.47).

There are also interesting examples of *non-holonomic additional constraints* imposed on affine motion. However, to describe them we need an additional concept of an affine quasi-velocity, i.e.,

$$
\Omega := \frac{d\varphi}{dt}\varphi^{-1} \in \mathrm{L}(V), \qquad \Omega^{i}{}_{j} = \frac{d\varphi^{i}{}_{A}}{dt}\left(\varphi^{-1}\right)^{A}{}_{j}.
$$

This object Ω is non-holonomic in the sense that there are no coordinates y satisfying the condition $\Omega = \dot{y}$. Its kinematical meaning is that it defines a gradient of *the Eulerian velocity field* of the affinely-constrained continuum, i.e., the material point passing the fixed spatial point x has the following translational velocity:

$$^{E}v^{i}(t,a) = \frac{\partial x^{i}}{\partial t}\left(t, a(t,x)\right) = \frac{dr^{i}}{dt} + \Omega^{i}{}_{j}\left(x^{j} - r^{j}\right).$$

In the instantaneous rest frame of the centre of mass, placed also at the instantaneous position of this centre in the physical space $\mathcal{M}$, we simply have

$$^{E}v^{i} = \Omega^{i}{}_{j}x^{j}.$$

Similarly, $\dot{\varphi}^{i}{}_{A}$ has to do with the gradient of *the Lagrangian velocity field* because the instantaneous velocity of the a-th particle, $a \in \mathcal{N}$, is as follows:

$$^{L}v(a)^{i} = \frac{dr^{i}}{dt} + \frac{d\varphi^{i}{}_{A}}{dt}\, a^{A}.$$

In certain problems it is also convenient to express the centre-of-mass translational velocity $v^{i} = \dot{r}^{i}$ in the co-moving terms, i.e.,

$$\widehat{v}^{A} = \left(\varphi^{-1}\right)^{A}{}_{i}v^{i}.$$

We can also use the quantities

$$\widehat{\Omega} := \varphi^{-1}\frac{d\varphi}{dt} \in \mathrm{L}(U), \qquad \widehat{\Omega}^{A}{}_{B} = \left(\varphi^{-1}\right)^{A}{}_{i}\frac{d\varphi^{i}{}_{B}}{dt}$$

that are the co-moving components of the quasi-velocity $\Omega^{i}{}_{j}$, i.e.,

$$\widehat{\Omega}^{A}{}_{B} = \left(\varphi^{-1}\right)^{A}{}_{i}\Omega^{i}{}_{j}\varphi^{j}{}_{B}.$$

From the geometrical point of view the quantities $\Omega^{i}{}_{j}$ and $\widehat{\Omega}^{A}{}_{B}$ are Lie-algebraic objects corresponding, respectively, to the right- and left-invariant vector fields on the linear group $\mathrm{GL}(3,\mathbb{R})$. They provide an affine counterpart of the rigid-body angular velocities, and in fact reduce to them when φ is confined to the manifold of isometries of $(U,\widehat{\eta})$ onto (V,g); then they become skew-symmetric respectively with respect to $\widehat{\eta}$ or g.

Then the internal affine moments may be interpreted as non-holonomic canonical momenta conjugated to these affine quasi-velocities, i.e.,

$$\pi^{A}{}_{i}\frac{d\varphi^{i}{}_{A}}{dt} = K^{i}{}_{j}\Omega^{j}{}_{i} = \widehat{K}^{A}{}_{B}\widehat{\Omega}^{B}{}_{A}.$$

The angular velocity of the affine motion and the deformation rate can be defined as follows:

$$\omega^i{}_j := \frac{1}{2}\left(\Omega^i{}_j - g^{il}\Omega^k{}_l g_{kj}\right), \qquad d^i{}_j := \frac{1}{2}\left(\Omega^i{}_j + g^{il}\Omega^k{}_l g_{kj}\right).$$

After g-lowering of the first index, the deformation rate tensor becomes totally symmetric, i.e.,

$$d_{ij} = g_{ik}d^k{}_j = \frac{1}{2}\left(g_{ik}\Omega^k{}_j + g_{jk}\Omega^k{}_i\right) = d_{ji},$$

and its co-moving components $\widehat{d}_{AB} = d_{ij}\varphi^i{}_A\varphi^j{}_B$ represent the strain rate, i.e.,

$$\widehat{d}_{AB} = \frac{1}{2}\frac{d\widehat{G}_{AB}}{dt}.$$

Thus, with the help of the above-defined quasi-velocities and quantities connected with them, the equations of motion of the affinely-rigid body (1.40) can be subject to two more non-holonomic additional constraints on the basis of the d'Alembert principle, i.e.,

- the another form of rigid-body constraints when the deformation rate vanishes, i.e., $d^i{}_j = 0$, but in spite of its non-holonomic form this constraints equation can be integrated to the finite condition $\widehat{G}_{AB} = g_{ij}\varphi^i{}_A\varphi^j{}_B = \widehat{\eta}_{AB}$ that defines the rigid-body configurations with equations of motion of the form (1.45),

- and the constraints of the rotation-less motion when the angular velocity of the affine motion vanishes, i.e., $\omega^i{}_j = 0$. It is interesting to note that this case has a distinguished geometrical interpretation because such constraints are essentially non-holonomic (comparing, e.g., with the rigid-body constraints) and do not impose any restrictions on the attainability of affine configurations. The equations of internal motion for this situation are given by the symmetric part of the second equation in (1.40), i.e.,

$$\varphi^{(i}{}_A\frac{d^2\varphi^{j)}{}_B}{dt^2}\widehat{J}^{AB} = N^{(ij)}.$$

Remark: the approximation of the rotation-less motion may provide a reasonable model of the dynamics of small inclusions in very viscous fluids.

As we have seen, Lie groups, e.g., the affine or linear ones, appear in a natural way as the configuration spaces of systems described with the help of generalized collective modes. Thus, it is very useful to develop the formal mathematical language for description of dynamical systems based on Lie groups and their homogeneous spaces as models of internal and collective degrees of freedom (see Appendix A). These models are realistic and quite often possess rigorous analytical solutions in terms of special functions and power series (probably, due to the analytical structure of Lie groups).

Chapter 2

Dynamical affine invariance

2.1 Dynamically invariant geodetic models

The affine models of collective and internal degrees of freedom are widely used by many authors not only in elastic problems but also in microphysical (molecules, atomic nuclei) and astrophysical ones. However, there it is only kinematics and the very geometry of degrees of freedom that is ruled by the affine group, whereas dynamics is based on the group of Euclidean motions. From this point of view, such models do not belong to the class of dynamical systems on Lie groups, e.g., in the sense of Arnold and Hermann. We will develop here some models which are ruled by affine or projective groups not only on the kinematical but also on the dynamical levels. This presents some interest from the purely geometrical point of view but some applications are also possible (defects in solids, collective nuclear models, and certain special problems in mechanics of deformable objects).

So, let us construct kinetic energy models, i.e., metric tensors on the configuration space, which are invariant under the spatial (left) or material (right) affine groups. Such models belong to the class of invariant geodetic systems with Lie-group-ruled degrees of freedom, which were investigated, e.g., by Arnold and Binz [3, 10]. The right affinely invariant problems may be considered as a very drastic discretization, i.e., reduction to a finite number of degrees of freedom, of the Arnold description of the ideal fluid as an infinite-dimensional Hamiltonian system on the group $\mathrm{SDiff}(n, \mathbb{R})$ of all volume-preserving diffeomorphisms.

Our kinetic energies are not based on the d'Alembert principle but only on the appropriate invariance demands. Some of such invariant geodetic models may describe elastic-like bounded vibrations even without any extra-introduced potential term. So to speak, interactions are encoded in the metric tensor on the configuration space, e.g., like in Maupertuis variational principle. We may expect some physical applications for such models in the collective nuclear dynamics, where there are no reasons to expect the d'Alembert model to work. Generally speaking, d'Alembert principle works when the collective motion is a "large" background perturbed by small negligible non-collective vibrations. In the droplet description of nuclei, the underlying non-collective micro-motion may be just "large" and collective modes may appear as some average kinematical characteristics of this hidden microscopic motion. Then the simplest procedure is to postulate some phenomenological model on the basis of invariance principle. Other applications may be expected in the theory of defects in solids and perhaps in dynamics of some macroscopic objects in fluids, e.g., gas bubbles.

Let us stress also that there are no geodetic models affinely-invariant (amorphous) simultaneously in the physical space $\mathcal{M}$ and in the material space $\mathcal{N}$ (see, e.g., [144]). This is because of the malicious non-semisimplicity of $\mathrm{GAf}(n,\mathbb{R})$, i.e., any twice covariant tensor field on $\mathrm{GAf}(n,\mathbb{R})$ simultaneously left and right invariant must be degenerate. The highest possible symmetries compatible with the non-singularity demand for the metric are the following:

- affine symmetry in the physical space $\mathcal{M}$ and metrical (Euclidean) symmetry in the material one $\mathcal{N}$, then we need no physical metric g,
- Euclidean symmetry in the physical space $\mathcal{M}$ and affine symmetry in the material one $\mathcal{N}$, then we need no material metric $\widehat{\eta}$.

Remark: although the second possibility is more physical, the first one also has the attractive features. For instance, then we have the situation very similar to the one in the general relativity, i.e., in the physical space $\mathcal{M}$ there is no fixed metric geometry at all and the components of the metric tensor are included in the physical degrees of freedom and dynamically coupled with matter distribution.

If we neglect translational motion, then there exist Hamiltonian geodetic systems on $\mathrm{GL}(n,\mathbb{R})$ invariant simultaneously under left and right regular translations but the underlying metrics are never positively-definite. Nevertheless, they may be physically applicable, e.g., in the theory of one-dimensional lattices (see, e.g., [178]).

2.1.1 General left invariant geodetic problems

Let us consider the left-invariant geodetic problem on the affine group with the total kinetic energy $T = T_{\rm tr} + T_{\rm int}$, where $T_{\rm int}$ is like (A.2), i.e.,

$$T_{\rm tr} = \frac{M}{2}\widehat{\eta}_{AB}\widehat{v}^A\widehat{v}^B = \frac{M}{2}C_{ij}v^iv^j, \qquad T_{\rm int} = \frac{1}{2}\widehat{\mathcal{L}}^B{}_A{}^D{}_C\widehat{\Omega}^A{}_B\widehat{\Omega}^C{}_D, \tag{2.1}$$

are the translational and internal parts of the kinetic energy, $\widehat{\mathcal{L}}^B{}_A{}^D{}_C$ are some constants, and

$$C_{ij} = \widehat{\eta}_{AB}\left(\varphi^{-1}\right)^A{}_i\left(\varphi^{-1}\right)^B{}_j$$

is the corresponding metric-like Cauchy deformation tensor of the physical space ("push-forward" of the metric $\widehat{\eta}$).

Such a general kinetic energy $T = T_{\rm tr} + T_{\rm int}$ defines the curved (non-Euclidean) metric on AfI$(\mathcal{N}, \mathcal{M})$. Moreover, in the physical space $\mathcal{M}$ we have only the transported from the material space $\mathcal{N}$ metric-like Cauchy deformation tensor $C = \left(\varphi^{-1}\right)^*\widehat{\eta}$, i.e., no naturally fixed metric in the physical space $\mathcal{M}$ is used and it may be equally the amorphous affine space as well as the metric space.

Let us also suppose that our problem is right-invariant under some maximal compact subgroup of GL$(U) \simeq$ GL$(n, \mathbb{R})$, then the internal kinetic energy have the following form:

$$T_{\rm int} = \frac{1}{2}\widehat{\eta}_{AB}\widehat{\Omega}^A{}_C\widehat{\Omega}^B{}_D\widehat{J}^{CD} + \frac{\alpha}{2}\widehat{\Omega}^A{}_B\widehat{\Omega}^B{}_A + \frac{\beta}{2}\widehat{\Omega}^A{}_A\widehat{\Omega}^B{}_B = \frac{1}{2}\widehat{\Xi}^B{}_A{}^D{}_C\widehat{\Omega}^A{}_B\widehat{\Omega}^C{}_D, \tag{2.2}$$

where

$$\widehat{\Xi}^B{}_A{}^D{}_C = \widehat{J}^{BD}\widehat{\eta}_{AC} + \alpha\delta^B{}_C\delta^D{}_A + \beta\delta^B{}_A\delta^D{}_C$$

is a generalized inertial tensor of the affinely-rigid body in the co-moving ("affinely-frozen" in the body) representation, in which it is constant in contrary to the laboratory representation. The last two terms in (2.2) are Casimir invariants on the total affine group, α and β are some constants. Also $\widehat{\Xi}$ has the following symmetry property in the pairs of its indices:

$$\widehat{\Xi}^B{}_A{}^D{}_C = \widehat{\Xi}^D{}_C{}^B{}_A.$$

In the following formulas we will need the spatial inertial quadrupole, i.e.,

$$J[\varphi]^{ij} = \varphi^i{}_K\varphi^j{}_L\widehat{J}^{KL}.$$

It is related to $\widehat{J}$ just as C^{-1} is to η. When the body is inertially isotropic, $J[\varphi]$ becomes proportional to the inverse Cauchy deformation tensor. Unlike the co-moving internal tensor $\widehat{J} \in U \otimes U$, the spatial inertial tensor $J[\varphi] \in V \otimes V$ is configuration-dependent, thus variable in time.

We may equally write the internal kinetic energy (2.2) in the following form:

$$T_{\rm int} = \frac{1}{2} C_{ij} \Omega^{i}{}_{k} \Omega^{j}{}_{l} J^{kl} + \frac{\alpha}{2} \Omega^{i}{}_{j} \Omega^{j}{}_{i} + \frac{\beta}{2} \Omega^{i}{}_{i} \Omega^{j}{}_{j} = \frac{1}{2} \Xi^{k}{}_{i}{}^{l}{}_{j} \Omega^{i}{}_{k} \Omega^{j}{}_{l}, \tag{2.3}$$

where

$$\Xi^{k}{}_{i}{}^{l}{}_{j} = J^{kl} C_{ij} + \alpha \delta^{k}{}_{j} \delta^{l}{}_{i} + \beta \delta^{k}{}_{i} \delta^{l}{}_{j}$$

is generalized inertial tensor in the laboratory representation. We can also introduce its mixed representations, e.g.,

$$\Xi^{A}{}_{i}{}^{B}{}_{j} = \widehat{\Xi}^{A}{}_{C}{}^{B}{}_{D} \left(\varphi^{-1}\right)^{C}{}_{i} \left(\varphi^{-1}\right)^{D}{}_{j}.$$

Let us define the canonical affine momenta corresponding to our affine velocities and the total affine momentum $P_{\rm tot}{}^{i}{}_{j} = r^{i} p_{j} + \Sigma^{i}{}_{j}$ as follows:

$$\begin{aligned}
p_{i} &= \frac{\partial T}{\partial v^{i}} = M C_{ij} v^{j}, & \widehat{p}_{A} &= \frac{\partial T}{\partial \widehat{v}^{A}} = p_{i} \varphi^{i}{}_{A} = M \widehat{\eta}_{AB} \widehat{v}^{B}, \\
\Sigma^{i}{}_{j} &= \frac{\partial T}{\partial \Omega^{j}{}_{i}} = \Xi^{i}{}_{j}{}^{l}{}_{k} \Omega^{k}{}_{l}, & \widehat{\Sigma}^{A}{}_{B} &= \frac{\partial T}{\partial \widehat{\Omega}^{B}{}_{A}} = \widehat{\Xi}^{A}{}_{B}{}^{D}{}_{C} \widehat{\Omega}^{C}{}_{D}, \\
\pi^{A}{}_{i} &= \frac{\partial T}{\partial \dot{\varphi}^{i}{}_{A}} = \Xi^{A}{}_{i}{}^{B}{}_{j} \dot{\varphi}^{j}{}_{B}, & P_{\rm tot}{}^{i}{}_{j} &= M r^{i} v^{k} C_{kj} + \Xi^{i}{}_{j}{}^{l}{}_{k} \Omega^{k}{}_{l}, \\
\widehat{P}_{\rm tot}{}^{A}{}_{B} &= (\varphi^{-1})^{A}{}_{i} P_{\rm tot}{}^{i}{}_{j} \varphi^{j}{}_{B} &=\;& M \widehat{r}^{A} \widehat{v}^{C} \widehat{\eta}_{CB} + \widehat{\Xi}^{A}{}_{B}{}^{C}{}_{D} \widehat{\Omega}^{D}{}_{C}
\end{aligned}$$

It is very convenient to introduce, beside the usual canonical momentum $\pi^{A}{}_{i}$, *the canonical affine spin* defined in two representations: the spatial and co-moving ones, i.e., $\Sigma \in {\rm L}(V)$ and $\widehat{\Sigma} \in {\rm L}(U)$. In terms of coordinates they are given by the following formulas:

$$\Sigma^{i}{}_{j} = \varphi^{i}{}_{A} \pi^{A}{}_{j}, \qquad \widehat{\Sigma}^{A}{}_{B} = \pi^{A}{}_{i} \varphi^{i}{}_{B}.$$

As previously Ω and $\widehat{\Omega}$, Σ and $\widehat{\Sigma}$ are connected through the following φ-similarity transformation:

$$\Sigma = \varphi \widehat{\Sigma} \varphi^{-1}, \qquad \Sigma^{i}{}_{j} = \varphi^{i}{}_{A} \widehat{\Sigma}^{A}{}_{B} \left(\varphi^{-1}\right)^{B}{}_{j}.$$

They are purely Hamiltonian quantities defined on the phase space; without any precisely defined Lagrangian or Hamiltonian we cannot relate them to generalized velocities. It is seen, however, that they are dual objects to affine velocities, i.e., they are non-holonomic canonical momenta conjugate to them in the sense of following pairing:

$$\langle \Sigma, \Omega \rangle = \left\langle \widehat{\Sigma}, \widehat{\Omega} \right\rangle := \operatorname{Tr}\left(\Sigma\Omega\right) = \operatorname{Tr}\left(\widehat{\Sigma}\widehat{\Omega}\right) = \pi^{A}{}_{i}\varphi^{i}{}_{A}.$$

This canonical isomorphism between two Lie algebras

$$\mathrm{GL}(U)' = \mathrm{L}(U), \qquad \mathrm{GL}(V)' = \mathrm{L}(V)$$

and their duals simplifies remarkably all formulas and considerations.

The quantities $\Sigma^{i}{}_{j}$ are Hamiltonian generators of $\mathrm{GL}(V)$ acting on $\mathrm{LI}(U,V)$ through the left translations:

$$\varphi \mapsto A\varphi, \qquad \varphi \in \mathrm{LI}(U,V), \qquad A \in \mathrm{GL}(V).$$

Similarly, $\widehat{\Sigma}^{A}{}_{B}$ generate right regular translations in the internal configuration space:

$$\varphi \mapsto \varphi B, \qquad \varphi \in \mathrm{LI}(U,V), \qquad B \in \mathrm{GL}(U).$$

In continuum mechanics these mappings are referred to as spatial and material transformations, respectively. In this case they include rotations and homogeneous deformations. Obviously, to use correctly such terms we must be given metric tensors in V and U. Then the g-antisymmetric part of Σ and the $\widehat{\eta}$-antisymmetric part of $\widehat{\Sigma}$ generate, respectively, spatial and material rigid rotations, and their symmetric parts generate deformations. The doubled antisymmetric parts are referred to as spin S and vorticity $\widehat{V}$ [37],

$$S^{i}{}_{j} = \Sigma^{i}{}_{j} - g^{ik}\Sigma^{l}{}_{k}g_{lj}, \qquad \widehat{V}^{A}{}_{B} = \widehat{\Sigma}^{A}{}_{B} - \widehat{\eta}^{AC}\widehat{\Sigma}^{D}{}_{C}\widehat{\eta}_{DB}.$$

If motion is not metrically-rigid, then $\widehat{V}$ is not a co-moving representation of S, i.e.,

$$S^{i}{}_{j} \neq \varphi^{i}{}_{A}\widehat{V}^{A}{}_{B}\left(\varphi^{-1}\right)^{B}{}_{j}.$$

The translational or orbital affine momentum with respect to some point $\mathcal{O} \in \mathcal{M}$ is defined as follows:

$$\Lambda(\mathcal{O})^{i}{}_{j} := r^{i}p_{j},$$

where r^{i} are Cartesian coordinates of the $\mathcal{O}$-radius vector of the current position of the centre of mass in the physical space $\mathcal{M}$. The total affine momentum with respect to $\mathcal{O}$ is given by the following expression:

$$P_{\mathrm{tot}}(\mathcal{O})^{i}{}_{j} := \Lambda(\mathcal{O})^{i}{}_{j} + \Sigma^{i}{}_{j}.$$

We see that $\Lambda(\mathcal{O})$ and $P_{\rm tot}(\mathcal{O})$ depend explicitly on the choice of $\mathcal{O}$. Unlike this, Σ is objective (in a fixed Galilean reference frame). There is a complete analogy with the properties of angular momentum, i.e., the doubled g-antisymmetric part of the above objects. The quantity $P_{\rm tot}(\mathcal{O})$ is a Hamiltonian generator of the group of affine transformations of $\mathcal{M}$ preserving $\mathcal{O}$, i.e., the $\mathcal{O}$-centred affine subgroup.

Let us consider Lagrangians $L = T - V(r, \varphi)$ corresponding to our kinetic energy T with velocity-independent potentials (no magnetic forces). The equations of motion for the translational and internal degrees of freedom are as follows:

$$\frac{dv^i}{dt} = \frac{1}{M} F^i + \left[\Omega^i{}_j + \left(C^{-1}\right)^{ik} \Omega^l{}_k C_{lj}\right] v^j, \tag{2.4}$$

$$\Xi^i{}_j{}^k{}_l \frac{d\Omega^l{}_k}{dt} = N^i{}_j - M v^i v^k C_{kj} + \left[\Omega^k{}_j \Xi^i{}_k{}^l{}_n - \Omega^i{}_k \Xi^k{}_j{}^l{}_n\right] \Omega^n{}_l, \tag{2.5}$$

or in the co-moving representation:

$$\frac{d\widehat{v}^A}{dt} = \frac{1}{M} \widehat{F}^A + \widehat{\eta}^{AB} \widehat{\Omega}^C{}_B \widehat{\eta}_{CD} \widehat{v}^D, \tag{2.6}$$

$$\begin{aligned} \widehat{\Xi}^A{}_B{}^C{}_D \frac{d\widehat{\Omega}^D{}_C}{dt} &= \widehat{N}^A{}_B - M \widehat{v}^A \widehat{v}^C \widehat{\eta}_{CB} \\ &+ \left[\widehat{\Omega}^E{}_B \widehat{\Xi}^A{}_E{}^C{}_D - \widehat{\Omega}^A{}_E \widehat{\Xi}^E{}_B{}^C{}_D\right] \widehat{\Omega}^D{}_C, \end{aligned} \tag{2.7}$$

where $\widehat{\eta}^{AB}$ is the reciprocal tensor to the metric tensor $\widehat{\eta}_{AB}$, i.e., $\widehat{\eta}^{AB} \widehat{\eta}_{BC} = \delta^A{}_C$,

$$F_i = -\frac{\partial V}{\partial r^i}, \qquad N^i{}_j = -\varphi^i{}_A \frac{\partial V}{\partial \varphi^j{}_A}$$

and

$$\widehat{F}_A = -\frac{\partial V}{\partial r^i} \varphi^i{}_A, \qquad \widehat{N}^A{}_B = -\frac{\partial V}{\partial \varphi^i{}_A} \varphi^i{}_B$$

are the total force and affine moment of forces in the laboratory and co-moving representations.

Remark: we have to stress that in this treatment we use different prescriptions for shifting the tensorial indices, i.e., various isomorphisms between contravariant and covariant objects. For example, the shifting of indices in the physical space is made with the help of the Cauchy deformation tensor C, e.g.,

$$F^i = \left(C^{-1}\right)^{ij} F_j, \qquad N^{ij} = N^i{}_k \left(C^{-1}\right)^{kj}.$$

If we introduce the kinematical affine spin (hypermomentum) in the laboratory representation as $K^{ij} = \Sigma^i{}_k \left(C^{-1}\right)^{kj}$ or equally in the co-moving representation as

$\widehat{K}^{AB} = \left(\varphi^{-1}\right)^{A}{}_{i}\left(\varphi^{-1}\right)^{B}{}_{j}K^{ij} = \widehat{\Sigma}^{A}{}_{C}\widehat{\eta}^{CB}$, then we can rewrite the previous equations (2.5) and (2.7) in the balance form:

$$\frac{dK^{ij}}{dt} = N^{ij} - Mv^{i}v^{j} + K^{ik}\left[\Omega^{j}{}_{k} + C_{kl}\Omega^{l}{}_{m}\left(C^{-1}\right)^{mj}\right], \tag{2.8}$$

$$\frac{d\widehat{K}^{AB}}{dt} = \widehat{N}^{AB} - M\widehat{v}^{A}\widehat{v}^{B} + \widehat{K}^{AC}\widehat{\eta}_{CD}\widehat{\Omega}^{D}{}_{E}\widehat{\eta}^{EB} - \widehat{\Omega}^{A}{}_{C}\widehat{K}^{CB}. \tag{2.9}$$

We may also introduce the internal angular momentum (spin) S^{ij} and so-called vorticity $\widehat{V}^{AB}$ as the doubled skew-symmetric part of K^{ij} and $\widehat{K}^{AB}$, respectively. Their balance equations can be also written as the doubled skew-symmetric parts of (2.8) and (2.9).

Performing the Legendre transformation we obtain the corresponding Hamiltonian in the following form:

$$H = \dot{r}^{i}p_{i} + \dot{\varphi}^{i}{}_{A}\pi^{A}{}_{i} - L = \mathcal{T}_{\text{tr}} + \mathcal{T}_{\text{int}} + V(r,\varphi),$$

where the translational and internal kinetic energy terms are as follows:

$$\mathcal{T}_{\text{tr}} = \frac{1}{2M}\widehat{\eta}^{AB}\widehat{p}_{A}\widehat{p}_{B} = \frac{1}{2M}\left(C^{-1}\right)^{ij}p_{i}p_{j},$$

$$\mathcal{T}_{\text{int}} = \frac{1}{2}\left(\widehat{\Xi}^{-1}\right)^{A}{}_{C}{}^{B}{}_{D}\widehat{\Sigma}^{C}{}_{A}\widehat{\Sigma}^{D}{}_{B} = \frac{1}{2}\left(\Xi^{-1}\right)^{i}{}_{A}{}^{j}{}_{B}\pi^{A}{}_{i}\pi^{B}{}_{j} = \frac{1}{2}\left(\Xi^{-1}\right)^{i}{}_{k}{}^{j}{}_{l}\Sigma^{k}{}_{i}\Sigma^{l}{}_{j}.$$

We have the following system of the non-zero basic Poisson brackets corresponding to the structure constants of $\mathrm{GAf}(n,\mathbb{R})$ and their concomitants:

$$\{r^{i}, p_{j}\} = \delta^{i}{}_{j}, \qquad \{r^{i}, \widehat{p}_{A}\} = \varphi^{i}{}_{A}, \tag{2.10}$$

$$\{\varphi^{i}{}_{A}, \pi^{B}{}_{j}\} = \delta^{i}{}_{j}\delta^{B}{}_{A}, \qquad \{\varphi^{i}{}_{A}, \widehat{\Sigma}^{B}{}_{C}\} = \delta^{B}{}_{A}\varphi^{i}{}_{C}, \tag{2.11}$$

$$\{\varphi^{i}{}_{A}, \Sigma^{j}{}_{k}\} = \delta^{i}{}_{k}\varphi^{j}{}_{A}, \qquad \{\Sigma^{j}{}_{k}, \left(\varphi^{-1}\right)^{A}{}_{i}\} = \delta^{j}{}_{i}\left(\varphi^{-1}\right)^{A}{}_{k}, \tag{2.12}$$

$$\{\widehat{p}_{A}, \Sigma^{i}{}_{j}\} = \varphi^{i}{}_{A}p_{j}, \qquad \{\Sigma^{i}{}_{j}, \Sigma^{k}{}_{l}\} = \delta^{i}{}_{l}\Sigma^{k}{}_{j} - \delta^{k}{}_{j}\Sigma^{i}{}_{l}, \tag{2.13}$$

$$\{\widehat{p}_{A}, \widehat{\Sigma}^{B}{}_{C}\} = \delta^{B}{}_{A}\widehat{p}_{C}, \qquad \{\widehat{\Sigma}^{A}{}_{B}, \widehat{\Sigma}^{C}{}_{D}\} = \delta^{C}{}_{B}\widehat{\Sigma}^{A}{}_{D} - \delta^{A}{}_{D}\widehat{\Sigma}^{C}{}_{B}, \tag{2.14}$$

$$\{\widehat{p}_{A}, V(r,\varphi)\} = \widehat{F}_{A}, \qquad \{p_{i}, V(r,\varphi)\} = F_{i}, \tag{2.15}$$

$$\{\widehat{\Sigma}^{A}{}_{B}, V(r,\varphi)\} = \widehat{N}^{A}{}_{B}, \qquad \{\Sigma^{i}{}_{j}, V(r,\varphi)\} = N^{i}{}_{j}, \tag{2.16}$$

$$\{\Sigma^{i}{}_{j}, C_{kl}\} = 2\delta^{i}{}_{(k}C_{l)j}, \qquad \{\Sigma^{i}{}_{j}, (C^{-1})^{kl}\} = -2(C^{-1})^{i(k}\delta^{l)}{}_{j}, \tag{2.17}$$

$$\{\widehat{\Sigma}^{A}{}_{B}, \widehat{G}_{CD}\} = -2\delta^{A}{}_{(C}\widehat{G}_{D)B}, \qquad \{\widehat{\Sigma}^{A}{}_{B}, (\widehat{G}^{-1})^{CD}\} = 2(\widehat{G}^{-1})^{A(C}\delta^{D)}{}_{B}. \tag{2.18}$$

The above Poisson brackets follow directly from the standard definition [3, 47, 57], i.e.,

$$\{F, G\} := \frac{\partial F}{\partial q^{\alpha}}\frac{\partial G}{\partial p_{\alpha}} - \frac{\partial F}{\partial p_{\alpha}}\frac{\partial G}{\partial q^{\alpha}},$$

where q^α are generalized coordinates and p_α are their conjugate canonical momenta. In our model q^α are given by r^i, $\varphi^i{}_A$, and p_α by p_i, $\pi^A{}_i$. In applications it is sufficient to remember that

$$\{q^\alpha, q^\beta\} = 0, \qquad \{p_\alpha, p_\beta\} = 0, \qquad \{q^\alpha, p_\beta\} = \delta^\alpha{}_\beta,$$

that Poisson bracket is bilinear (over constant reals $\mathbb{R}$), skew-symmetric, i.e.,

$$\{F, G\} = -\{G, F\},$$

satisfies the Jacobi identity, i.e.,

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0,$$

and finally that

$$\{F, H(G_1, \ldots, G_k)\} = \sum_{p=1}^{k} H_{,p}(G_1, \ldots, G_k)\{F, G_p\},$$

where commas before indices denote the partial derivatives. The formerly-quoted Poisson brackets together with the above rules are sufficient for all calculations concerning equations of motion and their analysis.

With the help of these Poisson brackets we may rewrite the equations of motion in the Hamiltonian form (the procedure based on Poisson brackets and canonical formalism is very often more easy and computationally less embarrassing than the one directly using the Euler-Lagrange equations):

$$\begin{aligned}
\frac{dp_i}{dt} = \{p_i, H\} &= F_i, \\
\frac{d\widehat{p}_A}{dt} = \{\widehat{p}_A, H\} &= \widehat{F}_A + \widehat{p}_C \left(\widehat{\Xi}^{-1}\right)^C{}_A{}^D{}_B \widehat{\Sigma}^B{}_D, \\
\frac{d\Sigma^i{}_j}{dt} = \{\Sigma^i{}_j, H\} &= N^i{}_j - \frac{1}{M}\left(C^{-1}\right)^{ik} p_k p_j, \\
\frac{d\widehat{\Sigma}^A{}_B}{dt} = \{\widehat{\Sigma}^A{}_B, H\} &= \widehat{N}^A{}_B - \frac{1}{M}\widehat{\eta}^{AC}\widehat{p}_C\widehat{p}_B \\
&+ \left[\widehat{\Sigma}^A{}_C \left(\widehat{\Xi}^{-1}\right)^C{}_B{}^E{}_D - \left(\widehat{\Xi}^{-1}\right)^A{}_C{}^E{}_D \widehat{\Sigma}^C{}_B\right] \widehat{\Sigma}^D{}_E, \\
\frac{dP_{\text{tot}}{}^i{}_j}{dt} = \{P_{\text{tot}}{}^i{}_j, H\} &= N_{\text{tot}}{}^i{}_j, \\
\frac{d\widehat{P}_{\text{tot}}{}^A{}_B}{dt} = \{\widehat{P}_{\text{tot}}{}^A{}_B, H\} &= \widehat{N}_{\text{tot}}{}^A{}_B \\
&+ \left[\widehat{P}_{\text{tot}}{}^A{}_C \left(\widehat{\Xi}^{-1}\right)^C{}_B{}^E{}_D - \left(\widehat{\Xi}^{-1}\right)^A{}_C{}^E{}_D \widehat{P}_{\text{tot}}{}^C{}_B\right] \Sigma^D{}_E.
\end{aligned}$$

where

$$N_{\mathrm{tot}}{}^{i}{}_{j} = r^{i}F_{j} + N^{i}{}_{j}, \qquad \widehat{N}_{\mathrm{tot}}{}^{A}{}_{B} = \widehat{r}^{A}\widehat{F}_{B} + \widehat{N}^{A}{}_{B}.$$

For the kinematical affine spin (hypermomentum) we have:

$$\begin{aligned}
\frac{dK^{ij}}{dt} &= \{K^{ij}, H\} = N^{ij} - \frac{1}{M}\left(C^{-1}\right)^{ik} p_{k}p_{l}\left(C^{-1}\right)^{lj} \\
&+ K^{is}\left[\left(\Xi^{-1}\right)^{j}{}_{s}{}^{n}{}_{l} + C_{sm}\left(\Xi^{-1}\right)^{m}{}_{k}{}^{n}{}_{l}\left(C^{-1}\right)^{kj}\right]K^{lp}C_{pn}, \\
\frac{d\widehat{K}^{AB}}{dt} &= \{\widehat{K}^{AB}, H\} = \widehat{N}^{AB} - \frac{1}{M}\widehat{\eta}^{AC}\widehat{p}_{C}\widehat{p}_{D}\widehat{\eta}^{DB} \\
&+ \left[\widehat{K}^{AC}\widehat{\eta}_{CL}\left(\widehat{\Xi}^{-1}\right)^{L}{}_{D}{}^{E}{}_{F}\widehat{\eta}^{DB} - \left(\widehat{\Xi}^{-1}\right)^{A}{}_{C}{}^{E}{}_{F}\widehat{K}^{CB}\right]\widehat{K}^{FH}\widehat{\eta}_{HE}.
\end{aligned}$$

The constants of motion on the total affine group in the absence of external forces, i.e., when $F_{i} = 0$ and $N^{i}{}_{j} = 0$, are only p_{i} and $P_{\mathrm{tot}}{}^{i}{}_{j}$. If we "freeze" the translational degrees of freedom, $\widehat{p}_{A}$ and $\Sigma^{i}{}_{j}$ become also the constants of motion.

2.1.2 General right invariant geodetic problems

Similarly, we can consider the right-invariant geodetic problems under the total affine group $\mathrm{GAf}(n, \mathbb{R})$.

Remark: although, strictly speaking, we have not "affine material invariance", but rather "linear material invariance" because there is no translational invariance in the material space $\mathcal{N}$. The centre of mass is fixed there and it reduces the symmetry to the centro-affine one.

Then the total kinetic energy $T = T_{\mathrm{tr}} + T_{\mathrm{int}}$, where T_{int} is like (A.3), can be written as follows:

$$T_{\mathrm{tr}} = \frac{M}{2}g_{ij}v^{i}v^{j} = \frac{M}{2}\widehat{G}_{AB}\widehat{v}^{A}\widehat{v}^{B}, \qquad T_{\mathrm{int}} = \frac{1}{2}\mathcal{R}^{j}{}_{i}{}^{l}{}_{k}\Omega^{i}{}_{j}\Omega^{k}{}_{l}, \tag{2.19}$$

where $\mathcal{R}^{j}{}_{i}{}^{l}{}_{k}$ are some constants, and

$$\widehat{G}_{AB} = g_{ij}\varphi^{i}{}_{A}\varphi^{j}{}_{B}$$

is the corresponding metric-like Green deformation tensor of the material space ("pull-back" of the metric g).

Again this general kinetic energy $T = T_{\mathrm{tr}} + T_{\mathrm{int}}$ defines the curved (non-Euclidean) metric on $\mathrm{AfI}(\mathcal{N}, \mathcal{M})$. Moreover, in the material space $\mathcal{N}$ we have only the transported from the physical space $\mathcal{M}$ metric-like Green deformation tensor

$\widehat{G} = \varphi^* g$, i.e., no naturally fixed metric in the material space N is used and it may be equally the amorphous affine space as well as the metric space.

Let us also suppose that our problem is left-invariant under some maximal compact subgroup of $\mathrm{GL}(V) \simeq \mathrm{GL}(n, \mathbb{R})$, then the internal kinetic energy have the following form:

$$T_{\mathrm{int}} = \frac{1}{2} g_{ij} \Omega^i{}_k \Omega^j{}_l J^{kl} + \frac{\alpha}{2} \Omega^i{}_j \Omega^j{}_i + \frac{\beta}{2} \Omega^i{}_i \Omega^j{}_j = \frac{1}{2} \Theta^k{}_i{}^l{}_j \Omega^i{}_k \Omega^j{}_l, \tag{2.20}$$

or, equivalently,

$$T_{\mathrm{int}} = \frac{1}{2} \widehat{G}_{AB} \widehat{\Omega}^A{}_C \widehat{\Omega}^B{}_D \widehat{J}^{CD} + \frac{\alpha}{2} \widehat{\Omega}^A{}_B \widehat{\Omega}^B{}_A + \frac{\beta}{2} \widehat{\Omega}^A{}_A \widehat{\Omega}^B{}_B = \frac{1}{2} \widehat{\Theta}^C{}_A{}^D{}_B \widehat{\Omega}^A{}_C \widehat{\Omega}^B{}_D,$$

where

$$\begin{aligned} \Theta^k{}_i{}^l{}_j &= J^{kl} g_{ij} + \alpha \delta^k{}_j \delta^l{}_i + \beta \delta^k{}_i \delta^l{}_j, \\ \widehat{\Theta}^C{}_A{}^D{}_B &= \widehat{J}^{CD} \widehat{G}_{AB} + \alpha \delta^C{}_B \delta^D{}_A + \beta \delta^C{}_A \delta^D{}_B \end{aligned}$$

are the generalized inertial tensor of the affinely-rigid body in the laboratory and co-moving representations, respectively. It has the following property of symmetry in the pairs of indices:

$$\Theta^k{}_i{}^l{}_j = \Theta^l{}_j{}^k{}_i.$$

Remark: we see that in any above-mentioned representation the generalized inertial tensor Θ is not simply a constant tensor unless we suppose that our problem is *spatially isotropic*, i.e., $J^{kl} = J g^{kl}$. So, for simplicity reasons, the following general discussion is based on this assumption. Then the tensor Θ in the laboratory representation is constant in opposition to ones in the co-moving and mixed representations.

The canonical momenta dual to the generalized velocities and the total affine momentum $P_{\mathrm{tot}}{}^i{}_j = r^i p_j + \Sigma^i{}_j$ are as follows:

$$\begin{aligned} p_i = \frac{\partial T}{\partial v^i} &= M g_{ij} v^j, & \widehat{p}_A = \frac{\partial T}{\partial \widehat{v}^A} &= p_i \varphi^i{}_A = M \widehat{G}_{AB} \widehat{v}^B, \\ \Sigma^i{}_j = \frac{\partial T}{\partial \Omega^j{}_i} &= \varphi^i{}_A \pi^A{}_j = \Theta^i{}_j{}^k{}_l \Omega^l{}_k, & \widehat{\Sigma}^A{}_B = \frac{\partial T}{\partial \widehat{\Omega}^B{}_A} &= \pi^A{}_i \varphi^i{}_B = \widehat{\Theta}^A{}_B{}^C{}_D \widehat{\Omega}^D{}_C, \\ \pi^A{}_i = \frac{\partial T}{\partial \dot{\varphi}^i{}_A} &= \Theta^A{}_i{}^B{}_j \dot{\varphi}^j{}_B, & P_{\mathrm{tot}}{}^i{}_j &= M r^i v^k g_{kj} + \Theta^i{}_j{}^k{}_l \Omega^l{}_k, \\ \widehat{P}_{\mathrm{tot}}{}^A{}_B = (\varphi^{-1})^A{}_i P_{\mathrm{tot}}{}^i{}_j \varphi^j{}_B &= M \widehat{r}^A \widehat{v}^C \widehat{G}_{CB} + \widehat{\Theta}^A{}_B{}^C{}_D \widehat{\Omega}^D{}_C. \end{aligned}$$

Let us consider Lagrangians $L = T - V(r, \varphi)$ corresponding to our kinetic energy with velocity-independent potentials (no magnetic forces). The equations of motion for the translational and internal degrees of freedom are as follows:

$$\frac{dv^i}{dt} = \frac{1}{M}F^i, \tag{2.21}$$

$$\Theta^i{}_j{}^k{}_l\frac{d\Omega^l{}_k}{dt} = N^i{}_j + \left[\Omega^i{}_k\Theta^k{}_j{}^l{}_n - \Omega^k{}_j\Theta^i{}_k{}^l{}_n\right]\Omega^n{}_l, \tag{2.22}$$

or in the co-moving representation:

$$\frac{d\widehat{v}^A}{dt} = \frac{1}{M}\widehat{F}^A - \widehat{\Omega}^A{}_B\widehat{v}^B, \tag{2.23}$$

$$\widehat{\Theta}^A{}_B{}^C{}_D\frac{d\widehat{\Omega}^D{}_C}{dt} = \widehat{N}^A{}_B + \left[\widehat{\Omega}^A{}_C\widehat{\Theta}^C{}_B{}^E{}_D - \widehat{\Omega}^C{}_B\widehat{\Theta}^A{}_C{}^E{}_D\right]\widehat{\Omega}^D{}_E. \tag{2.24}$$

Let us remind that now the shifting of indices in the material space is made with the help of the Green deformation tensor $\widehat{G}$, e.g.,

$$\widehat{F}^A = \left(\widehat{G}^{-1}\right)^{AB}\widehat{F}_B, \qquad \widehat{N}^{AB} = \widehat{N}^A{}_C\left(\widehat{G}^{-1}\right)^{CB}.$$

If we introduce the kinematical affine spin (hypermomentum) in the laboratory representation as $K^{ij} = \Sigma^i{}_k g^{kj}$ or equally in the co-moving representation as $\widehat{K}^{AB} = \widehat{\Sigma}^A{}_C\left(\widehat{G}^{-1}\right)^{CB}$, then we can rewrite the previous equations of motion (2.22) and (2.24) in the balance form:

$$\frac{dK^{ij}}{dt} = N^{ij} + \Omega^i{}_lK^{lj} - K^{ik}g_{kl}\Omega^l{}_sg^{sj}, \tag{2.25}$$

$$\frac{d\widehat{K}^{AB}}{dt} = \widehat{N}^{AB} - \widehat{K}^{AC}\left[\widehat{\Omega}^B{}_C + \widehat{G}_{CD}\widehat{\Omega}^D{}_E\left(\widehat{G}^{-1}\right)^{EB}\right]. \tag{2.26}$$

We may also introduce the internal angular momentum (spin) S^{ij} and vorticity $\widehat{V}^{AB}$ as the doubled skew-symmetric part of K^{ij} and $\widehat{K}^{AB}$, respectively. Their balance equations can be written as the doubled skew-symmetric part of (2.25) and (2.26).

Performing the Legendre transformation we obtain the corresponding Hamiltonian in the following form:

$$H = \dot{r}^ip_i + \dot{\varphi}^i{}_A\pi^A{}_i - L = \mathcal{T}_{\rm tr} + \mathcal{T}_{\rm int} + V(r,\varphi),$$

where the translational and internal kinetic energy terms are as follows:

$$\mathcal{T}_{\rm tr} = \frac{1}{2M}g^{ij}p_ip_j = \frac{1}{2M}\left(\widehat{G}^{-1}\right)^{AB}\widehat{p}_A\widehat{p}_B,$$

$$\mathcal{T}_{\rm int} = \frac{1}{2}\left(\Theta^{-1}\right)^i{}_k{}^j{}_l\Sigma^k{}_i\Sigma^l{}_j = \frac{1}{2}\left(\Theta^{-1}\right)^i{}_A{}^j{}_B\pi^A{}_i\pi^B{}_j = \frac{1}{2}\left(\widehat{\Theta}^{-1}\right)^C{}_A{}^D{}_B\widehat{\Sigma}^A{}_C\widehat{\Sigma}^B{}_D.$$

With the help of Poisson brackets (2.10)-(2.18) we can rewrite the equations of motion in the Hamiltonian form:

$$\begin{aligned}
\frac{dp_i}{dt} = \{p_i, H\} &= F_i,\\
\frac{d\widehat{p}_A}{dt} = \{\widehat{p}_A, H\} &= \widehat{F}_A + \widehat{p}_C \left(\widehat{\Theta}^{-1}\right)^C{}_A{}^D{}_B \widehat{\Sigma}^B{}_D,\\
\frac{d\Sigma^i{}_j}{dt} = \{\Sigma^i{}_j, H\} &= N^i{}_j + \left[\left(\Theta^{-1}\right)^i{}_k{}^m{}_n \Sigma^k{}_j - \Sigma^i{}_k \left(\Theta^{-1}\right)^k{}_j{}^m{}_n\right] \Sigma^n{}_m,\\
\frac{d\widehat{\Sigma}^A{}_B}{dt} = \{\widehat{\Sigma}^A{}_B, H\} &= \widehat{N}^A{}_B,\\
\frac{dP_{\text{tot}}{}^i{}_j}{dt} = \{P_{\text{tot}}{}^i{}_j, H\} &= N_{\text{tot}}{}^i{}_j + \frac{1}{M} g^{ik} p_k p_j\\
&+ \left[\left(\Theta^{-1}\right)^i{}_k{}^m{}_n \Sigma^k{}_j - \Sigma^i{}_k \left(\Theta^{-1}\right)^k{}_j{}^m{}_n\right] \Sigma^n{}_m,\\
\frac{d\widehat{P}_{\text{tot}}{}^A{}_B}{dt} = \{\widehat{P}_{\text{tot}}{}^A{}_B, H\} &= \widehat{N}_{\text{tot}}{}^A{}_B + \frac{1}{M}\left(\widehat{G}^{-1}\right)^{AC} \widehat{p}_C \widehat{p}_B\\
&+ \left[\widehat{r}^A \widehat{p}_C \left(\widehat{\Theta}^{-1}\right)^C{}_B{}^D{}_E - \left(\widehat{\Theta}^{-1}\right)^A{}_C{}^D{}_E \widehat{r}^C \widehat{p}_B\right] \widehat{\Sigma}^E{}_D.
\end{aligned}$$

For the kinematical affine spin (hypermomentum) we have:

$$\begin{aligned}
\frac{dK^{ij}}{dt} &= \{K^{ij}, H\} = N^{ij}\\
&+ \left[\left(\Theta^{-1}\right)^i{}_l{}^m{}_n K^{lj} - K^{is} g_{sl} \left(\Theta^{-1}\right)^l{}_k{}^m{}_n g^{kj}\right] K^{np} g_{pm},\\
\frac{d\widehat{K}^{AB}}{dt} &= \{\widehat{K}^{AB}, H\} = \widehat{N}^{AB}\\
&- \widehat{K}^{AC}\left[\left(\widehat{\Theta}^{-1}\right)^B{}_C{}^F{}_M + \widehat{G}_{CD}\left(\widehat{\Theta}^{-1}\right)^D{}_E{}^F{}_M \left(\widehat{G}^{-1}\right)^{EB}\right] \widehat{K}^{MN} \widehat{G}_{NF}.
\end{aligned}$$

The constants of motion on the total affine group in the absence of external forces, i.e., when $F_i = 0$ and $N^i{}_j = 0$, are only p_i, $\widehat{\Sigma}^A{}_B$. If we "freeze" the translational degrees of freedom, then $\widehat{p}_A$ is also the constant of motion.

2.1.3 Two-polar decomposition

Raising the first indices of above-defined symmetric positively-definite Green and Cauchy deformation tensors, respectively, with the help of $\widehat{\eta}$ and g, we obtain the mixed tensors $\widehat{G}[\varphi] \in U \otimes U^*$, $C[\varphi] \in V \otimes V^*$ with eigenvalues λ_a, λ_a^{-1}, $a = \overline{1, n}$. It is also convenient to use the quantities Q^a or q^a that are connected to λ_a as follows: $Q^a = \exp(q^a) = \sqrt{\lambda_a}$. The diagonal matrix $D = \operatorname{diag}(Q^1, \ldots, Q^n)$ is identified with the linear mapping $D : \mathbb{R}^n \to \mathbb{R}^n$.

Remark: in dynamical models based on the d'Alembert principle the quantities Q^a and their conjugate momenta P_a are more convenient that q^a and p_a. The latter ones are useful in models with affinely-invariant kinetic energies.

The configuration $\varphi \in \mathrm{LI}(U, V)$ may be characterized by D, i.e., by the system of fundamental stretchings $Q^a = \exp(q^a)$, and by the systems of eigenvectors $R_a \in U$, $L_a \in V$ of $\widehat{G}$, C normalized, respectively, in the sense of $\widehat{\eta}$ and g, i.e., we have that

$$\widehat{G} R_a = \lambda_a R_a = \exp\left(2q^a\right) R_a, \qquad C L_a = \lambda_a^{-1} L_a = \exp\left(-2q^a\right) L_a.$$

Obviously, when the spectrum is non-degenerate, then R_a, L_a are uniquely defined (up to re-ordering) and pair-wise orthogonal, i.e.,

$$\widehat{\eta}(R_a, R_b) = \widehat{\eta}_{CD} R^C{}_a R^D{}_b = \delta_{ab} = g_{ij} L^i{}_a L^j{}_b = g(L_a, L_a).$$

Such a situation is generic, thus, when at some time instant $t \in \mathbb{R}$ the configuration $\varphi(t)$ corresponds to degenerate situation, then $L_a(t)$, $R_a(t)$ may be also uniquely defined due to the continuity demand.

The elements of the corresponding dual bases will be denoted respectively by $R^a \in U^*$, $L^a \in V^*$. When necessary, to avoid misunderstandings, we shall indicate explicitly the dependence of the above quantities on φ, e.g., $q^a[\varphi]$, $R_a[\varphi]$, $L_a[\varphi]$, etc.

So, the Green and Cauchy deformation tensors may be respectively expressed as follows:

$$\begin{aligned}
\widehat{G}[\varphi] &= \sum_a \lambda_a[\varphi] R^a[\varphi] \otimes R^a[\varphi] = \sum_a \exp\left(2q^a[\varphi]\right) R^a[\varphi] \otimes R^a[\varphi], \\
C[\varphi] &= \sum_a \lambda_a^{-1}[\varphi] L^a[\varphi] \otimes L^a[\varphi] = \sum_a \exp\left(-2q^a[\varphi]\right) L^a[\varphi] \otimes L^a[\varphi].
\end{aligned}$$

In this way the configuration φ may be identified with the triple of fictitious objects, i.e., two rigid bodies in U and V with configurations represented, respectively, by orthonormal frames $R \in \mathrm{F}(U, \widehat{\eta})$, $L \in \mathrm{F}(V, g)$ and a one-dimensional n-particle system with coordinates q^a (or Q^a). Even for non-degenerate spectra of $\widehat{G}[\varphi]$, $C[\varphi]$ this representation is not unique because the labels a under the summation signs may be simultaneously permuted without affecting φ itself. For degenerate spectra this representation becomes continuously non-unique in a similar (although much stronger) way as, e.g., spherical coordinates at $r = 0$.

Let us observe that the linear frames $L = (\ldots, L_a, \ldots)$ and $R = (\ldots, R_a, \ldots)$ may be, as usual, identified with linear isomorphisms $L : \mathbb{R}^n \to V$ and $R : \mathbb{R}^n \to U$.

Similarly, their dual co-frames $\widetilde{L} = (\ldots, L^a, \ldots)$ and $\widetilde{R} = (\ldots, R^a, \ldots)$ are equivalent to isomorphisms $L^{-1} : V \to \mathbb{R}^n$ and $R^{-1} : U \to \mathbb{R}^n$. Identifying the diagonal matrix $D = \mathrm{diag}(\ldots, Q_a, \ldots)$ with a linear isomorphism $D : \mathbb{R}^n \to \mathbb{R}^n$, we may finally represent the given configuration as follows:

$$\varphi = LDR^{-1},$$

this is a geometric description of what is sometimes referred to as the two-polar decomposition [138, 145, 152, 160, 165, 178].

Strictly speaking, in continuum mechanics, when the orientation of the body is constant during any admissible motion (no mirror-reflections), one has to fix some pattern orientations in U, V and admit only orientation-preserving mappings φ. And then the non-connected sets of all orthonormal frames $\mathrm{F}(U, \widehat{\eta})$, $\mathrm{F}(V, g)$ are to be replaced by their connected submanifolds $\mathrm{F}^+(U, \widehat{\eta})$, $\mathrm{F}^+(V, g)$ of positively oriented frames.

Obviously, the spatial and material orientation-preserving isometries affect only the L- and R-gyroscopes on the left. Indeed, if $A \in \mathrm{SO}(V, g)$, $B \in \mathrm{SO}(U, \widehat{\eta})$, then $L \mapsto AL$, $R \mapsto BR$ result in $\varphi \mapsto A\varphi B^{-1}$. Their Hamiltonian generators, spin and minus-vorticity (i.e., respectively V- and U-spatial canonical spins) have identical Poisson-commutation rules.

For any of the mentioned rigid bodies, one can define in the usual way the angular velocity in two representations. One should stress that both V and U are from this point of view interpreted as "physical spaces". Their corresponding "material" ones are both identified with $\mathbb{R}^n$. The "co-moving" and "current" representations $\widehat{\chi} \in \mathrm{SO}(n, \mathbb{R})'$, $\chi \in \mathrm{SO}(V, g)'$ for the L-top are respectively given as follows:

$$\widehat{\chi}^a{}_b := \left\langle L^a, \frac{dL_b}{dt} \right\rangle = L^a{}_i \frac{dL^i{}_b}{dt}, \qquad \chi^i{}_j = \frac{dL^i{}_a}{dt} L^a{}_j.$$

The corresponding objects $\widehat{\vartheta} \in \mathrm{SO}(n, \mathbb{R})'$, $\vartheta \in \mathrm{SO}(U, \widehat{\eta})'$ for the R-top are defined by analogous formulas:

$$\widehat{\vartheta}^a{}_b := \left\langle R^a, \frac{dR_b}{dt} \right\rangle = R^a{}_K \frac{dR^K{}_b}{dt}, \qquad \vartheta^K{}_L = \frac{dR^K{}_a}{dt} R^a{}_L.$$

In certain problems it is convenient to use non-holonomic velocities $\dot{q}^a$, $\widehat{\chi}^a{}_b$, $\widehat{\vartheta}^a{}_b$ or $\dot{q}^a$, $\chi^i{}_j$, $\vartheta^A{}_B$. Similarly, non-holonomic conjugate momenta p_a, $\widehat{\rho}^a{}_b$, $\widehat{\tau}^a{}_b$ or p_a, $\rho^i{}_j$, $\tau^A{}_B$ are used, where again $\widehat{\rho}, \widehat{\tau} \in \mathrm{SO}(n, \mathbb{R})'$, $\rho \in \mathrm{SO}(V, g)'$, $\tau \in \mathrm{SO}(U, \widehat{\eta})'$. The pairing between non-holonomic momenta and velocities is given by the following expression:

$$\begin{aligned}\langle(\rho,\tau,p),(\chi,\vartheta,\dot{q})\rangle &= p_a\dot{q}^a+\frac{1}{2}\mathrm{Tr}\,(\rho\chi)+\frac{1}{2}\mathrm{Tr}\,(\tau\vartheta)\\ &= p_a\dot{q}^a+\frac{1}{2}\mathrm{Tr}\,(\widehat{\rho}\widehat{\chi})+\frac{1}{2}\mathrm{Tr}\left(\widehat{\tau}\widehat{\vartheta}\right)=\left\langle\left(\widehat{\rho},\widehat{\tau},p\right),\left(\widehat{\chi},\widehat{\vartheta},\dot{q}\right)\right\rangle.\end{aligned}$$

Remark: this system of notations is slightly redundant because ρ and τ as Hamiltonian generators of transformations

$$\varphi\mapsto A\varphi,\qquad \varphi\mapsto\varphi B^{-1},\qquad A\in\mathrm{SO}(V,g),\qquad B\in\mathrm{SO}(U,\widehat{\eta}),$$

coincide, respectively, with the Hamiltonian generators of left (spatial) and right (material) rigid rotations, i.e., with spin and minus-vorticity: $\rho=S$, $\tau=-V$.

The "co-moving" objects $\widehat{\rho}$, $\widehat{\tau}$ generate transformations

$$L\mapsto LA,\qquad R\mapsto RB,\qquad A,B\in\mathrm{SO}(n,\mathbb{R}),$$

and express the quantities ρ, τ in terms of the reference frames given, respectively, by the principal axes of the Cauchy and Green deformation tensors,

$$\rho=\widehat{\rho}^a{}_bL_a\otimes L^b,\qquad \tau=\widehat{\tau}^a{}_bR_a\otimes R^b.$$

If the linear spaces V and U are both identified with $\mathbb{R}^n$ and $\mathrm{LI}(U,V)$ with $\mathrm{GL}(n,\mathbb{R})$, then L and R in the two-polar splitting $\varphi=LDR^{-1}$ become elements of $\mathrm{SO}(n,\mathbb{R})$ and D, as previously, is a diagonal matrix with positive elements.

The two-polar decomposition is a by-product of the polar decomposition of $\mathrm{GL}^+(n,\mathbb{R})$, i.e.,

$$\varphi=UA,$$

where $U\in\mathrm{SO}(n,\mathbb{R})$, thus, $U^T=U^{-1}$, and $A=A^T$ is a symmetric positively-definite matrix. It is well-known that this decomposition is unique, whereas the two-polar one is charged with some multi-valuedness. Then the Green and Cauchy deformation tensors are represented as follows:

$$\widehat{G}=\varphi^T\varphi=A^2,\qquad C=\left(\varphi^{-1}\right)^T\varphi^{-1}=UA^{-2}U^{-1}.$$

One can also use the reversed polar decomposition, i.e.,

$$\varphi=BU,\qquad U\in\mathrm{SO}(n,\mathbb{R}),\qquad B=UAU^{-1}=B^T.$$

Then $\widehat{G}=U^{-1}B^2U$, $C=B^{-2}$. The two-polar decomposition is achieved by the orthogonal diagonalization of the matrix A, i.e., $A=VDV^{-1}$, where $V\in\mathrm{SO}(n,\mathbb{R})$. Then we have that $L=UV$ and $R=V$.

The polar splitting was described above in an over-simplified standard way, namely, the linear spaces U and V were identified with $\mathbb{R}^n$ and $\mathrm{LI}(U,V)$ with $\mathrm{GL}(n,\mathbb{R})$. Let us remind that in continuum mechanics the connected components of $\mathrm{LI}(U,V)$ and $\mathrm{GL}(n,\mathbb{R})$ are used as configuration spaces, i.e., the manifold of orientation-preserving isomorphisms $\mathrm{LI}^+(U,V)$ (it is assumed here that some orientations in U and V are fixed) and $\mathrm{GL}^+(n,\mathbb{R})$. It is also instructive to see what the both polar splittings are from the geometric point of view, when U and V are distinct linear spaces, non-identified with $\mathbb{R}^n$.

As mentioned above, when metric tensors $\widehat{\eta} \in U^* \otimes U^*$ and $g \in V^* \otimes V^*$ are fixed, then any configuration $\varphi \in \mathrm{LI}(U,V)$ with non-degenerate spectra of deformation tensors gives rise to the pair of orthonormal bases $L_a[\varphi] \in V$ and $R_a[\varphi] \in U$, $a = \overline{1,n}$. There exists exactly one isometry $U[\varphi] : U \to V$ such that $U[\varphi] \circ R_a[\varphi] = L_a[\varphi]$. Obviously, the isometry property is meant in the sense that $\widehat{\eta} = U[\varphi]^* g$, i.e., analytically $\widehat{\eta}_{AB} = g_{ij} U[\varphi]^i{}_A U[\varphi]^j{}_B$. Geometrical meaning of the polar decomposition is as follows:

$$\varphi = U[\varphi]A[\varphi] = B[\varphi]U[\varphi],$$

where the automorphisms $A[\varphi] \in \mathrm{GL}(U)$ and $B[\varphi] \in \mathrm{GL}(V)$ are symmetric, respectively, in the $\widehat{\eta}$- and g-sense, i.e.,

$$\widehat{\eta}\left(A[\varphi]x, y\right) = \widehat{\eta}\left(x, A[\varphi]y\right), \qquad g\left(B[\varphi]w, z\right) = g\left(w, B[\varphi]z\right)$$

for arbitrary $x, y \in U$ and $w, z \in V$. They are also positively definite, i.e.,

$$\widehat{\eta}(A[\varphi]x, x) > 0, \qquad g(B[\varphi]w, w) > 0$$

for arbitrary non-null $x \in U$ and $w \in V$.

In spite of the non-uniqueness contained in $L[\varphi]$ and $R[\varphi]$, the mappings $U[\varphi]$, $A[\varphi]$, and $B[\varphi]$ are unique. And the symmetric parts are obtained from each other by the $U[\varphi]$-intertwining, i.e., $B[\varphi] = U[\varphi]A[\varphi]U[\varphi]^{-1}$.

Remark: in mechanics of discrete affine systems we are free to admit such orientation-reversing isometries U and symmetric mappings A, B that are not necessarily positively definite.

Hence, returning to our dynamical affine models, it appears that the most adequate description of internal degrees of freedom is that based on the two-polar decomposition. Then the Cauchy and Green deformation tensors become as follows:

$$C = LD^{-2}L^T, \qquad \widehat{G} = RD^2R^T.$$

and affine quasi-velocities Ω and $\widehat{\Omega}$ can be rewritten in the following form:

$$\Omega = L\omega D^{-1}L^{T}, \qquad \widehat{\Omega} = RD^{-1}\omega R^{T},$$

where

$$\omega = \widehat{\chi}D + \dot{D} - D\widehat{\vartheta},$$

and instead of using the generalized velocities of fictitious gyroscopic motions, i.e., $\dot{L}$ and $\dot{R}$, we employ the co-moving representation of non-holonomic angular velocities, i.e.,

$$\widehat{\chi} = L^{T}\dot{L}, \qquad \widehat{\vartheta} = R^{T}\dot{R},$$

which are skew-symmetric, i.e.,

$$\widehat{\chi}^{T} = -\widehat{\chi}, \qquad \widehat{\vartheta}^{T} = -\widehat{\vartheta}.$$

So, the left affine and right orthogonal invariant $\left(\widehat{J}^{AB} = J\widehat{\eta}^{AB}\right)$ translational and internal kinetic energies may be rewritten as follows:

$$\begin{aligned}
T^{\rm left}_{\rm tr} &= \frac{M}{2}\mathrm{Tr}\left(LD^{-2}L^{T}\dot{r}\dot{r}^{T}\right),\\
T^{\rm left}_{\rm int} &= \frac{J+\alpha}{2}\mathrm{Tr}\left(\dot{D}^{2}D^{-2}\right) + \frac{\beta}{2}\left[\mathrm{Tr}\left(\dot{D}D^{-1}\right)\right]^{2}\\
&+ (J-\alpha)\mathrm{Tr}\left(\widehat{\chi}D\widehat{\vartheta}D^{-1}\right) - \frac{J}{2}\mathrm{Tr}\left(\widehat{\chi}D^{2}\widehat{\chi}D^{-2}\right) + \frac{\alpha}{2}\mathrm{Tr}\left(\widehat{\chi}^{2}\right) - \frac{J-\alpha}{2}\mathrm{Tr}\left(\widehat{\vartheta}^{2}\right),
\end{aligned}$$

and the canonical affine momenta corresponding to our affine velocities as follows:

$$\begin{aligned}
\zeta^{\rm left} &= \frac{\partial T}{\partial \dot{D}} = (J+\alpha)\dot{D}D^{-2} + \beta D^{-1}\mathrm{Tr}\left(\dot{D}D^{-1}\right),\\
\widehat{\rho}^{\rm left} &= \frac{\partial T}{\partial \widehat{\chi}} = \alpha\widehat{\chi} + \frac{J-\alpha}{2}\left[D\widehat{\vartheta}D^{-1} + D^{-1}\widehat{\vartheta}D\right] - \frac{J}{2}\left[D^{2}\widehat{\chi}D^{-2} + D^{-2}\widehat{\chi}D^{2}\right],\\
\widehat{\tau}^{\rm left} &= \frac{\partial T}{\partial \widehat{\vartheta}} = (\alpha-J)\widehat{\vartheta} + \frac{J-\alpha}{2}\left[D\widehat{\chi}D^{-1} + D^{-1}\widehat{\chi}D\right].
\end{aligned}$$

Similarly, the right affine and left orthogonal invariant $(J^{ij} = Jg^{ij})$ internal kinetic energies may be rewritten as follows:

$$\begin{aligned}
T^{\rm right}_{\rm int} &= \frac{J+\alpha}{2}\mathrm{Tr}\left(\dot{D}^{2}D^{-2}\right) + \frac{\beta}{2}\left[\mathrm{Tr}\left(\dot{D}D^{-1}\right)\right]^{2}\\
&+ (J-\alpha)\mathrm{Tr}\left(\widehat{\chi}D\widehat{\vartheta}D^{-1}\right) - \frac{J}{2}\mathrm{Tr}\left(\widehat{\vartheta}D^{2}\widehat{\vartheta}D^{-2}\right) + \frac{\alpha}{2}\mathrm{Tr}\left(\widehat{\vartheta}^{2}\right) - \frac{J-\alpha}{2}\mathrm{Tr}\left(\widehat{\chi}^{2}\right).
\end{aligned}$$

The canonical momenta dual to the generalized velocities are as follows:

$$\begin{aligned}
\zeta^{\rm right} &= \frac{\partial T}{\partial \dot{D}} = (J+\alpha)\dot{D}D^{-2} + \beta D^{-1}\mathrm{Tr}\left(\dot{D}D^{-1}\right),\\
\widehat{\rho}^{\rm right} &= \frac{\partial T}{\partial \widehat{\chi}} = (\alpha-J)\widehat{\chi} + \frac{J-\alpha}{2}\left[D\widehat{\vartheta}D^{-1} + D^{-1}\widehat{\vartheta}D\right],\\
\widehat{\tau}^{\rm right} &= \frac{\partial T}{\partial \widehat{\vartheta}} = \alpha\widehat{\vartheta} + \frac{J-\alpha}{2}\left[D\widehat{\chi}D^{-1} + D^{-1}\widehat{\chi}D\right] - \frac{J}{2}\left[D^{2}\widehat{\vartheta}D^{-2} + D^{-2}\widehat{\vartheta}D^{2}\right].
\end{aligned}$$

2.2 Materially or spatially isotropic models

It is useful to consider some isotropic models. For the materially isotropic $\widehat{\mathcal{L}}$-models the quantity $\widehat{\mathcal{L}}^{A}{}_{B}{}^{C}{}_{D}$ is a linear combination of tensors $\widehat{\eta}^{AC}\widehat{\eta}_{BD}$, $\delta^{A}{}_{D}\delta^{C}{}_{B}$, and $\delta^{A}{}_{B}\delta^{C}{}_{D}$. Similarly, for the spatially isotropic $\mathcal{R}$-models the tensor $\mathcal{R}^{i}{}_{j}{}^{k}{}_{l}$ is a linear combination of terms $g^{ik}g_{jl}$, $\delta^{i}{}_{l}\delta^{k}{}_{j}$, and $\delta^{i}{}_{j}\delta^{k}{}_{l}$.

Let us describe it more precisely for the situation when we neglect also Casimir invariants, i.e., the last two terms with the coefficients α and β in the internal kinetic energy expressions (2.2) or (2.20).

2.2.1 Left affine and right orthogonal invariant problems

If our general left affinely invariant models are right invariant under the orthogonal subgroup $\mathrm{SO}(U,\widehat{\eta}) \simeq \mathrm{SO}(n,\mathbb{R})$, i.e., they are *materially isotropic*, then we can put $\widehat{J}^{AB} = J\widehat{\eta}^{AB}$ in the afore-mentioned formulae.

So, the internal part of the kinetic energy left invariant under the total affine group $\mathrm{GAf}(n,\mathbb{R})$ and right invariant under the orthogonal subgroup $\mathrm{SO}(n,\mathbb{R})$ can be written as follows:

$$T_{\mathrm{int}} = \frac{J}{2}\widehat{\eta}_{AB}\widehat{\Omega}^{A}{}_{C}\widehat{\Omega}^{B}{}_{D}\widehat{\eta}^{CD} = \frac{J}{2}C_{ij}\dot{\varphi}^{i}{}_{A}\dot{\varphi}^{j}{}_{B}\widehat{\eta}^{AB} = \frac{J}{2}C_{ij}\Omega^{i}{}_{k}\Omega^{j}{}_{l}(C^{-1})^{kl}. \tag{2.27}$$

The canonical affine momenta corresponding to our affine velocities and the total affine momentum $P_{\mathrm{tot}}{}^{i}{}_{j}$ are then as follows:

$$\begin{aligned}
p_i &= \frac{\partial T}{\partial v^i} = MC_{ij}v^j, & \widehat{p}_A &= \frac{\partial T}{\partial \widehat{v}^A} = p_i\varphi^{i}{}_{A} = M\widehat{\eta}_{AB}\widehat{v}^B,\\
\Sigma^{i}{}_{j} &= \frac{\partial T}{\partial \Omega^{j}{}_{i}} = \varphi^{i}{}_{A}\pi^{A}{}_{j} = J\left[\Omega^{T_C}\right]^{i}{}_{j}, & \widehat{\Sigma}^{A}{}_{B} &= \frac{\partial T}{\partial \widehat{\Omega}^{B}{}_{A}} = \pi^{A}{}_{i}\varphi^{i}{}_{B} = J\left[\widehat{\Omega}^{T}\right]^{A}{}_{B},\\
\pi^{A}{}_{i} &= \frac{\partial T}{\partial \dot{\varphi}^{i}{}_{A}} = J\widehat{\eta}^{AB}\dot{\varphi}^{j}{}_{B}C_{ji}, & P_{\mathrm{tot}}{}^{i}{}_{j} &= Mr^iv^kC_{kj} + J\left[\Omega^{T_C}\right]^{i}{}_{j},\\
\widehat{P}_{\mathrm{tot}}{}^{A}{}_{B} &= (\varphi^{-1})^{A}{}_{i}P_{\mathrm{tot}}{}^{i}{}_{j}\varphi^{j}{}_{B} &=\ & M\widehat{r}^A\widehat{v}^C\widehat{\eta}_{CB} + J\left[\widehat{\Omega}^{T}\right]^{A}{}_{B},
\end{aligned}$$

where we have defined the C-transposition rule, i.e.,

$$\left[(\cdot)^{T_C}\right]^{i}{}_{j} = \left(C^{-1}\right)^{ik}(\cdot)^{l}{}_{k}C_{lj}.$$

The kinematical affine spin (hypermomentum) in the laboratory and co-moving representations have then the following expressions:

$$\begin{aligned}
K^{ij} &= \Sigma^{i}{}_{k}\left(C^{-1}\right)^{kj} = J\left(C^{-1}\right)^{ik}\Omega^{j}{}_{k} = J\varphi^{i}{}_{A}\widehat{\eta}^{AB}\dot{\varphi}^{j}{}_{B},\\
\widehat{K}^{AB} &= \widehat{\Sigma}^{A}{}_{C}\widehat{\eta}^{CB} = J\widehat{\eta}^{AC}\widehat{\Omega}^{B}{}_{C}.
\end{aligned}$$

The Lagrange equations of motion may be written in the following form:

$$\frac{dv^i}{dt} = \frac{1}{M}F^i + \left[\Omega + \Omega^{T_C}\right]^i{}_j v^j, \tag{2.28}$$

$$\frac{d\Omega^i{}_j}{dt} = \frac{1}{J}\left[N^{T_C}\right]^i{}_j - \frac{M}{J}v^i v^k C_{kj} + \left[\Omega^{T_C}, \Omega\right]^i{}_j, \tag{2.29}$$

or in the co-moving representation:

$$\frac{d\widehat{v}^A}{dt} = \frac{1}{M}\widehat{F}^A + \left[\widehat{\Omega}^T\right]^A{}_B \widehat{v}^B, \tag{2.30}$$

$$\frac{d\widehat{\Omega}^A{}_B}{dt} = \frac{1}{J}\left[\widehat{N}^T\right]^A{}_B - \frac{M}{J}\widehat{v}^A\widehat{v}^C\widehat{\eta}_{CB} + \left[\widehat{\Omega}^T, \widehat{\Omega}\right]^A{}_B, \tag{2.31}$$

where $[\cdot,\cdot]$ denotes the commutator.

It is convenient to rewrite the equations (2.29) and (2.31) in the balance form. For the kinematical affine spin (hypermomentum) we have the following expressions:

$$\frac{dK^{ij}}{dt} = N^{ij} - \frac{1}{M}\left(C^{-1}\right)^{ik} p_k p_l \left(C^{-1}\right)^{lj} + \frac{2}{J}K^{ik}C_{kl}K^{(lj)},$$

$$\frac{d\widehat{K}^{AB}}{dt} = \widehat{N}^{AB} - \frac{1}{M}\widehat{\eta}^{AC}\widehat{p}_C\widehat{p}_D\widehat{\eta}^{DB} + \frac{1}{J}\left[\widehat{K}^{AC}\widehat{K}^{BD} - \widehat{K}^{CA}\widehat{K}^{DB}\right]\widehat{\eta}_{CD},$$

and for internal spin and vorticity the balance equations are as follows:

$$\frac{dS^{ij}}{dt} = 2N^{[ij]} + \frac{1}{J}\left[S^{ik}K^{(lj)} + K^{(ik)}S^{lj}\right]C_{kl},$$

$$\frac{d\widehat{V}^{AB}}{dt} = 2\widehat{N}^{[AB]},$$

where $K^{(ij)}$ is the symmetric part of K^{ij}, $N^{[ij]}$ and $\widehat{N}^{[AB]}$ are the skew-symmetric parts of N^{ij} and $\widehat{N}^{AB}$, respectively.

In the two-polar decomposition $\varphi = LDR^T$, the translational and internal parts of the total kinetic energy may be rewritten as follows:

$$T_{\rm tr} = \frac{M}{2}\mathrm{Tr}\left(LD^{-2}L^T\dot{r}\dot{r}^T\right),$$

$$T_{\rm int} = \frac{J}{2}\mathrm{Tr}\left(\dot{D}^2D^{-2}\right) + J\mathrm{Tr}\left(\widehat{\chi}D\widehat{\vartheta}D^{-1}\right) - \frac{J}{2}\mathrm{Tr}\left(\widehat{\chi}D^2\widehat{\chi}D^{-2}\right) - \frac{J}{2}\mathrm{Tr}\left(\widehat{\vartheta}^2\right).$$

The canonical momenta can be written in the following form:

$$\zeta = \frac{\partial T}{\partial \dot{D}} = J\dot{D}D^{-2},$$

$$\widehat{\tau} = \frac{\partial T}{\partial \widehat{\vartheta}} = J\left[\mathrm{Asym}\left(D\widehat{\chi}D^{-1}\right) - \widehat{\vartheta}\right],$$

$$\widehat{\rho} = \frac{\partial T}{\partial \widehat{\chi}} = J\mathrm{Asym}\left(D\widehat{\vartheta}D^{-1} - D^2\widehat{\chi}D^{-2}\right).$$

The kinematical affine spin (hypermomentum), spin, and vorticity are as follows:

$$K = JLD\omega^T L^T, \qquad \widehat{K} = JR\omega^T D^{-1} R^T,$$
$$S = JLD\theta D L^T, \qquad \widehat{V} = JR\theta R^T,$$

where

$$\theta = -\frac{2\widehat{\tau}}{J} = \omega^T D^{-1} - D^{-1}\omega = 2\widehat{\vartheta} - D\widehat{\chi}D^{-1} - D^{-1}\widehat{\chi}D.$$

The equations of motion (2.28)–(2.31) in the two-polar decomposition can be rewritten in the following form:

$$\begin{aligned}
\frac{dv^i}{dt} &= \frac{1}{M}\left[LD^2L^T\right]^{ij} F_j \\
&+ \left[L\left(2\dot{D}D^{-1} + \widehat{\chi} - D^2\widehat{\chi}D^{-2}\right)L^T\right]^i{}_j \dot{r}^j, \\
\frac{d\widehat{\chi}}{dt} + \ddot{D}D^{-1} - D\frac{d\widehat{\vartheta}}{dt}D^{-1} &= \frac{1}{J}D^2L^TN^TLD^{-2} - \frac{M}{J}L^T\dot{r}\dot{r}^TLD^{-2} \\
&+ \dot{D}^2D^{-2} + \left[\dot{D}D^{-1}, 2\widehat{\chi} - D\widehat{\vartheta}D^{-1}\right] \\
&+ \left[\widehat{\chi} + \dot{D}D^{-1} - D\widehat{\vartheta}D^{-1}, D^2\widehat{\chi}D^{-2}\right].
\end{aligned}$$

Of course, the second equation could be obtained as a composition of the Lagrange equations of motion for the independent variables D, L, and R, which have, respectively, the following form:

$$\begin{aligned}
\ddot{D} = &- \frac{1}{J}D\frac{\partial V}{\partial D}D - \frac{M}{J}\mathrm{Sym}\left(L^T\dot{r}\dot{r}^TLD^{-1}\right) + \dot{D}^2D^{-1} \\
&+ \mathrm{Sym}\left(\left[D\widehat{\vartheta}D^{-1}, \widehat{\chi}\right] + \left[\widehat{\chi}D, D\widehat{\chi}D^{-2}\right]\right)D, \\
\mathrm{Asym}\left(D\frac{d\widehat{\vartheta}}{dt}D^{-1} - D^2\frac{d\widehat{\chi}}{dt}D^{-2}\right) &= \frac{M}{J}\mathrm{Asym}\left(D^{-2}L^T\dot{r}\dot{r}^TL\right) \\
&+ \mathrm{Asym}\left(\left[D\widehat{\vartheta}D^{-1}, \widehat{\chi} + \dot{D}D^{-1}\right]\right) \\
&+ \mathrm{Asym}\left(\left[\widehat{\chi} + 2\dot{D}D^{-1}, D^2\widehat{\chi}D^{-2}\right]\right), \\
\mathrm{Asym}\left(D\frac{d\widehat{\chi}}{dt}D^{-1}\right) - \frac{d\widehat{\vartheta}}{dt} &= \mathrm{Asym}\left(\left[D\widehat{\chi}D^{-1}, \widehat{\vartheta} + \dot{D}D^{-1}\right]\right).
\end{aligned}$$

After performing the Legendre transformation the internal kinetic energy (2.27) may be rewritten in the following form:

$$\mathcal{T}_{\mathrm{int}} = \frac{1}{2J}\widehat{\eta}^{AB}\widehat{\Sigma}^C{}_A\widehat{\Sigma}^D{}_B\widehat{\eta}_{CD} = \frac{1}{2J}\left(C^{-1}\right)^{ij}\pi^A{}_i\pi^B{}_j\widehat{\eta}_{AB} = \frac{1}{2J}\left(C^{-1}\right)^{ij}\Sigma^k{}_i\Sigma^l{}_jC_{kl}.$$

With the help of the Poisson brackets we may rewrite the equations of motion in the Hamiltonian form as follows:

$$
\begin{aligned}
\frac{dp_i}{dt} = \{p_i, H\} &= F_i,\\
\frac{d\widehat{p}_A}{dt} = \{\widehat{p}_A, H\} &= \widehat{F}_A + \frac{1}{J}\left[\widehat{\Sigma}^T\right]^B{}_A \widehat{p}_B,\\
\frac{d\Sigma^i{}_j}{dt} = \{\Sigma^i{}_j, H\} &= N^i{}_j - \frac{1}{M}\left(C^{-1}\right)^{ik} p_k p_j,\\
\frac{d\widehat{\Sigma}^A{}_B}{dt} = \{\widehat{\Sigma}^A{}_B, H\} &= \widehat{N}^A{}_B - \frac{1}{M}\widehat{\eta}^{AC}\widehat{p}_C\widehat{p}_B + \frac{1}{J}\left[\widehat{\Sigma}, \widehat{\Sigma}^T\right]^A{}_B,\\
\frac{dP_{\rm tot}{}^i{}_j}{dt} = \{P_{\rm tot}{}^i{}_j, H\} &= N_{\rm tot}{}^i{}_j,\\
\frac{d\widehat{P}_{\rm tot}{}^A{}_B}{dt} = \{\widehat{P}_{\rm tot}{}^A{}_B, H\} &= \widehat{N}_{\rm tot}{}^A{}_B + \frac{1}{J}\left[\widehat{P}_{\rm tot}, \widehat{\Sigma}^T\right]^A{}_B.
\end{aligned}
$$

The constants of motion on the total affine group for the geodetic systems, i.e., in the absence of external forces $F_i = 0$ and $N^i{}_j = 0$, are only p_i, $P_{\rm tot}{}^i{}_j$, and $\widehat{V}^{AB}$. If we "froze" the translational degrees of freedom, then $\widehat{p}_A$ and $\Sigma^i{}_j$ are also the constants of motion.

Remark: note that even if the canonical momentum p_i is a constant of motion, the velocity v^i is not (even the direction of v^i is, in general, variable) because they are interrelated through the φ-dependent Cauchy deformation tensor. This phenomenon could be called the *"drunk missile effect"*.

2.2.2 Right affine and left orthogonal invariant problems

If our general right (centro)-affinely invariant models are also left invariant under the orthogonal subgroup $\mathrm{SO}(V, g) \simeq \mathrm{SO}(n, \mathbb{R})$, i.e., they are *spatially isotropic*, then we can put $J^{ij} = Jg^{ij}$ in the afore-mentioned formulae.

Then the internal kinetic energy can be written as follows:

$$
T_{\rm int} = \frac{J}{2} g_{ij}\Omega^i{}_k\Omega^j{}_l g^{kl} = \frac{J}{2} g_{ij}\dot{\varphi}^i{}_A\dot{\varphi}^j{}_B\left(\widehat{G}^{-1}\right)^{AB} = \frac{J}{2}\widehat{G}_{AB}\widehat{\Omega}^A{}_C\widehat{\Omega}^B{}_D\left(\widehat{G}^{-1}\right)^{CD}. \quad (2.32)
$$

The canonical affine momenta corresponding to our affine velocities and the total affine momentum $P_{\rm tot}{}^i{}_j$ are then as follows:

$$
\begin{aligned}
p_i &= \frac{\partial T}{\partial v^i} = M g_{ij} v^j, & \widehat{p}_A &= \frac{\partial T}{\partial \widehat{v}^A} = p_i\varphi^i{}_A = M\widehat{G}_{AB}\widehat{v}^B,\\
\Sigma^i{}_j &= \frac{\partial T}{\partial \Omega^j{}_i} = \varphi^i{}_A\pi^A{}_j = J\left[\Omega^T\right]^i{}_j, & \widehat{\Sigma}^A{}_B &= \frac{\partial T}{\partial \widehat{\Omega}^B{}_A} = \pi^A{}_i\varphi^i{}_B = J\left[\widehat{\Omega}^{T_G}\right]^A{}_B,
\end{aligned}
$$

$$\pi^A{}_i = \frac{\partial T}{\partial \dot{\varphi}^i{}_A} = J g_{ij} \dot{\varphi}^j{}_B \left(\widehat{G}^{-1}\right)^{BA}, \qquad P_{\rm tot}{}^i{}_j = M r^i v^k g_{kj} + J \left[\Omega^T\right]^i{}_j,$$
$$\widehat{P}_{\rm tot}{}^A{}_B = \left(\varphi^{-1}\right)^A{}_i P_{\rm tot}{}^i{}_j \varphi^j{}_B \quad = \quad M \widehat{r}^A \widehat{v}^C \widehat{G}_{CB} + J \left[\widehat{\Omega}^{T_G}\right]^A{}_B,$$

where we have defined the $\widehat{G}$-transposition rule, i.e.,

$$\left[(\cdot)^{T_G}\right]^A{}_B = \left(\widehat{G}^{-1}\right)^{AC} (\cdot)^D{}_C \widehat{G}_{DB}.$$

The kinematical affine spin (hypermomentum) in the laboratory and co-moving representations is as follows:

$$K^{ij} = \Sigma^i{}_k g^{kj} = J g^{ik} \Omega^j{}_k = J \varphi^i{}_A \left(\widehat{G}^{-1}\right)^{AB} \dot{\varphi}^j{}_B,$$
$$\widehat{K}^{AB} = \widehat{\Sigma}^A{}_C \left(\widehat{G}^{-1}\right)^{CB} = J \left(\widehat{G}^{-1}\right)^{AC} \widehat{\Omega}^B{}_C.$$

The equations of spatial motion and the evolution of internal degrees of freedom are as follows:

$$\frac{dv^i}{dt} = \frac{1}{M} F^i, \tag{2.33}$$
$$\frac{d\Omega^i{}_j}{dt} = \frac{1}{J} \left[N^T\right]^i{}_j + \left[\Omega, \Omega^T\right]^i{}_j. \tag{2.34}$$

Equivalently, in the co-moving representation we have the following expressions:

$$\frac{d\widehat{v}^A}{dt} = \frac{1}{M} \widehat{F}^A - \widehat{\Omega}^A{}_B \widehat{v}^B, \tag{2.35}$$
$$\frac{d\widehat{\Omega}^A{}_B}{dt} = \frac{1}{J} \left[\widehat{N}^{T_G}\right]^A{}_B + \left[\widehat{\Omega}, \widehat{\Omega}^{T_G}\right]^A{}_B. \tag{2.36}$$

We can rewrite the equations (2.34) and (2.36) as the balance equations for the kinematical spin, i.e.,

$$\frac{dK^{ij}}{dt} = N^{ij} - \frac{1}{J} \left[K^{ik} K^{jl} - K^{ki} K^{lj}\right] g_{kl},$$
$$\frac{d\widehat{K}^{AB}}{dt} = \widehat{N}^{AB} - \frac{2}{J} \widehat{K}^{AC} \widehat{G}_{CD} \widehat{K}^{(DB)},$$

or for the internal spin and vorticity, i.e.,

$$\frac{dS^{ij}}{dt} = 2N^{[ij]},$$
$$\frac{d\widehat{V}^{AB}}{dt} = 2\widehat{N}^{[AB]} - \frac{1}{J} \left[\widehat{V}^{AC} \widehat{K}^{(DB)} + \widehat{K}^{(AC)} \widehat{V}^{DB}\right] \widehat{G}_{CD}.$$

In the two-polar decomposition $\varphi = LDR^T$, the internal part of the total kinetic energy may be rewritten as follows:

$$T_{\rm int} = \frac{J}{2}{\rm Tr}\left(\dot{D}^2 D^{-2}\right) + J{\rm Tr}\left(\widehat{\chi} D \widehat{\vartheta} D^{-1}\right) - \frac{J}{2}{\rm Tr}\left(\widehat{\vartheta} D^2 \widehat{\vartheta} D^{-2}\right) - \frac{J}{2}{\rm Tr}\left(\widehat{\chi}^2\right).$$

The canonical momenta can be written in the following form:

$$\begin{aligned}
\zeta &= \frac{\partial T}{\partial \dot{D}} = J\dot{D}D^{-2}, \\
\widehat{\rho} &= \frac{\partial T}{\partial \widehat{\chi}} = J\left[{\rm Asym}\left(D\widehat{\vartheta}D^{-1}\right) - \widehat{\chi}\right], \\
\widehat{\tau} &= \frac{\partial T}{\partial \widehat{\vartheta}} = J{\rm Asym}\left(D\widehat{\chi}D^{-1} - D^2\widehat{\vartheta}D^{-2}\right).
\end{aligned}$$

The kinematical affine spin (hypermomentum), spin, and vorticity are as follows:

$$\begin{aligned}
K &= JLD^{-1}\omega^T L^T, & \widehat{K} &= JRD^{-2}\omega^T D^{-1}R^T \\
S &= JL\widetilde{\theta}L^T, & \widehat{V} &= JRD^{-1}\widetilde{\theta}D^{-1}R^T,
\end{aligned}$$

where

$$\widetilde{\theta} = \frac{2\widehat{\rho}}{J} = D^{-1}\omega^T - \omega D^{-1} = D\widehat{\vartheta}D^{-1} + D^{-1}\widehat{\vartheta}D - 2\widehat{\chi}.$$

The equations of motion (2.34), (2.36) in the two-polar decomposition can be rewritten in the following form:

$$\begin{aligned}
\frac{d\widehat{\chi}}{dt} + \ddot{D}D^{-1} - D\frac{d\widehat{\vartheta}}{dt}D^{-1} &= \frac{1}{J}L^T N^T L + \dot{D}^2 D^{-2} + \left[\widehat{\chi} - 2D\widehat{\vartheta}D^{-1}, \dot{D}D^{-1}\right] \\
&+ \left[\widehat{\chi} + \dot{D}D^{-1} - D\widehat{\vartheta}D^{-1}, D^{-1}\widehat{\vartheta}D\right].
\end{aligned}$$

This equation could be also obtained as a composition of the Lagrange equations of motion for the independent variables D, L, and R, which have, respectively, the following form:

$$\begin{aligned}
\ddot{D} &= -\frac{1}{J}D\frac{\partial V}{\partial D}D + \dot{D}^2 D^{-1} \\
&+ {\rm Sym}\left(\left[D\widehat{\vartheta}D^{-1}, \widehat{\chi}\right]D + \left[\widehat{\vartheta}D, D\widehat{\vartheta}D^{-2}\right]D\right), \\
{\rm Asym}\left(D\frac{d\widehat{\vartheta}}{dt}D^{-1}\right) - \frac{d\widehat{\chi}}{dt} &= {\rm Asym}\left(\left[D\widehat{\vartheta}D^{-1}, \widehat{\chi} + \dot{D}D^{-1}\right]\right), \\
{\rm Asym}\left(D\frac{d\widehat{\chi}}{dt}D^{-1} - D^2\frac{d\widehat{\vartheta}}{dt}D^{-2}\right) &= {\rm Asym}\left(\left[D\widehat{\chi}D^{-1}, \widehat{\vartheta} + \dot{D}D^{-1}\right]\right) \\
&+ {\rm Asym}\left(\left[\widehat{\vartheta} + 2\dot{D}D^{-1}, D^2\widehat{\vartheta}D^{-2}\right]\right).
\end{aligned}$$

After performing the Legendre transformation the internal kinetic energy (2.32) may be rewritten in the following form:

$$\mathcal{T}_{\text{int}} = \frac{1}{2J} g^{ij} \Sigma^{k}{}_{i} \Sigma^{l}{}_{j} g_{kl} = \frac{1}{2J} g^{ij} \pi^{A}{}_{i} \pi^{B}{}_{j} \widehat{G}_{AB} = \frac{1}{2J} \left(\widehat{G}^{-1} \right)^{AB} \widehat{\Sigma}^{C}{}_{A} \widehat{\Sigma}^{D}{}_{B} \widehat{G}_{CD}.$$

With the help of Poisson brackets we may rewrite the equations of motion as follows:

$$\begin{aligned}
\frac{dp_i}{dt} = \{p_i, H\} &= F_i, \\
\frac{d\widehat{p}_A}{dt} = \{\widehat{p}_A, H\} &= \widehat{F}_A + \frac{1}{J} \left[\widehat{\Sigma}^{T_G} \right]^{B}{}_{A} \widehat{p}_B, \\
\frac{d\Sigma^{i}{}_{j}}{dt} = \{\Sigma^{i}{}_{j}, H\} &= N^{i}{}_{j} + \frac{1}{J} \left[\Sigma^{T}, \Sigma \right]^{i}{}_{j}, \\
\frac{d\widehat{\Sigma}^{A}{}_{B}}{dt} = \{\widehat{\Sigma}^{A}{}_{B}, H\} &= \widehat{N}^{A}{}_{B}, \\
\frac{dP_{\text{tot}}{}^{i}{}_{j}}{dt} = \{P_{\text{tot}}{}^{i}{}_{j}, H\} &= N_{\text{tot}}{}^{i}{}_{j} + \frac{1}{M} g^{ik} p_k p_j + \frac{1}{J} \left[\Sigma^{T}, \Sigma \right]^{i}{}_{j}, \\
\frac{d\widehat{P}_{\text{tot}}{}^{A}{}_{B}}{dt} = \{\widehat{P}_{\text{tot}}{}^{A}{}_{B}, H\} &= \widehat{N}_{\text{tot}}{}^{A}{}_{B} \\
&+ \frac{1}{M} \left(\widehat{G}^{-1} \right)^{AC} \widehat{p}_C \widehat{p}_B + \frac{1}{J} \left[\widehat{P}_{\text{tot}} - \widehat{\Sigma}, \widehat{\Sigma}^{T_G} \right]^{A}{}_{B}.
\end{aligned}$$

The constants of motion on the total affine group for the geodetic systems are only p_i, $\widehat{\Sigma}^{A}{}_{B}$, and S^{ij}. If we "froze" the translational degrees of freedom, then $\widehat{p}_A$ and $\widehat{P}_{\text{tot}}{}^{A}{}_{B}$ are also the constants of motion.

2.2.3 Controls in dynamics of affinely-rigid bodies

The structure of our equations suggests some natural and geometrically distinguished ways of including control inputs. As a controlling agent we may take the asymmetric moment of forces N or $\widehat{N}$. According to this choice we have two kinds of control problems: the inner and outer problems [45, 146]. In the first one the co-moving Lagrangian components $\widehat{N}$ of the controlling momentum are directly manipulated quantities, e.g., $\widehat{N} = \widehat{N}_0 + U(t)$, where $\widehat{N}_0$ and $U(t)$ are the background and the control terms; the control vector depends only on time but not on the state variables. Such mathematical models describe situations where the controlling devices, e.g., reaction motors or thrust-based propeller motors, are frozen immovably in the material. In the out-steering problems there are spatial (Eulerian) components N of the controlling moment that are assumed to be directly manipulated quantities. Such models describe situations where the control forces are produced by external devices like servomotors, pull rods, etc., or by external physical fields.

Variational problems with non-linear non-holonomic constraints seem to suggest that inertia $\widehat{J}$ is also a promising physical agent in problems of control [142, 146]. Although this way of control is non-realistic when the translational motion in space is concerned (because it would be rather hard to manipulate with masses of moving objects), inertial tensors of affinely-rigid bodies can be relatively easy subjected to our influence. This kind of control is achieved by introducing additional "steering" degrees of freedom.

2.3 From affine to projectively-rigid body

The concept of the metrically-rigid body have played very important role in the theoretical and applied mechanics (see, e.g., [3]), mainly because our macroscopic environment is dominated by objects which are approximately rigid. But what would happen if we went forward and got rid of the metrical properties keeping the concept of rigidness? As a result of this "weakening" of our demands the concept of an affinely-rigid body as a medium the deformative behaviour of which is restricted to performing homogeneous deformations only appeared. Various applications of the affine concept are possible, e.g., in the theory of large oscillations of molecules, small mono-crystals, atomic nuclei, and even in the theory of elementary particles. In fact, an affinely-rigid body in an amorphous affine space is an obvious counterpart of the usual metrically-rigid body in a Euclidean space. But we need not to get rid the metric once and for all, we may introduce it in our consideration at any step, and that is what makes this approach attractive. For instance, to be able to introduce the notion of the kinetic energy, we should have some fixed Euclidean metric.

So, as the nest step we would like to consider even more "amorphous" case, which we may obtain from (1.36) by generalizing our affine constraints to the projective ones:

$$x^i(t,a) = \frac{A^i{}_B(t)a^B + b^i(t)}{c_D(t)a^D + d(t)}, \qquad Ad - bc \neq 0, \qquad i, B, D = \overline{1,n}. \tag{2.37}$$

Then at any fixed time $t \in \mathbb{R}$ the configuration space Q of our problem is identical with the projective group $\mathrm{Pr}(n,\mathbb{R}) \supset \mathrm{GAf}(n,\mathbb{R})$ (for $N = M = \mathbb{R}^n$), and such a system of material points is called the projectively-rigid body. Due to the isomorphism $\mathrm{Pr}(n,\mathbb{R}) \simeq \mathrm{SL}(n+1,\mathbb{R})$ we may rewrite those constraints (2.37) in n dimensions as the constraints defining an incompressible affinely-rigid body in $n+1$ dimensions, i.e.,

$$x^{\mu}(t,a) = \varphi^{\mu}{}_{\nu}(t)a^{\nu}, \tag{2.38}$$

where

$$\varphi = \begin{pmatrix} A & b \\ c & d \end{pmatrix}, \qquad \det\varphi = 1, \qquad \mu,\nu = \overline{0,n}.$$

Remark: the additional variable is called 0-th only for convenience reasons.

Hence, the new configuration space $\widetilde{Q}$ is identical with the unimodular linear group $\mathrm{SL}(n+1,\mathbb{R})$.

2.3.1 Projectively invariant geodetic problems

Let us consider the left-invariant geodetic problem on the projective group $\mathrm{Pr}(n,\mathbb{R})$ (or equivalently on the unimodular linear group $\mathrm{SL}(n+1,\mathbb{R})$), which is also right-invariant under the orthogonal subgroup $\mathrm{SO}(n+1,\mathbb{R})$:

$$T_{\mathrm{left}} = \frac{J}{2}\mathrm{Tr}\left(\widehat{\Omega}^T\widehat{\Omega}\right) + \frac{\alpha}{2}\mathrm{Tr}\left(\widehat{\Omega}^2\right) + \frac{\beta}{2}\left(\mathrm{Tr}\widehat{\Omega}\right)^2, \tag{2.39}$$

and the right-invariant geodetic problem on the projective group, which is left-invariant under the orthogonal subgroup:

$$T_{\mathrm{right}} = \frac{J}{2}\mathrm{Tr}\left(\Omega^T\Omega\right) + \frac{\alpha}{2}\mathrm{Tr}\left(\Omega^2\right) + \frac{\beta}{2}\left(\mathrm{Tr}\Omega\right)^2, \tag{2.40}$$

where J,α,β are generalized inertial constants, the second and third terms for both kinetic energies are identical and are the Casimir invariants.

Again the most adequate description of internal degrees of freedom is that based on the two-polar decomposition, i.e., we split the system of degrees of freedom into three subsystems: $\varphi = LDR^T$, where $L,R \in \mathrm{SO}(n+1,\mathbb{R})$ are special orthogonal matrices ($L^TL = R^TR = \mathbb{I}$, $\det L = \det R = 1$) and D is diagonal, positive, and $\det D = 1$. If we take $D^{\mu\mu} = \exp(q^{\mu})$, then $\sum_{\mu} q^{\mu} = 0$. In this way our system is formally represented as a composition of two $(n+1)$-dimensional fictitious rigid bodies (systems of principal axes of the Cauchy and Green deformation tensors) and n independent material points oscillating along the straight line $\mathbb{R}$. Orthogonal transformations L and R diagonalize the Cauchy and Green deformation tensors: $C = LD^{-2}L^T$ and $\widehat{G} = RD^2R^T$. If there is no coincidence of diagonal elements of D, i.e., if the spectra of C and $\widehat{G}$ are simple, then the two-polar decomposition is finitely non-unique: $\varphi = LDR^T = \widetilde{L}\widetilde{D}\widetilde{R}^T$, where $\widetilde{L} = LO_{\pi}$, $\widetilde{D} = O_{\pi}^{-1}DO_{\pi}$,

$\widetilde{R} = RO_\pi$, and O_π is an orthogonal representation of the permutation group with the restriction $O_\pi \operatorname{Diag}(q^1, \ldots, q^{n+1})\, O_\pi^{-1} = \operatorname{Diag}\big(q^{\pi(1)}, \ldots, q^{\pi(n+1)}\big)$, i.e., $S_{n+1} \ni \pi \mapsto O_\pi \in \mathrm{SO}(n+1, \mathbb{R})$. The components of matrices O_π are only 0 or ± 1, i.e., in any row and column there is only one non-vanishing element. If the spectrum of C and $\widehat{G}$ is non-simple, then the non-uniqueness of decomposition becomes continuous. An extreme situation occurs when D is completely degenerate (proportional to the unit matrix), then no meaning may be assigned separately to L and R, only LR^T is well-defined.

It is convenient to introduce non-holonomic variables adapted to the two-polar decomposition, i.e., velocity $\dot{q}$ and non-holonomic angular velocities of the L and R rigid bodies in the co-moving representation: $\widehat{\chi} = L^T \dot{L} = -\widehat{\chi}^T$ and $\widehat{\vartheta} = R^T \dot{R} = -\widehat{\vartheta}^T$. Then the kinetic energies (2.39) and (2.40) may be rewritten in the combined form (the upper expression is related to the left-invariant and the lower one to the right-invariant problems) as follows:

$$
\begin{aligned}
T_{\text{left/right}} &= \frac{J+\alpha}{2}\operatorname{Tr}\left(\dot{D}^2 D^{-2}\right) + (J-\alpha)\operatorname{Tr}\left(\widehat{\chi} D \widehat{\vartheta} D^{-1}\right) \\
&+ \frac{\alpha}{2}\operatorname{Tr}\left(\widehat{\chi}^2 + \widehat{\vartheta}^2\right) - \frac{J}{2}\operatorname{Tr}\left(\begin{array}{c} \widehat{\chi} D^2 \widehat{\chi} D^{-2} + \widehat{\vartheta}^2 \\ \widehat{\chi}^2 + \widehat{\vartheta} D^2 \widehat{\vartheta} D^{-2} \end{array}\right).
\end{aligned}
$$

We see that the constant β is absent in the kinetic energy expression $T_{\text{left/right}}$ because of the condition $\det D = 1 \sim \sum_\mu q^\mu = 0$. If we explicitly substitute this condition into the expression for the kinetic energy, i.e., $q^0 = -\sum_{i=1}^n q^i$, then we can rewrite the previous formula in the following form:

$$
\begin{aligned}
T_{\text{left/right}} &= \frac{J+\alpha}{2}\left[\sum_{i=1}^{n} \left(\dot{q}^i\right)^2 + \langle \dot{q} \rangle^2\right] + \sum_{i=1}^{n} V_{\text{left/right}}\left(\left|q^i + \langle q \rangle\right|, \widehat{\chi}_{0i}, \widehat{\vartheta}_{0i}\right) \\
&+ \frac{1}{2}\sum_{i,j=1}^{n} U_{\text{left/right}}\left(\left|q^i - q^j\right|, \widehat{\chi}_{ij}, \widehat{\vartheta}_{ij}\right), \qquad (2.41)
\end{aligned}
$$

where $\langle \circ \rangle = \sum_{j=1}^n \circ^j$, and the one-point and binary effective interaction potentials are as follows:

$$
\begin{aligned}
V_{\text{left/right}} &= (J-\alpha)\left[\left(\widehat{\chi}_{0i}\right)^2 + \left(\widehat{\vartheta}_{0i}\right)^2 - 2\widehat{\chi}_{0i}\widehat{\vartheta}_{0i}\operatorname{ch}\left(q^i + \langle q \rangle\right)\right] \\
&+ 2J\left\{\begin{array}{c} \widehat{\chi}_{0i}^2 \\ \widehat{\vartheta}_{0i}^2 \end{array}\right\}\operatorname{sh}^2\left(q^i + \langle q \rangle\right), \\
U_{\text{left/right}} &= (J-\alpha)\left[\left(\widehat{\chi}_{ij}\right)^2 + \left(\widehat{\vartheta}_{ij}\right)^2 - 2\widehat{\chi}_{ij}\widehat{\vartheta}_{ij}\operatorname{ch}\left(q^i - q^j\right)\right]
\end{aligned}
$$

$$+ \quad 2J \left\{ \begin{array}{c} \widehat{\chi}_{ij}^2 \\ \widehat{\vartheta}_{ij}^2 \end{array} \right\} \mathrm{sh}^2 \left(q^i - q^j \right).$$

Let us define the canonical affine momenta $(p, \widehat{\rho}, \widehat{\tau})$, which are conjugate to $\left(\dot{q}, \widehat{\chi}, \widehat{\vartheta}\right)$, respectively. Here $\widehat{\rho}$ and $\widehat{\tau}$ are skew-symmetric matrices expressing the co-moving representation of the angular momenta, respectively, of L and R rigid bodies. The Legendre transformation reads:

$$\begin{aligned}
p_i &= \frac{\partial T_{\text{left/right}}}{\partial \dot{q}^i} = (J+\alpha)\left[\dot{q}^i + \langle \dot{q} \rangle\right] \Longleftrightarrow \dot{q}^i = \frac{1}{J+\alpha}\left[p_i - \frac{1}{n+1}\langle p \rangle\right], \\
\widehat{\rho}_{0i} &= \frac{\partial T_{\text{left/right}}}{\partial \widehat{\chi}_{0i}} \\
&= 2(J-\alpha)\left[\widehat{\chi}_{0i} - \widehat{\vartheta}_{0i}\mathrm{ch}\left(q^i + \langle q \rangle\right)\right] + 4J \left\{ \begin{array}{c} \widehat{\chi}_{0i} \\ 0 \end{array} \right\} \mathrm{sh}^2\left(q^i + \langle q \rangle\right), \\
\widehat{\tau}_{0i} &= \frac{\partial T_{\text{left/right}}}{\partial \widehat{\vartheta}_{0i}} \\
&= 2(J-\alpha)\left[\widehat{\vartheta}_{0i} - \widehat{\chi}_{0i}\mathrm{ch}\left(q^i + \langle q \rangle\right)\right] + 4J \left\{ \begin{array}{c} 0 \\ \widehat{\vartheta}_{0i} \end{array} \right\} \mathrm{sh}^2\left(q^i + \langle q \rangle\right), \\
\widehat{\rho}_{ij} &= \frac{\partial T_{\text{left/right}}}{\partial \widehat{\chi}_{ij}} \\
&= 2(J-\alpha)\left[\widehat{\chi}_{ij} - \widehat{\vartheta}_{ij}\mathrm{ch}\left(q^i - q^j\right)\right] + 4J \left\{ \begin{array}{c} \widehat{\chi}_{ij} \\ 0 \end{array} \right\} \mathrm{sh}^2\left(q^i - q^j\right), \\
\widehat{\tau}_{ij} &= \frac{\partial T_{\text{left/right}}}{\partial \widehat{\vartheta}_{ij}} \\
&= 2(J-\alpha)\left[\widehat{\vartheta}_{ij} - \widehat{\chi}_{ij}\mathrm{ch}\left(q^i - q^j\right)\right] + 4J \left\{ \begin{array}{c} 0 \\ \widehat{\vartheta}_{ij} \end{array} \right\} \mathrm{sh}^2\left(q^i - q^j\right),
\end{aligned}$$

where $i < j$. Their non-vanishing basic Poisson brackets are as follows:

$$\begin{aligned}
\{q^i, p_j\} &= \delta^i{}_j, \\
\{\widehat{\rho}_{\mu\nu}, \widehat{\rho}_{\kappa\sigma}\} &= \widehat{\rho}_{\mu\sigma}\delta_{\kappa\nu} - \widehat{\rho}_{\kappa\nu}\delta_{\mu\sigma} + \widehat{\rho}_{\sigma\nu}\delta_{\mu\kappa} - \widehat{\rho}_{\mu\kappa}\delta_{\sigma\nu} \\
\{\widehat{\tau}_{\mu\nu}, \widehat{\tau}_{\kappa\sigma}\} &= \widehat{\tau}_{\mu\sigma}\delta_{\kappa\nu} - \widehat{\tau}_{\kappa\nu}\delta_{\mu\sigma} + \widehat{\tau}_{\sigma\nu}\delta_{\mu\kappa} - \widehat{\tau}_{\mu\kappa}\delta_{\sigma\nu}.
\end{aligned}$$

It is more convenient later on to use the auxiliary variables $M := -\widehat{\rho} - \widehat{\tau} = -M^T$ and $N := \widehat{\rho} - \widehat{\tau} = -N^T$ instead of the very $\widehat{\rho}$ and $\widehat{\tau}$:

$$M_{0i} = 4\left[(J-\alpha)\left(\widehat{\chi}_{0i} + \widehat{\vartheta}_{0i}\right)\mathrm{sh}^2\left(\frac{q^i + \langle q \rangle}{2}\right) - J \left\{ \begin{array}{c} \widehat{\chi}_{0i} \\ \widehat{\vartheta}_{0i} \end{array} \right\} \mathrm{sh}^2\left(q^i + \langle q \rangle\right)\right],$$

$$
\begin{aligned}
N_{0i} &= 4\left[(J-\alpha)\left(\widehat{\chi}_{0i}-\widehat{\vartheta}_{0i}\right)\mathrm{ch}^2\left(\frac{q^i+\langle q\rangle}{2}\right)+J\left\{\begin{array}{c}\widehat{\chi}_{0i}\\ -\widehat{\vartheta}_{0i}\end{array}\right\}\mathrm{sh}^2\left(q^i+\langle q\rangle\right)\right],\\
M_{ij} &= 4\left[(J-\alpha)\left(\widehat{\chi}_{ij}+\widehat{\vartheta}_{ij}\right)\mathrm{sh}^2\left(\frac{q^i-q^j}{2}\right)-J\left\{\begin{array}{c}\widehat{\chi}_{ij}\\ \widehat{\vartheta}_{ij}\end{array}\right\}\mathrm{sh}^2\left(q^i-q^j\right)\right],\\
N_{ij} &= 4\left[(J-\alpha)\left(\widehat{\chi}_{ij}-\widehat{\vartheta}_{ij}\right)\mathrm{ch}^2\left(\frac{q^i-q^j}{2}\right)+J\left\{\begin{array}{c}\widehat{\chi}_{ij}\\ -\widehat{\vartheta}_{ij}\end{array}\right\}\mathrm{sh}^2\left(q^i-q^j\right)\right],
\end{aligned}
$$

where $i<j$. They satisfy the following Poisson bracket rules:

$$
\begin{aligned}
\{M_{\mu\nu},M_{\kappa\sigma}\}=\{N_{\mu\nu},N_{\kappa\sigma}\} &= -M_{\mu\sigma}\delta_{\kappa\nu}+M_{\kappa\nu}\delta_{\mu\sigma}-M_{\sigma\nu}\delta_{\mu\kappa}+M_{\mu\kappa}\delta_{\sigma\nu},\\
\{M_{\mu\nu},N_{\kappa\sigma}\} &= -N_{\mu\sigma}\delta_{\kappa\nu}+N_{\kappa\nu}\delta_{\mu\sigma}-N_{\sigma\nu}\delta_{\mu\kappa}+N_{\mu\kappa}\delta_{\sigma\nu}.
\end{aligned}
$$

Geodetic Hamiltonians corresponding to our kinetic energies (2.41) may be written as follows:

$$
\begin{aligned}
\mathcal{T}_{\text{left/right}} &= \frac{1}{2(J+\alpha)}\left[\sum_{i=1}^{n}(p_i)^2-\frac{1}{n+1}\langle p\rangle^2\right] \qquad (2.42)\\
&+ \frac{1}{32(J-\alpha)}\sum_{\mu,\nu=0}^{n}\left[M_{\mu\nu}^2+N_{\mu\nu}^2\right]\pm\frac{J}{8(J^2-\alpha^2)}\sum_{\mu,\nu=0}^{n}M_{\mu\nu}N_{\mu\nu}\\
&+ \sum_{i=1}^{n}V_{\text{eff}}\left(\left|q^i+\langle q\rangle\right|,M_{0i},N_{0i}\right)+\frac{1}{2}\sum_{i,j=1}^{n}U_{\text{eff}}\left(\left|q^i-q^j\right|,M_{ij},N_{ij}\right),
\end{aligned}
$$

where even in the purely geodetic problems we have the "internal" effective interaction potentials:

$$
\begin{aligned}
V_{\text{eff}} &= \frac{1}{16(J+\alpha)}\left\{M_{0i}^2\mathrm{cth}^2\left(\frac{q^i+\langle q\rangle}{2}\right)+N_{0i}^2\mathrm{th}^2\left(\frac{q^i+\langle q\rangle}{2}\right)\right\},\\
U_{\text{eff}} &= \frac{1}{16(J+\alpha)}\left\{M_{ij}^2\mathrm{cth}^2\left(\frac{q^i-q^j}{2}\right)+N_{ij}^2\mathrm{th}^2\left(\frac{q^i-q^j}{2}\right)\right\}.
\end{aligned}
$$

The Hamilton equations of motion may be expressed in terms of Poisson brackets as follows:

$$
\begin{aligned}
\frac{dp_i}{dt} &= \{p_i,H\}\\
&= \frac{1}{4(J+\alpha)\mathrm{sh}\left(q^i+\langle q\rangle\right)}\left\{M_{0i}^2\mathrm{cth}^2\left(\frac{q^i+\langle q\rangle}{2}\right)-N_{0i}^2\mathrm{th}^2\left(\frac{q^i+\langle q\rangle}{2}\right)\right\}\\
&+ \sum_{j=1}^{n}\frac{1}{8(J+\alpha)\mathrm{sh}\left(q^i-q^j\right)}\left\{M_{ij}^2\mathrm{cth}^2\left(\frac{q^i-q^j}{2}\right)-N_{ij}^2\mathrm{th}^2\left(\frac{q^i-q^j}{2}\right)\right\},\\
\frac{dM_{\mu\nu}}{dt} &= \{M_{\mu\nu},M_{\kappa\sigma}\}\frac{\partial\mathcal{T}_{\text{left/right}}}{\partial M_{\kappa\sigma}}+\{M_{\mu\nu},N_{\kappa\sigma}\}\frac{\partial\mathcal{T}_{\text{left/right}}}{N_{\kappa\sigma}},\\
\frac{dN_{\mu\nu}}{dt} &= \{N_{\mu\nu},M_{\kappa\sigma}\}\frac{\partial\mathcal{T}_{\text{left/right}}}{\partial M_{\kappa\sigma}}+\{N_{\mu\nu},N_{\kappa\sigma}\}\frac{\partial\mathcal{T}_{\text{left/right}}}{N_{\kappa\sigma}},
\end{aligned}
$$

where the corresponding partial derivatives of the kinetic energies (2.42) have the following form:

$$\begin{aligned}
\frac{\partial \mathcal{T}_{\text{left/right}}}{\partial M_{0i}} &= \frac{\left[J\text{ch}\left(q^i + \langle q\rangle\right) - \alpha\right] M_{0i} \pm 2JN_{0i}}{8(J^2-\alpha^2)\text{sh}^2\left[\left(q^i + \langle q\rangle\right)/2\right]},\\
\frac{\partial \mathcal{T}_{\text{left/right}}}{\partial N_{0i}} &= \frac{\left[J\text{ch}\left(q^i + \langle q\rangle\right) - \alpha\right] N_{0i} \pm 2JM_{0i}}{8(J^2-\alpha^2)\text{sh}^2\left[\left(q^i + \langle q\rangle\right)/2\right]},\\
\frac{\partial \mathcal{T}_{\text{left/right}}}{\partial M_{ij}} &= \frac{\left[J\text{ch}\left(q^i - q^j\right) - \alpha\right] M_{ij} \pm 2JN_{ij}}{8(J^2-\alpha^2)\text{sh}^2\left[\left(q^i - q^j\right)/2\right]},\\
\frac{\partial \mathcal{T}_{\text{left/right}}}{\partial N_{ij}} &= \frac{\left[J\text{ch}\left(q^i - q^j\right) - \alpha\right] N_{ij} \pm 2JM_{ij}}{8(J^2-\alpha^2)\text{sh}^2\left[\left(q^i - q^j\right)/2\right]}.
\end{aligned}$$

2.3.2 Geodetic problems on the projective line

In n dimensions a projectively-rigid body is such a body all projective relations between constituents of which during any admissible motion are invariant, i.e., material straight lines remain straight lines and all cross-ratios of any four points placed on the same straight lines are constant. It is interesting that the cross-ratio of four points on the line, i.e.,

$$(x_1, x_2, x_3, x_4) = \frac{x_4 - x_1}{x_4 - x_2} : \frac{x_3 - x_1}{x_3 - x_2},$$

plays here the same role as the usual mutual ratio of segments for the affine and the distance for the metrical geometries (see, e.g., [116]). After choosing the appropriate homogeneous coordinates and adding to the consideration a set of non-proper points in the infinity such defined cross ratio is constant under the action of the whole projective group $\text{Pr}(n, \mathbb{R})$.

Let us consider now the very simple and in some sense trivial but nevertheless very illustrative example of the one-dimensional left- and right-invariant geodetic problems on the projective group $\text{Pr}(1, \mathbb{R}) \simeq \text{SL}(2, \mathbb{R})$. Then we have:

$$D = \begin{bmatrix} e^{-q} & 0 \\ 0 & e^{q} \end{bmatrix}, \qquad L = \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix}, \qquad R = \begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix},$$

$$\widehat{\chi} = L^T\dot{L} = \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}, \qquad \widehat{\vartheta} = R^T\dot{R} = \begin{bmatrix} 0 & -\mu \\ \mu & 0 \end{bmatrix}, \qquad \lambda = \dot{\gamma}, \qquad \mu = \dot{\delta}.$$

The kinetic energy (2.41) now has the following form:

$$\begin{aligned}
T^{n=1}_{\text{left/right}} &= (J+\alpha)\dot{q}^2 \\
&+ (J-\alpha)\begin{bmatrix} \mu - \text{ch}(2q)\lambda \\ \lambda - \text{ch}(2q)\mu \end{bmatrix}^2 + (J+\alpha)\text{sh}^2(2q)\begin{bmatrix} \lambda^2 \\ \mu^2 \end{bmatrix}. \qquad (2.43)
\end{aligned}$$

The canonical momenta $(p, \widehat{\rho}, \widehat{\tau})$ (or, equivalently, the auxiliary variables M, N) and corresponding velocities $(\dot{q}, \lambda, \mu)$ are connected by the Legendre transformation:

$$\begin{aligned}
p &= 2(J+\alpha)\dot{q},\\
\widehat{\rho}_{\mathrm{left/right}} &= 2(J-\alpha)\left[\lambda - \mu \mathrm{ch}(2q)\right] + 4J \left\{ \begin{array}{c} \lambda \\ 0 \end{array} \right\} \mathrm{sh}^2(2q),\\
\widehat{\tau}_{\mathrm{left/right}} &= 2(J-\alpha)\left[\mu - \lambda \mathrm{ch}(2q)\right] + 4J \left\{ \begin{array}{c} 0 \\ \mu \end{array} \right\} \mathrm{sh}^2(2q),\\
M_{\mathrm{left/right}} &= 2(J-\alpha)(\lambda+\mu)\mathrm{sh}^2 q - 4J \left\{ \begin{array}{c} \lambda \\ \mu \end{array} \right\} \mathrm{sh}^2(2q),\\
N_{\mathrm{left/right}} &= 2(J-\alpha)(\lambda-\mu)\mathrm{ch}^2 q + 4J \left\{ \begin{array}{c} \lambda \\ -\mu \end{array} \right\} \mathrm{sh}^2(2q).
\end{aligned}$$

Geodetic Hamiltonians corresponding to our kinetic energies (2.43) are as follows:

$$\begin{aligned}
\mathcal{T}^{n=1}_{\mathrm{left/right}} &= \frac{p^2}{4(J+\alpha)}\\
&+ \frac{1}{4(J-\alpha)} \left\{ \begin{array}{c} \widehat{\tau}^2 \\ \widehat{\rho}^2 \end{array} \right\} + \frac{1}{4(J+\alpha)\mathrm{sh}^2(2q)} \left[\begin{array}{c} \widehat{\rho} + \mathrm{ch}(2q)\widehat{\tau} \\ \widehat{\tau} + \mathrm{ch}(2q)\widehat{\rho} \end{array} \right]^2\\
&= \frac{p^2}{4(J+\alpha)} + \frac{M^2+N^2}{16(J-\alpha)} \pm \frac{JMN}{4(J^2-\alpha^2)} + V^{n=1}_{\mathrm{eff}}(q, M, N),
\end{aligned}$$

where the one-dimensional effective potential and corresponding effective force are as follows:

$$\begin{aligned}
V^{n=1}_{\mathrm{eff}}(q, M, N) &= \frac{M^2 \mathrm{cth}^2 q + N^2 \mathrm{th}^2 q}{16(J+\alpha)},\\
F^{n=1}_{\mathrm{eff}} := -\frac{\partial V^{n=1}_{\mathrm{eff}}}{\partial q} &= \frac{M^2 \mathrm{cth}^2 q - N^2 \mathrm{th}^2 q}{4(J+\alpha)\mathrm{sh}(2q)}.
\end{aligned}$$

The canonical momenta $\widehat{\rho}$ and $\widehat{\tau}$ (or, equivalently, M and N) in the one-dimensional case are the constants of motion, so the Newton equation of motion of the fictitious particle on the line $\mathbb{R}$ in the "internal" effective potential $V^{n=1}_{\mathrm{eff}}$ are as follows:

$$\ddot{q} = \frac{1}{2(J+\alpha)} F^{n=1}_{\mathrm{eff}} = \frac{M^2 \mathrm{cth}^2 q - N^2 \mathrm{th}^2 q}{8(J+\alpha)^2 \mathrm{sh}(2q)}, \tag{2.44}$$

Assigning some special values to such constants of motion as energy E and canonical momenta M, N we can write the first-order differential equation on the q variable as follows:

$$\dot{q}^2 = \frac{1}{J+\alpha} E^{\pm}_{\mathrm{eff}} - \frac{M^2 \mathrm{cth}^2 q + N^2 \mathrm{th}^2 q}{16(J+\alpha)^2},$$

where

$$E_{\rm eff}^{\pm} = E - \frac{M^2+N^2}{16(J-\alpha)} \mp \frac{JMN}{4(J^2-\alpha^2)},$$

and finally we have the following solution of the equation of motion (2.44):

$$\begin{aligned} t(q) &= \int dq \frac{4(J+\alpha)}{\sqrt{16(J+\alpha)E_{\rm eff}^{\pm} - M^2{\rm cth}^2 q - N^2 {\rm th}^2 q}} \\ &= \frac{2(J+\alpha)}{\sqrt{\Theta}} \ln \left[M^2 - N^2 - \Theta {\rm ch}^2(2q) - 2\sqrt{\Theta}\ {\rm sh}\ q\sqrt{\Xi} \right], \end{aligned}$$

where the auxiliary symbols Θ and Ξ denotes, respectively, the following expressions:

$$\begin{aligned} \Theta &= 16(J+\alpha)E - 2J\frac{(M\pm N)^2}{J-\alpha}, \\ \Xi &= 16(J+\alpha)E_{\rm eff}^{\pm} - M^2{\rm cth}^2 q - N^2{\rm th}^2 q. \end{aligned}$$

Chapter 3

Quantization ideas

A fascinating feature of our models of affine collective dynamics is their extremely wide range of applications. It covers the nuclear and molecular dynamics, micromechanics of structured continua, perhaps nanostructure and defects phenomena, macroscopic elasticity and astrophysical phenomena like vibration of stars and clouds of cosmic dust. Obviously, microphysical applications must be based on the quantized version of the theory. And one is dealing then with a very curious convolution of quantum theory with mathematical methods of continuum mechanics. It is worth to mention that there were even attempts, mainly by Barut and Rączka [8], to describe the dynamics of strongly interacting elementary particles (hadrons) in terms of some peculiar, quantized continua. By the way, it is not excluded that the dynamics of cosmic objects like neutron stars must be also described in quantum terms. They are though giant nuclei, very exotic ones, because composed exclusively of neutrons (enormous "mass numbers" and vanishing "atomic numbers").

3.1 Classical background for quantization

First of all, let us briefly summarize the main above-described propositions about the classical dynamical affine invariance. Namely, the total kinetic energy (inertial metric tensor) is postulated in the following additive form:

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}}$$

(translational and internal parts). Left (spatially) affinely invariant expressions for the internal kinetic energy term T_{int} are constant-coefficients quadratic forms of $\widehat{\Omega}$ (see Appendix (A.2)). Right (materially) (centro-)affinely invariant ones are built in a similar way of Ω, i.e., (A.3). The doubly invariant $T_{\rm int}$ are combined of second-order Casimirs, i.e., (A.4) or (A.5):

$$T^{\rm aff-aff}_{\rm int} := \frac{A}{2}{\rm Tr}\left(\Omega^2\right) + \frac{B}{2}\left({\rm Tr}\ \Omega\right)^2 = \frac{A}{2}{\rm Tr}\left(\widehat{\Omega}^2\right) + \frac{B}{2}\left({\rm Tr}\ \widehat{\Omega}\right)^2, \qquad (3.1)$$

where A, B are some constants. Due to the semi-simplicity and non-compactness of ${\rm SL}(n,\mathbb{R})$, the kinetic energy $T^{\rm aff-aff}_{\rm int}$ is not positively definite, however, this does not exclude its physical usefulness. It is non-degenerate when $A + nB \neq 0$. The special case $A = 2n$, $B = -2$ corresponds just to the standard normalization of the Killing metric; its degeneracy is due to the non-semi-simplicity of ${\rm GL}(n,\mathbb{R})$.

3.1.1 Invariance of translational kinetic energies

Translational kinetic energies $T_{\rm tr}$ are never doubly invariant. The highest possible symmetries for mathematically reasonable models are those affine in the space and Euclidean in the material, or conversely. Thus, these two possibilities can by written in the following combined form:

$$\left\{\begin{array}{c} T^{\rm met-aff}_{\rm tr} \\ T^{\rm aff-met}_{\rm tr} \end{array}\right\} = \frac{M}{2}\left\{\begin{array}{c} g_{ij} \\ C_{ij} \end{array}\right\} v^i v^j = \frac{M}{2}\left\{\begin{array}{c} \widehat{G}_{AB} \\ \widehat{\eta}_{AB} \end{array}\right\} \widehat{v}^A \widehat{v}^B, \qquad (3.2)$$

where g, $\widehat{\eta}$ are respectively the spatial and material metric tensors (δ's in the orthonormal bases), and $\widehat{G} \in U^* \otimes U^*$, $C \in V^* \otimes V^*$ are respectively the Green and Cauchy deformation tensors,

$$\widehat{G}_{AB} = g_{ij}\varphi^i{}_A \varphi^j{}_B, \qquad C_{ij} = \widehat{\eta}_{AB}\left(\varphi^{-1}\right)^A{}_i \left(\varphi^{-1}\right)^B{}_j.$$

When there is no deformation, i.e., $\varphi \in {\rm LI}(U,\widehat{\eta}; V, g)$, then $\widehat{G} = \widehat{\eta}$, $C = g$. The corresponding deformation measures vanishing in the non-deformed state, i.e., Lagrange and Euler deformation tensors $\widehat{E} \in U^* \otimes U^*$, $e \in V^* \otimes V^*$ are given by the following expressions (see, e.g., [40, 41]):

$$\widehat{E} := \frac{1}{2}\left(\widehat{G} - \widehat{\eta}\right), \qquad e := \frac{1}{2}\left(g - C\right).$$

Remark: $\widehat{G}$ is independent of $\widehat{\eta}$ and may be defined even if the material space is purely affine, amorphous. Similarly, C is independent of g and is well-defined even if the physical space is metric-free. Therefore, the literally meant term "deformation" is better expressed by $\widehat{E}$, e than by $\widehat{G}$, C. However, in many formulas $\widehat{G}$, C are more natural and convenient. Deformation tensors behave under the action of isometries in a very peculiar way, namely, for any $A \in \mathrm{O}(V, g)$, $B \in \mathrm{O}(U, \widehat{\eta})$, we have:

$$\widehat{G}[A\varphi]_{KL} = \widehat{G}[\varphi]_{KL}, \qquad \widehat{G}[\varphi B]_{KL} = \widehat{G}[\varphi]_{CD} B^{C}{}_{K} B^{D}{}_{L},$$
$$C[A\varphi]_{ij} = C[\varphi]_{ab} \left(A^{-1}\right)^{a}{}_{i} \left(A^{-1}\right)^{b}{}_{j}, \qquad C[\varphi B]_{ij} = C[\varphi]_{ij}.$$

By the way, these formulas are valid for any $A \in \mathrm{GL}(V)$, $B \in \mathrm{GL}(U)$. The above invariance rules imply the vanishing of their Poisson-brackets respectively with spin $S^{i}{}_{j}$ or vorticity $\widehat{V}^{A}{}_{B}$, i.e.,

$$\{\widehat{G}_{KL}, S^{i}{}_{j}\} = 0, \qquad \{C_{ij}, \widehat{V}^{A}{}_{B}\} = 0,$$

and similarly for $\widehat{E}_{KL}$, e_{ij}.

3.1.2 Deformation invariants

Deformation invariants are important mechanical quantities. They are scalar measures of deformation, i.e., they do not contain any information concerning the orientation of deformation (its principal axes) in the physical or material space. They may be chosen in various ways but in an n-dimensional space exactly n of them may be functionally independent. The particular choice of n basic invariants depends on the considered problem and on the computational details. When non-specified, they will be denoted by $\mathcal{K}_a$, $a = \overline{1, n}$.

Let us define the mixed deformation tensors

$$\widehat{G} \in U \otimes U^{*}, \qquad C \in V \otimes V^{*}, \qquad \widehat{E} \in U \otimes U^{*}, \qquad e \in V \otimes V^{*},$$

namely,

$$\widehat{G}^{A}{}_{B} := \widehat{\eta}^{AC} \widehat{G}_{CB}, \quad C^{i}{}_{j} := g^{ik} C_{kj}, \quad \widehat{E}^{A}{}_{B} := \widehat{\eta}^{AC} \widehat{E}_{CB}, \quad e^{i}{}_{j} := g^{ik} e_{kj}.$$

A class of possible and geometrically natural choices of $\mathcal{K}_a$ is given by the following expressions:

$$\mathrm{Tr}\left(\widehat{G}^{k}\right), \qquad \mathrm{Tr}\left(C^{k}\right), \qquad \mathrm{Tr}\left(\widehat{E}^{k}\right), \qquad \mathrm{Tr}\left(e^{k}\right), \qquad k = \overline{1, n}.$$

In certain problems it is convenient to use the following eigenequations:

$$\det\left[\widehat{G}^{A}{}_{B} - \lambda \delta^{A}{}_{B}\right] = 0, \qquad \det\left[C^{i}{}_{j} - \lambda \delta^{i}{}_{j}\right] = 0,$$
$$\det\left[\widehat{E}^{A}{}_{B} - \lambda \delta^{A}{}_{B}\right] = 0, \qquad \det\left[e^{i}{}_{j} - \lambda \delta^{i}{}_{j}\right] = 0.$$

These are n-th order algebraic (polynomial) equations with respect to λ. Their solutions provide one of possible choices of basic invariants. Another, very convenient one is given by coefficients at λ^p, $p = \overline{0,(n-1)}$ [40, 41] (the coefficient at λ^n is standard and equals one). Deformation invariants are non-sensitive with respect to spatial and material isometries, i.e., for any $A \in \mathrm{O}(V,g)$, $B \in \mathrm{O}(U,\widehat{\eta})$ we have that

$$\mathcal{K}_a\left[A\varphi B\right] = \mathcal{K}_a\left[\varphi\right].$$

This implies the following obvious Poisson brackets:

$$\{\mathcal{K}_a, S^i{}_j\} = \{\mathcal{K}_a, \widehat{V}^A{}_B\} = 0.$$

In certain computational problems, but also in theoretical analysis, it is very convenient to use quantities $Q^a = \sqrt{\lambda_a}$, where λ_a are solutions of the above eigenequations, or $q^a = \ln Q^a$, i.e., $Q^a = \exp(q^a)$. The eigenvalues of C are equal to

$$\left(\lambda_a\right)^{-1} = \left(Q^a\right)^{-2} = \exp\left(-2q^a\right).$$

Any function F on the configuration space which depends on φ only through the deformation invariants is doubly isotropic, i.e., for any $A \in \mathrm{O}(V,g)$, $B \in \mathrm{O}(U,\widehat{\eta})$, $\varphi \in \mathrm{LI}(U,\widehat{\eta};V,g)$ it satisfies the following condition:

$$F(A\varphi B) = F(\varphi).$$

All such functions have vanishing Poisson brackets with spin and vorticity, i.e.,

$$\{F, S^i{}_j\} = \{F, \widehat{V}^A{}_B\} = 0.$$

3.1.3 Dynamical internal affinely invariant models

Any left and right invariant twice covariant field on the affine group is degenerate, thus, non-applicable as a kinetic energy model. The above-quoted models of T_{tr} show the highest reasonable invariance and fix our attention on the models of T_{int} with the same symmetry properties, i.e., metric-affine and affine-metric ones:

$$\begin{aligned}\left\{\begin{array}{c} T_{\mathrm{int}}^{\mathrm{met-aff}} \\ T_{\mathrm{int}}^{\mathrm{aff-met}} \end{array}\right\} &= \frac{I}{2}\mathrm{Tr}\left\{\begin{array}{c} \Omega^T\Omega \\ \widehat{\Omega}^T\widehat{\Omega} \end{array}\right\} + T_{\mathrm{int}}^{\mathrm{aff-aff}} \qquad (3.3)\\ &= \frac{I}{2}\mathrm{Tr}\left\{\begin{array}{c} \Omega^T\Omega \\ \widehat{\Omega}^T\widehat{\Omega} \end{array}\right\} + \frac{A}{2}\mathrm{Tr}\left\{\begin{array}{c} \Omega^2 \\ \widehat{\Omega}^2 \end{array}\right\} + \frac{B}{2}\left(\mathrm{Tr}\left\{\begin{array}{c} \Omega \\ \widehat{\Omega} \end{array}\right\}\right)^2.\end{aligned}$$

In some open domain of inertial constants $(I, A, B) \in \mathbb{R}^3$ the above expressions are positively definite, thus, "good" kinetic energies.

The usual kinetic energy compatible with the d'Alembert principle has the following form:

$$T = T_{\rm tr} + T_{\rm int} = \frac{M}{2} g_{ij} v^i v^j + \frac{1}{2} g_{ij} \dot{\varphi}^i{}_A \dot{\varphi}^j{}_B \widehat{J}^{AB},$$

where the inertial parameters are not primary ones but obtained from the measure μ describing the co-moving mass distribution:

$$M = \int d\mu(a), \qquad \widehat{J}^{AB} = \int a^A a^B d\mu(a), \qquad \int a^A d\mu(a) = 0$$

(the total mass and second-order moment of its distribution; $\widehat{J}^{AB} = I\widehat{\eta}^{AB}$ in the materially isotropic case). Without an appropriate potential term the above T is non-viable as a realistic Lagrangian because the generic deformative motion would be non-bounded and passing through singularities. Unlike this, in [175] it was shown that geodetic models (3.1), (3.3) based on curved inertial metrics predict an open family of bounded and an open family of non-bounded orbits on $\mathrm{SL}(n, \mathbb{R})$. It is only dilatational motion that needs extra stabilization.

From now on, let us neglect translational motion. Legendre transformation may be expressed in any of the following convenient forms:

$$\Sigma^i{}_j = \frac{\partial T_{\rm int}}{\partial \Omega^j{}_i}, \qquad \widehat{\Sigma}^A{}_B = \frac{\partial T_{\rm int}}{\partial \widehat{\Omega}^B{}_A}.$$

The laboratory and co-moving affine spins Σ, $\widehat{\Sigma}$ are, respectively, Hamiltonian generators (momentum mappings) of the left and right regular translations on $\mathrm{GL}(n, \mathbb{R})$. Geodetic Hamiltonians corresponding to (3.1), (3.3) have the following forms:

$$\mathcal{T}^{\rm aff-aff}_{\rm int} = \frac{1}{2A} \mathrm{Tr} \left\{ \begin{array}{c} \Sigma^2 \\ \widehat{\Sigma}^2 \end{array} \right\} - \frac{B}{2A(A+nB)} \left(\mathrm{Tr} \left\{ \begin{array}{c} \Sigma \\ \widehat{\Sigma} \end{array} \right\} \right)^2,$$

$$\left\{ \begin{array}{c} \mathcal{T}^{\rm met-aff}_{\rm int} \\ \mathcal{T}^{\rm aff-met}_{\rm int} \end{array} \right\} = \frac{1}{2\widetilde{I}} \mathrm{Tr} \left\{ \begin{array}{c} \Sigma^T \Sigma \\ \widehat{\Sigma}^T \widehat{\Sigma} \end{array} \right\} + \frac{1}{2\widetilde{A}} \mathrm{Tr} \left\{ \begin{array}{c} \Sigma^2 \\ \widehat{\Sigma}^2 \end{array} \right\} + \frac{1}{2\widetilde{B}} \left(\mathrm{Tr} \left\{ \begin{array}{c} \Sigma \\ \widehat{\Sigma} \end{array} \right\} \right)^2,$$

where

$$\widetilde{I} = \frac{I^2 - A^2}{I}, \qquad \widetilde{A} = \frac{A^2 - I^2}{A}, \qquad \widetilde{B} = -\frac{(I+A)(I+A+nB)}{B}.$$

One can also use the following convenient representation:

$$\mathcal{T}^{\rm aff-aff}_{\rm int} = \frac{1}{2A} C(2) - \frac{B}{2A(A+nB)} C(1)^2, \tag{3.4}$$

$$\left\{ \begin{array}{c} \mathcal{T}^{\rm met-aff}_{\rm int} \\ \mathcal{T}^{\rm aff-met}_{\rm int} \end{array} \right\} = \frac{C(2)}{2\alpha} + \frac{C(1)^2}{2\beta} + \frac{1}{2\mu} \left\{ \begin{array}{c} \|S\|^2 \\ \|\widehat{V}\|^2 \end{array} \right\}, \tag{3.5}$$

where the Casimir invariants on the $\mathrm{GL}(n,\mathbb{R})$ are as follows:

$$C(k) = \mathrm{Tr}\left(\Sigma^k\right) = \mathrm{Tr}\left(\widehat{\Sigma}^k\right),$$

and

$$S = \Sigma - \Sigma^T, \qquad \widehat{V} = \widehat{\Sigma} - \widehat{\Sigma}^T$$

are spin and vorticity (generators of spatial and material rotations), $\|S\|$ and $\|\widehat{V}\|$ are their magnitudes, i.e.,

$$\|S\|^2 = -\frac{1}{2}\mathrm{Tr}\left(S^2\right) \geq 0, \qquad \|\widehat{V}\|^2 = -\frac{1}{2}\mathrm{Tr}\left(\widehat{V}^2\right) \geq 0,$$

and α, β, and μ are constants:

$$\alpha := I + A, \qquad \beta := -\frac{(I+A)(I+A+nB)}{B}, \qquad \mu := \frac{I^2 - A^2}{I}.$$

Obviously, $\mathcal{T}^{\mathrm{aff-aff}}_{\mathrm{int}}$ is obtained by putting $I = 0$, i.e., $\mu = \infty$.

3.1.4 Splitting into isochoric and dilatational motions

If we identify analytically U and V with $\mathbb{R}^n$ and $\mathrm{LI}(U,V)$ with $\mathrm{GL}(n,\mathbb{R})$, then it is clear that the connected component of unity, i.e., $\mathrm{GL}^+(n,\mathbb{R})$, becomes the direct product $\mathrm{GL}^+(n,\mathbb{R}) \simeq \mathrm{SL}(n,\mathbb{R}) \times \exp(\mathbb{R}) = \mathrm{SL}(n,\mathbb{R}) \times \mathbb{R}^+$, where the second group factor is obviously meant in the multiplicative sense, i.e., as $\mathrm{GL}^+(1,\mathbb{R})$. It describes *pure dilatations*, whereas $\mathrm{SL}(n,\mathbb{R})$ refers to *the isochoric motion.* Without this identification, $\mathrm{LI}(U,V)$ may be represented as the Cartesian product of any of the aforementioned leaves (of mutually non-compressed configurations) and the multiplicative group $\mathbb{R}\backslash\{0\}$. If some volume-standards ν_U, $\nu_V = \Delta(\varphi)\nu_U$, where $\Delta(\varphi)$ is the volume extension ratio, (e.g., metric-based ones ν_η, ν_g) and orientations are fixed in U and V, then $\mathrm{LI}^+(U,V)$, i.e., the manifold of orientation-preserving isomorphisms, is identified with the product $\mathrm{SLI}(U,\nu_U;V,\nu_V)\times\exp(\mathbb{R})$, where, obviously, the first term consists of transformations φ for which $\Delta(\varphi) = 1$, i.e., $q(\varphi) = 0$ if instead of the volume extension ratio we use the linear size extension ration

$$D(\varphi) = \sqrt[n]{\Delta(\varphi)} = \exp\left[q(\varphi)\right].$$

Such a formulation is more correct from the point of view of geometrical purity, however, for our purposes the analytical identification of $\mathrm{LI}^+(U,V)$ with $\mathrm{GL}^+(n,\mathbb{R}) \simeq \mathrm{SL}(n,\mathbb{R}) \times \exp(\mathbb{R}) = \mathrm{SL}(n,\mathbb{R}) \times \mathbb{R}^+$ is sufficient and, as a matter of fact, more convenient.

Thus, any matrix $\varphi \in \mathrm{GL}^{+}(n, \mathbb{R})$ is uniquely represented as $\varphi = \exp(q)\Phi$, where $\Phi \in \mathrm{SL}(n, \mathbb{R})$. Obviously, we have that $\det \varphi = \exp(nq)$. It is convenient to introduce the following shear velocities:

$$\omega := \frac{d\Phi}{dt}\Phi^{-1}, \qquad \widehat{\omega} := \Phi^{-1}\frac{d\Phi}{dt}.$$

Obviously, ω, $\widehat{\omega} \in \mathrm{SL}(n, \mathbb{R})'$, i.e., they are trace-less. Then affine velocities may be expressed as follows:

$$\Omega = \omega + \frac{1}{n}\mathrm{Tr}\,(\Omega)\,\mathbb{I}_n = \omega + \frac{dq}{dt}\mathbb{I}_n, \qquad \widehat{\Omega} = \widehat{\omega} + \frac{1}{n}\mathrm{Tr}\left(\widehat{\Omega}\right)\mathbb{I}_n = \widehat{\omega} + \frac{dq}{dt}\mathbb{I}_n,$$

where, obviously, $\mathbb{I}_n$ denotes the unit $n \times n$ matrix.

Analogously, the affine spin splits as follows:

$$\Sigma = \sigma + \frac{p}{n}\mathbb{I}_n, \qquad \widehat{\Sigma} = \widehat{\sigma} + \frac{p}{n}\mathbb{I}_n,$$

where σ, $\widehat{\sigma} \in \mathrm{SL}(n, \mathbb{R})'$, and p denotes the dilatational canonical momentum. The velocity-momentum pairing becomes as follows:

$$\mathrm{Tr}\,(\Sigma\Omega) = \mathrm{Tr}\,(\sigma\omega) + p\dot{q} = \mathrm{Tr}\,(\widehat{\sigma}\widehat{\omega}) + p\dot{q} = \mathrm{Tr}\left(\widehat{\Sigma}\widehat{\Omega}\right).$$

Poisson-bracket relations for σ-components are based on the structure constants of $\mathrm{SL}(n, \mathbb{R})$. The same is obviously true for $\widehat{\sigma}$ with the proviso that the signs are reversed. The mixed $\{\sigma, \widehat{\sigma}\}$ brackets do vanish. Obviously, $\{q, p\} = 1$, and the quantities describing dilatations, i.e., q and p, Poisson-commute with those describing shears, i.e., Φ, σ, and $\widehat{\sigma}$.

The doubly-invariant "kinetic energy" (3.1) is a superposition of the isochoric and dilatational terms, i.e.,

$$T^{\mathrm{aff-aff}}_{\mathrm{int}} = \frac{A}{2}\mathrm{Tr}\left(\omega^2\right) + \frac{n(A + nB)}{2}\dot{q}^2 = T^{\mathrm{aff-aff}}_{\mathrm{int-sh}} + T^{\mathrm{aff-aff}}_{\mathrm{int-dil}}.$$

Performing the Legendre transformation,

$$\sigma = A\omega, \qquad p = n(A + nB)\dot{q},$$

we obtain the following geodetic Hamiltonian:

$$\mathcal{T}^{\mathrm{aff-aff}}_{\mathrm{int}} = \frac{1}{2A}\mathrm{Tr}\left(\sigma^2\right) + \frac{p^2}{2n(A + nB)} = \mathcal{T}^{\mathrm{aff-aff}}_{\mathrm{int-sh}} + \mathcal{T}^{\mathrm{aff-aff}}_{\mathrm{int-dil}}. \qquad (3.6)$$

In these expressions the quantities ω, σ may be replaced by their co-moving representations $\widehat{\omega}$, $\widehat{\sigma}$. More generally, including into consideration also the metrical-affine and affine-metrical models, we can write that

$$\mathcal{T}^{\text{aff}-\text{aff}}_{\text{int}} = \frac{C_{\text{SL}(n,\mathbb{R})}(2)}{2A} + \frac{p^2}{2n(A+nB)}, \tag{3.7}$$

$$\left\{ \begin{array}{c} \mathcal{T}^{\text{met}-\text{aff}}_{\text{int}} \\ \mathcal{T}^{\text{aff}-\text{met}}_{\text{int}} \end{array} \right\} = \frac{C_{\text{SL}(n,\mathbb{R})}(2)}{2(I+A)} + \frac{p^2}{2n(I+A+nB)} + \frac{I}{2(I^2-A^2)} \left\{ \begin{array}{c} \|S\|^2 \\ \|\widehat{V}\|^2 \end{array} \right\}, \tag{3.8}$$

where the second-order Casimir invariant on the $\text{SL}(n,\mathbb{R})$ is as follows:

$$C_{\text{SL}(n,\mathbb{R})}(2) = \text{Tr}\left(\sigma^2\right) = \text{Tr}\left(\widehat{\sigma}^2\right).$$

It is seen that $\mathcal{T}^{\text{met}-\text{aff}}_{\text{int}}$ and $\mathcal{T}^{\text{aff}-\text{met}}_{\text{int}}$ differ merely by geometrically interesting $\|S\|^2$- and $\|\widehat{V}\|^2$-correction terms from the doubly invariant $\mathcal{T}^{\text{aff}-\text{aff}}_{\text{int}}$ in which A is to be replaced by $(I+A)$.

Classical equations may be analyzed and solved with the use of Poisson brackets and exponential mapping.

3.1.5 Stabilizing dilatations

Besides the geodetic affinely-invariant models, it is of great interest to consider some potential models $H = T + V(r,\varphi)$. In this thesis we concentrate mainly on the highly-symmetrical models that are spatially and/or materially isotropic. It is natural to assume that the potential V is compatible with these invariance properties of the kinetic term. This means that if the potential V is invariant under internal spatial (material) rotations, then it depends on the configurations φ only through the Green (Cauchy) deformation tensor $\widehat{G}$ (C). If it is both spatially and materially isotropic in internal degrees of freedom, then it depends on φ only through the deformation invariants $\mathcal{K}_a$, e.g., parameterized by $(q^1,\ldots,q^n)$. In the special case of invariance of the potential V under the spatial and material volume-preserving groups $\text{SL}(V)$ and $\text{SL}(U)$, it depends on φ only through the determinant $\det\varphi$ or, equivalently, it is a function of $q = \left(q^1 + \cdots + q^n\right)/n$, i.e., the "centre of mass" of logarithmic deformation invariants q^i.

In the case of above-described splitting into the pure dilatations and isochoric motion, the dilatational and shear-rotational degrees of freedom are mutually independent (there is no interaction between them),

$$H = H_{\text{sh}} + H_{\text{dil}} = \mathcal{T}_{\text{sh}} + \mathcal{T}_{\text{dil}} + V\left(q^1,\ldots,q^n\right),$$

and then it suggests us to concentrate on the potentials where these degrees of freedom are explicitly separated, i.e.,

$$V\left(q^1,\ldots,q^n\right)=V_{\text{dil}}(q)+V_{\text{sh}}\left(\ldots,q^i-q^j,\ldots\right).$$

The most natural scheme for the shear potential V_{sh} is that of "binary interactions" between the deformation invariants, i.e.,

$$V_{\text{sh}}=\frac{1}{2}\sum_{i\neq j}V_{ij}\left(\left|q^i-q^j\right|\right),$$

where V_{ij} depends only on the relative positions of deformation invariants $|q^i-q^j|$.

Let us stress that the incompressible dynamical affine models may encode the dynamics of elastic vibrations without any extra potential term used because their general solutions contain open subset of bounded motions [174, 175]. If we have no imposed any incompressibility constraints, then the only possibility of stabilizing dilatations is to include some extra potential preventing the unlimited expansion to the infinite size and asymptotic contraction to the point-like object. There is plenty of such phenomenological modelling potentials, e.g.,

$$V_{\text{dil}}=\frac{\kappa}{8}\left(D^2+\frac{1}{D^2}-2\right)=\frac{\kappa}{8}\left(\text{ch}(2q)-1\right),\qquad \kappa>0.$$

Obviously, this potential is positively infinite at $q=\mp\infty$ ($D=0$, $D=+\infty$) and has the global stable equilibrium at $q=0$ ($D=1$), where it behaves as the harmonic oscillator: $V_{\text{dil}}(q)\approx\kappa q^2/2$ for $q\approx 0$. For strongly extended bodies it also behaves harmonically in the D-variable sense. Another phenomenological model would be just the global form of the harmonic oscillator, i.e.,

$$V_{\text{dil}}(q)=\frac{\kappa}{2}q^2.$$

One can also try to use some toy models predicting "dissociation" of the body (its unlimited size-expansion), unlimited collapse, or both of them above some threshold of the total dilatational energy, e.g.,

$$V_{\text{dil}}(q)=\frac{\kappa}{2}\left(\text{th}^2 q-1\right).$$

In certain problems it may be reasonable to use phenomenological models preventing contraction but admitting dissociation.

In quantized version of the theory one can stabilize dilatations in an easy way with the use of the q-variable potential well (perhaps with the infinite walls) concentrated around $q=0$ ($D=1$).

3.1.6 Two-polar splitting and lattice-like structures

It is very convenient and instructive to express our Hamiltonians, kinetic energies and configuration metrics in terms of the two-polar splitting. Let us introduce some auxiliary quantities

$$M = -\widehat{\rho} - \widehat{\tau}, \qquad N = \widehat{\rho} - \widehat{\tau}.$$

Their basic Poisson brackets, i.e.,

$$\begin{aligned}
\{q^a, p_b\} &= \delta^a{}_b,\\
\{q^a, M^c{}_d\} &= \{p_a, M^c{}_d\} = \{q^a, N^c{}_d\} = \{p_a, N^c{}_d\} = 0,\\
\{M_{ab}, M_{cd}\} &= \{N_{ab}, N_{cd}\} = M_{cb}g_{ad} - M_{ad}g_{cb} + M_{ac}g_{db} - M_{db}g_{ac},\\
\{M_{ab}, N_{cd}\} &= N_{cb}g_{ad} - N_{ad}g_{cb} + N_{ac}g_{db} - N_{db}g_{ac},
\end{aligned}$$

follow, obviously, from those for $\widehat{\rho}$ and $\widehat{\tau}$, i.e.,

$$\begin{aligned}
\{\widehat{\rho}_{ab}, \widehat{\rho}_{cd}\} &= -\widehat{\rho}_{cb}g_{ad} + \widehat{\rho}_{ad}g_{cb} - \widehat{\rho}_{ac}g_{db} + \widehat{\rho}_{db}g_{ac},\\
\{\widehat{\tau}_{ab}, \widehat{\tau}_{cd}\} &= -\widehat{\tau}_{cb}g_{ad} + \widehat{\tau}_{ad}g_{cb} - \widehat{\tau}_{ac}g_{db} + \widehat{\tau}_{db}g_{ac},\\
\{\widehat{\rho}_{ab}, \widehat{\tau}_{cd}\} &= 0,
\end{aligned}$$

which are based on the structure constants of $\mathrm{SO}(n,\mathbb{R})$ because $\widehat{\rho}$ and $\widehat{\tau}$ are Hamiltonian generators of $\mathrm{SO}(n,\mathbb{R})$.

One can easily show that the second-order Casimir invariant $C(2)$ occurring in the main terms of our affine kinetic Hamiltonians has the following form:

$$C(2) = \sum_a p_a^2 + \frac{1}{16}\sum_{a,b}\frac{(M^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}} - \frac{1}{16}\sum_{a,b}\frac{(N^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}. \tag{3.9}$$

Obviously, M and N are antisymmetric in the Kronecker-delta sense, i.e.,

$$M^a{}_b = -g^{al}M^k{}_l g_{kb} = -M_b{}^a, \qquad N^a{}_b = -g^{al}N^k{}_l g_{kb} = -N_b{}^a.$$

The first term in (3.9) may be suggestively decomposed into the "relative" and "centre-of-mass" parts as follows:

$$\frac{1}{2n}\sum_{a,b}(p_a - p_b)^2 + \frac{p^2}{n}.$$

Obviously, the first-order Casimir invariant coincides with p, i.e.,

$$C(1) = p = \sum_a p_a. \tag{3.10}$$

For geodetic systems (and for more general systems with potentials V depending only on deformation invariants) the spin $S=\rho$ and vorticity $V=\tau$ are constants of motion and may be used for extracting from equations of motion some information concerning the general solution. Unlike this the quantities $\widehat{\rho}$ and $\widehat{\tau}$, thus, also M and N, fail to be constants of motion except the special case $n=2$, when the rotation group is Abelian. However, on the level of qualitative analysis, the expression (3.9) based on $\widehat{\rho}$ and $\widehat{\tau}$ is more convenient because it does not involve L, R-variables, i.e., the rotational degrees of freedom of deformation tensors.

Let us observe that the kinetic energy based on the second-order Casimir invariant (3.9) has the characteristic lattice structure, i.e.,

$$\mathcal{T}_{\rm latt}=\frac{C(2)}{2A}=\frac{1}{2A}\sum_{a}p_a^2+\frac{1}{32A}\sum_{a,b}\frac{(M^a{}_b)^2}{{\rm sh}^2\frac{q^a-q^b}{2}}-\frac{1}{32A}\sum_{a,b}\frac{(N^a{}_b)^2}{{\rm ch}^2\frac{q^a-q^b}{2}}. \qquad (3.11)$$

This expression resembles structurally the hyperbolic Sutherland n-body system on the straight line [160, 166, 174, 175, 178]. Positions of the fictitious material points are given by deformation invariants q^a. The "particles" have identical masses and are indistinguishable. Unlike the hyperbolic Sutherland system, the coupling amplitudes $M^a{}_b$, $N^a{}_b$ are not equal and constant. Instead of this, they are dynamical variables on the equal footing with q^a, p_a. The negative N-contribution to $\mathcal{T}_{\rm latt}$ describes the attractive forces between lattice points, whereas the positive M-term corresponds to repulsion. Under the appropriate initial conditions we have stable bounded vibrations without any use of the potential energy term. Therefore, the non-definiteness of $\mathcal{T}_{\rm latt}$ is not only non-embarrassing, but just desirable as a tool for describing "elastic" vibrations on the basis of purely geodetic models.

The internal kinetic energies (3.4), (3.5) using the expression (3.9) may be rewritten as follows:

$$\begin{aligned}\mathcal{T}^{\rm aff-aff}_{\rm int} &= \frac{1}{2A}C(2)-\frac{B}{2A(A+nB)}C(1)^2\\ &= \frac{1}{4nA}\sum_{a,b}(p_a-p_b)^2+\frac{1}{2n(A+nB)}p^2\\ &+ \frac{1}{32A}\sum_{a,b}\frac{(M^a{}_b)^2}{{\rm sh}^2\frac{q^a-q^b}{2}}-\frac{1}{32A}\sum_{a,b}\frac{(N^a{}_b)^2}{{\rm ch}^2\frac{q^a-q^b}{2}},\end{aligned} \qquad (3.12)$$

$$\left\{\begin{array}{c}\mathcal{T}^{\rm met-aff}_{\rm int}\\ \mathcal{T}^{\rm aff-met}_{\rm int}\end{array}\right\}=\mathcal{T}^{\rm aff-aff}_{\rm int}\left[A\mapsto I+A\right]+\frac{I}{2\left(I^2-A^2\right)}\left\{\begin{array}{c}\|S\|^2\\ \|\widehat{V}\|^2\end{array}\right\}. \qquad (3.13)$$

Comparing these expressions with those decomposed into the isochoric and dilatational parts, i.e., with (3.7), (3.8), we see that the second-order Casimir invariant on $\mathrm{SL}(n,\mathbb{R})$ may be rewritten in the following equivalent form:

$$C_{\mathrm{SL}(n,\mathbb{R})}(2) = \frac{1}{2n}\sum_{a,b}(p_a - p_b)^2 + \frac{1}{16}\sum_{a,b}\frac{(M^a{}_b)^2}{\mathrm{sh}^2\frac{q^a-q^b}{2}} - \frac{1}{16}\sum_{a,b}\frac{(N^a{}_b)^2}{\mathrm{ch}^2\frac{q^a-q^b}{2}}. \tag{3.14}$$

This expression is very suggestive because it expresses the quantity $C_{\mathrm{SL}(n)}(2)$ and the corresponding contribution to $\mathcal{T}_{\mathrm{int}}$, i.e., the metric tensor on the manifold of incompressible motions, as the sum of $n(n-1)/2$ two-dimensional clusters, i.e., $\mathbb{R}^2$-coordinate planes in $\mathbb{R}^n$. Incompressibility is expressed by the fact that the invariants q^a and their conjugate momenta p_a enter the above formula through the shape-describing differences $\left(q^a - q^b\right)$, i.e., the ratios Q^a/Q^b, and $p_a - p_b$. This expression may be very convenient when studying invariant geodetic models on the projective group $\mathrm{Pr}(n,\mathbb{R})$, i.e., when dealing with the mechanics of projectively-rigid bodies, because the projective group $\mathrm{Pr}(n,\mathbb{R})$ may be identified in a standard way with $\mathrm{SL}(n+1,\mathbb{R})$.

For the doubly-isotropic d'Alembert model the two-polar splitting leads to the following kinetic Hamiltonian term:

$$\mathcal{T}^{\mathrm{d'A}}_{\mathrm{int}} = \frac{1}{2I}\sum_{a}P_a^2 + \frac{1}{8I}\sum_{a,b}\frac{(M^a{}_b)^2}{(Q^a - Q^b)^2} + \frac{1}{8I}\sum_{a,b}\frac{(N^a{}_b)^2}{(Q^a + Q^b)^2}. \tag{3.15}$$

It is purely repulsive on the level of Q-variables, thus, without any potential term it is non-realistic as a model of elastic vibrations. It is related to the Calogero-Moser lattices similarly as the previous models show some kinship with the hyperbolic Sutherland lattices [17, 96, 97, 152, 160, 165, 178, 186].

3.1.7 Compactification of deformation invariants

If we consider the situation when $\mathrm{GL}(n,\mathbb{R})$ is replaced by $\mathrm{U}(n)$, i.e., another and completely opposite real form of $\mathrm{GL}(n,\mathbb{C})$, then it gives us the compactification of the deformation invariants q^a, i.e., we take them modulo 2π (n-dimensional torus) and put formally $Q^a = \exp(iq^a)$. The Lie algebra $\mathrm{U}(n)'$ consists of anti-Hermitian matrices. Then the positively-definite kinetic energies may be postulated in the following form:

$$T^{\mathrm{aff-aff}}_{\mathrm{int}} = -\frac{A}{2}\mathrm{Tr}\left(\Omega^2\right) - \frac{B}{2}\left(\mathrm{Tr}\Omega\right)^2 = \frac{A}{2}\mathrm{Tr}\left(\Omega^+\Omega\right) + \frac{B}{2}\mathrm{Tr}\left(\Omega^+\right)\mathrm{Tr}\left(\Omega\right), \tag{3.16}$$

where, as previously, $\Omega = \dot{\varphi}\varphi^{-1}$, and $A, B > 0$. Obviously, in this expression for the kinetic energy, Ω may be as well replaced by $\widehat{\Omega} = \varphi^{-1}\dot{\varphi}$.

In the two-polar decomposition $\varphi = LDR^{-1}$, where $L, R \in \mathrm{SO}(n,\mathbb{R})$, and $D = \mathrm{diag}(\ldots, \exp(iq^a), \ldots)$, one obtains for that the second-order Casimir invariant on the total U(2) is as follows:

$$C_{\mathrm{U}(n)}(2) = \sum_a p_a^2 + \frac{1}{16}\sum_{a,b}\frac{(M^a{}_b)^2}{\sin^2\frac{q^a-q^b}{2}} + \frac{1}{16}\sum_{a,b}\frac{(N^a{}_b)^2}{\cos^2\frac{q^a-q^b}{2}}. \tag{3.17}$$

We see that it resembles the usual Sutherland lattice structure for q-particles. Just as previously, it may be convenient to use the splitting into SU(n)- and U(1)-terms, i.e.,

$$\begin{aligned} C_{\mathrm{U}(1)}(2) &= \frac{p^2}{n}, \\ C_{\mathrm{SU}(n)}(2) &= \frac{1}{2n}\sum_{a,b}(p_a - p_b)^2 + \frac{1}{16}\sum_{a,b}\frac{(M^a{}_b)^2}{\sin^2\frac{q^a-q^b}{2}} + \frac{1}{16}\sum_{a,b}\frac{(N^a{}_b)^2}{\cos^2\frac{q^a-q^b}{2}}. \end{aligned}$$

Then the geodetic motion is bounded because U(n) is compact, and the affine kinetic energies may be rewritten as follows:

$$\begin{aligned} \mathcal{T}^{\mathrm{aff-aff}}_{\mathrm{int}} &= \frac{1}{4nA}\sum_{a,b}(p_a - p_b)^2 + \frac{1}{2n(A+nB)}p^2 \\ &+ \frac{1}{32A}\sum_{a,b}\frac{(M^a{}_b)^2}{\sin^2\frac{q^a-q^b}{2}} + \frac{1}{32A}\sum_{a,b}\frac{(N^a{}_b)^2}{\cos^2\frac{q^a-q^b}{2}}, \end{aligned} \tag{3.18}$$

$$\left\{\begin{array}{c} \mathcal{T}^{\mathrm{met-aff}}_{\mathrm{int}} \\ \mathcal{T}^{\mathrm{aff-met}}_{\mathrm{int}} \end{array}\right\} = \mathcal{T}^{\mathrm{aff-aff}}_{\mathrm{int}}\left[A \mapsto I + A\right] + \frac{I}{2\left(I^2 - A^2\right)}\left\{\begin{array}{c} \|S\|^2 \\ \|\widehat{V}\|^2 \end{array}\right\}. \tag{3.19}$$

The binary structures of $C_{\mathrm{SL}(n,\mathbb{R})}(2)$ and $C_{\mathrm{SU}(n)}(2)$ and their dependence on the variables q^a, p_a through their differences $q^a - q^b$, $p_a - p_b$ is geometrically interesting in itself. The splitting into SL$(2,\mathbb{R})$- and SU(2)-clusters corresponding to all possible coordinate planes $\mathbb{R}^2$ in $\mathbb{R}^n$ may be also analytically helpful. However, some sophisticated mathematical techniques would be necessary then, like, e.g., the Dirac procedure for degenerate (constrained) systems. The point is that, in general, different clusters are not analytically independent. And any procedure based on some ordering of variables destroys the explicit binary structure and makes the structure of $\mathcal{T}^{\mathrm{aff-aff}}_{\mathrm{int}}$ rather obscure.

Remark: at the end of this classical preliminaries, let us stress once more the very interesting fact, namely, that the general solution of $C(2)$-based geodetic models contains as a particular subfamily the general solution of the mentioned Calogero-Moser and Sutherland models. It is obtained by putting $N^a{}_b = 0$, and all $M^a{}_b$ with $b \neq a$ equal to some fixed constant M.

3.2 Quantum description of affine models

3.2.1 Problems concerning quantization

There are, obviously, many delicate problems concerning quantization which cannot be discussed here and, fortunately, do not interfere directly with the main subjects of our analysis. Nevertheless, we mention briefly some of them. Strictly speaking, wave functions are not scalars but complex densities of the weight 1/2 so that the bilinear expression $\overline{\Psi}\Psi$ is a real scalar density of weight one, thus, a proper object for describing probability distributions [82]. But in all realistic models, and the our one is not an exception, the configuration space is endowed with some Riemannian structure. And this enables one to factorize scalar (and tensor) densities into products of scalars (tensors) and some standard densities built of the metric tensor. Therefore, the wave function may be finally identified with the complex scalar field (multi-component one when there are internal degrees of freedom).

There are also some arguments for modifying $\mathbf{T}$ by some scalar term proportional to the curvature scalar. Of course, such a term may be always formally interpreted as some correction potential. And besides, here we usually deal with Riemannian manifolds of the constant Riemannian curvature, and then such additional terms result merely in the over-all shifting of energy levels.

In Riemann manifolds the Levi-Civita affine connection preserves the scalar product; because of this, the operator ∇_μ is formally anti-self-adjoint and operators

$$\frac{\hbar}{i}\nabla_\mu, \qquad \mathbf{T} = -\frac{\hbar^2}{2}\Gamma^{\mu\nu}\nabla_\mu\nabla_\nu$$

are formally self-adjoint. They are, however, differential operators, thus, the difficult problem of self-adjoint extensions appears. And besides, being differential operators, they are unbounded in the usual sense, thus, their spectral analysis also becomes a difficult and delicate subject. All such problems will be neglected and considered in the zeroth-order approximation of the mathematical rigor, just as it is usually done in practical physical applications. This is also justified by the fact that, as a rule, our first-order differential operators generate some well-definite global transformation groups admitting a lucid geometrical interpretation. It is typical that in such situations all subtle problems on the level of functional analysis, like the common domains, etc., may be successfully solved.

Therefore, from now on we will proceed in a "physical" way and all terms like "self-adjoint", "Hermitian", and so on will be used in a rough way characteristic for physical papers and applied mathematics.

We shall deal almost exclusively with stationary problems when the Hamilton operator $\mathbf{H}$ is time-independent, thus, the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \mathbf{H}\psi$$

will be replaced by its stationary form, i.e., by the eigenequation $\mathbf{H}\Psi = E\Psi$, where, obviously,

$$\psi = \exp\left(-\frac{i}{\hbar}Et\right)\Psi$$

and Ψ is a time-independent wave function on the configuration space.

3.2.2 Multi-valuedness of wave functions

There is another delicate point concerning fundamental aspects of quantization which, however, may be of some importance and will be analyzed later on. Namely, it is claimed in all textbooks of quantum mechanics that wave functions solving reasonable Schrödinger equations must satisfy strong regularity conditions, and first of all they must be well-defined one-valued functions all over the configuration space, in addition, continuous together with their derivatives. This demand is mathematically essential in the theory of Sturm-Liouville equations and besides it has to do with quantization or, more precisely, discrete spectra of certain physical quantities. By the way, these two things are not independent.

However, there are certain arguments that some physical systems may admit *multi-valued wave functions*. Although the one-valuedness of wave functions is well-motivated in $\mathbb{R}^n$ but it is not necessarily so in multiply connected manifolds with finite homotopy groups. Then one can use the universal covering manifold of the configuration space, in our case simply the universal covering group $\overline{\mathrm{GL}\,(n,\mathbb{R})}$, as a proper, so to speak, "hidden" configuration space. Physically, to preserve the usual Born statistical interpretation, we must only demand the squared modulus $\overline{\Psi}\Psi$ to be one-valued because it represents the probability distribution of detecting a system in various regions of the configuration space, thus, it has to be projectable from the covering manifold to the original configuration space. Nevertheless, the class of acceptable wave functions Ψ is essentially wider. For the wave function Ψ itself it is enough to be "locally" one-valued and sufficiently smooth on the universal covering manifold $\overline{Q}$, and then it may be essentially non-projectable, i.e., multi-valued from the point of view of the original configuration space Q. This may lead to a consistent quantum mechanics, perhaps with some kind of *superselection rules*.

That is the fact, e.g., in the quantum mechanics of rigid bodies, which is sometimes expected to be a good model of spin of the elementary particles [5, 6, 7]. The configuration space of the rigid body without translational motion may be identified with the proper orthogonal group $\mathrm{SO}(3,\mathbb{R})$ ($\mathrm{SO}(n,\mathbb{R})$ in n dimensions), obviously, when some reference orientation and Cartesian coordinates are fixed. But it is well-known that $\mathrm{SO}(3,\mathbb{R})$ is doubly-connected (and so is $\mathrm{SO}(n,\mathbb{R})$ for any $n > 3$). Its covering group is $\mathrm{SU}(2)$ ($\mathrm{Spin}(n)$ for any $n > 3$). Therefore, the two-valued wave functions possessing two different signs at the same $\mathrm{SO}(3,\mathbb{R})$-point are admitted [7]. Obviously, they are single-valued on $\mathrm{SU}(2)$. So, it is really an instructive exercise, and perhaps also a promising physical hypothesis, to develop the rigid top theory with $\mathrm{SU}(2)$ as configuration space [5, 6, 7].

In affinely-rigid body mechanics we are dealing with a similar situation, namely, $\mathrm{GL}(3,\mathbb{R})$ and $\mathrm{SL}(3,\mathbb{R})$ (more generally, $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{R})$ for $n \geq 3$) are doubly-connected. This topological property is simply inherited from the corresponding one for $\mathrm{SO}(3,\mathbb{R})$ ($\mathrm{SO}(n,\mathbb{R})$) on the basis of the polar decomposition [8, 206, 207]. Therefore, the standard quantization procedure in such manifolds should be modified by using wave amplitudes defined on the covering manifolds $\overline{\mathrm{GL}\,(n,\mathbb{R})}$, $\overline{\mathrm{SL}\,(n,\mathbb{R})}$. By the way, some difficulty and mathematical curiosity appears then because these covering groups are non-linear (do not admit faithful realizations in terms of finite-dimensional matrices).

3.2.3 Quantized dynamical affine models

Integration elements corresponding to Haar measures α, λ on the affine and linear groups are given respectively as follows:

$$\begin{aligned} d\alpha(r,\varphi) &= (\det\varphi)^{-n-1}\, dr^1 \cdots dr^n d\varphi^1{}_1 \cdots d\varphi^n{}_n, \\ d\lambda(\varphi) &= (\det\varphi)^{-n}\, d\varphi^1{}_1 \cdots d\varphi^n{}_n. \end{aligned}$$

Expressing the measure λ in terms of the two-polar decomposition $\varphi = LDR^{-1}$ we obtain the following expression:

$$d\lambda(\varphi) = d\lambda\,(L; q^a; R) = P_\lambda dq^1 \cdots dq^n d\mu(L) d\mu(R),$$

where μ denotes the Haar measure on the orthogonal group $\mathrm{SO}(n,\mathbb{R})$, and

$$P_{\lambda} = \prod_{i\neq j} \left|\mathrm{sh}\left(q^{i} - q^{j}\right)\right|. \tag{3.20}$$

The Haar measure on the internal configuration space of the isochoric (incompressible) affinely-rigid body may be expressed in terms of the Dirac distribution function as follows:

$$d\lambda_{\mathrm{SL}(n,\mathbb{R})}(\varphi) = P_{\lambda}\delta\left(q^{1} + \cdots + q^{n}\right)dq^{1}\cdots dq^{n}d\mu(L)d\mu(R).$$

The quantum mechanics of the affinely-rigid body is formulated in the Hilbert spaces

$$\mathrm{L}^{2}\left(\mathrm{GAf}\left(n,\mathbb{R}\right) \simeq \mathrm{GL}\left(n,\mathbb{R}\right) \times_{s} \mathbb{R}^{n}, \alpha\right), \qquad \mathrm{L}^{2}\left(\mathrm{GL}\left(n,\mathbb{R}\right), \lambda\right)$$

respectively for systems with and without translational degrees of freedom. Kinetic energy operator $\mathbf{T}$ has the following standard form (A.9):

$$\mathbf{T} = -\frac{\hbar^{2}}{2}\Delta,$$

where Δ is the Laplace-Beltrami operator based on the metric tensor underlying the classical kinetic energy (A.10). The direct computation of Δ is rather complicated and the resulting formula is completely non-readable. However, the group structure enables one to express $\mathbf{T}$ in terms of differential operators

$$\mathbf{\Sigma}^{i}{}_{j} = -i\hbar\varphi^{i}{}_{A}\frac{\partial}{\partial\varphi^{j}{}_{A}}, \qquad \widehat{\mathbf{\Sigma}}^{A}{}_{B} = -i\hbar\varphi^{i}{}_{B}\frac{\partial}{\partial\varphi^{i}{}_{A}}$$

generating left and right regular translations. They are operators of laboratory and co-moving components of the affine spin. Regular translations are unitary in the sense of scalar product based on the Haar measure. Therefore, $\mathbf{\Sigma}$ and $\widehat{\mathbf{\Sigma}}$ are formally self-adjoint. Of course, being differential operators, they are unbounded, thus, they are not Hermitian in the literal mathematical sense. Nevertheless, they are good physical observables.

The usual spin and vorticity operators are respectively given by

$$\mathbf{S}^{a}{}_{b} := \mathbf{\Sigma}^{a}{}_{b} - g^{ac}g_{bd}\mathbf{\Sigma}^{d}{}_{c}, \qquad \widehat{\mathbf{V}}^{A}{}_{B} := \widehat{\mathbf{\Sigma}}^{A}{}_{B} - \widehat{\eta}^{AC}\widehat{\eta}_{BD}\widehat{\mathbf{\Sigma}}^{D}{}_{C}.$$

Another important quantity is the canonical momentum conjugate to the dilatational coordinate q. On the quantum level it is represented by the following formally self-adjoint operator:

$$\mathbf{p} = -i\hbar\frac{\partial}{\partial q}.$$

It is also convenient to use the deviatoric (shear) parts of the affine spin,

$$\mathbf{s}^a{}_b := \mathbf{\Sigma}^a{}_b - \frac{\mathbf{p}}{n}\delta^a{}_b, \qquad \widehat{\mathbf{s}}^A{}_B := \widehat{\mathbf{\Sigma}}^A{}_B - \frac{\mathbf{p}}{n}\delta^A{}_B,$$

and obviously, $\mathbf{p} = \mathbf{C}(1) = \mathbf{\Sigma}^a{}_a = \widehat{\mathbf{\Sigma}}^A{}_A$.

The kinetic energy operators corresponding to the above-described classical models of internal kinetic energies are simply obtained by replacing the classical quantities $\Sigma^a{}_b$, $\widehat{\Sigma}^A{}_B$ by the above operators $\mathbf{\Sigma}^a{}_b$, $\widehat{\mathbf{\Sigma}}^A{}_B$ without any attention to be paid to the ordering problem (just because of the group-theoretic interpretation of these quantities):

$$\begin{aligned}
\mathbf{T}^{\mathrm{aff-aff}}_{\mathrm{int}} &= \frac{1}{2A}\left\{\begin{array}{c}\mathbf{\Sigma}^i{}_j\mathbf{\Sigma}^j{}_i\\ \widehat{\mathbf{\Sigma}}^A{}_B\widehat{\mathbf{\Sigma}}^B{}_A\end{array}\right\} - \frac{B}{2A(A+nB)}\left\{\begin{array}{c}\mathbf{\Sigma}^i{}_i\mathbf{\Sigma}^j{}_j\\ \widehat{\mathbf{\Sigma}}^A{}_A\widehat{\mathbf{\Sigma}}^B{}_B\end{array}\right\},\\
\left\{\begin{array}{c}\mathbf{T}^{\mathrm{met-aff}}_{\mathrm{int}}\\ \mathbf{T}^{\mathrm{aff-met}}_{\mathrm{int}}\end{array}\right\} &= \frac{1}{2\widetilde{I}}\left\{\begin{array}{c}g_{ij}\mathbf{\Sigma}^i{}_k\mathbf{\Sigma}^j{}_l g^{kl}\\ \widehat{\eta}_{AB}\widehat{\mathbf{\Sigma}}^A{}_C\widehat{\mathbf{\Sigma}}^B{}_D\widehat{\eta}^{CD}\end{array}\right\}\\
&+ \frac{1}{2\widetilde{A}}\left\{\begin{array}{c}\mathbf{\Sigma}^i{}_j\mathbf{\Sigma}^j{}_i\\ \widehat{\mathbf{\Sigma}}^A{}_B\widehat{\mathbf{\Sigma}}^B{}_A\end{array}\right\} + \frac{1}{2\widetilde{B}}\left\{\begin{array}{c}\mathbf{\Sigma}^i{}_i\mathbf{\Sigma}^j{}_j\\ \widehat{\mathbf{\Sigma}}^A{}_A\widehat{\mathbf{\Sigma}}^B{}_B\end{array}\right\}.
\end{aligned}$$

Due to the group-theoretical structure of the above objects as generators, the classical splitting of the kinetic energy into incompressible (shear-rotational) and dilatational parts, i.e., (3.7) and (3.8), remains literally valid on the quantum level:

$$\begin{aligned}
\mathbf{T}^{\mathrm{aff-aff}}_{\mathrm{int}} &= \frac{\mathbf{C}_{\mathrm{SL}(n,\mathbb{R})}(2)}{2A} + \frac{\mathbf{p}^2}{2n(A+nB)},\\
\left\{\begin{array}{c}\mathbf{T}^{\mathrm{met-aff}}_{\mathrm{int}}\\ \mathbf{T}^{\mathrm{aff-met}}_{\mathrm{int}}\end{array}\right\} &= \frac{\mathbf{C}_{\mathrm{SL}(n,\mathbb{R})}(2)}{2(I+A)} + \frac{\mathbf{p}^2}{2n(I+A+nB)} + \frac{I}{2(I^2-A^2)}\left\{\begin{array}{c}\|\mathbf{S}\|^2\\ \|\widehat{\mathbf{V}}\|^2\end{array}\right\},
\end{aligned}$$

where the quantum operators corresponding to the classical second-order Casimir invariant on $\mathrm{SL}(n,\mathbb{R})$ and magnitudes of the spin and vorticity are as follows:

$$\mathbf{C}_{\mathrm{SL}(n,\mathbb{R})}(2) = \mathrm{Tr}\left(\mathbf{s}^2\right) = \mathrm{Tr}\left(\widehat{\mathbf{s}}^2\right), \quad \|\mathbf{S}\|^2 = -\frac{1}{2}\mathrm{Tr}\left(\mathbf{S}^2\right), \quad \|\widehat{\mathbf{V}}\|^2 = -\frac{1}{2}\mathrm{Tr}\left(\widehat{\mathbf{V}}^2\right).$$

On the basis of the classical discussion in [174, 175] we may suggest that on the quantum level the above kinetic energies restricted to $\mathrm{SL}(n,\mathbb{R})$ have both discrete and continuous spectra and predicts the bounded oscillatory solutions even if no extra potential on $\mathrm{SL}(n,\mathbb{R})$ is used.

There are $\mathrm{GL}(n,\mathbb{R})$-problems where the separation of the isochoric $\mathrm{SL}(n,\mathbb{R})$-terms is not necessary, sometimes it is even undesirable. Then it is more convenient to use the quantized version of (3.4) and (3.5), i.e.,

$$\begin{aligned}
\mathbf{T}^{\rm aff-aff}_{\rm int} &= \frac{1}{2A}\mathbf{C}(2) - \frac{B}{2A(A+nB)}\mathbf{p}^2,\\
\left\{\begin{array}{c}\mathbf{T}^{\rm met-aff}_{\rm int}\\ \mathbf{T}^{\rm aff-met}_{\rm int}\end{array}\right\} &= \frac{\mathbf{C}(2)}{2\alpha} + \frac{\mathbf{p}^2}{2\beta} + \frac{1}{2\mu}\left\{\begin{array}{c}\|\mathbf{S}\|^2\\ \|\widehat{\mathbf{V}}\|^2\end{array}\right\},
\end{aligned}$$

where the second-order Casimir operator on the total $\mathrm{GL}(n,\mathbb{R})$ is given by the following expression:

$$\mathbf{C}(2) := \mathbf{\Sigma}^a{}_b\mathbf{\Sigma}^b{}_a = \widehat{\mathbf{\Sigma}}^A{}_B\widehat{\mathbf{\Sigma}}^B{}_A.$$

In particular, if the inertial constant B vanishes, then the affine-affine model $\mathbf{T}^{\rm aff-aff}_{\rm int}$ may be interpreted in terms of one-dimensional multi-body problems in the sense of Calogero, Moser, Sutherland [123, 177].

Just as in the classical case, on $\mathrm{GL}(n,\mathbb{R})$ we have to use some dilatations-stabilizing potential $V_{\rm dil}(q)$ must be introduced if the system has to possess bound states. As previously, harmonic oscillator and potential well are the simplest and most convincing models, at least in nuclear physics. For more general doubly isotropic potentials $V(q^1,\ldots,q^n)$ depending only on deformation invariants, there is no possibility of avoiding differential equations (with the help of ladder procedures). Nevertheless, the problem is then still remarkably simplified in comparison with the general case, because the quantum dynamics of deformation invariants is autonomous (in this respect the quantum problem is in a sense simpler than the classical one). The procedure is based then on the two-polar decomposition, which by the way is also very convenient on the level of purely geodetic models. In certain problems, e.g., spatially isotropic but materially anisotropic ones, the polar decomposition is also convenient.

Similarly, the corresponding expressions for the translational kinetic energies (3.2) have the following forms:

$$\left\{\begin{array}{c}\mathbf{T}^{\rm met-aff}_{\rm tr}\\ \mathbf{T}^{\rm aff-met}_{\rm tr}\end{array}\right\} = \frac{1}{2m}\left\{\begin{array}{c}g^{ij}\\ \left(C^{-1}\right)^{ij}\end{array}\right\}\mathbf{p}_i\mathbf{p}_j = \frac{m}{2}\left\{\begin{array}{c}\left(\widehat{G}^{-1}\right)^{AB}\\ \widehat{\eta}^{AB}\end{array}\right\}\widehat{\mathbf{p}}_A\widehat{\mathbf{p}}_B,$$

where $\mathbf{p}_i$, $\widehat{\mathbf{p}}_A$ are the linear momentum operators respectively in laboratory and co-moving representations, i.e.,

$$\mathbf{p}_i = -i\hbar\frac{\partial}{\partial r^i}, \qquad \widehat{\mathbf{p}}_A = \varphi^i{}_A\mathbf{p}_i = -i\hbar\varphi^i{}_A\frac{\partial}{\partial r^i}.$$

As mentioned, there are no affine-affine models of $\mathbf{T}_{\mathrm{tr}}$, and therefore, no affine-affine models of $\mathbf{T}$. The corresponding "metric tensors" on $\mathrm{GAf}(n, \mathbb{R})$ would have to be singular.

3.2.4 Two-polar splitting in the quantum case

Let us consider the case of two-polar decomposition on the quantum level. Unfortunately, the automatical replacement of classical group-theoretical quantities by seemingly natural operators does not work any longer. This means that although p may be automatically substituted by $\mathbf{p} = -i\hbar\partial/\partial q$, the quantities p_a cannot be replaced by $-i\hbar\partial/\partial q^a$ and $\sum_a p_a^2$ cannot be "quantized" to the usual $\mathbb{R}^2$-Laplace operator expression, i.e.,

$$-\hbar^2\Delta[q^a] = -\hbar^2 \sum_a \frac{\partial^2}{\partial q^{a2}}.$$

This is because the additive translations of logarithmic deformation invariants are not geometrically fundamental operations. Fortunately, there are no problems with the spin and vorticity operators $\mathbf{S}^i{}_j = \mathbf{r}^i{}_j$, $\widehat{\mathbf{V}}^A{}_B = -\mathbf{t}^A{}_B$ and with operators $\widehat{\mathbf{r}}^a{}_b$, $\widehat{\mathbf{t}}^a{}_b$. Again this is because of their group-theoretical interpretation, namely, the spin and vorticity generate respectively spatial and material rotations and the operators $\widehat{\mathbf{r}}^a{}_b$, $\widehat{\mathbf{t}}^a{}_b$ are their projections onto the principal axes of the Cauchy and Green deformation tensors, i.e.,

$$\widehat{\mathbf{r}}^a{}_b = L^a{}_i L^j{}_b \mathbf{S}^i{}_j, \qquad \widehat{\mathbf{t}}^a{}_b = -R^a{}_A R^B{}_b \widehat{\mathbf{V}}^A{}_B$$

(the ordering of operators just as written here). Just as in the classical theory, $\widehat{\mathbf{r}}^a{}_b$, $\widehat{\mathbf{t}}^a{}_b$ are generators (in the quantum-Poisson-bracket sense) of the right action of $\mathrm{SO}(n, \mathbb{R})$ on the quantities $L : \mathbb{R}^n \to V$, $R : \mathbb{R}^n \to U$, namely,

$$L \mapsto LU, \qquad R \mapsto RU.$$

Just as $\widehat{\mathbf{r}}^a{}_b$, $\widehat{\mathbf{t}}^a{}_b$, the operators $\mathbf{S}^i{}_j$, $\widehat{\mathbf{V}}^A{}_B$ act only on generalized coordinates x^μ, y^μ parameterizing respectively L and R (some Euler angles, rotation vectors, first-kind canonical coordinates, and so on).

Just as in classical theory, it is convenient to introduce the following operators:

$$\mathbf{M}^a{}_b := -\widehat{\mathbf{r}}^a{}_b - \widehat{\mathbf{t}}^a{}_b, \qquad \mathbf{N}^a{}_b := \widehat{\mathbf{r}}^a{}_b - \widehat{\mathbf{t}}^a{}_b.$$

Commutation relations for operators $\mathbf{S}^i{}_j$, $\widehat{\mathbf{V}}^A{}_B$, $\widehat{\mathbf{r}}^a{}_b$, $\widehat{\mathbf{t}}^a{}_b$, $\mathbf{M}^a{}_b$, $\mathbf{N}^a{}_b$ are directly isomorphic with those for the generators of $\mathrm{SO}(n, \mathbb{R})$ and are expressed in a straightforward way in terms of the structure constants of $\mathrm{SO}(n, \mathbb{R})$.

Then the kinetic energy operators for the affine models (3.12), (3.13) are given as follows:

$$\begin{aligned}\mathbf{T}^{\text{aff}-\text{aff}}_{\text{int}} &= -\frac{\hbar^2}{2A}\mathbf{D}_\lambda + \frac{\hbar^2 B}{2A(A+nB)}\frac{\partial^2}{\partial q^2} \\ &+ \frac{1}{32A}\sum_{a,b}\frac{\left(\mathbf{M}^a{}_b\right)^2}{\operatorname{sh}^2\frac{q^a-q^b}{2}} - \frac{1}{32A}\sum_{a,b}\frac{\left(\mathbf{N}^a{}_b\right)^2}{\operatorname{ch}^2\frac{q^a-q^b}{2}}, \qquad (3.21)\end{aligned}$$

$$\left\{\begin{array}{c}\mathbf{T}^{\text{met}-\text{aff}}_{\text{int}}\\ \mathbf{T}^{\text{aff}-\text{met}}_{\text{int}}\end{array}\right\} = \mathbf{T}^{\text{aff}-\text{aff}}_{\text{int}}\left[A\mapsto I+A\right] + \frac{1}{2\mu}\left\{\begin{array}{c}\|\mathbf{S}\|^2\\ \|\widehat{\mathbf{V}}\|^2\end{array}\right\}, \qquad (3.22)$$

where

$$\mathbf{D}_\lambda = \frac{1}{P_\lambda}\sum_a\frac{\partial}{\partial q^a}P_\lambda\frac{\partial}{\partial q^a} = \sum_a\frac{\partial^2}{\partial q^{a2}} + \sum_a\frac{\partial \ln P_\lambda}{\partial q^a}\frac{\partial}{\partial q^a} \qquad (3.23)$$

and P_λ is given by (3.20). It is seen that $\mathbf{D}_\lambda$ differs from the $\mathbb{R}^n$-Laplace operator $\sum_a \partial^2/\partial q^{a2}$ by some first-order differential operator. This is because of the mentioned breakdown of the naive classical analogy between p_a and $(\hbar/i)\partial/\partial q^a$. The reason of this breakdown lies in that that the additive translations

$$q^a \mapsto q^a + u^a$$

do not preserve the measures λ and α. Because of this their argument-wise action on wave functions is not unitary in $\mathrm{L}^2\left(\mathrm{GL}\left(n,\mathbb{R}\right),\lambda\right)$ and $\mathrm{L}^2\left(\mathrm{GAf}\left(n,\mathbb{R}\right),\alpha\right)$.

Quite similarly, the quantum version of the doubly-isotropic d'Alembert model (3.15) may be written as follows:

$$\mathbf{T}^{\text{d'A}}_{\text{int}} = -\frac{\hbar^2}{2I}\mathbf{D}_l + \frac{1}{8I}\sum_{a,b}\frac{\mathbf{M}^2_{ab}}{(Q^a-Q^b)^2} + \frac{1}{8I}\sum_{a,b}\frac{\mathbf{N}^2_{ab}}{(Q^a+Q^b)^2}, \qquad (3.24)$$

where

$$\mathbf{D}_l = \frac{1}{P_l}\sum_a\frac{\partial}{\partial Q^a}P_l\frac{\partial}{\partial Q^a} = \sum_a\frac{\partial^2}{\partial Q^{a2}} + \sum_a\frac{\partial \ln P_l}{\partial Q^a}\frac{\partial}{\partial Q^a}, \qquad (3.25)$$

the weight factor P_l is given by the following expression:

$$P_l = \prod_{i\neq j}\left(Q^{i2}-Q^{j2}\right) = \prod_{i\neq j}\left(Q^i+Q^j\right)\left(Q^i-Q^j\right),$$

and, as we remember, $Q^a = \exp\left(q^a\right)$.

Finally, the kinetic energy operators which correspond to the classical expressions (3.18), (3.19) for affine systems on $\mathrm{U}(n)$, i.e., those with the "compactified" deformation invariants, have the following form:

$$\begin{aligned}\mathbf{T}^{\text{aff}-\text{aff}}_{\text{int}} &= -\frac{\hbar^2}{2A}\mathbf{D}_U + \frac{\hbar^2 B}{2A(A+nB)}\frac{\partial^2}{\partial q^2}\\ &+ \frac{1}{32A}\sum_{a,b}\frac{(\mathbf{M}^a{}_b)^2}{\sin^2\frac{q^a-q^b}{2}} + \frac{1}{32A}\sum_{a,b}\frac{(\mathbf{N}^a{}_b)^2}{\cos^2\frac{q^a-q^b}{2}}, \qquad (3.26)\end{aligned}$$

$$\left\{\begin{array}{c}\mathbf{T}^{\text{met}-\text{aff}}_{\text{int}}\\ \mathbf{T}^{\text{aff}-\text{met}}_{\text{int}}\end{array}\right\} = \mathbf{T}^{\text{aff}-\text{aff}}_{\text{int}}\left[A\mapsto I+A\right] + \frac{1}{2\mu}\left\{\begin{array}{c}\|\mathbf{S}\|^2\\ \|\widehat{\mathbf{V}}\|^2\end{array}\right\}, \qquad (3.27)$$

where

$$\mathbf{D}_U = \frac{1}{P_U}\sum_a\frac{\partial}{\partial q^a}P_U\frac{\partial}{\partial q^a} = \sum_a\frac{\partial^2}{\partial(q^a)^2} + \sum_a\frac{\partial\ln P_U}{\partial q^a}\frac{\partial}{\partial q^a}, \qquad (3.28)$$

and the weight factor P_U is given by the following expression:

$$P_U = \prod_{a\neq b}\left|\sin(q^a-q^b)\right|.$$

The corresponding Haar measure on $\mathrm{U}(n)$ is given by the following expression:

$$d\lambda_U\left(L;q^a;R\right) = P_U dq^1\cdots dq^n d\mu(L)d\mu(R),$$

where μ, as previously, denotes the Haar measure on $\mathrm{SO}(n,\mathbb{R})$.

3.2.5 Polar and two-polar expansions of wave functions

The configuration spaces of affinely-rigid body, i.e., roughly speaking (if translational motion is neglected) $\mathrm{GL}^+(n,\mathbb{R})$ or $\mathrm{SL}(n,\mathbb{R})$, are doubly-connected, and the problem of physically admissible two-valued wave functions also appears here. There is, however, some difficulty, namely, that their universal covering groups $\overline{\mathrm{GL}^+(n,\mathbb{R})}$ and $\overline{\mathrm{SL}(n,\mathbb{R})}$ are nonlinear, i.e., they do not possess faithful realizations in terms of finite matrices. The nonlinearity of the mentioned coverings implies, in particular, that affine spinors (half-objects) must be either infinite-dimensional or ruled by nonlinear realizations of $\overline{\mathrm{GL}^+(n,\mathbb{R})}$ or $\overline{\mathrm{SL}(n,\mathbb{R})}$ as abstract groups constructed with the help of loops in $\mathrm{GL}^+(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{R})$.

However, in quantum mechanics of affinely-rigid bodies the construction of multi-valued wave functions may be analytically overcome with the use of polar and two-polar splittings. Let us begin from the first one,

$$\varphi = UA = BU = \left(UAU^{-1}\right)U,$$

where $U \in \mathrm{SO}(n, \mathbb{R})$, and $A, B \in \mathrm{Symm}^{+}(n, \mathbb{R})$, i.e., they are symmetric and positively definite (and in the case of $\mathrm{SL}(n, \mathbb{R})$ their determinants equal one). The splitting is unique and, because of this, $\mathrm{GL}^{+}(n, \mathbb{R})$ as a manifold (but not as a group) may be identified with the Cartesian products

$$\mathrm{SO}(n, \mathbb{R}) \times \mathrm{Symm}^{+}(n, \mathbb{R}) \qquad \text{or} \qquad \mathrm{Symm}^{+}(n, \mathbb{R}) \times \mathrm{SO}(n, \mathbb{R}),$$

where the manifold $\mathrm{Symm}^{+}(n, \mathbb{R})$ is topologically diffeomorphic with $\mathbb{R}^{n(n+1)/2}$ ($\mathbb{R}^6$ in the physical three-dimensional case). Therefore, the covering manifold may be identified with

$$\mathrm{Spin}(n) \times \mathrm{Symm}^{+}(n, \mathbb{R}) \qquad \text{or} \qquad \mathrm{Symm}^{+}(n, \mathbb{R}) \times \mathrm{Spin}(n).$$

In the physical three-dimensional case, these splittings become

$$\mathrm{SU}(2) \times \mathbb{R}^6 \qquad \text{or} \qquad \mathbb{R}^6 \times \mathrm{SU}(2).$$

Topological non-triviality is absorbed here by the factor $\mathrm{SO}(3, \mathbb{R})$ (in general by $\mathrm{SO}(n, \mathbb{R})$) and covered by $\mathrm{SU}(2)$ (in general by $\mathrm{Spin}(n)$). Therefore, the admissible multi-valued wave functions may be expanded as follows:

$$\Psi(u, A) = \sum_{s} \sum_{m,m'=-s}^{s} C^{s}{}_{mm'}(A) \mathcal{D}^{s}{}_{mm'}(u),$$

where s are non-negative integers or positive half-integers, the summation over m, k is performed in steps by one, and $\mathcal{D}^s$ are matrices of unitary irreducible representations of $\mathrm{SU}(2)$. For integer and half-integer values of s they satisfy, respectively, the following conditions:

$$\mathcal{D}^{s}_{mm'}(-u) = \pm \mathcal{D}^{s}_{mm'}(u)$$

for any $u \in \mathrm{SU}(2)$; obviously $\pm u$ project onto the same element of $\mathrm{SO}(3, \mathbb{R})$. But $\overline{\Psi}\Psi$ must be one-valued on $\mathrm{GL}^{+}(3, \mathbb{R})$, therefore, a kind of superselection rule appears according to which states with half-integer and integer s cannot be superposed with each other. This is a toy model of the fermionic and bosonic sectors.

Therefore, in any admissible Ψ, $C^{s}{}_{mm'} = 0$ either for all non-negative integer or for all positive half-integer s. More rigorously, we would have to write

$$\Psi(u, A) = \sum_{\sigma=1}^{\infty} \sum_{\mu,\kappa=0}^{\sigma} C^{\frac{\sigma}{2}}{}_{(-\frac{\sigma}{2}+\mu),(-\frac{\sigma}{2}+\kappa)}(A) \mathcal{D}^{\frac{\sigma}{2}}{}_{(-\frac{\sigma}{2}+\mu),(-\frac{\sigma}{2}+\kappa)}(u)$$

for the half-integer spin ("fermionic") situations or

$$\Psi(u, A) = \sum_{s=0}^{\infty} \sum_{\mu,\kappa=0}^{2s} C^{s}{}_{(-s+\mu),(-s+\kappa)}(A)\mathcal{D}^{s}{}_{(-s+\mu),(-s+\kappa)}(u)$$

for the integer spin ("bosonic") situations. These formulas are valid with summation over all indices meant in steps by one. If $\overline{\Psi}\Psi$ is to be one-valued probability distribution, then the superposing between indicated subspaces of function series is forbidden (a kind of superselection rule), and the admissible Hamiltonians must exclude any transitions between them.

In the highly-symmetric models it is more convenient to use the two-polar decomposition. It consists of diagonalization of A or B, e.g.,

$$A = RDR^{-1}, \qquad R \in \mathrm{SO}\,(n, \mathbb{R})\,, \qquad D \in \mathrm{Diag}\,(\mathbb{R}^n)\,.$$

Assigning $L := UR \in \mathrm{SO}(n, \mathbb{R})$, we have finally that

$$\varphi = LDR^{-1}.$$

In this way $\varphi \in \mathrm{GL}^{+}\,(n, \mathbb{R})$ is identified with a triplet

$$(L; D(q); R) \in \mathrm{SO}\,(n, \mathbb{R}) \times \mathbb{R}^n \times \mathrm{SO}\,(n, \mathbb{R})\,.$$

Then, according to the Peter-Weyl theorem, the wave functions on $\mathrm{GL}^{+}(n, \mathbb{R})$ may be expanded in L, R-variables with respect to matrix elements of irreducible representations of the compact group $\mathrm{SO}(n, \mathbb{R})$. Obviously, the expansion coefficients depend on deformation invariants. In general, we have that

$$\Psi(\varphi) = \Psi(L, D, R) = \sum_{\alpha,\beta\in\Omega} \sum_{m,n=1}^{N(\alpha)} \sum_{k,l=1}^{N(\beta)} \mathcal{D}^{\alpha}_{mn}(L) f^{\alpha\beta}_{\substack{nk\\ml}}(D)\mathcal{D}^{\beta}_{kl}\left(R^{-1}\right), \tag{3.29}$$

where Ω denotes the set of equivalence classes of unitary irreducible representations of $\mathrm{SO}(n, \mathbb{R})$, $N(\alpha)$ is the dimension of the α-th representation class (it is finite because $\mathrm{SO}(n, \mathbb{R})$ is compact), and $\mathcal{D}^{\alpha}$ is the $N(\alpha) \times N(\alpha)$ matrices of irreducible representations. For many classical groups $\mathcal{D}^{\alpha}$ are explicitly known (at least in terms of some well-investigated special functions). The argument D of f is the system of deformation invariants, e.g., $(q^1, \ldots, q^n)$.

The non-uniqueness of the two-polar decomposition implies that the deformation invariants $(q^1, \ldots, q^n)$ are very complicated indistinguishable parastatistical "particles" on the real axis $\mathbb{R}$. The point is that the reduced amplitudes $f^{\alpha\beta}$ as functions of $(q^1, \ldots, q^n)$ must satisfy certain conditions due to which the resulting Ψ as a function of (L, D, R) does not distinguish triplets (L, D, R) representing the same configuration φ, i.e., $\Psi(L_1, D_1, R_1) = \Psi(L_2, D_2, R_2)$ if $L_1 D_1 R_1^{-1} = L_2 D_2 R_2^{-1}$.

Let us describe this non-uniqueness explicitly. So, let $K \in \mathrm{O}(n, \mathbb{R})$ denote the finite group of orthogonal matrices which have exactly one non-vanishing entry in every row and column; obviously, these entries equal to ± 1. The subgroup

$$K^+ := K \cap \mathrm{SO}\,(n, \mathbb{R})$$

consists of afore-defined matrices with the determinants equal to $+1$. It is easy to see that the groups K, K^+ have respectively $(2n)n!$ and $(n)n!$ elements. For any $U \in K$, the similarity transformation

$$\mathrm{Diag}\,(\mathbb{R}^n) \ni D \mapsto U^{-1} D U \in \mathrm{Diag}\,(\mathbb{R}^n)$$

results in a permutation of diagonal elements of D:

$$\left(Q^1, \ldots, Q^n\right) \mapsto \left(Q^{\pi_U(1}, \ldots, Q^{n)}\right),$$

i.e.,

$$\left(q^1, \ldots, q^n\right) \mapsto \left(q^{\pi_U(1}, \ldots, q^{n)}\right).$$

The assignment

$$K \ni U \mapsto \pi_U \in \mathrm{S}^{(n)}$$

is a $(2n : 1)$-epimorphism of K onto the permutation group $\mathrm{S}^{(n)}$. Restricting it to K^+, we obtain an $(n : 1)$-epimorphism.

Let $\mathrm{GL}^{+(n)}\,(n, \mathbb{R})$ denote the subset of $\mathrm{GL}^+\,(n, \mathbb{R})$ with the simple spectra of deformation tensors, and $M^{(n)}$ be the corresponding subset of $\mathrm{SO}(n, \mathbb{R}) \times \mathbb{R}^n \times \mathrm{SO}(n, \mathbb{R})$ consisting of such triplets $(L; q^1, \ldots, q^n; R)$ that all q^i's are pairwise disjoint. Let K^+ act on $M^{(n)}$ according to the following rule:

$$\left(L; q^1, \ldots, q^n; R\right) \mapsto \left(LU; q^{\pi_U(1}, \ldots, q^{n)}; RU\right).$$

Let us denote the corresponding transformation group of $M^{(n)}$ by $H^{(n)}$. It is clear that $\mathrm{GL}^{+(n)}\,(n, \mathbb{R})$ is diffeomorphic with $M^{(n)}/H^{(n)}$, i.e.,

$$\mathrm{GL}^{+(n)}\,(n, \mathbb{R}) \simeq M^{(n)}/H^{(n)}.$$

Of course, this situation of non-degenerate spectra is a generic one, so this is the main part of the multi-valuedness of the two-polar decomposition.

To describe the covering group $\overline{\mathrm{GL}^+(n,\mathbb{R})}$ one should use the following auxiliary manifold:

$$\mathrm{Spin}(n) \times \mathbb{R}^n \times \mathrm{Spin}(n),$$

i.e., for $n = 3$,

$$\mathrm{SU}(2) \times \mathbb{R}^3 \times \mathrm{SU}(2).$$

Let $\tau : \mathrm{Spin}(n) \to \mathrm{SO}(n,\mathbb{R})$ denote the canonical projection and $\overline{K^+} \subset \mathrm{Spin}(n)$ denote the $(2n)n!$-element subgroup $\tau^{-1}(K^+)$. The manifold $M^{(n)}$ introduced above is covered by $\overline{M^{(n)}}$, i.e., the subset of such triplets

$$\left(l; q^1, \ldots, q^n; r\right) \in \mathrm{Spin}(n) \times \mathbb{R}^n \times \mathrm{Spin}(n)$$

that all q^i's are pairwise disjoint. And $\overline{K^+}$ induces on $\overline{M^{(n)}}$ the transformation group $\overline{H^{(n)}}$ the action of which is given by the following rule:

$$\left(l; q^1, \ldots, q^n; r\right) \mapsto \left(lu; q^{\pi_{\tau(u)}(1}, \ldots, q^{n)}; ru\right),$$

where $u \in \mathrm{Spin}(n)$. The corresponding generic part of the configuration space is given by the following quotient manifold:

$$Q^{(n)} \simeq \overline{M^{(n)}} / \overline{H^{(n)}}.$$

In situations when there are coincidences of q^i's, i.e., when the spectra of deformations tensors are degenerate, description is more complicated. Such configurations are non-generic, and they need some special treatment. Let us present a brief scheme for this.

Let $M^{(k;p_1,\ldots,p_n)}$ be the set of such triplets $(L; q^1, \ldots, q^n; R)$ that there are only k different q^i's, every one with a multiplicity factor p_σ and

$$\sum_{\sigma=1}^{k} p_\sigma = n.$$

Let us take the transformation group $H^{(k;p_1,\ldots,p_n)}$ with its action on $M^{(k;p_1,\ldots,p_n)}$ as

$$\left(L; q^1, \ldots, q^n; R\right) \mapsto \left(LU; q^{\pi_U(1}, \ldots, q^{n)}; RU\right),$$

where U runs over the group generated by K and the subgroup $W^{(k;p_1,\ldots,p_n)} \subset \mathrm{SO}(n,\mathbb{R})$ composed of k $p_\sigma \times p_\sigma$ blocks, every one in the corresponding $\mathrm{SO}(p_\sigma,\mathbb{R})$. The generic subset $Q^{(k;p_1,\ldots,p_n)}$ is given by the following quotient:

$$M^{(k;p_1,\ldots,p_n)} / H^{(k;p_1,\ldots,p_n)}.$$

When the half-integer spin is to be taken into account, we must consider the manifold $\overline{M^{(k;p_1,\ldots,p_n)}}$ consisting of triplets $(l;q^1,\ldots,q^n;r)$, where $l,r\in \mathrm{Spin}(n)$, and $(q^1,\ldots,q^n)$ are degenerate as above. Then

$$\overline{H^{(k;p_1,\ldots,p_n)}} = \tau^{-1}\left(H^{(k;p_1,\ldots,p_n)}\right)\subset \mathrm{Spin}\left(n,\mathbb{R}\right),$$

and the manifold of the corresponding degenerate configuration is the quotient

$$\overline{M^{(k;p_1,\ldots,p_n)}}\Big/\overline{H^{(k;p_1,\ldots,p_n)}}$$

taken with respect to the action

$$\left(l;q^1,\ldots,q^n;r\right)\mapsto\left(lu;q^{\pi_{\tau(u)}(1},\ldots,q^{n)};ru\right),$$

where $u\in\overline{H^{(k;p_1,\ldots,p_n)}}$. When $k<n$, i.e., at least one multiplicity factor is nontrivial, the group $\overline{H^{(k;p_1,\ldots,p_n)}}$ is continuous, and the resulting quotient is lower-dimensional. In the physical case $n=3$, we have obviously only two possibilities of nontrivial blocks, $\mathrm{SO}(2,\mathbb{R})\times$ $\mathrm{SO}(1,\mathbb{R})$ and the total $\mathrm{SO}(3,\mathbb{R})$ (respectively two of q's or all of them equal); obviously, $\mathrm{SO}(1,\mathbb{R})=\{1\}$.

3.2.6 Algebraization procedure

The operators $\mathbf{S}^i{}_j$, $\widehat{\mathbf{V}}^A{}_B$, $\widehat{\mathbf{r}}^a{}_b$, $\widehat{\mathbf{t}}^a{}_b$ when acting on functions $\mathcal{D}^\alpha{}_{mk}$ may be replaced by some standard algebraic operations. This enables one to reduce the Schrödinger equation for the wave functions Ψ depending on n^2 variables $\varphi^i{}_A$ to some eigenproblems for the multi-component amplitudes $f^{\alpha\beta}$ depending only on the n deformation invariants q^a. Therefore, in a sense, the problem may be reduced to the Cartan subgroup of diagonal matrices φ (the maximal Abelian subgroup in $\mathrm{GL}(n,\mathbb{R})$).

It is clear that in geodetic models, or in models with doubly isotropic potentials, m and l in the Peter-Weyl expansion (3.29) are "good" quantum numbers. In other words, the spin and vorticity operators $\mathbf{S}^i{}_j$, $\widehat{\mathbf{V}}^A{}_B$ do commute with the Hamilton operator $\mathbf{H}$. The same concerns the representation labels $\alpha,\beta\in\Omega$, i.e., the systems of eigenvalues for the Casimir operators of the groups $\mathrm{SO}(V,g)$, $\mathrm{SO}(U,\widehat{\eta})$ acting argument-wise on the wave functions. Let us remind that these Casimirs are given by the following expressions:

$$\begin{aligned}\mathbf{C}_{\mathrm{SO}(V,g)}(p) &= \mathbf{S}^i{}_k\mathbf{S}^k{}_m\cdots\mathbf{S}^r{}_z\mathbf{S}^z{}_i,\\ \mathbf{C}_{\mathrm{SO}(U,\widehat{\eta})}(p) &= \widehat{\mathbf{V}}^A{}_K\widehat{\mathbf{V}}^K{}_M\cdots\widehat{\mathbf{V}}^R{}_Z\widehat{\mathbf{V}}^Z{}_A,\end{aligned}$$

where there are p operator multipliers in every expression, $p \leq n$, and the above Casimir invariants vanish trivially for the odd values of p.

In this situation, it is convenient to keep α, β, m, l fixed and use the following reduced amplitudes:

$$\Psi^{\alpha\beta}_{ml}(L, D, R) = \sum_{n=1}^{N(\alpha)} \sum_{k=1}^{N(\beta)} \mathcal{D}^{\alpha}_{mn}(L) f^{\alpha\beta}_{nk}(D) \mathcal{D}^{\beta}_{kl}\left(R^{-1}\right). \tag{3.30}$$

So, in the physical case $n = 3$ we have, obviously, the standard form of $\mathrm{SO}(n, \mathbb{R})$-Casimirs, i.e.,

$$\mathbf{C}_{\mathrm{SO}(V,g)}(2) = \mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2 = \widehat{\mathbf{r}}_1^2 + \widehat{\mathbf{r}}_2^2 + \widehat{\mathbf{r}}_3^2 = \mathbf{C}_{\mathrm{SO}(3,\mathbb{R})}(2), \tag{3.31}$$

$$\mathbf{C}_{\mathrm{SO}(U,\widehat{\eta})}(2) = \widehat{\mathbf{V}}_1^2 + \widehat{\mathbf{V}}_2^2 + \widehat{\mathbf{V}}_3^2 = \widehat{\mathbf{t}}_1^2 + \widehat{\mathbf{t}}_2^2 + \widehat{\mathbf{t}}_3^2 = \mathbf{C}_{\mathrm{SO}(3,\mathbb{R})}(2). \tag{3.32}$$

Then Ω is the set of non-negative integer, α, β are traditionally denoted by symbols like $s, j = 0, 1, 2, \ldots$, etc., $N(s) = 2s + 1$, $N(j) = 2j + 1$, and the indices (m, n), (k, l) are considered as jumping by 1, respectively, from $-s$ to s and from $-j$ to j. Thus, the expansion (3.29) is written according to the mentioned conventions as

$$\Psi(\varphi) = \Psi(L, D, R) = \sum_{s,j=0}^{\infty} \sum_{m,n=-s}^{s} \sum_{k,l=-j}^{j} \mathcal{D}^{s}_{mn}(L) f^{sj}_{\substack{nk\\ml}}(D) \mathcal{D}^{j}_{kl}\left(R^{-1}\right), \tag{3.33}$$

and, similarly, the reduced amplitudes (3.30) are written as follows:

$$\Psi^{sj}_{ml}(L, D, R) = \sum_{n=-s}^{s} \sum_{k=-j}^{j} \mathcal{D}^{s}_{mn}(L) f^{sj}_{nk}(D) \mathcal{D}^{j}_{kl}\left(R^{-1}\right), \tag{3.34}$$

where $\mathcal{D}^s$ are the Wigner matrices of $(2s+1)$-dimensional irreducible representations of the three-dimensional rotation group (they are well-known special functions of mathematical physics).

Due to the peculiarity of the dimension three, where skew-symmetric tensors may be identified with axial vectors, it is more convenient instead of (3.31) and (3.32) to use the following magnitudes:

$$\|\mathbf{S}\|^2 = -\frac{1}{2}\mathbf{S}^a{}_b\mathbf{S}^b{}_a, \qquad \|\widehat{\mathbf{V}}\|^2 = -\frac{1}{2}\widehat{\mathbf{V}}^A{}_B\widehat{\mathbf{V}}^B{}_A.$$

Then

$$\|\mathbf{S}\|^2 = \mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2, \qquad \|\widehat{\mathbf{V}}\|^2 = \widehat{\mathbf{V}}_1^2 + \widehat{\mathbf{V}}_2^2 + \widehat{\mathbf{V}}_3^2,$$

where

$$\mathbf{S}_a = \frac{1}{2}\varepsilon_{ab}{}^c\mathbf{S}^b{}_c, \qquad \widehat{\mathbf{V}}_A = \frac{1}{2}\varepsilon_{AB}{}^C\widehat{\mathbf{V}}^B{}_C.$$

The raising and lowering of indices is meant here in the sense of orthonormal coordinates (Kronecker-delta trivial operation). The same convention is used for $\widehat{\mathbf{r}}^a{}_b$, $\widehat{\mathbf{t}}^a{}_b$, i.e.,

$$\widehat{\mathbf{r}}_a = \frac{1}{2}\varepsilon_{ab}{}^c\widehat{\mathbf{r}}^b{}_c, \qquad \widehat{\mathbf{t}}_a = \frac{1}{2}\varepsilon_{ab}{}^c\widehat{\mathbf{t}}^b{}_c.$$

Obviously, the amplitudes Ψ^{sj}_{ml} are eigenfunctions of rotational Casimir invariants, i.e.,

$$\|\mathbf{S}\|^2\Psi^{sj}_{ml} = \|\widehat{\mathbf{r}}\|^2\Psi^{sj}_{ml} = \hbar^2 s(s+1)\Psi^{sj}_{ml}, \tag{3.35}$$

$$\|\widehat{\mathbf{V}}\|^2\Psi^{sj}_{ml} = \|\widehat{\mathbf{t}}\|^2\Psi^{sj}_{ml} = \hbar^2 j(j+1)\Psi^{sj}_{ml}. \tag{3.36}$$

Let us denote the corresponding eigenvalues as follows:

$$C(s,2) = \hbar^2 s(s+1), \qquad C(j,2) = \hbar^2 j(j+1), \tag{3.37}$$

where s, j are non-negative integers or non-negative integers and positive half-integers when $\mathrm{GL}^+(3,\mathbb{R})$ and $\mathrm{SL}(3,\mathbb{R})$ are replaced by their coverings $\overline{\mathrm{GL}^+(3,\mathbb{R})}$ and SU(2).

According to the tradition, we often use such a basis that Ψ^{sj}_{ml} are also eigenfunctions of the third components of rotational generators, i.e.,

$$\mathbf{S}_3\Psi^{sj}_{ml} = \hbar m\Psi^{sj}_{ml}, \qquad \widehat{\mathbf{V}}_3\Psi^{sj}_{ml} = \hbar l\Psi^{sj}_{ml}.$$

And, obviously, if the values n, k in the superposition (3.34) are kept fixed, we have that

$$\widehat{\mathbf{r}}_3\Psi^{sj}_{\substack{ml\\nk}} = \hbar n\Psi^{sj}_{\substack{ml\\nk}}, \qquad \widehat{\mathbf{t}}_3\Psi^{sj}_{\substack{ml\\nk}} = \hbar k\Psi^{sj}_{\substack{ml\\nk}}.$$

Let us use the exponential formulas for the parameterization of the elements $W(\omega)$ of the groups $\mathrm{SO}(V,g)$ and $\mathrm{SO}(U,\widehat{\eta})$, i.e.,

$$W(\omega) = \exp\left(\frac{1}{2}\omega^i{}_j E^j{}_i\right), \qquad W(\omega) = \exp\left(\frac{1}{2}\widehat{\omega}^A{}_B\widehat{E}^B{}_A\right),$$

where $E^i{}_j$, $\widehat{E}^A{}_B$ are basic matrices of Lie algebras $\mathrm{SO}(V,g)'$, $\mathrm{SO}(U,\widehat{\eta})'$:

$$\left(E^i{}_j\right)^k{}_l = \delta^i{}_l\delta^k{}_j - g^{ik}g_{jl}, \qquad \left(\widehat{E}^A{}_B\right)^C{}_D = \delta^A{}_D\delta^C{}_B - \widehat{\eta}^{AC}\widehat{\eta}_{BD}.$$

The skew-symmetry of $\omega\omega^i{}_j$ and $\widehat{\omega}^A{}_B$ in the above exponential formulas is meant respectively as follows:

$$\omega^i{}_j = -g^{ik}\omega^l{}_k g_{lj}, \qquad \widehat{\omega}^A{}_B = -\widehat{\eta}^{AC}\widehat{\omega}^D{}_C\widehat{\eta}_{DB}.$$

The group $\mathrm{SO}(n,\mathbb{R})$ may also be parameterized by the first-kind canonical coordinates ω, namely,

$$W(\omega) = \exp\left(\frac{1}{2}\omega^{a}{}_{b}E^{b}{}_{a}\right),$$

where the basic matrices of the Lie algebra $\mathrm{SO}(n,\mathbb{R})'$ are given by

$$\left(E^{b}{}_{a}\right)^{c}{}_{d} = \delta^{b}{}_{d}\delta^{c}{}_{a} - \delta^{bc}\delta_{ad},$$

and the matrix ω is skew-symmetric in the "cosmetic" Kronecker sense. Therefore, independent coordinates may be chosen as $\omega^{a}{}_{b}$, $a < b$, or conversely. However, for the symmetry reasons it is more convenient to use the representation with the summation extended over all possible $\omega^{a}{}_{b}$.

Then the matrices of irreducible representations $\mathcal{D}^{\alpha}$ are given by

$$\mathcal{D}^{\alpha}\left(L(l)\right) = \exp\left(\frac{1}{2}l^{a}{}_{b}M^{\alpha b}{}_{a}\right), \qquad \mathcal{D}^{\alpha}\left(R(r)\right) = \exp\left(\frac{1}{2}r^{a}{}_{b}M^{\alpha b}{}_{a}\right),$$

where l and r denote the ω-parameters, respectively, for the L- and R-factors of the two-polar decomposition. The anti-Hermitian matrices M^{α} can be expressed by the Hermitian ones S^{α} as follows:

$$S^{\alpha a}{}_{b} = \frac{\hbar}{i}M^{\alpha a}{}_{b}.$$

The commutation rules for $M^{\alpha a}{}_{b}$ are expressed through the structure constants of $\mathrm{SO}(n,\mathbb{R})$, i.e.,

$$[M^{s}{}_{ab}, M^{s}{}_{cd}] = -g_{ad}M^{s}{}_{cb} + g_{cb}M^{s}{}_{ad} - g_{bd}M^{s}{}_{ac} + g_{ac}M^{s}{}_{bd},$$

and therefore,

$$\frac{1}{i\hbar}[S^{j}{}_{ab}, S^{j}{}_{cd}] = g_{ad}S^{j}{}_{cb} - g_{cb}S^{j}{}_{ad} + g_{bd}S^{j}{}_{ac} - g_{ac}S^{j}{}_{bd}.$$

In the physical three-dimensional case we may introduce the matrices

$$S^{j}{}_{a} := \frac{1}{2}\varepsilon_{a}{}^{bc}S^{j}{}_{bc},$$

and then we obviously have the following commutation rules:

$$\frac{1}{i\hbar}[S^{j}{}_{a}, S^{j}{}_{b}] = \varepsilon_{ab}{}^{c}S^{j}{}_{c}.$$

From the fact that $\mathcal{D}^{\alpha}$ are unitary irreducible representations and the operators $(i/\hbar)\mathbf{S}^{k}{}_{l}$, $(i/\hbar)\widehat{\mathbf{V}}^{A}{}_{B}$, $(i/\hbar)\widehat{\mathbf{r}}^{a}{}_{b}$, $(i/\hbar)\widehat{\mathbf{t}}^{a}{}_{b}$ are infinitesimal generators of left and right orthogonal actions on the (L,R)-variables it follows immediately that

$$\begin{aligned} \mathbf{S}^{i}{}_{j}\Psi^{\alpha\beta} &= S^{\alpha i}{}_{j}\Psi^{\alpha\beta}, & \widehat{\mathbf{r}}^{a}{}_{b}\Psi^{\alpha\beta} &= \mathcal{D}^{\alpha}(L)S^{\alpha a}{}_{b}f^{\alpha\beta}(D)\mathcal{D}^{\beta}\left(R^{-1}\right), \\ \widehat{\mathbf{V}}^{A}{}_{B}\Psi^{\alpha\beta} &= \Psi^{\alpha\beta}S^{\beta A}{}_{B}, & \widehat{\mathbf{t}}^{a}{}_{b}\Psi^{\alpha\beta} &= \mathcal{D}^{\alpha}(L)f^{\alpha\beta}(D)S^{\beta a}{}_{b}\mathcal{D}^{\beta}\left(R^{-1}\right). \end{aligned}$$

Therefore, spin and vorticity act on the wave amplitudes $\Psi^{\alpha\beta}$ as a whole, and in a purely algebraic way. On the other hand, to describe in an algebraic way the action of $\widehat{\mathbf{r}}^{a}{}_{b}$, $\widehat{\mathbf{t}}^{a}{}_{b}$, one must extract from $\Psi^{\alpha\beta}$ the reduced amplitudes $f^{\alpha\beta}(q^{1},\ldots,q^{n})$. And it is only this amplitude that is affected by the action of $\widehat{\mathbf{r}}^{a}{}_{b}$, $\widehat{\mathbf{t}}^{a}{}_{b}$ according to the following rules:

$$\widehat{\mathbf{r}}^{a}{}_{b}: \quad f^{\alpha\beta} \mapsto S^{\alpha a}{}_{b}f^{\alpha\beta}, \qquad \widehat{\mathbf{t}}^{a}{}_{b}: \quad f^{\alpha\beta} \mapsto f^{\alpha\beta}S^{\beta a}{}_{b}.$$

It is very convenient to use the following notation:

$$\overrightarrow{S^{\alpha}}{}^{a}{}_{b}f^{\alpha\beta} := S^{\alpha a}{}_{b}f^{\alpha\beta}, \qquad \overleftarrow{S^{\beta}}{}^{a}{}_{b}f^{\alpha\beta} := f^{\alpha\beta}S^{\beta a}{}_{b}.$$

We assumed that the representations $\mathcal{D}^{\alpha}$ of $\mathrm{SO}(n,\mathbb{R})$ are irreducible, therefore, the matrices

$$C^{\alpha}(p) = S^{\alpha a}{}_{b}S^{\alpha b}{}_{c}\cdots S^{\alpha u}{}_{w}S^{\alpha w}{}_{a}$$

(with p factors) are proportional to the $N(\alpha)\times N(\alpha)$ identity matrices, i.e.,

$$C^{\alpha}(p) = \left(\frac{\hbar}{i}\right)^{p} C(\alpha,p)\mathbb{I}_{N(\alpha)}, \tag{3.38}$$

where the numbers $C(\alpha,p)$ are eigenvalues of the corresponding Casimir operators built of the generators of the left and right regular translations on $\mathrm{SO}(n,\mathbb{R})$.

In particular, in the physical three-dimensional case we have

$$\begin{aligned} \|\mathbf{S}\|^{2}\Psi^{sj} = \|\widehat{\mathbf{r}}\|^{2}\Psi^{sj} &= \hbar^{2}s(s+1)\Psi^{sj}, & \mathbf{S}_{a}\Psi^{sj} &= S^{s}{}_{a}\Psi^{sj}, \\ \|\widehat{\mathbf{V}}\|^{2}\Psi^{sj} = \|\widehat{\mathbf{t}}\|^{2}\Psi^{sj} &= \hbar^{2}j(j+1)\Psi^{sj}, & \widehat{\mathbf{V}}_{a}\Psi^{sj} &= \Psi^{sj}S^{j}{}_{a}, \end{aligned}$$

where $S^{s}{}_{a}$ are standard Wigner matrices of the angular momentum with the squared magnitude $\hbar^{2}s(s+1)$. Multiplying them by $(i/\hbar)$ we obtain standard bases of irreducible representations of the Lie algebra $\mathrm{SO}(3,\mathbb{R})'$. For the standard Wigner representation the following expressions are also true:

$$\mathbf{S}_{3}\Psi^{sj}_{ml} = \hbar m\Psi^{sj}_{ml}, \qquad \widehat{\mathbf{V}}_{3}\Psi^{sj}_{ml} = \hbar l\Psi^{sj}_{ml}.$$

Similarly, the action of $\widehat{\mathbf{r}}$, $\widehat{\mathbf{t}}$ operators is represented by the following operations on the reduced amplitudes:

$$\widehat{\mathbf{r}}_a : \quad f^{sj} \mapsto S^s{}_a f^{sj} = \overrightarrow{S^s}{}_a f^{sj}, \qquad \widehat{\mathbf{t}}_a : \quad f^{sj} \mapsto f^{sj} S^j{}_a = \overleftarrow{S^j}{}_a f^{sj}.$$

In particular,

$$\widehat{\mathbf{r}}_3 f^{sj}_{ml} = \hbar m f^{sj}_{ml}, \qquad \widehat{\mathbf{t}}_3 f^{sj}_{ml} = \hbar l f^{sj}_{ml}.$$

3.2.7 Potential case

We restrict ourselves to Hamiltonians of the form $\mathbf{H} = \mathbf{T} + \mathbf{V}$ with some doubly-isotropic potentials $V\left(q^1, \ldots, q^n\right)$, in particular, with some dilatation-stabilizing potentials $V(q)$ (affinely-invariant geodetic incompressible models). Then the action of operators $\mathbf{M}^a{}_b$ and $\mathbf{N}^a{}_b$ become algebraic and standard, and the stationary Schrödinger equation, i.e., energy eigenproblem

$$\mathbf{H}\Psi = E\Psi,$$

splits into family of eigenproblems for the reduced multi-component amplitudes $f^{\alpha\beta}$ (they are partial differential equations involving q^a-variables only):

$$\mathbf{H}^{\alpha\beta} f^{\alpha\beta} = E^{\alpha\beta} f^{\alpha\beta},$$

where $f^{\alpha\beta}$ for any $\alpha, \beta \in \Omega$ is an $N(\alpha) \times N(\beta)$ matrix depending on $(q^1, \ldots, q^n)$. In a consequence of the double (spatial and material) isotropy, this problem is $N(\alpha) \times N(\beta)$-fold degenerate, i.e., for every component of $f^{\alpha\beta}$ there exists an $N(\alpha) \times N(\beta)$-dimensional subspace of solutions. Let us remind that in the primary symbols $f^{\alpha\beta}_{\substack{nk\\ml}}$ the indices m, l just label the degeneracy of solutions for every $f^{\alpha\beta}_{nk}$. The reduced Hamiltonians $\mathbf{H}^{\alpha\beta}$ is an $N(\alpha) \times N(\beta)$-matrix of second-order differential operators, i.e.,

$$\mathbf{H}^{\alpha\beta} = \mathbf{T}^{\alpha\beta} + \mathbf{V},$$

where $\mathbf{V}$ denotes a dilatation-stabilizing or general doubly-isotropic potential, and $\mathbf{T}^{\alpha\beta}$ denotes the kinetic energy operator. It is one of the previous ones restricted to the corresponding (α, β)-subspace.

Therefore, for the affine-affine, metric-affine, and affine-metric models we have, respectively,

$$\mathbf{T}^{\alpha\beta}_{\text{aff}-\text{aff}} f^{\alpha\beta} = - \frac{\hbar^2}{2A}\mathbf{D}_\lambda f^{\alpha\beta} + \frac{1}{32A}\sum_{a,b} \frac{\left(\overleftarrow{S^\beta}{}^a{}_b - \overrightarrow{S^\alpha}{}^a{}_b\right)^2}{\operatorname{sh}^2 \frac{q^a - q^b}{2}} f^{\alpha\beta}$$

$$- \quad \frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^{a}{}_{b}+\overrightarrow{S^{\alpha}}{}^{a}{}_{b}\right)^{2}}{\mathrm{ch}^{2}\frac{q^{a}-q^{b}}{2}}f^{\alpha\beta}+\frac{\hbar^{2}B}{2A(A+nB)}\frac{\partial^{2}}{\partial q^{2}}f^{\alpha\beta},$$

$$\left\{\begin{array}{c}\mathbf{T}^{\alpha\beta}_{\mathrm{met-aff}}\\ \mathbf{T}^{\alpha\beta}_{\mathrm{aff-met}}\end{array}\right\}f^{\alpha\beta} = \mathbf{T}^{\alpha\beta}_{\mathrm{aff-aff}}\left[A\mapsto I+A\right]f^{\alpha\beta}+\frac{1}{2\mu}\left\{\begin{array}{c}C(\alpha,2)\\ C(\beta,2)\end{array}\right\}f^{\alpha\beta},$$

where $C(\alpha,2)$ and $C(\beta,2)$ are the α-th and β-th eigenvalues of the Casimirs $\|\mathbf{S}\|^2$ and $\|\mathbf{V}\|^2$, respectively, Obviously, for the physical dimension $n=3$, we have that $f^{\alpha\beta}=f^{sj}$ and $C(s,2)=\hbar^2 s(s+1)$, $C(j,2)=\hbar^2 j(j+1)$.

It is so as if the doubly affine background $\mathbf{T}^{\alpha\beta}_{\mathrm{aff-aff}}$, i.e., the kinetic energy affinely-invariant both in the physical and material space, was responsible for some fundamental part of the spectra, perturbated by some internal rotations of the body itself or of the deformation axes. This perturbation and the resulting splitting of energy levels becomes remarkable when μ is small, i.e., when the inertial constants I, A differ slightly. The suggestive terms

$$\frac{\hbar^2}{2\mu}s(s+1), \qquad \frac{\hbar^2}{2\mu}j(j+1)$$

as contributions to energy levels are very interesting and seem to be supported by experimental data in various ranges of physical phenomena.

Let us quote the corresponding form of $\mathbf{T}^{\alpha\beta}_{\mathrm{d'A}}$ for the quantized d'Alembert model:

$$\mathbf{T}^{\alpha\beta}_{\mathrm{d'A}}f^{\alpha\beta}=-\frac{\hbar^{2}}{2I}\mathbf{D}_{l}f^{\alpha\beta}+\frac{1}{8I}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^{a}{}_{b}-\overrightarrow{S^{\alpha}}{}^{a}{}_{b}\right)^{2}}{\left(Q^{a}-Q^{b}\right)^{2}}f^{\alpha\beta}+\frac{1}{8I}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^{a}{}_{b}+\overrightarrow{S^{\alpha}}{}^{a}{}_{b}\right)^{2}}{\left(Q^{a}+Q^{b}\right)^{2}}f^{\alpha\beta}.$$

Finally, for affine systems on $\mathrm{U}(n)$ we have the following expressions:

$$\mathbf{T}^{\alpha\beta}_{\mathrm{aff-aff}}f^{\alpha\beta} = -\frac{\hbar^{2}}{2A}\mathbf{D}_{U}f^{\alpha\beta}+\frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^{a}{}_{b}-\overrightarrow{S^{\alpha}}{}^{a}{}_{b}\right)^{2}}{\sin^{2}\frac{q^{a}-q^{b}}{2}}f^{\alpha\beta}$$

$$+ \quad \frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S^{\beta}}{}^{a}{}_{b}+\overrightarrow{S^{\alpha}}{}^{a}{}_{b}\right)^{2}}{\cos^{2}\frac{q^{a}-q^{b}}{2}}f^{\alpha\beta}+\frac{\hbar^{2}B}{2A(A+nB)}\frac{\partial^{2}}{\partial q^{2}}f^{\alpha\beta},$$

$$\left\{\begin{array}{c}\mathbf{T}^{\alpha\beta}_{\mathrm{met-aff}}\\ \mathbf{T}^{\alpha\beta}_{\mathrm{aff-met}}\end{array}\right\}f^{\alpha\beta} = \mathbf{T}^{\alpha\beta}_{\mathrm{aff-aff}}\left[A\mapsto I+A\right]f^{\alpha\beta}+\frac{1}{2\mu}\left\{\begin{array}{c}C(\alpha,2)\\ C(\beta,2)\end{array}\right\}f^{\alpha\beta}.$$

In this way the problem has been successfully reduced from n^2 internal degrees of freedom (physically 9, sometimes 4) to the n purely deformative degrees of freedom (physically 3, sometimes 2). The price one pays for that is the use of multi-component wave functions subject to the strange parastatistical conditions in the reduced q^a-variables. The particular values of labels α, β and the corresponding matrices $S^{\alpha a}{}_b$, $S^{\beta a}{}_b$ describe the influence of quantized rotational degrees of freedom on the quantized dynamics of deformation invariants. It is interesting that on the classical level there is no simple way to perform such a dynamical reduction to the deformation invariants.

3.2.8 Three-dimensional physical case

Let us now just concentrate on the physical case $n = 3$. In our highly-symmetric geodetic models the projections of spin onto some space-fixed z-axis and the projection of vorticity onto some body-fixed z'-axis are good quantum numbers. Our wave functions may be expanded as follows:

$$\Psi(u, q, v) = \sum_{s,j} \sum_{m,n=-s}^{s} \sum_{k,l=-j}^{j} \mathcal{D}^{s}_{mn}(u) f^{sj}_{\substack{nk\\ml}}(q) \mathcal{D}^{j}_{kl}\left(v^{-1}\right), \tag{3.39}$$

where $u, v \in \mathrm{SU}(2)$, and q is here an abbreviation for (q^1, q^2, q^3), (m, n) and (k, l) run over the range from $-s$ to s and $-j$ to j, respectively, in integer steps, whereas s, j are non-negative integers starting from 0 or positive half-integers starting from 1/2. But, just as in rigid-body mechanics, there is a superselection rule. Namely, if $|\Psi|^2$ is to be one-valued, then either s, j must be simultaneously half-integer or simultaneously integer. The reduced invariant-dependent amplitudes $f^{sj}\left(q^1, q^2, q^3\right)$ vanish in the mixed case, i.e., if $(j - s)$ is half-integer. For the case of degenerate triplets (q^1, q^2, q^3) the f-amplitudes must be chosen in such a way as not to distinguish triplets (u, q, v) describing equivalent configurations.

When there is no external potential, i.e., in purely geodetic models, it is convenient to restrict ourselves to expansions with fixed values of s, j, m, and l:

$$\Psi^{sj}_{ml}(u, q, v) = \sum_{n=-s}^{s} \sum_{k=-j}^{j} D^{s}_{mn}(u) f^{sj}_{nk}(q) D^{j}_{kl}\left(v^{-1}\right). \tag{3.40}$$

Everything said above applies to $\mathrm{SL}(3, \mathbb{R})$-geodetic situations when dilatations are stabilized with the help of some potential $V_{\mathrm{dil}}(q)$ (now $q = (q^1 + q^2 + q^3)/3$), or even to more general non-geodetic situations when the potential energy is non-trivial but depends only on deformation invariants, $V_{\mathrm{dil}}(q^1, q^2, q^3)$.

For fixed s, j the reduced amplitude f^{sj} is a q^i-dependent $(2s + 1) \times (2j + 1)$ matrix. It satisfies the family of reduced Schrödinger eigenequations for energy levels, i.e., $\mathbf{H}^{sj} f^{sj} = E^{sj} f^{sj}$. For our dynamical affine models we have that

$$\begin{aligned}
\mathbf{H}^{sj}_{\mathrm{aff-aff}} f^{sj} &= -\frac{\hbar^2}{2A}\mathbf{D}_\lambda f^{sj} + \frac{\hbar^2 B}{2A(A+nB)}\frac{\partial^2}{\partial q^2} f^{sj} \\
&+ \frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S}^{j}_{ab} - \overrightarrow{S}^{s}_{ab}\right)^2}{\mathrm{sh}^2\frac{\left(q^a - q^b\right)}{2}} f^{sj} \\
&- \frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S}^{j}_{ab} + \overrightarrow{S}^{s}_{ab}\right)^2}{\mathrm{ch}^2\frac{\left(q^a - q^b\right)}{2}} f^{sj} + \mathbf{V}_{\mathrm{dil}} f^{sj}, \\
\left\{\begin{array}{c} \mathbf{H}^{sj}_{\mathrm{met-aff}} \\ \mathbf{H}^{sj}_{\mathrm{aff-met}} \end{array}\right\} f^{sj} &= \mathbf{H}^{sj}_{\mathrm{aff-aff}}\left[A \mapsto I + A\right] f^{sj} + \frac{\hbar^2}{2\mu}\left\{\begin{array}{c} s(s+1) \\ j(j+1) \end{array}\right\} f^{sj},
\end{aligned}$$

where $\mathbf{V}_{\mathrm{dil}}$ denotes the dilatations-stabilizing potential. The structure of equations does not change when, besides of $q = (q^1 + \cdots + q^n)/n$, $\mathbf{V}_{\mathrm{dil}}$ depends also on other deformation invariants (eigenvalues of deformation tensors). The terms with ch^2- and sh^2-denominators describe respectively the effective attraction (acting even without any potential term) and repulsion.

Remark: sometimes it is convenient to eliminate the first-order differential operators from the expression for $\mathbf{D}_\lambda$, then we use the modified deformation amplitude:

$$\phi = \sqrt{P_\lambda} f.$$

The resulting Schrödinger equation is analogous to the above one with the difference that $\mathbf{D}_\lambda$ is replaced by the following expression:

$$-\frac{\hbar}{2m}\frac{1}{P_\lambda^2} + \frac{\hbar^2}{4m}\frac{1}{P_\lambda}\sum_a \left(\frac{\partial P_\lambda}{\partial q^a}\right)^2 + \sum_a \frac{\partial^2}{\partial (q^a)^2},$$

i.e., the usual $\mathbb{R}^n$-Laplacean modified by some extra auxiliary "potential" term. The scalar product representation is then also modified in an appropriate way.

The simplest possible situation is when $s = j = 0$, i.e., purely scalar amplitude f^{00}. Then the Hamilton operator reduces to

$$-\frac{\hbar^2}{2\alpha}\mathbf{D}_\lambda f^{00} - \frac{\hbar^2}{2\beta}\frac{\partial^2 f^{00}}{\partial q^2} + \mathbf{V}_{\mathrm{dil}} f^{00}.$$

If we admit half-integers, then the next simple situation is $s = j = 1/2$. Then

$$S^{1/2}{}_a = \frac{\hbar}{2}\sigma_a,$$

i.e., spin and vorticity are represented by Pauli matrices σ_a multiplied by $\hbar/2$. Therefore,

$$\left(S^{1/2}{}_a\right)^2 = \frac{\hbar^2}{4}\mathbb{I}_2,$$

where $\mathbb{I}_2$ is the unit 2×2 matrix. Obviously, in two dimensions, when the covering kernel is isomorphic with $\mathbb{Z}$, there is no half-integer angular momentum.

Let us recall that in the exceptional case $n = 3$ the bi-indices (a, b) may be replaced by the dual indices c, where $a \neq c \neq b$, namely,

$$S^j_a = \frac{1}{2}\epsilon_{abc}S^j_{bc}, \qquad S^j_{ab} = \epsilon_{abc}S^j_c.$$

Then after some calculations it may be shown that two terms in $\mathbf{H}^{sj}$ controlled by the factor $1/32\alpha$ has the following explicit form:

$$\begin{aligned} &\frac{(S^s_1)^2 f^{sj} - 2S^s_1 f^{sj} S^j_1 + f^{sj}(S^j_1)^2}{16\alpha\ \mathrm{sh}^2\left[\left(q^2 - q^3\right)/2\right]} + \frac{(S^s_2)^2 f^{sj} - 2S^s_2 f^{sj} S^j_2 + f^{sj}(S^j_2)^2}{16\alpha\ \mathrm{sh}^2\left[\left(q^1 - q^3\right)/2\right]} \\ +&\frac{(S^s_3)^2 f^{sj} - 2S^s_3 f^{sj} S^j_3 + f^{sj}(S^j_3)^2}{16\alpha\ \mathrm{sh}^2\left[\left(q^1 - q^2\right)/2\right]} - \frac{(S^s_1)^2 f^{sj} + 2S^s_1 f^{sj} S^j_1 + f^{sj}(S^j_1)^2}{16\alpha\ \mathrm{ch}^2\left[\left(q^2 - q^3\right)/2\right]} \\ -&\frac{(S^s_2)^2 f^{sj} + 2S^s_2 f^{sj} S^j_2 + f^{sj}(S^j_2)^2}{16\alpha\ \mathrm{ch}^2\left[\left(q^1 - q^3\right)/2\right]} - \frac{(S^s_3)^2 f^{sj} + 2S^s_3 f^{sj} S^j_3 + f^{sj}(S^j_3)^2}{16\alpha\ \mathrm{ch}^2\left[\left(q^1 - q^2\right)/2\right]}. \end{aligned}$$

Depending on the relationship between s and j, the $\mathrm{SL}(3,\mathbb{R})$-geodetic spectrum is discrete (bounded states) or continuous. The same is true for the total $\mathrm{GL}(3,\mathbb{R})$-dynamics, when an appropriate dilatations-stabilizing potential is used. Standard terms $(\hbar^2/2\mu)s(s+1)$ and $(\hbar^2/2\mu)j(j+1)$ appearing, respectively, in the metric-affine and affine-metric models as some corrections to the standard affine-affine spectrum are very interesting. They seem to be confirmed by experimental data concerning nuclear and hadronic energetic (mass) spectra. The controlling quantum numbers s and j have to do with spin and probably isospin properties.

The doubly-isotropic d'Alembert model in three dimensions has the following form:

$$\begin{aligned} \mathbf{T}^{sj}_{\mathrm{d'A}} f^{sj} = &- \frac{\hbar^2}{2I}\mathbf{D}_l f^{sj} + \frac{1}{4I}\sum_{a=1}^{3}\varepsilon^a{}_{bc}\frac{(S^s{}_a)^2 f^{sj} - 2S^s{}_a f^{sj} S^j{}_a + f^{sj}(S^j{}_a)^2}{(Q^b - Q^c)^2} \\ &+ \frac{1}{4I}\sum_{a=1}^{3}\varepsilon^a{}_{bc}\frac{(S^s{}_a)^2 f^{sj} + 2S^s{}_a f^{sj} S^j{}_a + f^{sj}(S^j{}_a)^2}{(Q^b + Q^c)^2}. \end{aligned}$$

3.2.9 Planar geodetic case

Finally, let us briefly describe the two-dimensional situation, i.e., "Flatland" [1], which is also of some physical interest. Obviously, it may have some direct physical applications when we deal with flat molecules or other structural elements. Besides of it, the two-dimensional models shed some light on the general situation and enable one to make it more comprehensible and lucid. There are some very exceptional features of the dimension $n = 2$. They are very peculiar, in a sense pathological. But nevertheless the resulting simplifications generate some ideas and hypotheses concerning the general dimension. Of course, later on they must be verified on the independent basis.

The one-dimensional group of planar rotations $\mathrm{SO}(2,\mathbb{R})$ is Abelian, therefore, $\widehat{\rho} = \rho = S$, $\widehat{\tau} = \tau = -\widehat{V}$. In the doubly-isotropic models S and $\widehat{V}$ are constants of motion and so are in the two-dimensional case $\widehat{\rho}$, $\widehat{\tau}$, M, N. It is not the case for $n > 2$, where, as always in isotropic models, S, $\widehat{V}$ are constants of motion but $\widehat{\rho}$, $\widehat{\tau}$ do not equal S, $-\widehat{V}$ and are non-constant. But it is exactly the use of $\widehat{\rho}$, $\widehat{\tau}$ and their combinations M, N that simplifies the problem and leads to a partial separation of variables. In the two-dimensional space these things coincide and the problem may be effectively reduced to the dynamics of two-deformation invariants both on the classical and quantum level.

Therefore, the two-polar decomposition $\varphi = LDR^{-1}$ may be parameterized in the standard way, i.e.,

$$D = \begin{bmatrix} Q^1 & 0 \\ 0 & Q^2 \end{bmatrix} = \begin{bmatrix} \exp\left(q^1\right) & 0 \\ 0 & \exp\left(q^2\right) \end{bmatrix},$$

$$L = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \qquad R = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix},$$

$$S = p_\alpha \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \widehat{V} = p_\beta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where p_α and p_β are canonical momenta conjugate to α and β, respectively. It is convenient to introduce the following new variables:

$$q := \frac{q^1 + q^2}{2}, \qquad x := q^2 - q^1,$$

their conjugate momenta are, respectively,

$$p = p_1 + p_2, \qquad y = \frac{p_2 - p_1}{2}.$$

The Haar measure on $\mathrm{GL}(2,\mathbb{R})$ is given by the following expression:

$$d\lambda\left(\alpha; q^1, q^2; \beta\right) = \left|\mathrm{sh}\left(q^1 - q^2\right)\right| d\alpha d\beta dq^1 dq^2,$$

i.e.,

$$d\lambda(\alpha; q, x; \beta) = |\text{sh } x|\, d\alpha d\beta dx dq.$$

The Fourier expansion of wave functions with respect to α and β is as follows:

$$\Psi(\alpha; q, x; \beta) = \sum_{m,n\in\mathbf{Z}} f^{mn}(q, x) e^{im\alpha} e^{in\beta}.$$

The reduced Hamiltonians corresponding to our dynamical affine models are as follows:

$$\begin{aligned}
\mathbf{H}^{mn}_{\text{aff}-\text{aff}} f^{mn} &= -\frac{\hbar^2}{2A}\mathbf{D}_\lambda f^{mn} + \frac{\hbar^2 B}{2A(A+2B)}\frac{\partial^2}{\partial q^2} f^{mn} \\
&+ \frac{\hbar^2 (n-m)^2}{16A\ \text{sh}^2(x/2)} f^{mn} - \frac{\hbar^2 (n+m)^2}{16A\ \text{ch}^2(x/2)} f^{mn} + \mathbf{V}_{\text{dil}}(q) f^{mn}, \\
\left\{ \begin{array}{c} \mathbf{H}^{mn}_{\text{met}-\text{aff}} \\ \mathbf{H}^{mn}_{\text{aff}-\text{met}} \end{array} \right\} f^{mn} &= \mathbf{H}^{mn}_{\text{aff}-\text{aff}} \left[A \mapsto I + A\right] f^{mn} + \frac{\hbar^2}{2\mu}\left\{ \begin{array}{c} m^2 \\ n^2 \end{array} \right\} f^{mn},
\end{aligned}$$

where

$$\mathbf{D}_\lambda = \frac{1}{|\text{sh } x|}\frac{\partial}{\partial x}\left(|\text{sh } x| \frac{\partial}{\partial x} \right).$$

As for purely incompressible motion, there exist both bounded and continuous spectra depending on the relationship between quantum numbers n, m, i.e., for the x-sector of the above operators there exists discrete spectrum if $|n+m| > |n-m|$, i.e., if $mn > 0$.

Finally, let us quote the corresponding formulas for the quantized doubly-isotropic d'Alembert model. In the two-polar coordinates the Lebesgue measure element is given by the following expression:

$$dl\left(\alpha; Q^1, Q^2; \beta\right) = P_l\left(Q^1, Q^2\right) d\alpha d\beta dQ^1 dQ^2,$$

where

$$P_l = \left| \left(Q^1\right)^2 - \left(Q^2\right)^2 \right| = \left| \left(Q^1 + Q^2\right)\left(Q^1 - Q^2\right) \right|.$$

The reduced amplitudes f^{mn} satisfy the eigenequations

$$\mathbf{H}^{mn}_{\text{d'A}} f^{mn} = \mathbf{T}^{mn}_{\text{d'A}} f^{mn} + \mathbf{V}\left(Q^1, Q^2\right) f^{mn} = E^{mn} f^{mn} \tag{3.41}$$

with

$$\mathbf{T}^{mn}_{\text{d'A}} f^{mn} = -\frac{\hbar^2}{2I}\mathbf{D}_l f^{mn} + \frac{\hbar^2 m^2}{4I\left(Q^1 - Q^2\right)^2} f^{mn} + \frac{\hbar^2 n^2}{4I\left(Q^1 + Q^2\right)^2} f^{mn},$$

where

$$\mathbf{D}_l = \frac{1}{P_l}\frac{\partial}{\partial Q^1}\left(P_l \frac{\partial}{\partial Q^1}\right) + \frac{1}{P_l}\frac{\partial}{\partial Q^2}\left(P_l \frac{\partial}{\partial Q^2}\right).$$

Chapter 4

Internal symmetries in field theories

One of the main subjects of this thesis is the analysis of models of collective and internal degrees of freedom. We have started from mechanical problems and the special stress was laid on affine model of collective and internal modes. Incidentally, the distinction between collective and internal modes of extended (in particular, continuous) mechanical objects is not always based on some principal physics. Quite often, when dealing with very small objects we have no experimental abilities to get precisely into their structure and just for simplicity we describe as essentially internal motion something that later on, on the basis of better experimental abilities, turns out to be the relative motion in an extended system. Usually the common mathematical feature is some geometric basis of collective and internal modes. It is typical that kinematics is ruled by groups somehow related to the geometry of the physical space and some other spaces appearing in the theory. It is so in mechanics of rigid and affinely-rigid bodies; as it has already been mentioned, the projective and conformal groups are also interesting from this point of view. There are some more delicate points concerning the dynamics. In rigid-body mechanics, the geodetic dynamical models invariant under the full kinematical group are often used. In traditional approaches to the affine modes (Bogoyavlenski, Eringen, etc. [13, 40, 41, 42, 43]) it is only kinematics, but not dynamics, that is affinely invariant. Unlike this, we constructed dynamically affinely-invariant geodetic models. The idea that the kinematical and dynamical groups coincide on the fundamental level is physically very promising. It is reasonable to expect that something like the restriction of the kinematical group to some proper dynamical subgroup appears due to some mechanism like the spontaneous symmetry breaking, when analyzing solutions close to some background solution, but not on the level of fundamental laws [14, 193, 194].

This is just the procedure known from the modern theory of fundamental physical fields. And as a matter of fact, there are many common features between this theory and mechanics of extended mechanical systems. Fundamental fields have internal degrees of freedom ruled by various orthogonal, unitary, Lorentz, conformal, and affine groups. The conformal group, used in the relativistic field theory, is distinguished mathematically by many properties. It is the largest group preserving light cones, at the same time it is the smallest semisimple group containing Poincaré group. The non-relativistic counterpart of this conformal group is often applied in mechanics of collective mechanical modes. This was the reason why we discuss here some problems connected with field theories ruled by the pseudo-unitary group $\mathrm{SU}(2,2)$, i.e., the universal covering group of the conformal group. We hope that the obtained results may be applicable in condensed matter theory either, e.g., as a basis of some generalized approaches to superfluidity.

The linear groups $\mathrm{GL}(3,\mathbb{R})$, $\mathrm{GL}(4,\mathbb{R})$ were suggested by Ne'eman, Hehl, and others [60, 59, 65, 150, 152, 157, 159, 164] as invariance groups underlying some hypothetic fundamental physics. And there is a link between fundamental physics and mechanics of generalized microstructured, more precisely, micromorphic continua. One can formulate field theories for the fields of linear frames (aholonomic frames) in a manifold, invariant under the internal $\mathrm{GL}(3,\mathbb{R})$ or $\mathrm{GL}(4,\mathbb{R})$ symmetry and the "external" group of spatial diffeomorphisms (general covariance). And this is at the same time a candidate for some new fundamental field theory, but also a completely new model of the relativistic micromorphic continua [147, 164]. Such models are mathematically interesting and at the same time physically useful, e.g., in the defect theory in solids [189]. Roughly speaking, they are micromorphic continua with affinely invariant dynamics (not only kinematics).

4.1 $\mathrm{U}(2,2)$-invariant spinorial geometrodynamics

4.1.1 Standard generally-relativistic Dirac theory

Generally-relativistic Dirac theory deals with a triple of mutually interacting objects [156, 157, 164], i.e., the bispinor matter wave Ψ^r and two geometrodynamical quantities, namely, the tetrad field $e^\mu{}_A$ and the $\mathrm{SL}(2,\mathbb{C})$-ruled bispinor connection

$\omega^r{}_{s\mu}$, which defines the following covariant differentiation of bispinors:

$$D_\mu \Psi^r = \partial_\mu \Psi^r + \omega^r{}_{s\mu} \Psi^s.$$

The target spaces of e and Ψ, i.e., $\mathbb{R}^4$ and $\mathbb{C}^4$, are endowed with the following geometric structures:

- $\mathbb{R}^4$ is a Minkowskian space with the scalar product η, which has the following analytical form:
$$\eta_{AB} = \mathrm{diag}(1, -1, -1, -1).$$
The tetrad field e and the internal metric η define the metric tensor g on the space-time manifold $\mathcal{M}$, i.e.,
$$g_{\mu\nu} := \eta_{AB} e^A{}_\mu e^B{}_\nu.$$

- in $\mathbb{C}^4$ we fix a neutral-signature Hermitian form G, i.e.,
$$G_{\bar{r}s} = \mathrm{diag}(1, 1, -1, -1),$$
which defines the Dirac conjugation of bispinors:
$$\widetilde{\Psi}_r := \overline{\Psi}^{\bar{s}} G_{\bar{s}r}.$$

Within the matrix algebra $\mathrm{L}(4, \mathbb{C})$ we fix a quadruplet of G-Hermitian Dirac matrices γ^A satisfying the Clifford anticommutation rules, i.e.,

$$\gamma^A \gamma^B + \gamma^B \gamma^A = 2\eta^{AB} \mathbb{I}_4.$$

The pair "tetrad field-bispinor connection" (e, ω) induces the Einstein-Cartan affine connection:

$$\Gamma^\lambda{}_{\mu\nu} := e^\lambda{}_A \Gamma^A{}_{B\nu} e^B{}_\mu + e^\lambda{}_A e^A{}_{\mu,\nu},$$

where $e^\lambda{}_A$ are components of the dual cotetrad field, i.e., $e^A{}_\mu e^\mu{}_B = \delta^A{}_B$,

$$\Gamma^A{}_{B\mu} := \frac{1}{2} \mathrm{Tr} \left(\gamma^A \omega_\mu \gamma_B \right),$$

and the shifting of Greek and capital Latin indices is meant respectively in the g- and η-sense.

Then, the matter Lagrangian is given by the following expression:

$$\mathcal{L}_{\mathrm{mat}} \left(\Psi, e, \omega \right) = \frac{i}{2} e^\mu{}_A \gamma^{Ar}{}_s \left(\widetilde{\Psi}_r D_\mu \Psi^s - D_\mu \widetilde{\Psi}_r \Psi^s \right) \sqrt{|g|} - m \widetilde{\Psi}_r \Psi^r \sqrt{|g|}. \tag{4.1}$$

The quantities Ψ^r, $e^A{}_\mu$, and $\Gamma^A{}_{B\mu}$ (or, equivalently, $\omega^r{}_{s\mu}$) are independent dynamical variables of the theory. The wave field Ψ belongs to the material sector, whereas the pair (e, ω) describe the geometrodynamical degrees of freedom. A few choices of geometrodynamical Lagrangians are logically consistent and compatible with experimental data. The simplest of them, used in Einstein-Cartan theory, is proportional to the curvature scalar $R(\Gamma, g)$ built of Γ and g, i.e.,

$$\mathcal{L}^{\mathrm{EC}}_{\mathrm{geom}}(e, \omega) = \frac{1}{k} g^{\mu\nu} R(\Gamma)^\alpha{}_{\mu\alpha\nu} \sqrt{|g|}. \tag{4.2}$$

The cosmological term may be also added to the geometrodynamical Lagrangian, i.e.,

$$\mathcal{L}^{\mathrm{cosm}}_{\mathrm{geom}}(e, \omega) = \Lambda \sqrt{|g|}. \tag{4.3}$$

There are also some more sophisticated models, e.g., admitting the Yang-Mills terms quadratic in curvature, i.e.,

$$\mathcal{L}^{\mathrm{YM}}_{\mathrm{geom}}(e, \omega) = \frac{1}{l} R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\kappa\lambda} g^{\mu\kappa} g^{\nu\lambda} \sqrt{|g|}, \tag{4.4}$$

or algebraic Weitzenböck terms quadratic in torsion $S^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{[\mu\nu]}$, i.e.,

$$\begin{aligned} \mathcal{L}^{\mathrm{W}}_{\mathrm{geom}}(e, \omega) &= A g_{\alpha\beta} S^\alpha{}_{\mu\nu} S^\beta{}_{\kappa\lambda} g^{\mu\kappa} g^{\nu\lambda} \sqrt{|g|} \\ &+ B g_{\mu\nu} S^\alpha{}_{\beta\mu} S^\beta{}_{\alpha\nu} \sqrt{|g|} + C g_{\mu\nu} S^\alpha{}_{\alpha\mu} S^\beta{}_{\beta\nu} \sqrt{|g|}. \end{aligned} \tag{4.5}$$

In the above formulas k, Λ, l, A, B, and C are some constants.

The total Lagrangian consists of some (or all) of the above-described terms:

$$\mathcal{L}(\Psi, e, \omega) = \mathcal{L}_{\mathrm{mat}}(\Psi, e, \omega) + \mathcal{L}_{\mathrm{geom}}(e, \omega).$$

This scheme is a kind of gauge theory. The matter field Ψ^r is a cross-section of an associate bundle with the standard fibre $\mathbb{C}^4$, the principal bundle is ruled by $\mathrm{SL}(2, \mathbb{C})$, i.e., the covering group of $\mathrm{SO}(1,3)^\uparrow$, the cotetrad field $e^A{}_\mu$, or rather its $2:1$ spinorial covering field, is a reference frame, i.e., a cross-section of the corresponding principal bundle, $\omega^r{}_{s\mu}$ is a connection form on the principal bundle, the connection form $\Gamma^A{}_{B\mu}$ takes values in the algebras $\mathrm{SO}(1,3)' \simeq \mathrm{SL}(2,\mathbb{C})' \simeq$. This theory obeys the general rules of well-established gauge schemes, and in the specially-relativistic approximation, when the tetrad is holonomic and the metric flat, i.e., $e^A{}_{\mu,\nu} - e^A{}_{\nu,\mu} = 0$ and $R^\alpha{}_{\beta\mu\nu} = 0$, it is perfectly confirmed by experimental data. Nevertheless, it evokes some principal objections:

- the tetrad filed e enters the matter Lagrangian through the differential one-form $e^{r}{}_{s\mu} := g_{\mu\nu} e^{\nu}{}_{A} \gamma^{Ar}{}_{s}$ with values in the $\mathbb{R}$-linear span of Dirac matrices. This linear subspace of the space of all G-Hermitian matrices is fixed once for all and used as the value-space of $e^{r}{}_{s\mu}$ at all space-time points. This is a global, rigidly-fixed structure that drastically violates the local paradigm of gauge theories. In a sense, it is an action-at-distance concept. It would be much more compatible with the local philosophy of gauge theories if we admit the linear mappings $e^{r}{}_{s\mu}(x)$ to be general injections of $T_x\mathcal{M}$ into the space of G-Hermitian operators in $\mathbb{C}^4$.

- in typical gauge theories of fundamental interactions (Salam-Weinberg model, chromodynamics, and so on) the reference frame never occurs explicitly as a dynamical variable. Field equations are imposed on associate bundle objects (matter) and connections in principal bundles (interaction). Unlike this, in spinor theory the tetrad field is an important dynamical variable from the gravitational sector, i.e., we cannot avoid them when constructing Lagrangians. In this situation there appears a temptation to modify the theory in such a way as to turn the cotetrad into a gauge field of some kind.

- the explicit use of the internal metric G in the construction of Lagrangian suggests that it is rather the total pseudo-unitary group $\mathrm{U}(4, G) \simeq \mathrm{U}(2, 2)$ than its injected subgroup $\mathrm{SL}(2, \mathbb{C})$ that should be used as a proper group of physical symmetries. In fact, $\mathrm{SU}(2, 2)$ is used for a long time in conformal field theory and twistor models because it is the covering group of the conformal group $\mathrm{Co}(1, 3)$ of the Minkowskian space $(\mathbb{R}^4, \eta)$. However, without serious and complicated modifications this approach is applicable only to the massless particles in the Minkowskian space-time. Moreover, although in this treatment field equations are invariant under $\mathrm{SU}(2, 2)$ combined with the conformal action on the wave function's argument, but the Lagrangian itself is not invariant. Thus, the resulting symmetries are non-Noetherian and do not lead to conservation laws.

- there is something mysterious in the reliance of the Lagrangian (4.1) upon the vector densities
$$J^{r}{}_{s\mu} := \left(D_\mu \widetilde{\Psi}_s \Psi^r - \widetilde{\Psi}_s D_\mu \Psi^r \right) \sqrt{|g|},$$
which structurally remind the typical bosonic currents implied by the Noether

theorem. To interpret ${J^r}_{s\mu}$ in such a way we must postulate some symmetry group and an appropriate Lagrangian. The algebraic structure of ${J^r}_{s\mu}$ suggests the group $\mathrm{U}(2,2)$ and the Klein-Gordon Lagrangian for Ψ with G used as an internal metric.

4.1.2 Second-order model with internal $\mathbf{U}(2,2)$ symmetry

Let us consider a generally-relativistic system [156, 157] which consists of the normal-hyperbolic metric field $g_{\mu\nu}$ and the quadruplet of complex scalar fields Ψ^r defined on the four-dimensional space-time manifold $\mathcal{M}$, which is not endowed with any absolute geometry except the very differential structure. On the contrary, the target space $\mathbb{C}^4$ is endowed with the absolute geometry based on a fixed Hermitian form G with the neutral signature $(++--)$. The second target space, i.e., the algebra $\mathrm{L}(4,\mathbb{C})$ of complex matrices appears as the faithful irreducible realization of the complexified Clifford algebra for the standard Minkowskian space $(\mathbb{R}^4,\eta)$, where $\eta_{AB} = \mathrm{diag}(1,-1,-1,-1)$. The algebra $\mathrm{L}(4,\mathbb{C})$ and the complex group $\mathrm{GL}(4,\mathbb{C}) \subset \mathrm{L}(4,\mathbb{C})$ provide a general framework for describing internal symmetries. The G-shift of indices enables one to construct the Dirac conjugation $\widetilde{\Psi}_r := \overline{\Psi}^{\bar{s}} G_{\bar{s}r}$, which is an antilinear isomorphism of $\mathbb{C}^4$ onto its dual $\mathbb{C}^{4*} \simeq \mathbb{C}^4$. The scalar product $G(u,v) = G_{\bar{r}s}\bar{u}^{\bar{r}}v^s$ gives rise to the pseudo-unitary group $\mathrm{U}(4,G) \simeq \mathrm{U}(2,2) \subset \mathrm{GL}(4,\mathbb{C})$. The corresponding Lie algebra $\mathrm{U}(4,G)' \simeq \mathrm{U}(2,2)' \subset \mathrm{L}(4,\mathbb{C})$ consists of matrices A which are G-anti-Hermitian, i.e., they satisfy $G(Au,v) = -G(u,Av)$ for any $u,v \in \mathbb{C}^4$. The imaginary unit multiple $i\mathrm{U}(2,2)'$ of $\mathrm{U}(2,2)$ consists of G-Hermitian matrices, in particular, Dirac matrices belong to this class.

Hence, we have three kinds of independent dynamical variables: the matter wave amplitude $\Psi^r(x^\mu)$, $\Psi : \mathcal{M} \to \mathbb{C}^4$, the normal-hyperbolic metric tensor $g_{\mu\nu}$, and the $\mathrm{U}(2,2)$-ruled connection ${\vartheta^r}_{s\mu}(x)$ on $\mathcal{M}$, locally represented as a $\mathrm{U}(2,2)'$-valued differential one-form, i.e.,

$$\mathcal{M} \ni x \mapsto \vartheta_x \in \mathrm{L}\left(T_x\mathcal{M}, \mathrm{U}(4,G)'\right).$$

The geometrodynamical sector is described by two field quantities g and ϑ. It is important that now there is no dynamical use of tetrad, affine connection, or $\mathrm{SL}(2,\mathbb{C})$-ruled spinor connection. Instead, all these quantities appear as byproducts of ϑ after the $\mathrm{SL}(2,\mathbb{C})$-reduction procedure is performed.

Local transformations $A : \mathcal{M} \to \mathrm{U}(2,2)$ act on the field quantities according to the following standard rules:

$$Ag = g, \tag{4.6}$$

$$(A\Psi)(x) = A(x)\Psi(x), \tag{4.7}$$

$$(A\vartheta)_x = A(x)\vartheta_x A^{-1}(x) - dA_x A(x)^{-1}. \tag{4.8}$$

Covariant differentiation of wave amplitudes is defined as follows:

$$\nabla_\mu \Psi = \partial_\mu \Psi + \rho \vartheta_\mu \Psi + \frac{q-\rho}{4} \mathrm{Tr}\left(\vartheta_\mu\right) \Psi,$$

where the coupling constants ρ and q correspond respectively to the subgroups $\mathrm{SU}(2,2)$ and $e^{\mathbb{R}}\mathbb{I}$. The curvature two-form ϕ depends only on the "semisimple" coupling constant ρ, i.e.,

$$\phi_{\mu\nu} = \partial_\mu \vartheta_\nu - \partial_\nu \vartheta_\mu + \rho[\vartheta_\mu, \vartheta_\nu].$$

The $\mathrm{U}(2,2)$-gauge-invariant matter Lagrangian is assumed in the Klein-Gordon form:

$$\mathcal{L}_{\mathrm{mat}}\left(\Psi, \vartheta, g\right) = \frac{b}{2} g^{\mu\nu} \nabla_\mu \widetilde{\Psi} \nabla_\nu \Psi \sqrt{|g|} - \frac{c}{2} \widetilde{\Psi} \Psi \sqrt{|g|}, \tag{4.9}$$

where b, c are some constants. This is the only reasonable model locally invariant under $\mathrm{U}(2,2)$ because Dirac-like models based on first-order differential equations are incompatible with our choice of degrees of freedom (we have no tetrad or any other vector-valued differential one-form transforming under $A : \mathcal{M} \to \mathrm{U}(2,2)$ according to a homogeneous-linear rule). The gauge-invariant Noether current corresponding to the $\mathrm{U}(2,2)$-symmetry is given by the following expression:

$$J\left(\Psi, \vartheta, g\right)^r{}_{s\mu} := \frac{b}{2}\left(\Psi^r \nabla_\mu \widetilde{\Psi}_s - \nabla_\mu \Psi^r \widetilde{\Psi}_s\right) \sqrt{|g|}.$$

Just as in electrodynamics, it is algebraically equivalent to derivatives of the Lagrangian $\mathcal{L}_{\mathrm{mat}}$ with respect to the gauge potential, i.e.,

$$\frac{\partial \mathcal{L}_{\mathrm{mat}}\left(\Psi, \vartheta, g\right)}{\partial \vartheta^r{}_{s\mu}} = \rho J^s{}_r{}^\mu + \frac{q-\rho}{4} J^z{}_z{}^\mu \delta^s{}_r.$$

The most reasonable dynamical model for the connection ϑ bases on the Yang-Mills Lagrangian, i.e.,

$$\mathcal{L}_{\mathrm{YM}}\left(\vartheta, g\right) = \frac{a}{4} \mathrm{Tr}\left(\phi_{\mu\nu}\phi_{\kappa\lambda}\right) g^{\mu\kappa} g^{\nu\lambda} \sqrt{|g|} - \frac{a'}{4} \mathrm{Tr}\left(\phi_{\mu\nu}\right) \mathrm{Tr}\left(\phi_{\kappa\lambda}\right) g^{\mu\kappa} g^{\nu\lambda} \sqrt{|g|}, \tag{4.10}$$

where a and a' are some constants depending on the choice of units, they refer respectively to the subgroups $\mathrm{SU}(2,2)$ and $e^{\mathbb{R}}\mathbb{I}$ of $\mathrm{U}(2,2)$.

There are a few possibilities for the choice of the dynamical term for g. Let us quote three most natural of them:

- just as in the Palatini model, there is no separate Lagrangian for g. The total Lagrangian reduces to $\mathcal{L}_{\mathrm{mat}}(\Psi, \vartheta, g) + \mathcal{L}_{\mathrm{YM}}(\vartheta, g)$, and the metric tensor enters it in a purely algebraic way. Nevertheless, g is a dynamical variable subject to the variational procedure. Then the usual gravitational constant of Einstein theory is proportional to the inverse of a.

- we use the Hilbert-Einstein model, i.e.,

$$\mathcal{L}_{\mathrm{HE}}(g) = -lR(g)\sqrt{|g|} + \Lambda\sqrt{|g|}, \tag{4.11}$$

  where l and Λ are some constants, and $R(g)$ denotes the scalar curvature of g. Formally, the parameter Λ has the cosmological-constant status, however, with no a priori restrictions on its sign. Obviously, putting $l = 0$ and $\Lambda = 0$, we obtain the above-described Palatini-like model.

- or g may be a byproduct of something else, e.g., it may be some vector-valued differential one-form E (generalized cotetrad) on $\mathcal{M}$ which transforms under $A : \mathcal{M} \to \mathrm{U}(2,2)$ according to a homogeneous rule. Then it is reasonable to assume the Lagrangian in the quadratic form with respect to the ϑ-covariant differential of E.

Later on we do not precise any particular choice, i.e.,

$$\mathcal{L}(\Psi, \vartheta, g) = \mathcal{L}_{\mathrm{mat}}(\Psi, \vartheta, g) + \mathcal{L}_{\mathrm{YM}}(\vartheta, g) + \mathcal{L}_{\mathrm{HE}}(g).$$

The resulting Euler-Lagrange equations may be concisely written in the following form:

$$g^{\mu\nu}\nabla_{\mu}[g]\nabla_{\nu}[g]\Psi + \frac{c}{b}\Psi = 0, \tag{4.12}$$

$$\chi^{\mu\nu}{}_{;\nu} + \rho\left[\vartheta_{\nu}, \chi^{\mu\nu}\right] = \rho J^{\mu} + \frac{q-\rho}{4}\mathrm{Tr}\left(J^{\mu}\right)\mathbb{I}_4, \tag{4.13}$$

$$l\left(R(g)^{\mu\nu} - \frac{1}{2}R(g)g^{\mu\nu}\right) = -\frac{\Lambda}{2} + \frac{1}{2}T^{\mu\nu}, \tag{4.14}$$

where $\nabla_{\mu}[g]$ denotes the total covariant differentiation unifying the internal ϑ-connection (internal r-indices) an the external Levi-Civita connection $\{g\}$ (spatio-temporal μ-indices), the semicolon denotes the Levi-Civita covariant differentiation, $\chi^{\mu\nu}$ is the gauge field momentum, i.e.,

$$\chi^{\mu\nu} := \frac{\partial \mathcal{L}_{\mathrm{YM}}}{\partial \vartheta_{\mu,\nu}} = -a\phi^{\mu\nu}\sqrt{|g|} - a'\mathrm{Tr}\left(\phi^{\mu\nu}\right)\mathbb{I}_4\sqrt{|g|},$$

$R(g)^{\mu\nu}$ denotes the Ricci tensor of g, and $T^{\mu\nu}$ is the metrical energy-momentum tensor of (Ψ, ϑ), i.e.,

$$T^{\mu\nu} := -\frac{2}{\sqrt{|g|}}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}.$$

4.1.3 Correspondence with standard theory

To discuss the correspondence with the generally-covariant Dirac theory and the Einstein-Cartan-type geometrodynamics, we have to expand all internal quantities with respect to a basis adapted to an appropriate monomorphism of $\mathrm{SL}(2,\mathbb{C})$ into $\mathrm{U}(2,2)$ (this monomorphism corresponds to the standard injection of the proper Lorentz group $\mathrm{SO}(1,3)^{\uparrow}$ into the conformal group $\mathrm{Co}(1,3)$). Let γ^A, $A = \overline{0,3}$, be a quadruplet of Dirac matrices adapted to the Hermitian form G, thus, $i\gamma^A \in \mathrm{U}(2,2)'$. It is well-known that complexified Clifford algebra $\mathrm{L}(4,\mathbb{C})$ is generated by Dirac matrices. The most natural γ-adapted basis contains the following standard matrices:

$$\gamma^5 = -\gamma_5 = -\gamma^0\gamma^1\gamma^2\gamma^3, \qquad {}^A\gamma = i\gamma^A\gamma^5 = -i\gamma^5\gamma^A,$$
$$\Sigma^{AB} = \frac{1}{4}\left(\gamma^A\gamma^B - \gamma^B\gamma^A\right) = -\Sigma^{BA}.$$

The quadruplet of ${}^A\gamma$'s obeys the Clifford rules with the reversed signature $(-{+}{+}{+})$, i.e.,

$$\gamma^A\gamma^B + \gamma^B\gamma^A = -2\eta^{AB}\mathbb{I}_4.$$

The Lie algebra $\mathrm{U}(2,2)'$ is an $\mathbb{R}$-linear shell of the following matrices:

$$i\gamma^A, \qquad i{}^A\gamma, \qquad \Sigma^{AB}, \qquad i\gamma^5, \qquad i\mathbb{I}_4.$$

Hence, the connection form can be expand as follows:

$$\vartheta_\mu = \frac{1}{2\rho}\widetilde{\Gamma}^{AB}{}_\mu\Sigma_{AB} + \frac{1}{4\rho}Q_\mu\frac{1}{i}\gamma^5 + A_\mu i\mathbb{I}_4 + e^A{}_\mu i\tau_A + f_{A\mu}i\xi^A,$$

where

$$\tau_A := \frac{1}{2}\left(\gamma_A + {}_A\gamma\right), \qquad \xi_A := \frac{1}{2}\left(\gamma_A - {}_A\gamma\right).$$

If $\det\left[e^A{}_\mu\right] \neq 0$, then $e^A{}_\mu$ may play the cotetrad role of the $\mathrm{SL}(2,\mathbb{C})$ theory, $f_{A\mu}$ is an auxiliary cotetrad and both of them transform homogeneously under $\mathrm{GL}(2,\mathbb{C})$. In this case, $\widetilde{\Gamma}^{AB}{}_\mu$ are expected to be nonholonomic components of some affine Einstein-Cartan connection, Q_μ is a candidate for the Weyl covector, and finally,

$$\Gamma^A{}_{B\mu} := \widetilde{\Gamma}^A{}_{B\mu} + \frac{1}{2}Q_\mu\delta^A{}_B$$

seem to be nonholonomic components of the corresponding Einstein-Cartan-Weyl connection. With the help of cotetrads e and f we can define two natural affine connections, namely,

$$\begin{aligned}\Gamma(e)^{\lambda}{}_{\mu\nu} &:= e^{\lambda}{}_{A}\Gamma^{A}{}_{B\nu}e^{B}{}_{\mu} + e^{\lambda}{}_{A}e^{A}{}_{\mu,\nu},\\ \Gamma(f)^{\lambda}{}_{\mu\nu} &:= -f_{A\mu}\Gamma^{A}{}_{B\nu}f^{\lambda B} + f^{\lambda A}f_{A\mu,\nu},\end{aligned}$$

and two Dirac-Einstein metrics, i.e.,

$$h(e,\eta)_{\mu\nu} := \eta_{AB}e^{A}{}_{\mu}e^{B}{}_{\nu}, \qquad h(f,\eta)_{\mu\nu} := \eta^{AB}f_{A\mu}f_{B\nu}.$$

Let us start with the purely geometrodynamical sector, i.e., when $\Psi = 0$. Let us substitute into the Euler-Lagrange equation the following Dirac-Einstein Ansatz:

$$f_{A\mu} = k\eta_{AB}e^{B}{}_{\mu}, \quad g_{\mu\nu} = ph(e)_{\mu\nu}, \quad Q_{\mu} = A_{\mu} = 0, \quad S(e)^{\lambda}{}_{\mu\nu} = S(f)^{\lambda}{}_{\mu\nu} = 0,$$

where k and p are some constants, and $S(e)$, $S(f)$ are torsions respectively of $\Gamma(e)$ and $\Gamma(f)$. We obtain the beautiful compatibleness of our system of Euler-Lagrange equations with the above conditions, and then it reduces to the following form:

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = -12\frac{\rho^2 k}{p}g^{\mu\nu},$$

where $R^{\mu\nu}$ denotes the Ricci tensor built of g. But the proper correspondence with Dirac theory is attained when $k = 1$ and $p = 1$. Then we have the standard Einstein equations with a kind of cosmological term proportional to ρ^2 (the coupling constant which refers to the subgroup $\mathrm{SU}(2,2)$). This is a very important link between the microscopic quantum phenomena and the phenomena on the cosmic scale.

In the matter sector, if we substitute to the wave equation the Dirac-Einstein Ansatz with $p = k = 1$, then we obtain

$$e^{\mu}{}_{A}i\gamma^{A}\left(D_{\mu} + S^{\nu}{}_{\nu\mu}\right)\Psi - \frac{4b\rho^2 - c}{2b\rho}\Psi + \frac{1}{2\rho}g^{\mu\nu}D_{\mu}[g]D_{\nu}[g]\Psi = 0,$$

where D_{μ} is the $\mathrm{SL}(2,\mathbb{C})$-part of the $\mathrm{U}(2,2)$-covariant differentiation, and $D_{\mu}[g]$ unifies the previous differentiation with the Levi-Civita differentiation and with the $\Gamma^{A}{}_{B\mu}$-differentiation.

The first two terms perfectly correspond with the Dirac theory in the Einstein-Cartan space. The question is, however, whether the third d'Alembert term does not destroy completely this correspondence because differential equations are structurally unstable with respect to cancelling their highest-order terms.

The easiest way to answer this question is to discuss the specially-relativistic limit, i.e., when $e^{\mu}{}_{A} = \delta^{\mu}{}_{A}$, $\Gamma^{A}{}_{B\mu} = 0$, and $g_{\mu\nu} = \eta_{\mu\nu}$. Then we obtain the following Klein-Gordon-Dirac equation:

$$i\gamma^{\mu}\partial_{\mu}\Psi - \frac{4b\rho^2 - c}{2b\rho}\Psi + \frac{1}{2\rho}\partial^{\mu}\partial_{\mu}\Psi = 0. \tag{4.15}$$

4.2 Klein-Gordon-Dirac equation

So, we consider in more details the Klein-Gordon-Dirac equation (4.15), i.e., a linear differential equation with constant coefficients obtained by superposing Dirac and d'Alembert operators, which appears from the U(2, 2)-ruled gauge model of spinorial geometrodynamics in a natural and logical way. Another kind of motivation comes from a standard model of electroweak interactions with its mysterious pairing of fundamental fermions.

Let us consider the density of Klein-Gordon-Dirac Lagrangian in the following form:

$$\mathcal{L} = ug^{\mu\nu}\partial_{\mu}\overline{\Psi}\partial_{\nu}\Psi + \frac{vi}{2}\left(\overline{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - \partial_{\mu}\overline{\Psi}\gamma^{\mu}\Psi\right) - w\overline{\Psi}\Psi, \tag{4.16}$$

where $g^{\mu\nu}$ is a metric tensor, which in the specially-relativistic limit it equals $\eta^{\mu\nu}$, i.e., a flat metric tensor on a space-time manifold $\mathcal{M}$ with constant coefficients and a signature $(+, -, -, -)$, $\overline{\Psi} = \Psi^{+}\gamma^{0}$ introduces the rule of Dirac conjugation of bispinors, and u, v, w are some real constants.

Since the density of the Dirac Lagrangian

$$\mathcal{L} = \frac{i}{2}\left(\overline{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - \partial_{\mu}\overline{\Psi}\gamma^{\mu}\Psi\right) - m\overline{\Psi}\Psi$$

contains only the first order derivatives from $\overline{\Psi}$ and Ψ, the action $S = \int \mathcal{L}d^4x$ can have neither minimum nor maximum and the principle of the least action $\delta S = 0$ defines only a stationary point but not the extremum of the action integral [9]. In our opinion, adding d'Alembert operator into the Dirac theory is not only interesting by itself but can also help solve these difficulties in the process of deducing the Dirac equation from the variation principle.

4.2.1 Klein-Gordon-Dirac equations of motion

Lagrange equations of motion for Klein-Gordon-Dirac Lagrangian take the following form:

$$vi\gamma^{\mu}\partial_{\mu}\Psi - w\Psi \;=\; ug^{\mu\nu}\partial_{\mu}\partial_{\nu}\Psi,$$

$$vi\partial_\mu \overline{\Psi}\gamma^\mu + w\overline{\Psi} \;\; = \;\; -ug^{\mu\nu}\partial_\mu\partial_\nu\overline{\Psi}.$$

As it is said in [157], such an equation does not correspond to any irreducible representation of Poincaré group, and in this sense it is not admitted by Wigner-Bargmann classification as a relativistic wave equation for elementary particles. Nevertheless, there are no principle obstacles against considering a continuous dynamical system ruled by Klein-Gordon-Dirac equation.

In the momentum representation, i.e., when

$$\Psi(x) = \int e^{-ip_\mu x^\mu}\varphi(p)\,dp,$$

we obtain the equations of motion as follows:

$$v\gamma^\mu p_\mu \varphi(p) - w\varphi(p) = -ug^{\mu\nu}p_\mu p_\nu \varphi(p).$$

Since $g^{\mu\nu}p_\mu p_\nu = p^2$, and in the case when $v \neq 0$, we can rewrite this equation as follows:

$$\gamma^\mu p_\mu \varphi(p) = m\varphi(p), \tag{4.17}$$

which formally looks like Dirac equation with the mass

$$m^2 = p^2 = \left(\frac{w - up^2}{v}\right)^2.$$

Thus, we may write that the general solution of Klein-Gordon-Dirac equation is a superposition of two Dirac plane harmonic waves with masses

$$m_\pm = \frac{1}{\sqrt{2}\,|u|}\sqrt{v^2 + 2uw \pm |v|\sqrt{v^2 + 4uw}}.$$

For the existence of real non-negative (non-tachyonic situation) solutions for m^2 we should have

$$(v^2 + 4uw \geq 0) \wedge (v^2 + 2uw \geq 0),$$

i.e.,

$$(uw \geq 0,\ \forall v) \vee \left(uw < 0,\ v^2 \geq 4\,|uw|\right).$$

To complete our consideration we add the analysis of such a situation presented in [157]. The appearance of two mass shells in a general solution of Klein-Gordon-Dirac equation does not have to be so embarrassing as it could seem because of the following reasons:

- If the splitting of masses $\triangle m = m_+ - m_-$ is large, then, in normal conditions, it may be difficult to excite the states with the mass m_+ because the frequency spectrum of external perturbations will have to contain frequencies of the order $(m_+ - m_-)c^2/h$, e.g., if $u \to 0$, then $m_- \to |w| / |v|$ and $m_+ \to \infty$ (cf. this with the idea of Pauli-Villars-Rayski regularization[108]).

- It is not excluded that the superposition of states with two masses might be just desirable, e.g., one could try to explain in this way the mysterious kinship between heavy leptons and their neutrinos or the corresponding pairing between quarks. If there is no algebraic term, $w = 0$, then $m_- = 0$ and $m_+ = |v| / |u|$. Thus, in spite of a purely differential character of Klein-Gordon-Dirac equation, massive states appear and are paired with the massless ones.

- For special values of u, v, w, i.e., when $v^2 + 4uw = 0$, the mass gap vanishes and $m_- = m_+ = |w| / |u|$, then the Klein-Gordon-Dirac equation is exactly reduced to the Dirac equation.

Thus, for the solution of Dirac equation we may write the expansion in eigenfunctions in accordance with the superposition principle in the following form:

$$\begin{aligned} \Psi(x) &= \sum_{s=1,2} \int d\mu(m,\vec{p}) \left(e^{-ipx} u_{\vec{p}}^{s,m} a_{\vec{p}}^{s,m} + e^{ipx} v_{\vec{p}}^{s,m} b_{\vec{p}}^{+s,m} \right) \\ &+ \sum_{s=1,2} \int d\mu(M,\vec{p}) \left(e^{-ipx} u_{\vec{p}}^{s,M} a_{\vec{p}}^{s,M} + e^{ipx} v_{\vec{p}}^{s,M} b_{\vec{p}}^{+s,M} \right), \end{aligned} \tag{4.18}$$

$$\begin{aligned} \overline{\Psi}(x) &= \sum_{r=1,2} \int d\mu(m,\vec{p}) \left(e^{ipx} \overline{u}_{\vec{p}}^{r,m} a_{\vec{p}}^{+r,m} + e^{-ipx} \overline{v}_{\vec{p}}^{r,m} b_{\vec{p}}^{r,m} \right) \\ &+ \sum_{r=1,2} \int d\mu(M,\vec{p}) \left(e^{ipx} \overline{u}_{\vec{p}}^{r,M} a_{\vec{p}}^{+r,M} + e^{-ipx} \overline{v}_{\vec{p}}^{r,M} b_{\vec{p}}^{r,M} \right), \end{aligned} \tag{4.19}$$

where $M = m_+$ and $m = m_-$, the normalized measure of these integrals is

$$d\mu(m,\vec{p}) = \frac{m d^3 p}{(2\pi)^3 E_{\vec{p}}^m},$$

where

$$E_{\vec{p}}^m = p_0 = \sqrt{m^2 + \vec{p}^{\,2}},$$

and $u_{\vec{p}}^{s,m}$ and $v_{\vec{p}}^{r,m}$ are the amplitudes of plane harmonic waves with positive and negative frequencies (Dirac bispinors), which we may write in the following form:

$$u_{\vec{p}}^{s,m} = \frac{1}{\sqrt{2m\left(m + E_{\vec{p}}^m\right)}} \begin{pmatrix} \left(m + E_{\vec{p}}^m\right) \omega^s \\ \vec{\sigma}\vec{p}\omega^s \end{pmatrix},$$

$$v_{\vec{p}}^{r,m}=\frac{1}{\sqrt{2m\left(m+E_{\vec{p}}^{m}\right)}}\left(\begin{array}{c}\vec{\sigma}\vec{p}\omega^{r}\\ \left(m+E_{\vec{p}}^{m}\right)\omega^{r}\end{array}\right),$$

where ω^s is Dirac 3-spinor that satisfies the following normalization condition:

$$\omega^{+s}\omega^{r}=\delta^{sr}.$$

The multiplication rules for Dirac bispinors with the same mass (m or M) are as follows (the second table of multiplication rules may be obtained by the substitution of M instead of m):

	$u_{\vec{p}}^{s,m}$	$v_{\vec{p}}^{s,m}$	$u_{-\vec{p}}^{s,m}$	$v_{-\vec{p}}^{s,m}$
$(\overline{u})_{\vec{p}}^{r,m}$	δ^{rs}	0	$E_{\vec{p}}^{m}\delta^{rs}/m$	$-p^{A}\sigma_{A}^{rs}/m$
$(\overline{v})_{\vec{p}}^{r,m}$	0	$-\delta^{rs}$	$p^{A}\sigma_{A}^{rs}/m$	$-E_{\vec{p}}^{m}\delta^{rs}/m$
$u_{\vec{p}}^{+r,m}$	$E_{\vec{p}}^{m}\delta^{rs}/m$	$p^{A}\sigma_{A}^{rs}/m$	δ^{rs}	0
$v_{\vec{p}}^{+r,m}$	$p^{A}\sigma_{A}^{rs}/m$	$E_{\vec{p}}^{m}\delta^{rs}/m$	0	δ^{rs}

, (4.20)

where $p^A\sigma_A^{rs}$ are matrix elements of the matrix

$$\left(\begin{array}{cc}p_z & p_x-ip_y\\ p_x+ip_y & -p_z\end{array}\right),$$

which is given as a scalar product of Dirac vector-matrix $\vec{\sigma}$ and the 3-momentum $\vec{p}$.

For bispinors with different masses m and M we may write the multiplication rules in the following tables:

	$u_{\vec{p}}^{s,M}$	$v_{\vec{p}}^{s,M}$	$u_{-\vec{p}}^{s,M}$	$v_{-\vec{p}}^{s,M}$
$(\overline{u})_{\vec{p}}^{r,m}$	$A\delta^{rs}$	$-Bp^{A}\sigma_{A}^{rs}$	$C\delta^{rs}$	$-Dp^{A}\sigma_{A}^{rs}$
$(\overline{v})_{\vec{p}}^{r,m}$	$Bp^{A}\sigma_{A}^{rs}$	$-A\delta^{rs}$	$Dp^{A}\sigma_{A}^{rs}$	$-C\delta^{rs}$
$u_{\vec{p}}^{+r,m}$	$C\delta^{rs}$	$Dp^{A}\sigma_{A}^{rs}$	$A\delta^{rs}$	$Bp^{A}\sigma_{A}^{rs}$
$v_{\vec{p}}^{+r,m}$	$Dp^{A}\sigma_{A}^{rs}$	$C\delta^{rs}$	$Bp^{A}\sigma_{A}^{rs}$	$A\delta^{rs}$

(4.21)

and

	$u_{\vec{p}}^{s,m}$	$v_{\vec{p}}^{s,m}$	$u_{-\vec{p}}^{s,m}$	$v_{-\vec{p}}^{s,m}$
$(\overline{u})_{\vec{p}}^{r,M}$	$A\delta^{rs}$	$Bp^{A}\sigma_{A}^{rs}$	$C\delta^{rs}$	$-Dp^{A}\sigma_{A}^{rs}$
$(\overline{v})_{\vec{p}}^{r,M}$	$-Bp^{A}\sigma_{A}^{rs}$	$-A\delta^{rs}$	$Dp^{A}\sigma_{A}^{rs}$	$-C\delta^{rs}$
$u_{\vec{p}}^{+r,M}$	$C\delta^{rs}$	$Dp^{A}\sigma_{A}^{rs}$	$A\delta^{rs}$	$-Bp^{A}\sigma_{A}^{rs}$
$v_{\vec{p}}^{+r,M}$	$Dp^{A}\sigma_{A}^{rs}$	$C\delta^{rs}$	$-Bp^{A}\sigma_{A}^{rs}$	$A\delta^{rs}$

, (4.22)

where the coefficients are as follows:

$$\begin{aligned} A &= \frac{(m+p_0)(M+P_0)-p^2}{2\sqrt{mM}\sqrt{(m+p_0)(M+P_0)}} > 0, \\ B &= \frac{M+P_0-m-p_0}{2\sqrt{mM}\sqrt{(m+p_0)(M+P_0)}} \geq 0, \\ C &= \frac{(m+p_0)(M+P_0)+p^2}{2\sqrt{mM}\sqrt{(m+p_0)(M+P_0)}} > 0, \\ D &= \frac{M+P_0+m+p_0}{2\sqrt{mM}\sqrt{(m+p_0)(M+P_0)}} > 0, \end{aligned}$$

and $p_0 = E^m_{\vec{p}}$, $P_0 = E^M_{\vec{p}}$. In the case of equal masses $m = M$ we obtain the following limit values:

$$A = 1, \qquad B = 0, \qquad C = \frac{1}{m}E^m_{\vec{p}}, \qquad D = \frac{1}{m}.$$

4.2.2 Lagrange formalism

To apply Lagrange formalism we calculate the following derivatives:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \Psi_{,\mu}} &= ug^{\mu\lambda}\overline{\Psi}_{,\lambda} + \frac{vi}{2}\overline{\Psi}\gamma^\mu, \\ \frac{\partial \mathcal{L}}{\partial \overline{\Psi}_{,\mu}} &= ug^{\mu\lambda}\Psi_{,\lambda} - \frac{vi}{2}\gamma^\mu\Psi. \end{aligned}$$

Then the energy-momentum tensor and 4-current are as follows:

$$\begin{aligned} t^\mu_\nu &= \overline{\Psi}_{,\nu}\frac{\partial \mathcal{L}}{\partial \overline{\Psi}_{,\mu}} + \frac{\partial \mathcal{L}}{\partial \Psi_{,\mu}}\Psi_{,\nu} - \mathcal{L}\delta^\mu_\nu \\ &= ug^{\mu\lambda}\left(\overline{\Psi}_{,\lambda}\Psi_{,\nu} + \overline{\Psi}_{,\nu}\Psi_{,\lambda}\right) + \frac{vi}{2}\left(\overline{\Psi}\gamma^\mu\Psi_{,\nu} - \overline{\Psi}_{,\nu}\gamma^\mu\Psi\right), \qquad (4.23) \\ j^\mu &= i\left(\overline{\Psi}\frac{\partial \mathcal{L}}{\partial \overline{\Psi}_{,\mu}} + \frac{\partial \mathcal{L}}{\partial \Psi_{,\mu}}\Psi\right) = uig^{\mu\lambda}\left(\overline{\Psi}\Psi_{,\lambda} - \overline{\Psi}_{,\lambda}\Psi\right) + v\overline{\Psi}\gamma^\mu\Psi_{,\nu}. \qquad (4.24) \end{aligned}$$

The term $\mathcal{L}\delta^\mu_\nu$ may be reduced with the help of Klein-Gordon-Dirac equation to the 4-divergence but the density of Lagrangian is determined only to the accuracy of the 4-divergence of space coordinates and time function, so we may neglect this term. This is in accordance with the fact, that the density of Dirac Lagrangian on the solutions of Dirac equation equals 0 [9, 14]. Then we may obtain the forms of Hamiltonian, the 3-momentum and total charge from (4.23-4.24):

$$H = \int t_{00} d^3x = \int \left\{2u\overline{\Psi}_{,0}\Psi_{,0} + \frac{vi}{2}\left(\Psi^+\Psi_{,0} - \Psi^+_{,0}\Psi\right)\right\} d^3x, \qquad (4.25)$$

$$P_i = \int t_{0i} d^3x \;=\; \int \left\{ u\left(\overline{\Psi}_{,0}\Psi_{,i} + \overline{\Psi}_{,i}\Psi_{,0}\right) + \frac{vi}{2}\left(\Psi^+\Psi_{,i} - \Psi^+_{,i}\Psi\right) \right\} d^3x, \quad (4.26)$$

$$Q = \int j^0 d^3x \;=\; \int \left\{ ui\left(\overline{\Psi}\Psi_{,0} - \overline{\Psi}_{,0}\Psi\right) + v\Psi^+\Psi \right\} d^3x, \quad (4.27)$$

where $i = \overline{1,3}$. From the form of the total charge $Q(\psi) = \langle \psi, \psi \rangle$ we may obtain a rule of the scalar product of two different wave functions $\psi(x)$ and $\varphi(x)$:

$$\begin{aligned}
\langle \psi, \varphi \rangle &= \frac{1}{4}\left[Q(\psi + \varphi) - Q(\psi - \varphi) - iQ(\psi + i\varphi) + iQ(\psi - i\varphi)\right] \\
&= ui \int \left(\overline{\psi}\varphi_{,0} - \overline{\psi}_{,0}\varphi\right) + v \int \psi^+ \varphi d^3x \\
&= u \langle \psi, \varphi \rangle_{KG} + v \langle \psi, \varphi \rangle_D ,
\end{aligned}$$

i.e., it is the superposition of Klein-Gordon and Dirac scalar products.

4.2.3 Canonical formalism

Now if we define the field momenta π_Ψ and $\pi_{\overline{\Psi}}$ as follows:

$$\begin{aligned}
\pi_\Psi = \frac{\partial \mathcal{L}}{\partial \Psi_{,0}} &= u\overline{\Psi}_{,0} + \frac{vi}{2}\Psi^+, \\
\pi_{\overline{\Psi}} = \frac{\partial \mathcal{L}}{\partial \overline{\Psi}_{,0}} &= u\Psi_{,0} - \frac{vi}{2}\gamma^0\Psi,
\end{aligned}$$

then from the canonical form for Hamiltonian

$$H = \int \left\{ \pi_\Psi \Psi_{,0} + \overline{\Psi}_{,0}\pi_{\overline{\Psi}} - \mathcal{L} \right\} d^3x$$

we may obtain the same expression as in (4.25).

After the substitution of Ψ and $\overline{\Psi}$ in the expressions for Hamiltonian (4.25), 3-momentum (4.26), and total charge (4.27) by their expansions in plane harmonic waves (4.18), we obtain that

$$\begin{aligned}
H &= \int \frac{d^3p}{(2\pi)^3} \sum_s F\left(a^{+s,m}_{\vec{p}} a^{s,m}_{\vec{p}} - b^{s,m}_{\vec{p}} b^{+s,m}_{\vec{p}}\right) + G(M, M) + I(m \Leftrightarrow M) \quad (4.28) \\
&+ \int \frac{d^3p}{(2\pi)^3} \sum_{r,s} p^A \sigma^{rs}_A J\left(a^{+r,m}_{\vec{p}} b^{+s,m}_{-\vec{p}} - b^{r,m}_{\vec{p}} a^{s,m}_{-\vec{p}}\right) + K(M, M) + L(m \Leftrightarrow M), \\
\vec{P} &= \int \frac{d^3p}{(2\pi)^3} \vec{p} \sum_s N\left(a^{+s,m}_{\vec{p}} a^{s,m}_{\vec{p}} - b^{s,m}_{\vec{p}} b^{+s,m}_{\vec{p}}\right) + O(M, M) + R(m \Leftrightarrow M) \\
&+ \int \frac{d^3p}{(2\pi)^3} \sum_{r,s} S\left(a^{+r,m}_{\vec{p}} b^{+s,M}_{-\vec{p}} - b^{r,m}_{\vec{p}} a^{s,M}_{-\vec{p}} - a^{+r,M}_{\vec{p}} b^{+s,m}_{-\vec{p}} + b^{r,M}_{\vec{p}} a^{s,m}_{-\vec{p}}\right), \quad (4.29)
\end{aligned}$$

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s T \left(a^{+s,m}_{\vec{p}} a^{s,m}_{\vec{p}} + b^{s,m}_{\vec{p}} b^{+s,m}_{\vec{p}} \right) + V(M, M) + U(m \Leftrightarrow M)$$
$$+ \int \frac{d^3p}{(2\pi)^3} \sum_{r,s} W \left(a^{+r,m}_{\vec{p}} b^{+s,M}_{-\vec{p}} + b^{r,m}_{\vec{p}} a^{s,M}_{-\vec{p}} - a^{+r,M}_{\vec{p}} b^{+s,m}_{-\vec{p}} - b^{r,M}_{\vec{p}} a^{s,m}_{-\vec{p}} \right), \quad (4.30)$$

where (M, M) means the same term as the first one in the row but with masses M in the brackets instead of m, $(m \Leftrightarrow M) = (m, M) + (M, m)$, and the coefficients are as follows:

$$F = m(2mu + v), \qquad G = M(2Mu + v),$$
$$J = 2mu, \qquad I = mM \left(2Au + \frac{Cv}{2} \frac{E^m_{\vec{p}} + E^M_{\vec{p}}}{E^m_{\vec{p}} E^M_{\vec{p}}} \right),$$
$$K = 2Mu, \qquad L = mM \left(2Du + \frac{Bv}{2} \frac{E^m_{\vec{p}} - E^M_{\vec{p}}}{E^m_{\vec{p}} E^M_{\vec{p}}} \right),$$
$$N = \frac{m\,(2mu - v)}{E^m_{\vec{p}}}, \qquad O = \frac{M\,(2Mu - v)}{E^M_{\vec{p}}},$$
$$R = \frac{mM \left(Au \left[E^m_{\vec{p}} + E^M_{\vec{p}} \right] - Cv \right)}{E^m_{\vec{p}} E^M_{\vec{p}}}, \qquad S = \frac{mM \left(Du \left[E^m_{\vec{p}} - E^M_{\vec{p}} \right] - Bv \right)}{E^m_{\vec{p}} E^M_{\vec{p}}},$$
$$T = \frac{m\,(2mu + v)}{E^m_{\vec{p}}}, \qquad V = \frac{M\,(2Mu + v)}{E^M_{\vec{p}}},$$
$$U = \frac{mM \left(Au \left[E^m_{\vec{p}} + E^M_{\vec{p}} \right] + Cv \right)}{E^m_{\vec{p}} E^M_{\vec{p}}}, \qquad W = \frac{mM \left(Du \left[E^m_{\vec{p}} - E^M_{\vec{p}} \right] + Bv \right)}{E^m_{\vec{p}} E^M_{\vec{p}}}.$$

In the case of equal masses $m = M = |w| / |u|$ when u, v, w have special values, i.e., $v^2 + 4uw = 0$, we may obtain the following limit values for the Hamiltonian, 3-momentum, and total charge:

$$H = 4m\,(2mu + v) \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a^{+s,m}_{\vec{p}} a^{s,m}_{\vec{p}} - b^{s,m}_{\vec{p}} b^{+s,m}_{\vec{p}} \right)$$
$$+ 8mu \int \frac{d^3p}{(2\pi)^3} \sum_{r,s} p^A \sigma^{rs}_A \left(a^{+r,m}_{\vec{p}} b^{+s,m}_{-\vec{p}} - b^{r,m}_{\vec{p}} a^{s,m}_{-\vec{p}} \right), \quad (4.31)$$
$$\vec{P} = 4m\,(2mu - v) \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{E^m_{\vec{p}}} \sum_s \left(a^{+s,m}_{\vec{p}} a^{s,m}_{\vec{p}} + b^{s,m}_{\vec{p}} b^{+s,m}_{\vec{p}} \right), \quad (4.32)$$
$$Q = 4m\,(2mu + v) \int \frac{d^3p}{(2\pi)^3 E^m_{\vec{p}}} \sum_s \left(a^{+s,m}_{\vec{p}} a^{s,m}_{\vec{p}} + b^{s,m}_{\vec{p}} b^{+s,m}_{\vec{p}} \right). \quad (4.33)$$

We may consider another limit case when $w = 0$, therefore, $m = 0$ and $M = |v| / |u|$. In this case all coefficients in (4.28)-(4.30), except G, K, O, and V, are equal to 0. Then the Hamiltonian, 3-momentum, and total charge contain only terms which describe the states with mass M and have the same form as in (4.31)-(4.33) but with the following substitutions of coefficients before the integrals:

$$\begin{aligned} 4m(2mu+v) &\rightarrow 4M(2Mu+v), \\ 8mu &\rightarrow 8Mu, \\ 4m(2mu-v) &\rightarrow 4M(2Mu-v). \end{aligned}$$

4.2.4 Quantization remarks

After the quantization we may consider $a_{\vec{p}}^{s,\circ}$, $a_{\vec{p}}^{+s,\circ}$ and $b_{\vec{p}}^{s,\circ}$, $b_{\vec{p}}^{+s,\circ}$ as operators of creation and annihilation of a particle with the mass m or M, 3-momentum $\vec{p}$, and spin s, i.e., $\mathbf{a}_{\vec{p}}^{s,\circ}$, $\mathbf{a}_{\vec{p}}^{+s,\circ}$ and $\mathbf{b}_{\vec{p}}^{s,\circ}$, $\mathbf{b}_{\vec{p}}^{+s,\circ}$. We may find the commutation laws for these operators from the expressions for the Hamilton, the 3-momentum, and total charge operators (as in [9]), which are obtained from (4.28)-(4.30) by substitution of $a_{\vec{p}}^{s,\circ}$, $a_{\vec{p}}^{+s,\circ}$ and $b_{\vec{p}}^{s,\circ}$, $b_{\vec{p}}^{+s,\circ}$ with operators $\mathbf{a}_{\vec{p}}^{s,\circ}$, $\mathbf{a}_{\vec{p}}^{+s,\circ}$ and $\mathbf{b}_{\vec{p}}^{s,\circ}$, $\mathbf{b}_{\vec{p}}^{+s,\circ}$. The eigenvalues of operators $\mathbf{a}_{\vec{p}}^{+s,\circ}\mathbf{a}_{\vec{p}}^{s,\circ}$ and $\mathbf{b}_{\vec{p}}^{+s,\circ}\mathbf{b}_{\vec{p}}^{s,\circ}$ are equal to the positive numbers $N_{\vec{p}}^{s,\circ}$ and $\overline{N}_{\vec{p}}^{s,\circ}$, which are the numbers of particles and anti-particles with the mass m or M, 3-momentum $\vec{p}$, and spin s. From the condition of positivity of the energy (the eigenvalue of Hamilton operator) and the conservation law of the total charge (4.30) we may obtain anti-commutation laws for the following operators (the second set with M instead of m):

$$\{\mathbf{a}_{\vec{p}}^{r,m},\ \mathbf{a}_{\vec{p}}^{+s,m}\} = \delta^{rs}, \qquad \{\mathbf{b}_{\vec{p}}^{r,m},\ \mathbf{b}_{\vec{p}}^{+s,m}\} = \delta^{rs}. \tag{4.34}$$

This means that we may consider the particles which are described by the Klein-Gordon-Dirac wave function (4.18) as fermions. There arises the question: "Why our wave function does not describe any bosons in spite of the fact that our Klein-Gordon-Dirac equation contains Klein-Gordon term?" One of possible answers may be that the wave function (4.18) is not complete because we have obtained our equation (4.17) with the essential restriction $v \neq 0$, which means that any proper passage from (4.16) to Klein-Gordon Lagrangian is impossible.

In spite of dealing with a superposition of d'Alembert and Dirac operators, our model has nothing to do with the supersymmetric mixing of spinors and bosons. It is based on the $\mathrm{U}(2,2)$-gauge formulation of gravitation that is some modification of the Poincaré-gauge theory of gravitation, simply the Poincaré group (or rather its $\mathrm{SL}(2,\mathbb{C})\times\mathbb{R}^4$-covering) is replaced by the $\mathrm{SU}(2,2)$-covering of the conformal group. Similarity to the Seiberg-Witten model is superficial.

4.3 Green function formalism

The case considered here is to a certain extent complementary to the previous one, i.e., the restrictions on correlations between the coefficients of Klein-Gordon-Dirac equation, which are necessary to the existence of real non-negative (non-tachyonic situation) solutions for the mass of the corresponding 4-dimensional particle interpretation, define such a situation which was excluded from consideration previously. In other words, we are describing the case with the dominating Klein-Gordon term as contrary to the case in [157, 164, 170] with the dominating Dirac term.

4.3.1 Green function for Klein-Gordon-Dirac equation

Let us consider the Klein-Gordon-Dirac equation

$$iv\gamma^{\mu}\partial_{\mu}\Psi - w\Psi = ug^{\mu\nu}\partial_{\mu}\partial_{\nu}\Psi, \quad \mu = \overline{0,3}, \tag{4.35}$$

derivable from the corresponding Klein-Gordon-Dirac Lagrangian

$$\mathcal{L} = ug^{\mu\nu}\partial_{\mu}\overline{\Psi}\partial_{\nu}\Psi + \frac{iv}{2}\left(\overline{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - \partial_{\mu}\overline{\Psi}\gamma^{\mu}\Psi\right) - w\overline{\Psi}\Psi, \tag{4.36}$$

where u, v, and w are some real constants, γ^{μ} are Dirac matrices, which satisfy the condition $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$, $g^{\mu\nu}$ is the metric tensor, which in the special-relativistic limit equals $\eta^{\mu\nu}$, i.e, the flat metric tensor on space-time manifold M with the signature $(+, -, -, -)$ and constant coefficients, $\overline{\Psi} = \Psi^{+}\gamma^{0}$ is the Dirac conjugated wave function. For completeness of description we should add the following initial conditions:

$$\Psi(x^{\mu})\big|_{t=t_0} = \Psi_0(\vec{r}, t_0), \qquad \left.\frac{\partial\Psi(x^{\mu})}{\partial t}\right|_{t=t_0} = \Phi_0(\vec{r}, t_0). \tag{4.37}$$

From the equation (4.35) we can define the Klein-Gordon-Dirac operator $\mathbf{K}_{GD}$ as follows:

$$\mathbf{K}_{GD} = ug^{\mu\nu}\partial_{\mu}\partial_{\nu} - iv\gamma^{\mu}\partial_{\mu} + w \equiv ug^{\mu\nu}\left(\partial_{\mu} - ia_{\mu}\right)\left(\partial_{\nu} - ia_{\nu}\right) + \widetilde{w}, \tag{4.38}$$

where

$$a_{\mu} = (v/2u)g_{\mu\nu}\gamma^{\nu}, \qquad \widetilde{w} = w + ug^{\mu\nu}a_{\mu}a_{\nu} - iug^{\mu\nu}(\partial_{\mu}a_{\nu}).$$

In the special-relativistic limit we have

$$\widetilde{w} = w + v^2/4u.$$

We can define the momenta $\mathbf{p}_\mu = -i\partial_\mu$ (we use the natural system of units, i.e., $e = c = \hbar = 1$), then

$$\mathbf{K}_{GD} = -ug^{\mu\nu}\left(\mathbf{p}_\mu - a_\mu\right)\left(\mathbf{p}_\nu - a_\nu\right) + \widetilde{w} \equiv -u\mathbf{K}_G, \tag{4.39}$$

where

$$\mathbf{K}_G = g^{\mu\nu}\left(\mathbf{p}_\mu - a_\mu\right)\left(\mathbf{p}_\nu - a_\nu\right) + m^2$$

is some Klein-Gordon operator for the 3-dimensional particle of the mass

$$m^2 = -\frac{\widetilde{w}}{u} = \text{(in special relativistic limit)} = -\frac{(4uw + v^2)}{4u^2}$$

in the external field a_μ. For the existence of real non-negative (non-tachyonic situation) solutions for m^2 we should have in the special-relativistic limit

$$(uw < 0) \ \wedge \ \left(v^2 \leq 4|uw|\right).$$

As it was already said, this situation is complementary to the conditions in [157, 164, 170], i.e.,

$$(uw \geq 0, \ \forall v) \ \vee \ \left(uw < 0, \ v^2 \geq 4|uw|\right).$$

Now we can define the Green function for the Klein-Gordon-Dirac equation as follows:

$$\mathbf{K}_{GD}D(x, x_0) = \delta^{(4)}(x - x_0)$$

or, equivalently,

$$\left(\mathbf{H}_x + \frac{m}{2}\right)D(x, x_0) = -\frac{1}{2mu}\delta^{(4)}(x - x_0), \tag{4.40}$$

where

$$\mathbf{H}_x = \frac{1}{2m}g^{\mu\nu}(x)\left[\mathbf{p}_\mu - a_\mu(x)\right]\left[\mathbf{p}_\nu - a_\nu(x)\right]$$

is the Hamiltonian operator for the 4-dimensional particle in the external field a_μ. On this stage we consider a general form of the metric tensor $g^{\mu\nu}(x)$, i.e., non-constant one, then the external field $a_\mu(x)$ is also non-constant and can be formally interpreted as some "electro-magnetic" field. This analogy is, of course, only superficial and is broken in the special-relativistic case, when $g^{\mu\nu}(x)$ becomes $\eta^{\mu\nu}$ and $a_\mu(x)$ becomes constant field a_μ. Later on we will consider only the special-relativistic situation.

The Green function can be obtained formally from (4.40) as follows:

$$D(x, x_0) = -\frac{1}{2mu}\left(\mathbf{H}_x + \frac{m}{2}\right)^{-1}\delta^{(4)}(x - x_0). \tag{4.41}$$

We can use the Feynman representation of the inverse operator (as, e.g., in [11]), i.e.,

$$\mathbf{B}^{-1} = \pm\frac{1}{i}\int_0^\infty ds e^{-\epsilon s} e^{\pm is\mathbf{B}}, \qquad \epsilon \to 0, \tag{4.42}$$

for rewriting (4.41) in the following form:

$$D(x, x_0) = -\frac{i}{2mu}\int_0^\infty ds e^{-is\frac{m}{2}} Q(x, s; x_0, 0), \tag{4.43}$$

where s is an evolution parameter (the proper time),

$$Q(x, s; x_0, 0) = e^{-is\mathbf{H}_x}\delta^{(4)}(x - x_0)$$

is the Green function for the corresponding stationary Schrödinger equation in 4-dimensional space:

$$i\frac{\partial\varphi(x;s)}{\partial s} = \mathbf{H}_x\varphi(x;s), \qquad \mathbf{H}_x = \frac{1}{2m}\eta^{\mu\nu}\left(\mathbf{p}_\mu - a_\mu\right)\left(\mathbf{p}_\nu - a_\nu\right), \tag{4.44}$$

which we can obtain performing the differentiation over s of $Q(x, s; x_0, 0)$:

$$\left(i\frac{\partial}{\partial s} - \mathbf{H}_x\right) Q(x, s; x_0, 0) = i\delta^{(4)}(x - x_0)\delta(s). \tag{4.45}$$

For obtaining the explicit expression for the Green function $Q(x, s; x_0, 0)$ we can use the following equality for operators [11, 184]:

$$e^{\frac{\alpha}{2}\mathbf{A}^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\xi \exp\left(-\frac{1}{2}\xi^2 \pm \xi\sqrt{\alpha}\mathbf{A}\right), \tag{4.46}$$

and then

$$Q(x, s; x_0, 0) = -\frac{im^2}{4\pi^2 s^2}\exp\left\{\frac{im}{2s}\left[(t - t_0)^2 - (\vec{r} - \vec{r}_0)^2\right] + ia_\mu(x^\mu - x_0^\mu)\right\}. \tag{4.47}$$

Finally, after substituting (4.47) in (4.43) and introducing a new variable $\xi = m/s$, the total Green function $D(x, x_0)$ has the following form:

$$D(x, x_0) = -\frac{G(y, z)}{8\pi^2 u}\exp\left\{ia_\mu(x^\mu - x_0^\mu)\right\},$$

where

$$G(y, z) = \int_0^\infty d\xi \exp\left\{-\frac{1}{2}\left(y\xi + \frac{z}{\xi}\right)\right\}, \tag{4.48}$$

and

$$y = i\left[(\vec{r} - \vec{r}_0)^2 - (t - t_0)^2\right], \qquad z = im^2.$$

For performing the integration procedure in (4.48) we can construct formally the differential equation a solution of which is our function $G(y,z)$. It turns out that it is the modified Bessel equation

$$\frac{\partial^2 G}{\partial z^2} - \frac{y}{4z}G = 0,$$

which has the following solution (the condition for mixed second derivatives, i.e.,

$$\frac{\partial^2 G(y,z)}{\partial y \partial z} = \frac{\partial^2 G(y,z)}{\partial z \partial y} = \frac{1}{4}G(y,z),$$

is already taken into account):

$$G(y,z) = a\delta(-iy) + b\sqrt{\frac{z}{y}}Z_1(\sqrt{yz}),$$

where a, b are constants, $Z_1(z)$ is either the modified Bessel function of the first kind $I_1(z)$ or the second kind $K_1(z)$ [67].

We can notice that for a mass-less "particle" we have

$$\begin{aligned} G(y,0) &= \int_0^\infty d\xi \exp\left\{-\frac{y\xi}{2}\right\} \\ &= \text{(Sohotskyi formulae)} = 2\pi\delta(-iy) + 2\frac{\mathcal{P}}{y}, \end{aligned}$$

where $\mathcal{P}/y$ is a generalized function (just like δ-function), and the symbol $\mathcal{P}$ itself stands for the integration in the main meaning [184]. For the modified Bessel functions we can write the approximate formulae

$$I_1(z) \approx z/2, \qquad K_1(z) \approx 1/z \qquad \text{for} \qquad |z| \ll 1.$$

Hence, we can choose the modified Bessel function of the second kind K_1 and our constants a and b are as follows: $a = 2\pi$, $b = 2$. Then we have that

$$\sqrt{yz} = \begin{cases} m\sqrt{(t-t_0)^2 - (\vec{r}-\vec{r}_0)^2}, & \text{if} \quad (t-t_0)^2 > (\vec{r}-\vec{r}_0)^2, \\ \\ im\sqrt{(\vec{r}-\vec{r}_0)^2 - (t-t_0)^2}, & \text{if} \quad (t-t_0)^2 < (\vec{r}-\vec{r}_0)^2. \end{cases}$$

So for the latter case it is convenient to use not the modified Bessel functions K_1 but the Hankel functions $H_1^{(1)}$ or $H_1^{(2)}$, i.e., the Bessel functions of the third kind, which are interrelated as follows:

$$\frac{2}{\pi}K_1(ix) = -H_1^{(1)}(-x) = -H_1^{(2)}(x).$$

Finally, the Green function $D(x,x_0)$ has the following form:

$$D(x,x_0) = -\frac{\exp\{ia_\mu(x^\mu - x_0^\mu)\}}{4\pi u}\Big[\delta\left((t-t_0)^2 - (\vec{r}-\vec{r}_0)^2\right) + \tag{4.49}$$

$$+ \left\{\begin{array}{ll} \frac{m}{\pi}\frac{K_1\ m\sqrt{(t-t_0)^2-(\vec{r}-\vec{r}_0)^2}}{\sqrt{(t-t_0)^2-(\vec{r}-\vec{r}_0)^2}}, & \text{if} \quad (t-t_0)^2 > (\vec{r}-\vec{r}_0)^2 \\ -\frac{im}{2}\frac{H_1^{(2)}\ m\sqrt{(\vec{r}-\vec{r}_0)^2-(t-t_0)^2}}{\sqrt{(\vec{r}-\vec{r}_0)^2-(t-t_0)^2}}, & \text{if} \quad (t-t_0)^2 < (\vec{r}-\vec{r}_0)^2 \end{array}\right.\Bigg].$$

4.3.2 Structure of general solution

Now let us consider the two-component wave function

$$\overrightarrow{\Psi} = (\Psi_1, \Psi_2)^T,$$

where

$$\Psi_1 = \Psi, \qquad \Psi_2 = i\frac{\partial\Psi}{\partial t}.$$

Then our second-order differential equation (4.35) (in the special-relativistic case) becomes the system of two first-order ones:

$$i\frac{\partial\Psi_1}{\partial t} = \Psi_2, \tag{4.50}$$

$$i\frac{\partial\Psi_2}{\partial t} = \left(-\triangle + i\frac{v}{u}\gamma^j\nabla_j + \frac{w}{u}\right)\Psi_1 - \frac{v}{u}\gamma^0\Psi_2. \tag{4.51}$$

In the symbolic way we may rewrite the previous equations as follows:

$$i\frac{\partial\overrightarrow{\Psi}}{\partial t} = \mathbf{L}_{\vec{r}}(t)\overrightarrow{\Psi}, \tag{4.52}$$

where

$$\mathbf{L}_{\vec{r}}(t) = \begin{bmatrix} 0 & 1 \\ -\triangle + \frac{iv}{u}\gamma^j\nabla_j + \frac{w}{u} & -\frac{v}{u}\gamma^0 \end{bmatrix}.$$

The initial conditions (4.37) for this symbolic Schrödinger equation can be rewritten as follows:

$$\overrightarrow{\Psi}_0(\vec{r},t_0) = \left(\Psi_0(\vec{r},t_0), \Phi_0(\vec{r},t_0)\right)^T.$$

Then we can use the T-exponent method for describing the Green function of Schrödinger equation [11]. First of all, the differential equation (4.52) can be rewritten as the integro-differential one:

$$\overrightarrow{\Psi}(\vec{r},t) = \overrightarrow{\Psi}_0(\vec{r},t_0) + \frac{1}{i}\int_{t_0}^{t} dt'\mathbf{L}_{\vec{r}}(t')\overrightarrow{\Psi}(\vec{r},t'). \tag{4.53}$$

Using the iteration method we can solve this equation and introduce the Green function in the matrix form as follows:

$$\overrightarrow{\Psi}(\vec{r},t) = \int d^3\vec{r}_0 \mathbf{G}(\vec{r},t;\vec{r}_0,t_0)\overrightarrow{\Psi}_0(\vec{r}_0,t_0), \tag{4.54}$$

where

$$\mathbf{G}(\vec{r},t;\vec{r}_0,t_0) = \theta(t-t_0)\delta(\vec{r}-\vec{r}_0)T\mathrm{exp}\left\{-i\int_{t_0}^{t} dt'\mathbf{L}_{\vec{r}}(t')\right\}. \tag{4.55}$$

The T-exponent operator here is understood as a series:

$$T\mathrm{exp}\left\{-i\int_{t_0}^{t} dt'\mathbf{L}_{\vec{r}}(t')\right\} =$$

$$1+\sum_{n\geq 1}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n \mathbf{L}_{\vec{r}}(t_1)\mathbf{L}_{\vec{r}}(t_2)\ldots\mathbf{L}_{\vec{r}}(t_n), \tag{4.56}$$

and the symbol T itself stands for the chronological multiplication of operators, e.g., for two operators $\mathbf{A}(t)$ and $\mathbf{B}(t')$ we have the following rule:

$$T\left(\mathbf{A}(t)\mathbf{B}(t')\right) = \theta(t-t')\mathbf{A}(t)\mathbf{B}(t') + \theta(t'-t)\mathbf{B}(t')\mathbf{A}(t). \tag{4.57}$$

The equation (4.54) can be rewritten in the explicit form:

$$\begin{aligned}\Psi(\vec{r},t) &= \int d^3\vec{r}_0\left[G_{11}(\vec{r},t;\vec{r}_0,t_0)\Psi_0(\vec{r}_0,t_0) + iG_{12}(\vec{r},t;\vec{r}_0,t_0)\Phi_0(\vec{r}_0,t_0)\right], \\ i\frac{\partial\Psi(\vec{r},t)}{\partial t} &= \int d^3\vec{r}_0\left[G_{21}(\vec{r},t;\vec{r}_0,t_0)\Psi_0(\vec{r}_0,t_0) + iG_{22}(\vec{r},t;\vec{r}_0,t_0)\Phi_0(\vec{r}_0,t_0)\right],\end{aligned} \tag{4.58}$$

i.e., for defining $\Psi(\vec{r},t)$ in any time instant t it is necessary to know only the Green function matrix components $G_{11}(\vec{r},t;\vec{r}_0,t_0)$ and $G_{12}(\vec{r},t;\vec{r}_0,t_0)$. With the help of differentiation over t in (4.55) we can find the following equation for the Green function $\mathbf{G}$:

$$\frac{\partial\mathbf{G}}{\partial t} = -i\mathbf{L}_{\vec{r}}(t)\mathbf{G} + \delta(t-t_0)\delta(\vec{r}-\vec{r}_0)\mathbb{I}, \tag{4.59}$$

where $\mathbb{I}$ is the identity matrix. In the explicit form it is a system of four equations:

$$i\frac{\partial G_{11}}{\partial t} = G_{21} + i\delta(t-t_0)\delta(\vec{r}-\vec{r}_0), \tag{4.60}$$

$$i\frac{\partial G_{12}}{\partial t} = G_{22}, \tag{4.61}$$

$$i\frac{\partial G_{21}}{\partial t} = \left[-\triangle + \frac{iv}{u}\gamma^j\nabla_j + \frac{w}{u}\right]G_{11} - \frac{v}{u}\gamma^0 G_{21}, \tag{4.62}$$

$$i\frac{\partial G_{22}}{\partial t} = \left[-\triangle + \frac{iv}{u}\gamma^j\nabla_j + \frac{w}{u}\right]G_{12} - \frac{v}{u}\gamma^0 G_{22} + i\delta(t-t_0)\delta(\vec{r}-\vec{r}_0). \tag{4.63}$$

From the equations (4.60) and (4.62) we can see that the matrix component G_{11} satisfies the following equation:

$$\mathbf{K}_{GD} G_{11} = u\delta(\vec{r}-\vec{r}_0)\frac{\partial}{\partial t}\delta(t-t_0) - iv\gamma^0\delta(\vec{r}-\vec{r}_0)\delta(t-t_0). \tag{4.64}$$

Equivalently, composing (4.61) and (4.63) we obtain that

$$\mathbf{K}_{GD} G_{12} = -iu\delta(\vec{r}-\vec{r}_0)\delta(t-t_0). \tag{4.65}$$

The initial conditions for these equations, which can be obtained from (4.58), are as follows:

$$G_{11}(\vec{r},t_0;\vec{r}_0,t_0) = \delta(\vec{r}-\vec{r}_0), \qquad G_{12}(\vec{r},t_0;\vec{r}_0,t_0) = 0. \tag{4.66}$$

The G_{11} and G_{12} are not independent functions. If we use the properties of the δ-function in (4.64), then we may replace the differentiation over t by the differentiation over t_0. We can notice that the Klein-Gordon-Dirac operator $\mathbf{K}_{GD}$ and the operator $i\partial/\partial t_0 + (v/u)\gamma^0$ are commuting because they act on different variables. Then we can write that

$$G_{11}(\vec{r},t;\vec{r}_0,t_0) = \left[i\frac{\partial}{\partial t_0} + \frac{v}{u}\gamma^0\right] G_{12}(\vec{r},t;\vec{r}_0,t_0). \tag{4.67}$$

Moreover, the equation (4.65) for the Green function $G_{12}(\vec{r},t;\vec{r}_0,t_0)$ is almost the same as the equation (4.40) for the Klein-Gordon-Dirac equation Green function $D(\vec{r},t;\vec{r}_0,t_0)$. Hence, if we define the retarding Klein-Gordon-Dirac equation Green function

$$D_{\text{ret}}(\vec{r},t;\vec{r}_0,t_0) = \theta(t-t_0)D(\vec{r},t;\vec{r}_0,t_0), \tag{4.68}$$

then we can write that

$$G_{12}(\vec{r},t;\vec{r}_0,t_0) = -iuD_{\text{ret}}(\vec{r},t;\vec{r}_0,t_0).$$

Thus, the general solution of the Klein-Gordon-Dirac equation (4.35) (for the time instant $t > t_0$) with the initial conditions (4.37) is as follows:

$$\begin{aligned}\Psi(\vec{r},t) &= \int d^3\vec{r}_0 \left(u\frac{\partial}{\partial t_0} - iv\gamma^0\right) D_{\text{ret}}(\vec{r},t;\vec{r}_0,t_0)\Psi_0(\vec{r}_0,t_0) \\ &+ u\int d^3\vec{r}_0 D_{\text{ret}}(\vec{r},t;\vec{r}_0,t_0)\Phi_0(\vec{r}_0,t_0). \end{aligned} \tag{4.69}$$

Remark: we have found the general solution for Klein-Gordon-Dirac equation with the help of Green function method. The Klein-Gordon-Dirac operator has been reduced to some extent to the usual Klein-Gordon operator, i.e., we supposed that Klein-Gordon term in (4.35) was dominating in a sense. This was possible due to special relations between the coefficients of Klein-Gordon-Dirac equation. These relations are complementary to the ones in [170], where we have found the general solution for Klein-Gordon-Dirac equation as a superposition of two Dirac plane harmonic waves with different masses. In [157, 164] it has been shown that the appearance of two mass shells in the general solution not only is not undesirable but even can help explain, for example, a mysterious kinship between heavy leptons and their neutrinos or the corresponding pairing between quarks. Otherwise, the mass splitting $\triangle m = m_+ - m_-$ could be very large (then perhaps it is too difficult to excite the m_+-states) or very small (then perhaps it is below the present accuracy of our experiments) for not being found in the experimental way.

Philosophical remarks: nonlinearity and symmetry

We discussed certain models of internal and collective degrees of freedom based on the geometry of the physical space and some other spaces related to it. The special stress was laid on the use of affine, linear, orthogonal, projective, and (pseudo)unitary groups. We have shown that there is some methodological and also deeper physical similarity between extended systems of material points (both discrete and continuous) and physical fields including very fundamental ones. This is an example of the impact of geometry on mechanics and physics. At the same time models of this type are analytically and computationally effective because due to the analytic structure of Lie groups there often exist explicit solutions in terms of some standard special functions of mathematical physics, in particular, function series (first of all, power series). We have constructed models where just as in rigid-body mechanics or hydrodynamics of incompressible ideal fluids [2, 3, 10, 38, 39, 86] the dynamics is encoded in invariant geodetic models without the explicit use of potentials. This is an essential novelty in comparison with many known from literature models of affinely-rigid (pseudo-rigid) bodies, where the group of dynamical symmetries is a proper (and often remarkably "smaller") subgroup of the group underlying kinematics of degrees of freedom. In our models only dilatational motion must be then stabilized by some potential. Due to the fact that realistic solids and fluids are weakly compressible, this stabilization is usually achieved in a successful way by some relatively simple model (toy) potentials like harmonic oscillators with large elastic constants or potential wells; the latter are particularly (although not exclusively) useful in quantum problems. Incidentally, quantization is necessary when one aims at describing nanostructures, objects like fullerens, and other structural elements in the molecular or supra-molecular scale. We have formulated the main ideas of this quantization, in particular, certain reduction procedures have been obtained such that the dynamics of rotations of Green and Cauchy deformation tensors may be separated and described fully in terms of some known special functions. It is only the dynamics of deformation invariants that should be separately, explicitly solved. And the great deal of dynamics may be encoded in the very expression for the kinetic energy, more precisely in geodetic models invariant under the group of kinematical symmetries.

We have developed the system of Poisson brackets and commutator relations which enables one to analyze the classical and quantum problems in almost algebraic terms. These Poisson brackets and methods are to certain extent related to those formulated by Guillemin and Sternberg [55, 56] in their symplectic approach to collective modes. There is also some link between them and Chandrasekhar's method of virial coefficients in dynamics of macroscopic extended bodies [23] and Bohr-Mottelson's collective approach to nuclear dynamics [15]. Quite independently of these problems, interesting in themselves, we concentrate on models applicable in mechanics of structured media. We have also discussed some relations between our models and the theory of integrable one-dimensional chains, both classical and quantized. Certain simple problems concerning models of collective modes based on the projective group have also been discussed.

To the best of our knowledge, the models dynamically invariant under the affine and projective group are new. All models of affine modes developed in mechanics of structured bodies and in usual elastic problems have only affine kinematics but their dynamics was at most invariant under the isometry group.

As it has been already mentioned, affine models can be used for describing realistic continua in a rather alternative form to that developed by Eringen, Grot, Bressau, Cherny, and others. The moving micromorphic medium can be described then as the field of linear frames (tetrads) in the space-time manifold. Roughly speaking, the integral curves of the time-like legs of tetrads describe the motion of material points (they are their world lines), the spatial legs describe the dynamics of internal affine degrees of freedom. And the gap between mechanics and field theory becomes diffused. One deals with micromorphic ether, cosmic substratum describing relativistic fluids or elastic media. The same model may be alternatively interpreted as an alternative description of gravitation and the space-time dynamics. This is in a sense revival of the old theory of Hehl and Kröner of the space-time as relativistic continuum. And just as in these theories, there is some link with the theories of defects, as studied by Kröner, Hehl, Kondo, and others.

Conformal models of collective modes and certain ideas of superfluidity motivated our interest in using the group $\mathrm{U}(2,2)$ of pseudo-unitary transformations in the field theory. The obtained results are rather purely field-theoretic ones, nevertheless the relations with continuum mechanics are rather easily visible.

No doubt that linear theories with their superposition principle seem to be the simplest models of physical phenomena. Nevertheless, they are too poor to describe physical reality in an adequate way. Among others, they are free of essential self-interaction.

Nowadays, the nonlinear science embraces both sophisticated mathematical methods (like differential geometry, dynamical systems, functional analysis, nonlinear differential equations, solitons, etc.) and very practical applications (like struggling with pollution of the environment, salvation of the ecological equilibrium, and so on). The common denominator is the essential nonlinearity which arises from the geometrical structure of the problem and is connected to some symmetry problem. On the one hand, in the nonlinear problems, any knowledge about the symmetry group of the model or the symmetry demand extra imposed on searching of solutions help to obtain explicit analytical results or insure reliability of numerical calculations. On the other hand, it is often the case that the demand of invariance of the problem's dynamics under a sufficiently rich symmetry group leads to the nonlinearity of this model. There are various examples of such an interrelation between nonlinearity and symmetry.

- In linear electrodynamics stationary centrally-symmetric solutions of the field equations are singular at the symmetry centre and their total field energy is infinite. Interpreting such centres as point charges we obtain infinite electromagnetic masses.

- In realistic field theories underlying elementary particle physics the polynomial non-linearity usually appears. For instance, it is typical that Lagrangians have the quartic structure.

- Solitary waves which appear in various branches of fundamental and applied physics owe their existence to various kinds of nonlinearity (very often non-algebraic ones).

- General relativity is non-linear (although it is quasi-linear, at least in the gravitational sector). Although its equations are given by rational functions of field variables, Lagrangian themselves are not rational. In the Einstein theory for the first time we deal with the essential nonlinearity that is not only non-perturbative, i.e., it is not a small nonlinear correction to some dominant linear background, but also tightly connected with some symmetry demands, i.e., the demand of general covariance. Indeed, any Lagrangian theory invariant under the group of all diffeomorphisms have to be nonlinear (although, like Einstein theory, it may be quasi-linear).

- Nonlinearity of non-Abelian gauge theories have also to do with some symmetry group.
- In the mechanical study of affinely-rigid bodies, the invariance under the total affine group causes the nonlinearity of the geodetic motion and establishes some link with the theory of integrable lattices.
- Nonlinear theory of media with microstructure, internal degrees of freedom, defects, and collective modes are very important for material engineering.
- In the modern control theory based on the methods of differential geometry the nonlinearity is connected with Lie groups and symmetry problems, e.g., the controllability of the mechanical system depends on the structure of the relevant Lie group or Lie algebra. Modern methods developed by Brockett, Mayn, Millman, Süssman, and others are a generalization of the Kalman criteria for the systems with linear forces as regards control parameters.

One of the most important examples of the nonlinearity is the original Born-Infeld nonlinearity [154, 162] that was motivated by the demand to eliminate characteristic singularities of Maxwell theory like the infinite values of potentials and fields at the point source, and the infinite electrostatic energy of the field produced by a point charge (thus, the infinite electromagnetic mass of the point particle). There was also a tempting idea to repeat the success of general relativity and derive the equation of motion for point charges from the electromagnetic field equations. Unfortunately, the progress in this respect was rather limited. This is because of the fact that in general relativity the link between field equations and equations of motion is not only due to the nonlinearity itself but first of all due to the Bianchi identities which follow from the very special kind of nonlinearity implied by the general covariance. Due to its non-polynomial structure, the Born-Infeld theory is rather resistant to quantization attempts.

Although the amazing success of quantum field theory and renormalization techniques (even classical ones developed by Dirac) for some time reduced the interest in the Born-Infeld theory, nowadays interest in it is growing again in connection with strings, p-branes, alternative approaches to gravitation, etc.

Affinely invariant dynamical models of objects with affine degrees of freedom provide a very interesting example of the deep connection between physically interesting essential (non-perturbative) nonlinearity and geometrically motivated large symmetry groups. It is not accidental that such models turned out to be very intimately related to the theory of the known integrable lattices by Calogero-Moser, Sutherland, and others.

On the field-theoretic level this is also the case. There is a very tight relation between physically interesting generalized Born-Infeld type nonlinearities and the invariance under the "large" symmetry group, containing simultaneously the general covariance (invariance under the diffeomorphisms group) and the internal group of affine or conformal symmetries. There is a perspective of deep physical and mechanical applications for such non-perturbatively nonlinear models in mechanics of continua, including structured ones, dynamics of nano-structures, and nonlinear electrodynamics of continua. For example, one can expect applications in the theory of strong laser beams interacting with matter.

Appendix A

Dynamical systems on Lie groups

Let us briefly recall the general description of systems with group-theoretical degrees of freedom [2, 3, 132, 168]. We do not consider the general case of Hamiltonian systems with homogeneous spaces as configuration manifolds but just concentrate on the special case when the corresponding group G acts freely, i.e., when every point x has a trivial isotropy group H_x. More general systems on homogeneous spaces may be obtained by an appropriate quotient procedure.

For simplicity, we use the linear groups formalism because practically all Lie groups used in physics may be faithfully realized by finite-dimensional matrices. The only exceptions are covering groups of the real linear and unimodular groups, i.e., $\overline{\mathrm{GL}(n,\mathbb{R})}$ and $\overline{\mathrm{SL}(n,\mathbb{R})}$ (if $n = 2$, then we have infinite $\mathbb{Z}$-coverings, and if $n > 2$, then these are double coverings). This means that if we wanted to have linear representations of these groups, then they would be infinite-dimensional, and if we wanted to have finite-dimensional ones, then they would be non-linear realizations of $\overline{\mathrm{GL}(n,\mathbb{R})}$ or $\overline{\mathrm{SL}(n,\mathbb{R})}$. This fact was the reason of many confusions and misunderstandings in attempts of generalizing usual spinors onto affine framework so as to obtain the half-objects ruled by these groups.

A.1 Introducing geometrical objects

Hence, let us consider a mechanical system whose configuration space is identified with some (linear) Lie group G [168]. The first step of analysis is the theory of left- and right-invariant geodetic systems, i.e., such systems for which the Lagrangian and total energy are identical with the kinetic energy expression based on an appropriate Riemannian structure of G.

Motions are described as sufficiently regular curves

$$\mathbb{R} \ni t \mapsto g(t) \in G.$$

For any such curve, its tangent vectors, i.e., *generalized velocities*,

$$\dot{g}(t) \in T_{g(t)}G$$

may be transported to the Lie algebra $G' = T_eG$, where e is a neutral element of the group, with the help of right or left $g(t)^{-1}$-translations, resulting in quantities that may be called *quasi-velocities*, i.e.,

$$\Omega(t) := \dot{g}(t)g(t)^{-1}, \qquad \widehat{\Omega}(t) := g(t)^{-1}\dot{g}(t), \qquad \Omega = g\widehat{\Omega}g^{-1} = \mathrm{Ad}_g\widehat{\Omega}$$

(they are related through the adjoint transformation). If G is non-Abelian, then Ω and $\widehat{\Omega}$ are non-holonomic quasi-velocities, i.e., there are no generalized coordinates whose time derivatives are our quasi-velocities Ω and $\widehat{\Omega}$.

In this way, the tangent and cotangent bundles, i.e., TG and T^*G, may be identified in two canonical ways with the Cartesian products:

$$TG \simeq G \times G', \qquad T^*G \simeq G \times G'^*.$$

If G is a linear group, i.e.,

$$G \subset GL(W) \subset L(W)$$

for some linear space W (e.g., $\mathbb{R}^n$ or $\mathbb{C}^n$), then we can simply identify

$$L(W)^* \simeq L(W)$$

in the sense of pairing $\langle C, D\rangle = \mathrm{Tr}(CD)$. The dual of $G' \subset L(W)$ has the following form:

$$G'^* \simeq L(W)^*/\mathrm{An}G',$$

where $\mathrm{An}G'$ consists of functionals vanishing on G'. According to the above trace formula, $\mathrm{An}G'$ may be identified with some linear subspace $G'^{\perp}$ of $L(W)$, thus,

$$G'^* \simeq L(W)/G'^{\perp}.$$

In typical situations this quotient space may be canonically identified with some distinguished linear subspace of $L(W)$ consisting of natural representants of cosets. Then G'^* may be identified with G' itself.

The left and right regular translations, i.e.,

$$g \mapsto L_k(g) = kg, \qquad g \mapsto R_k(g) = gk,$$

affect the Lie-algebraic objects Ω and $\widehat{\Omega}$ according to the adjoint transformation or the invariance rule:

$$\begin{aligned} L_k &: \quad \Omega \mapsto k\Omega k^{-1} = \mathrm{Ad}_k\Omega, \qquad && \widehat{\Omega} \mapsto \widehat{\Omega}, \\ R_k &: \quad \Omega \mapsto \Omega, && \widehat{\Omega} \mapsto k^{-1}\widehat{\Omega}k = \mathrm{Ad}_k^{-1}\widehat{\Omega}. \end{aligned}$$

With the help of these quasi-velocities on the group G some right- and left-invariant vector fields X and Y can be defined, i.e.,

$$X_g[\Omega] := \Omega g, \qquad Y_g[\widehat{\Omega}] := g\widehat{\Omega}.$$

In canonical formalism, the dual objects $\Sigma, \widehat{\Sigma} \in G'^*$ are used as well. They are related to the canonical momenta $p \in T_g^*G$ and configurations g as follows:

$$\langle \Sigma, \Omega \rangle = \left\langle \widehat{\Sigma}, \widehat{\Omega} \right\rangle = \langle p, \dot{g} \rangle \,,$$

where the bracket symbol denotes evaluation of covectors on vectors. The above formula implies that

$$\Sigma = g\widehat{\Sigma}g^{-1} = \mathrm{Ad}_g^{*-1}\widehat{\Sigma},$$

where Ad_g^* is the adjoint of Ad_g. The objects Σ and $\widehat{\Sigma}$ are Hamiltonian generators of the groups of left and right regular translations, respectively. Transformation rules for them are as follows:

$$\begin{aligned} L_k &: \quad \Sigma \mapsto k\Sigma k^{-1} = \mathrm{Ad}_k^{*-1}\Sigma, \qquad && \widehat{\Sigma} \mapsto \widehat{\Sigma}, \\ R_k &: \quad \Sigma \mapsto \Sigma, && \widehat{\Sigma} \mapsto k^{-1}\widehat{\Sigma}k = \mathrm{Ad}_k^{*}\widehat{\Sigma}. \end{aligned}$$

With the help of Σ and $\widehat{\Sigma}$ we can define some right- and left-invariant covector fields, i.e., differential one-forms, A and B on the group G. If the aforementioned identification of G'^* with G' works, then

$$A_g[\Sigma] = g^{-1}\Sigma, \qquad B_g[\widehat{\Sigma}] = \widehat{\Sigma}g^{-1}.$$

Poisson brackets of Σ are expressed through the structure constants of the group G, those of $\widehat{\Sigma}$ have a reversed sign, and the mutual Poisson brackets of Σ- and $\widehat{\Sigma}$-components vanish because left and right regular translations mutually commute, i.e.,

$$\{\Sigma_\mu, \Sigma_\nu\} = C_{\mu\nu}{}^{\lambda}\Sigma_\lambda, \qquad \{\widehat{\Sigma}_\mu, \widehat{\Sigma}_\nu\} = -C_{\mu\nu}{}^{\lambda}\widehat{\Sigma}_\lambda, \qquad \{\Sigma_\mu, \widehat{\Sigma}_\nu\} = 0.$$

For any function f depending only on coordinates q, we have that

$$\{\Sigma_\mu, f\} = -L_\mu f, \qquad \{\widehat{\Sigma}_\mu, f\} = -R_\mu f,$$

where L_μ and R_μ are differential operators generating, respectively, left and right regular translations on the group G. Thus, if q^μ are canonical coordinates of the first kind on G, i.e., $g(q) = e^{q^\mu E_\mu}$, then

$$\frac{\partial}{\partial q^\mu} f\left(k(q)g\right)\bigg|_{q=0} = \left(L_\mu f\right)(g), \qquad \frac{\partial}{\partial q^\mu} f\left(gk(q)\right)\bigg|_{q=0} = \left(R_\mu f\right)(g),$$

and

$$[L_\mu, L_\nu] = C_{\mu\nu}{}^\lambda L_\lambda, \qquad [R_\mu, R_\nu] = -C_{\mu\nu}{}^\lambda R_\lambda, \qquad [L_\mu, R_\nu] = 0.$$

A.2 Geodetic systems on Lie groups

For a system with group-theoretical degrees of freedom, the kinetic energy T is equivalent to some Riemannian structure on the group G, i.e.,

$$T = \frac{1}{2}\Gamma_{\mu\nu}(q)\dot{q}^\mu \dot{q}^\nu, \tag{A.1}$$

where q^μ are some generalized coordinates on G. In general, the dynamical metric tensor Γ depends both on intrinsic geometry of the group G and on some physically motivated inertial parameters.

The theory of geodetic Hamiltonian systems on Lie groups developed by Hermann, Arnold, and others (see, e.g., [3, 62, 63]) deals with kinetic energies T (Riemannian metrics Γ) that are invariant under left or right (or both) regular translations on the group G.

The left-invariant geodetic systems on G are based on kinetic energies which are quadratic forms of $\widehat{\Omega}$ with constant coefficients, i.e.,

$$T_{\text{left}} = \frac{1}{2}\widehat{\mathcal{L}}^B{}_A{}^D{}_C \widehat{\Omega}^A{}_B \widehat{\Omega}^C{}_D, \tag{A.2}$$

where coefficients $\widehat{\mathcal{L}}$ are constant and symmetric in bi-indices $\left({}^B{}_A\right)$, $\left({}^D{}_C\right)$. This quadratic form is also assumed to be non-degenerate, although not necessarily positively definite. If $\widehat{\Omega}$ is a non-holonomic quasi-velocity, then the corresponding Riemannian structure on G is curved.

Similarly, the right-invariant geodetic systems on G are based on kinetic energies which are quadratic forms of Ω with constant coefficients, i.e.,

$$T_{\text{right}} = \frac{1}{2}\mathcal{R}^j{}_i{}^l{}_k \Omega^i{}_j \Omega^k{}_l, \tag{A.3}$$

where $\mathcal{R}$ is also constant and symmetric in bi-indices $\left(^{j}{}_{i}\right)$, $\left(^{l}{}_{k}\right)$. For non-holonomic quasi-velocities, the underlying metric tensor on the group G is also curved, i.e., essentially Riemannian.

Particularly interesting are highly-symmetric geodetic models when kinetic energies T and metric tensors Γ are simultaneously invariant under left and right regular translations. Then such kinetic energies are linear combinations of two basic second-order Casimir invariants, i.e.,

$$T_{\rm both} = \frac{A}{2}\Omega^{i}{}_{j}\Omega^{j}{}_{i} + \frac{B}{2}\Omega^{i}{}_{i}\Omega^{j}{}_{j} = \frac{A}{2}\widehat{\Omega}^{K}{}_{L}\widehat{\Omega}^{L}{}_{K} + \frac{B}{2}\widehat{\Omega}^{K}{}_{K}\widehat{\Omega}^{L}{}_{L}, \tag{A.4}$$

where A, B are some constants. Using invariant terms, we can say that such a kinetic energy $T_{\rm both}$ is a linear combination of two basic second-order Casimir invariants, i.e.,

$$T_{\rm both} = \frac{A}{2}{\rm Tr}\left(\Omega^{2}\right) + \frac{B}{2}\left({\rm Tr}\ \Omega\right)^{2} = \frac{A}{2}{\rm Tr}\left(\widehat{\Omega}^{2}\right) + \frac{B}{2}\left({\rm Tr}\ \widehat{\Omega}\right)^{2}. \tag{A.5}$$

It is easily seen that this expression is never positively-definite. The reason is that the maximal semisimple subgroups ${\rm SL}(V)$ and ${\rm SL}(U)$ (their determinants equal to unity) are non-compact, thus, the quadratic form ${\rm Tr}\left(\Omega^{2}\right) = {\rm Tr}\left(\widehat{\Omega}^{2}\right)$ has the hyperbolic signature $(n(n+1)/2\ +, n(n-1)/2\ -)$, where the positive contribution corresponds to the "non-compact" and the negative one to the "compact" dimensions in ${\rm GL}(V)$ and ${\rm GL}(U)$.

By the way, the above quadratic forms reduce to the Killing forms (Killing scalar products) on ${\rm L}(V)$ and ${\rm L}(U)$ [62, 63, 69] when $A = 2n$, $B = -2$. As ${\rm L}(V)$ and ${\rm L}(U)$ are non-semisimple, in this special unhappy case the scalar product (kinetic energy) is degenerate, thus, non-applicable in usual mechanical problems. The singularity consists of dilatational Lie algebras $\mathbb{R}{\rm Id}_{V}$, $\mathbb{R}{\rm Id}_{U}$. More generally, the same holds when $A = -Bn$. Paradoxically enough, non-degenerate forms (A.5) ($A \neq -Bn$) may be mechanically useful in spite of their non-definiteness.

The usual d'Alembert model, which is invariant under Abelian additive translations ${\rm LI}(U,V) \ni \varphi \mapsto \varphi + \alpha$, $\alpha \in {\rm L}(U,V)$ in $Q = \mathcal{M}\times\ {\rm LI}(U,V)$, is the special case of general models of the following form:

$$T_{\rm d'A} = \frac{1}{2}\mathcal{A}^{K}{}_{i}{}^{L}{}_{j}\frac{d\varphi^{i}{}_{K}}{dt}\frac{d\varphi^{j}{}_{L}}{dt}, \tag{A.6}$$

where $\mathcal{A}$ is constant and symmetric in bi-indices $\left(^{K}{}_{i}\right)$, $\left(^{L}{}_{j}\right)$.

Remark: the peculiarity of the internal part of the usual total kinetic energy

$$T = T_{\rm tr} + T_{\rm int-d'A} = \frac{m}{2}g_{ij}\frac{dx^{i}}{dt}\frac{dx^{i}}{dt} + \frac{1}{2}g_{ij}\frac{d\varphi^{i}{}_{A}}{dt}\frac{d\varphi^{i}{}_{B}}{dt}\widehat{J}^{AB} \tag{A.7}$$

within this class of the $\mathcal{A}$-based kinetic energies (A.6) is that there $\mathcal{A}$ factorizes, i.e., $\mathcal{A}^{K}{}_{i}{}^{L}{}_{j} = g_{ij}\widehat{J}^{KL}$, and it is invariant under the left action of $\mathrm{SO}(V, g)$ and the right action of $\mathrm{SO}(U, \widehat{J}^{-1})$ or, in particular, $\mathrm{SO}(U, \widehat{\eta})$, when the inertia is isotropic, i.e., $\widehat{J} = J\widehat{\eta}$. It is clear that the $\mathcal{A}$-based models of T_{int} are never invariant under $\mathrm{GL}(V)$ or $\mathrm{GL}(U)$, and the underlying metric on $\mathrm{LI}(U, V)$ is flat.

A.3 Potential models and equations of motion

The following discussion is basically the same for geodetic models when Lagrangian coincides with the kinetic energy term, i.e., $L = T$, and for such potential models when Lagrangian is a sum of the kinetic energy term and the potential that does not depend on the generalized velocities $\dot{q}$, i.e., $L = T + V(q)$. Then the Legendre transformation has the following form:

$$p_{\mu} = \frac{\partial L}{\partial \dot{q}^{\mu}} = \Gamma_{\mu\nu}\dot{q}^{\nu},$$

and the corresponding Hamiltonian is given by the following expression:

$$H = \mathcal{T} + V(q) = \frac{1}{2}\Gamma^{\mu\nu}p_{\mu}p_{\nu} + V(q), \qquad \text{(A.8)}$$

where $\Gamma^{\mu\nu}$ is a reciprocal tensor to the metric tensor $\Gamma_{\mu\nu}$, i.e., $\Gamma^{\mu\lambda}\Gamma_{\lambda\nu} = \delta^{\mu}{}_{\nu}$.

Using the Poisson brackets formalism, the equations of motion for dynamical systems with Hamiltonians of the form (A.8) can be written in the Hamilton-Poisson form:

$$\frac{df}{dt} = \{f, H\},$$

where f runs over some systems of coordinates in the phase-space manifold, i.e., some maximal system of functionally independent functions.

A.4 Quantization of classical geodetic systems

Let us consider the standard procedure of quantization of the classical geodetic models (A.1)-(A.6) or the classical potential models with the Hamiltonian (A.8) (see, e.g., [166, 168, 169, 173]).

As usual, the metric tensor Γ gives rise to the natural measure μ_{Γ} on the group G, and the Riemannian volume element based on this measure is as follows:

$$d\mu_{\Gamma}(q) = \sqrt{|\det[\Gamma_{\mu\nu}]|}dq^{1} \cdots dq^{f},$$

where f denotes the number of degrees of freedom, i.e., $f = \dim G$. For simplicity the square-root expression will be always denoted by $\sqrt{|\Gamma|}$. The mathematical framework of *Schrödinger quantization* is based on $L^2(G, \mu_\Gamma)$, i.e., the Hilbert space of complex-valued wave functions on the group G which are square-integrable in the μ_Γ-sense. Their scalar product is given by the following standard formula:

$$\langle \Psi_1 | \Psi_2 \rangle = \int \overline{\Psi_1(q)} \Psi_2(q) d\mu_\Gamma(q).$$

The classical kinetic energy expression (A.1) is replaced by the following operator:

$$\mathbf{T} = -\frac{\hbar^2}{2} \Delta(\Gamma), \tag{A.9}$$

where $\hbar$ denotes the ("crossed") Planck constant, and $\Delta(\Gamma)$ is the Laplace-Beltrami operator corresponding to the metric tensor Γ, i.e.,

$$\Delta(\Gamma) = \frac{1}{\sqrt{|\Gamma|}} \sum_{\mu,\nu} \partial_\mu \left(\sqrt{|\Gamma|} \Gamma^{\mu\nu} \partial_\nu \right) = \Gamma^{\mu\nu} \nabla_\mu \nabla_\nu. \tag{A.10}$$

Obviously, ∇_μ denotes the Levi-Civita covariant differentiation induced by the metric Γ. Therefore, the quantum kinetic energy operator $\mathbf{T}$ is formally obtained from the corresponding classical expression $\mathcal{T}$ (kinetic Hamiltonian) by the substitution of p_μ by the operator $\mathbf{p}_\mu = (\hbar/i)\nabla_\mu$, which is formally self-adjoint in $L^2(G, \mu_\Gamma)$.

If the classical problem is non-geodetic and some potential $V(q^1, \ldots, q^f)$ (the velocity-dependent generalized potentials are not considered here) is admitted, then the corresponding quantum Hamilton (energy) operator is given by the following expression:

$$\mathbf{H} = \mathbf{T} + \mathbf{V},$$

where the operator $\mathbf{V}$ acts on wave functions simply multiplying them by V, i.e., $(\mathbf{V}\Psi)(q) = V(q)\Psi(q)$. This is the reason why very often we do not distinguish graphically between $\mathbf{V}$ and V.

A.5 Some instructive examples

Let us quote a few examples of group-theoretical configuration spaces of collective modes and internal degrees of freedom [168]:

- $G = \mathrm{E}(n, \mathbb{R}) = \mathrm{SO}(n, \mathbb{R}) \times_s \mathbb{R}^n$ for *a metrically-rigid body.*

- $G = \mathrm{SO}(n, \mathbb{R})$ for the metrically-rigid body without translational motion. The quasi-velocities Ω and $\widehat{\Omega}$ are skew-symmetric, and their matrix elements are components of the angular velocity with respect to the space- and body-fixed reference frames, respectively. $\mathrm{SO}(n, \mathbb{R})'^{*}$ may be canonically identified with $\mathrm{SO}(n, \mathbb{R})'$ itself through the following trace formula:

  $$\langle A, B \rangle = \frac{1}{2} \mathrm{Tr}(AB).$$

  The skew-symmetric dual objects Σ and $\widehat{\Sigma}$ describe the rotational angular momentum in terms of the space- and body-fixed reference frames, respectively. Left regular translations describe the rigid body spatial rotations, whereas the right translations permute material points without affecting the body orientation in the physical space. The kinetic energy expression is as follows:

  $$T = -\frac{1}{2} \mathrm{Tr} \left(\widehat{\Omega}^2 \widehat{J} \right),$$

  where $\widehat{J}$ is a constant symmetric positively definite matrix describing the rotational inertia. This kinetic energy T is left-invariant, i.e., non-sensitive with respect to spacial rotations. It becomes also right-invariant when the top is spherical, i.e., $\widehat{J}$ is proportional to the identity matrix. In general, degeneracy of $\widehat{J}$ corresponds to the invariance with respect to the right action of certain subgroups of $\mathrm{SO}(n, \mathbb{R})$. If $n = 3$ and $\widehat{J}$ is once degenerate, then we deal with the symmetric top. The corresponding geodetic Hamiltonians can be written as follows:

  $$\mathcal{T} = -\frac{1}{2} \mathrm{Tr} \left(\widehat{\Sigma}^2 \widehat{J}^{-1} \right).$$

- $G = \mathrm{GAf}(n, \mathbb{R}) \simeq \mathrm{GL}(n, \mathbb{R}) \times_s \mathbb{R}^n$ for *an affinely-rigid (homogeneously-deformable) body.*

- $G = \mathrm{GL}(n, \mathbb{R})$ for the affinely-rigid body without translational motion. Then Lie algebra $G' = \mathrm{L}(n, \mathbb{R})$ and $G'^{*} \simeq \mathrm{L}(n, \mathbb{R})$.

- $G = \mathrm{SL}(n, \mathbb{R}) \times_s \mathbb{R}^n$ for *an incompressible affinely-rigid body.*

- $G = \mathrm{SL}(n, \mathbb{R})$ for the incompressible affinely-rigid body without translational motion. Then Lie algebra $G' = \mathrm{SL}(n, \mathbb{R})' \simeq \mathrm{SL}(n, \mathbb{R})'^{*}$ consists of all trace-less matrices.

- $G = \mathrm{Pr}(n, \mathbb{R}) \simeq \mathrm{SL}(n + 1, \mathbb{R})$ for *a projectively-rigid body.*

- $G = \mathrm{U}(n)$ for *a unitary-rigid body*. Then Lie algebra $G' = \mathrm{U}(n)' \simeq \mathrm{U}(n)'^*$ consists of all anti-hermitian matrices, i.e., ones satisfying: $A^\dagger = -A$.

- $G = \mathrm{GL}(n, \mathbb{C})$ for *a complexified affinely-rigid body*. Then Lie algebra $G' \simeq \mathrm{L}(n, \mathbb{C}) \simeq \mathrm{L}(n, \mathbb{C})^*$. The real Lie groups $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{U}(n)$ are two different (in a sense opposite) real forms of the same complex Lie group $\mathrm{GL}(n, \mathbb{C})$.

- $G = \mathrm{Diff}(\mathbb{R}^n)$ for *a compressible continuous medium*.

- $G = \mathrm{SDiff}(\mathbb{R}^n)$ for *an ideal incompressible fluid* [3]. This is an infinite-dimensional group that consists of all volume-preserving diffeomorphisms of $\mathbb{R}^n$ onto itself. Lie algebra G' consists of vector fields with the vanishing divergence. The geodetic Hamiltonian system on $\mathrm{SDiff}(\mathbb{R}^n)$ underlying the ideal fluid dynamics is right-invariant.

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Curriculum vitae

Vasyl Kovalchuk was born on December 8, 1973, to Vasyl and Hanna Kovalchuk in the town of Kamin-Kashyrski, Ukraine. In June 1991, he graduated from Kamin-Kashyrski State Secondary School #1 (certificate with honours).

In September 1991, he began studies at Lviv National University named after Ivan Franko in the city of Lviv, Ukraine. In June 1996, he successfully received a Master of Science Degree in Physics (diploma with honours) defending the thesis "Taking into account relativistic corrections for atomic interactions".

From November 1996 to October 1999, he took a post-graduate course in Theoretical Physics at Lviv National University. During that period, he conducted there practical lessons in Theoretical Mechanics, Electrodynamics, Thermodynamics, Statistical Physics, and colloquia in Quantum Mechanics for students of Physics Department.

In October 2000, he moved to Warsaw and continued his education at the Institute of Fundamental Technological Research of Polish Academy of Sciences towards a Doctor of Philosophy Degree in Mechanics. In February 2005, in connection with his PhD studies the Supervisor Programme (grant 4 T07A 032 28) "Nonlinear dynamics of collective deformation modes and its application in mechanics of media with micro- and nanostructures" financed by the Ministry of Scientific Research and Information Technology was initiated.