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ORIGINAL ARTICLE

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Hamiltonian and Lagrangian theory of viscoelasticity

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Abstract The viscoelastic relaxation modulus is a positive-definite function of time. This property alone allows the definition of a conserved energy which is a positive-definite quadratic functional of the stress and strain fields. Using the conserved energy concept a Hamiltonian and a Lagrangian functional are constructed for dynamic viscoelasticity. The Hamiltonian represents an elastic medium interacting with a continuum of oscillators. By allowing for multiphase displacement and introducing memory effects in the kinetic terms of the equations of motion a Hamiltonian is constructed for the visco-poroelasticity.

Keywords Viscoelasticity · Poroelasticity · Relaxation · Energy conservation · Hamiltonian · Lagrangian · Poisson bracket

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List of symbols

$]a, b[$—the open interval with ends at a, b;
$\dot{u} = \partial u/\partial t$;
f'—distributional derivative of f;
δ_a—Dirac delta measure with support at a;
$\mathscr{L}^2(\mathbb{R}_+; \Sigma; m)$, Σ: page 483;
$\langle \mathbf{a}, \mathbf{b}\rangle = a_{kl}\, b_{kl}$—scalar product on S;
$|\mathbf{a}| := \langle \mathbf{a}, \mathbf{a}\rangle^{1/2}$;
$\mathsf{A} \geq 0$ ($\mathsf{A} \in \Sigma$) is equivalent to $\langle \mathbf{v}, \mathsf{A}\,\mathbf{v}\rangle \geq 0$ for all $\mathbf{v} \in S$;
λ—the Lebesgue measure on $\Omega \subset \mathbb{R}^d$;
S—space of real symmetric tensors of rank 2;
$S_{\mathbb{C}} = S \oplus \mathrm{i}\, S$—space of complex symmetric rank 2 tensors;

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Σ, $\Sigma_{\mathbb{C}}$—space of symmetric operators on S, $S_{\mathbb{C}}$;
$\mathscr{I}$: page 483;
$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbb{C}} = \overline{a_{kl}}\, b_{kl}$—scalar product on $S_{\mathbb{C}}$;
da—surface area in $\mathbb{R}^d$.

1 Introduction

An explicit Hamiltonian and Lagrangian are constructed for general linear hereditary viscoelastic media. The underlying Poisson structure is defined in terms of canonical coordinates on an infinite-dimensional phase space. Compatibility of dissipation with Hamiltonian and Lagrangian mechanics is thus demonstrated. Contrary to a widespread belief in fluid dynamics and rheology ([29,34,20,21,15,6]), it is not necessary to amend the Poissonian formalism by an additional dissipative bracket in order to account for irreversibility and energy dissipation within the Hamiltonian framework. Since the model represents only the mechanical energy [9] it is expected that dissipation of mechanical energy should result in a decrease of energy if the system is isolated from energy supply. Energy conservation is achieved in thermo-viscoelasticity by bringing thermal variables into play. We demonstrate here that a reasonable conserved energy exists for purely mechanical viscoelastic systems without introducing thermal variables. The role of thermal energy is then taken over by a continuum of uncoupled oscillators driven by the elastic field. By a purely mathematical argument, using only the assumption that the relaxation modulus is a positive-definite function, we arrive at a stored energy E_{C} consisting of the energy of an elastic subsystem, the energy of a subsystem consisting of a continuum of uncoupled driven oscillators, and the coupling energy. The forces driving the oscillators are determined by the strain of the elastic subsystem. The sum of the kinetic energy and E_{C}, integrated over the volume of an isolated viscoelastic body, is conserved. An apparent contradiction between energy dissipation and energy conservation is thus resolved by the appearance of a matter subsystem (oscillators) in addition to the elastic field energy. The matter subsystem, represented by a one-parameter family of oscillators, is similar to the thermal reservoir [18] except for the absence of randomness. There is thus an analogy with an electromagnetic field interacting with matter.

Compatibility of dissipation with Hamiltonian and Lagrangian mechanics of finite-dimensional systems has been the subject of a long-standing debate [5,4,38,39,2,14]. Negative conclusions reported in [5] are based on excessively restrictive assumptions on the general form of the equations. On the positive side, Riewe's Hamiltonian fractional-order equations [38,39,12] provide an example of Hamiltonian and Lagrangian formalism for dissipative and dispersive media. Riewe's fractional-order equations formally resemble finite-dimensional Hamiltonian systems. Riewe's Hamiltonian is, however, not an integral of motion because the usual chain rule of differentiation does not apply to fractional-order derivatives. Riewe's approach is limited to fractional-order time derivatives and therefore is not applicable to viscoelasticity.

Explicit integrals of motion have recently been found for the damped oscillator [11]. They are in general not quadratic functionals and the search for them is somewhat ad hoc.

Although there is an abundant literature on classical Poisson and Lie–Poisson structures for nondissipative continuous media and weakly nonlocal completely integrable systems, including MHD, elasticity, and hydrodynamics [33,36,27,32], Poisson brackets for dissipative hereditary continuous media have not been constructed yet. An exception is the Hamiltonian theory of dielectric relaxation developed by Tip [42,43]. Tip's Hamiltonian theory is based on a reasoning which is roughly consistent with the method adopted in this paper, although it lacks a rigorous mathematical underpinning. Hamiltonian theory of dielectric media has proved very useful in quantization of the bulk dielectric interacting with individual molecules. Extending Tip's method to the total field momentum Stallinga [41] obtained an energy–momentum conservation in a generalized Minkowski formulation. A more general Hamiltonian theory of dissipative media, developed by Figotin and Schenker in [17], can be applied to dissipative and dispersive dielectric media after an appropriate reformulation of the Maxwell equations but it is not clear how it could be applied to viscoelastic media. It is based on the spectral characterization of material response in the framework of the Herglotz–Nevanlinna theory.

It is shown herein that ordinary hereditary linear viscoelasticity is consistent with energy conservation and can be formulated in the Hamiltonian framework with canonical Poisson brackets (i.e., in Darboux coordinates).

A conserved energy involving a quadratic stored energy functional E_{C} can be constructed under a very mild restriction on the response functions, satisfied by the viscoelastic behavior of every real material. In the purely mechanical model of viscoelasticity considered here the dissipated energy turns out to be stored in a continuum of driven oscillators. In the conserved energy formulation the driven oscillators play the role of the heat bath and, in fact, they have the spectral properties of the heat bath model of [28].

The construction of conserved energy depends on the assumption that the relaxation modulus is a positive-definite function. Hereditary constitutive equations in continuous media subject to relaxation processes involve Volterra operators which have positive-definite kernels. The last property can be justified by the fluctuation–dissipation theorem [30,13]. A spectral representation of positive definite functions implied by the Bochner theorem provides a systematic tool for the construction of quadratic integrals of motion for linear models of dissipative–dispersive continuous media.

Viscoelastic relaxation in real materials is additionally characterized by a positive relaxation time spectrum, which also ensures that the viscoelastic relaxation modulus is a positive-definite function [26]. Therefore viscoelastic relaxation moduli belong to a special subclass of positive-definite functions consisting of locally integrable completely monotone (LICM) functions. LICM relaxation moduli additionally admit a dissipative energy functional E_{M} that can be derived from the Bernstein theorem [24]. The dissipative energy functional involves Debye elements instead of oscillators. The dissipative energy functional plays a key role in the proof of existence and uniqueness of solutions of initial-value problems in visco- and thermo-viscoelastic media [23].

Our first step is to use Bochner's theorem to construct the stored energy component of the conserved energy. Hamiltonian formulation of viscoelasticity is then achieved by constructing generalized coordinates and the conjugate momenta for the conserved energy $E_{\mathrm{K}} + E_{\mathrm{C}}$. The Lagrangian theory of viscoelasticity follows by a Legendre transformation.

2 Formulation of linear dynamic viscoelasticity

Let S denote the space of symmetric rank-2 tensors on $\mathbb{R}^d$, and let Σ be the space of symmetric operators on S. The dimension of S is $D := d(d+1)/2$. The elements of Σ can be considered as rank-4 tensors with the minor and major symmetries known from the theory of elasticity. The convolution of a function $\mathsf{F} : [0, \infty[\rightarrow \Sigma$ and an S-valued function $\mathbf{g} : \mathbb{R} \rightarrow S$ is defined by the formula

$$\mathsf{F} * \mathbf{g}(t) = \int_0^\infty \mathsf{F}(s)\, \mathbf{g}(t-s)\, \mathrm{d}s \tag{1}$$

or

$$(\mathsf{F} * \mathbf{g})_{kl}(t) = \int_0^\infty F_{klmn}(s)\, g_{mn}(t-s)\, \mathrm{d}s \tag{2}$$

componentwise.

The viscoelastic medium occupies a domain $\Omega \subset \mathbb{R}^d$ and satisfies the momentum balance

$$\rho\, \dot{\mathbf{v}} = \operatorname{div} \boldsymbol{\sigma} + \mathbf{f} \tag{3}$$

as well as the constitutive equation

$$\boldsymbol{\sigma} = \mathsf{G} * \dot{\mathbf{e}} \tag{4}$$

where $\mathbf{v} = \dot{\mathbf{u}}$ is the particle velocity and $e_{kl} = (u_{k,l} + u_{l,k})/2$ is the strain.

$$\sigma_{kl}(t, x) = \int_0^\infty G_{klmn}(s, x)\, \dot{e}_{mn}(t-s, x)\, \mathrm{d}s, \tag{5}$$

where $G_{klmn}(s, x) = G_{mnkl}(s, x) = G_{lkmn}(s, x)$.

Assumption 1

$$\int_{-\infty}^{\infty} \langle \mathbf{f}(t), \mathsf{G} * \mathbf{f}(t) \rangle \, \mathrm{d}t \geq 0 \tag{6}$$

for all square integrable functions $\mathbf{f} : \mathbb{R} \rightarrow S$ with compact support.

We shall define a complex space $S_{\mathbb{C}} = S \oplus \mathrm{i}S$ of pairs $(\mathbf{v}, \mathbf{w}) \in S \oplus S$ such that multiplication by complex numbers is defined by $(a + \mathrm{i}b)\,(\mathbf{v}, \mathbf{w}) = (a\,\mathbf{v} - b\,\mathbf{w}, b\,\mathbf{v} + a\,\mathbf{w})$. By $\Sigma_{\mathbb{C}}$ we shall denote the complex linear space of complex linear mappings $S_{\mathbb{C}} \rightarrow S_{\mathbb{C}}$.

Assumption 1 asserts that G is a tensor-valued causal positive-definite function (or function of positive type in the terminology of [19]). A generalized version of the Bochner theorem implies that G is the Fourier transform of a tensor-valued Radon measure M satisfying some positivity property. We shall state the Bochner theorem in a form appropriate for our purposes:

Theorem 1 *If Assumption 1 is satisfied and the function* G *has a finite limit* $\mathsf{G}_0 := \lim_{t \rightarrow 0+} \mathsf{G}(t)$*, then there is a* $\Sigma_{\mathbb{C}}$*-valued Radon measure* M *satisfying the inequalities*:

$$\langle \mathbf{y}, \mathsf{M}([a, b])\,\mathbf{y} \rangle_{\mathbb{C}} \geq 0 \quad \forall \mathbf{y} \in S_{\mathbb{C}}, \quad \forall a, b \in \mathbb{R}, \quad a < b \tag{7}$$

such that

$$\mathsf{G}(s) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi s}\, \mathsf{M}(\mathrm{d}\xi), \qquad s \geq 0. \tag{8}$$

The inverse of (8) *is*

$$\mathsf{M}(\xi) = \frac{1}{2\pi} \left(\hat{\mathsf{G}}(\xi) + \hat{\mathsf{G}}(\xi)^{\dagger} \right), \tag{9}$$

where the circumflex denotes the inverse Fourier transform (in the distributions sense):

$$\hat{\mathsf{G}}(\xi) := \int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}\xi t}\, \mathsf{G}(t)\, \mathrm{d}t. \tag{10}$$

Theorem 1 follows from Theorem 16.2.7 in [19] by identifying the complex space $S_{\mathbb{C}}$ and its subspace S with the space $\mathbb{C}^D$ and its subspace $\mathbb{R}^D$, $D = d(d+1)/2$.

By Lemma A1 the tensor-valued Radon measure M is positive, it can be factored into a positive real-valued measure $m_1(\mathrm{d}\xi)$ and an m_1-integrable tensor-valued function N, which is the Radon–Nikodym derivative of M with respect to m_1:

$$\mathsf{M}(\mathrm{d}\xi) = \mathsf{N}(\xi)\, m_1(\mathrm{d}\xi).$$

Growth conditions satisfied by m_1 are rather complicated [40].

The last equation has to be modified to allow for inhomogeneous viscoelastic media:

$$\mathsf{M}(\mathrm{d}\xi, x) = \mathsf{N}(\xi, x)\, m_1(\mathrm{d}\xi, x). \tag{11}$$

Equation (8) can be expressed in terms of a Stieltjes integral

$$\mathsf{G}(s, x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi s}\, \mathsf{N}(\xi, x)\, \mathrm{d}_{\xi} \mu_1(\xi, x), \tag{12}$$

where $\mu_1(\xi, x) := m_1([0, \xi], x)$, $\mathrm{d}_{\xi}\, \mu_1(\xi, x) := m_1(\mathrm{d}\xi, x)$ and

$$\lim_{\xi \rightarrow \infty} \mu_1(\xi, x) = \mu_1^{\infty}(x) < \infty. \tag{13}$$

Equation (13) follows from the fact that $\mu_1^\infty(x) = m_1([0,\infty[\,,x) = |\mathsf{M}|([0,\infty[\,,x)$. The function $\mu_1(\cdot,x)$ is nondecreasing and continuous from the right, while the function $\mathsf{N}(\xi,x) \in \Sigma$ satisfies the inequalities

$$\mathsf{N}(\xi,x) \geq 0 \tag{14}$$

and $|\mathsf{N}(\xi,x)| \leq 1$ for $m_1 \times \lambda$-almost all $(\xi,x) \in \mathbb{R} \times \Omega$.

If we set

$$m(\{0\},x) := m_1(\{0\},x) \quad \text{a.a. in } \Omega, \tag{15}$$

$$m(\mathscr{U},x) := \frac{1}{2} m_1(\mathscr{U},x) \quad \text{for every measurable } \mathscr{U} \text{ such that } 0 \notin \mathscr{U}, \text{ a.e. in } \Omega \tag{16}$$

and $\mu(\xi,x) := m([0,\xi],x)$, then, by Lemma 2 (ii),

$$\mathsf{G}(s,x) = \int\limits_{[0,\infty[} \cos(s\,\xi)\,\mathsf{N}(\xi,x)\,m(\mathrm{d}\xi,x). \tag{17}$$

For the construction of the Hamiltonian we shall need an additional assumption:

Assumption 2 $\mathsf{N}(\xi,x)$ is invertible $m_1 \times \lambda$-almost everywhere.

Let

$$\mathbf{y}(t,x;\xi) := \int\limits_{-\infty}^{t} \mathrm{e}^{\mathrm{i}\xi(t-t')}\,\dot{\mathbf{e}}(t',x)\;\mathrm{d}t', \tag{18}$$

$$\mathbf{w}(t,x;\xi) := \int\limits_{-\infty}^{t} \cos(\xi(t-t'))\,\dot{\mathbf{e}}(t',x)\;\mathrm{d}t', \tag{19}$$

$$\mathbf{z}(t,x;\xi) := \int\limits_{-\infty}^{t} \sin(\xi(t-t'))\,\dot{\mathbf{e}}(t',x)\;\mathrm{d}t', \tag{20}$$

$$\boldsymbol{\zeta}(t,x;\xi) := (\mathbf{w}(t,x;\xi) - \mathbf{e}(t,x))/\xi. \tag{21}$$

The stress can be expressed in terms of the auxiliary field $\mathbf{w}$:

$$\boldsymbol{\sigma}(t,x) = \int\limits_0^\infty \left[\,\int\limits_{[0,\infty[} \cos(s\,\xi)\,\mathsf{N}(\xi,x)\,m(\mathrm{d}\xi,x)\right]\dot{\mathbf{e}}(t-s)\,\mathrm{d}s = \int\limits_{[0,\infty[} \mathsf{N}(\xi,x)\,\mathbf{w}(t,x;\xi)\,m(\mathrm{d}\xi,x) \tag{22}$$

using the Fubini theorem.

The auxiliary field $\mathbf{w}$ satisfies the differential equation

$$\ddot{\mathbf{w}}(t,x;\xi) + \xi^2\,\mathbf{w}(t,x;\xi) = \ddot{\mathbf{e}}(t,x) \tag{23}$$

while

$$\mathbf{z}(t,x;\xi) = \begin{cases} -(\dot{\mathbf{w}} - \dot{\mathbf{e}})/\xi & \text{if } \xi > 0, \\ 0 & \text{if } \xi = 0, \end{cases} \tag{24}$$

and $\mathbf{y} = \mathbf{w} + \mathrm{i}\,\mathbf{z}$. The dots denote derivatives with respect to the time t. The initial data for Eq. (23) can be determined from Eq. (19):

$$\mathbf{w}(0,x,\xi) = \int\limits_{-\infty}^{0} \cos(\xi\,t')\,\dot{\mathbf{e}}(t')\,\mathrm{d}t', \tag{25}$$

$$\dot{\mathbf{w}}(0,x,\xi) = \dot{\mathbf{e}}(0,x) + \xi \int\limits_{-\infty}^{0} \sin(\xi\,t')\,\dot{\mathbf{e}}(t')\,\mathrm{d}t'. \tag{26}$$

If $\lim_{t\to-\infty}\mathbf{e}(t,x)=0$, then

$$\mathbf{w}(t,x;0)=\mathbf{e}(t,x), \tag{27}$$

$$\mathbf{z}(t,x;0)=0. \tag{28}$$

The stored energy density is defined by the following formula

$$\begin{aligned} U(t,x) &:= \frac{1}{2}\int\limits_{[0,\infty[} \langle \mathbf{y}(t,x;\xi), \mathsf{N}(\xi,x)\,\mathbf{y}(t,x;\xi)\rangle_{\mathbb{C}}\, m(\mathrm{d}\xi) \\ &= \frac{1}{2}\int\limits_{[0,\infty[} \langle \mathbf{w}(t,x;\xi), \mathsf{N}(\xi,x)\,\mathbf{w}(t,x;\xi)\rangle\, m(\mathrm{d}\xi) + \frac{1}{2}\int\limits_{[0,\infty[} \langle \mathbf{z}(t,x;\xi), \mathsf{N}(\xi,x)\,\mathbf{z}(t,x;\xi)\rangle\, m(\mathrm{d}\xi) \\ &= \frac{1}{2}\langle \mathbf{e}(t,x), \mathsf{G}_\infty(x)\,\mathbf{e}(t,x)\rangle + \frac{1}{2}\int\limits_{]0,\infty[} \langle \mathbf{w}(t,x;\xi), \mathsf{N}(\xi,x)\,\mathbf{w}(t,x;\xi)\rangle\, m(\mathrm{d}\xi) \\ &\quad + \frac{1}{2}\int\limits_{]0,\infty[} \langle \mathbf{z}(t,x;\xi), \mathsf{N}(\xi,x)\,\mathbf{z}(t,x;\xi)\rangle\, m(\mathrm{d}\xi), \end{aligned} \tag{29}$$

where

$$\mathsf{G}_\infty(x) := \Delta\mu(x)\,\mathsf{N}(0,x) \tag{30}$$

and $\Delta\mu(x) := \mu(0+,x)-\mu(0-,x) = m(\{0\})$. The second line follows from the fact that N is a real and symmetric tensor-valued function.

Theorem 2

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \langle \boldsymbol{\sigma}, \dot{\mathbf{e}}\rangle$$

Proof Working out the derivative $\mathrm{d}U/\mathrm{d}t$ and noting that the terms involving $\mathrm{i}\,\xi\,\mathbf{y}$ cancel,

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{1}{2}\int\limits_{[0,\infty[} \langle \mathbf{y}(\xi), \mathsf{N}(\xi)\,\dot{\mathbf{y}}(\xi)\rangle_{\mathbb{C}}\, m(\mathrm{d}\xi) + \frac{1}{2}\int\limits_{[0,\infty[} \langle \dot{\mathbf{y}}(\xi), \mathsf{N}(\xi)\,\mathbf{y}(\xi)\rangle_{\mathbb{C}}\, m(\mathrm{d}\xi) = \langle \boldsymbol{\sigma}, \dot{\mathbf{e}}\rangle,$$

where the irrelevant arguments t, x have been suppressed. In the last line the symmetry of the operator N and Eq. (22) were used. □

The energy balance for a simple material

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\rho\mathbf{v}^2}{2} + \langle \boldsymbol{\sigma}, \dot{\mathbf{e}}\rangle + \operatorname{div}\mathbf{j} = \mathbf{f}^\top\,\mathbf{v}, \tag{31}$$

where $j^l = -\sigma^{kl}\,v_l$ is the energy flux density, implies that the total energy

$$E_{\mathrm{tot}}(t) = \int\limits_{\Omega} E(t,x)\,\lambda(\mathrm{d}x), \tag{32}$$

where $E(t,x)$ is the energy density

$$E := \frac{\rho\,\mathbf{v}^2}{2} + U, \tag{33}$$

satisfies the following energy balance

$$\frac{\mathrm{d}E_{\mathrm{tot}}}{\mathrm{d}t} = -\int\limits_{\partial\Omega} \mathbf{j}^\top\mathbf{n}\,\mathrm{d}a + \int\limits_{\Omega} \mathbf{f}^\top\,\mathbf{v}\,\lambda(\mathrm{d}x), \tag{34}$$

where $\mathbf{n}$ is the exterior unit normal on $\partial\Omega$.

If $\partial\Omega = \Sigma_1\cup\Sigma_2$, $\Sigma_1\cap\Sigma_2=\emptyset$, and $\mathbf{v}=0$ on Σ_1, $\boldsymbol{\sigma}\,\mathbf{n}=0$ on Σ_2, $\mathbf{f}=0$ in Ω, then the total energy E_{tot} is conserved.

3 Hamiltonian formulation of linear viscoelasticity

The generalized coordinates are the displacement vector $\mathbf{u}$ and the auxiliary variables $\boldsymbol{\zeta}(t, x; \xi) := \xi^{-1} (\mathbf{w}(t, x; \xi) - \mathbf{e}(t, x))$. Note that Eq. (23) is equivalent to the equation

$$\ddot{\boldsymbol{\zeta}}(t, x; \xi) + \xi^2 \boldsymbol{\zeta}(t, x; \xi) = -\xi\, \mathbf{e}(t, x). \tag{35}$$

The generalized momenta are defined by the equations

$$\mathbf{p}(t, x) = \rho(x)\, \mathbf{v}(t, x), \tag{36}$$

$$\mathbf{q}(t, x; \xi) = \mathsf{N}(\xi, x)\, \dot{\boldsymbol{\zeta}}(t, x; \xi). \tag{37}$$

The Hamiltonian $H(\mathbf{u}, \boldsymbol{\zeta}, \mathbf{p}, \mathbf{q})$ is the energy density E_{tot} expressed in terms of the generalized coordinates and momenta:

$$H(t) = \int_{\Omega} h(t, x)\, \lambda(\mathrm{d}x) \tag{38}$$

with

$$h = \frac{1}{2\rho}\, \mathbf{p}(x)^2 + \frac{1}{2} \langle \mathbf{e}(x), \mathsf{G}_0(x)\, \mathbf{e}(x) \rangle + \frac{1}{2} \int_{[0,\infty[} \xi^2 \langle \boldsymbol{\zeta}(x, \xi), \mathsf{N}(\xi)\, \boldsymbol{\zeta}(x, \xi) \rangle\, m(\mathrm{d}\xi, x)$$
$$+ \left\langle \int_{[0,\infty[} \xi\, \mathsf{N}(\xi, x)\, \boldsymbol{\zeta}(x, \xi)\, m(\mathrm{d}\xi, x), \mathbf{e}(x) \right\rangle + \frac{1}{2} \int_{[0,\infty[} \langle \mathbf{q}(\xi), \mathsf{N}(\xi, x)^{-1}\, \mathbf{q}(\xi) \rangle\, m(\mathrm{d}\xi, x), \tag{39}$$

where the argument t has been suppressed, and

$$\mathsf{G}_0(x) := \int_{[0,\infty[} \mathsf{N}(\xi, x)\, m(\mathrm{d}\xi, x). \tag{40}$$

In view of (14) and (40) the right-hand side of Eq. (39) is a positive-definite functional of $(\mathbf{p}, \mathbf{e}, \boldsymbol{\zeta}, \mathbf{q})$.

In addition to the displacement $\mathbf{u}$ and momentum $\mathbf{p}$ fields the arguments of the Hamiltonian H include the one-parameter family of fields $\boldsymbol{\zeta}$, $\mathbf{q}$ on the space

$$\mathscr{H}^1_m = \left\{ \mathbf{g} : [0, \infty[\times \Omega \to S \mid \int_{\Omega} \left[\int_{[0,\infty[} [\langle \mathbf{g}(\xi, x),\ \mathbf{g}(\xi, x) \rangle + \langle \nabla\, \mathbf{g}(\xi, x), \nabla\, \mathbf{g}(\xi, x) \rangle]\, m(\mathrm{d}\xi, x) \right] \lambda(\mathrm{d}x) < \infty \right\}.$$

Note that the variables

$$\boldsymbol{\zeta}(t, x; \xi) = -\frac{1}{\xi} \int_{-\infty}^{t} [1 - \cos(\xi\, t')]\, \dot{\mathbf{e}}(t', x)\, \mathrm{d}t'$$

and

$$\mathbf{q}(t, x; \xi) = -\frac{1}{\xi} [1 - \cos(\xi t)]\, \mathsf{N}(\xi, x)\, \dot{\mathbf{e}}(t, x) - \mathsf{N}(\xi, x) \int_{-\infty}^{t} \sin(\xi\, t')\, \dot{\mathbf{e}}(t', x)\, \mathrm{d}t'$$

vanish at $\xi = 0$. Therefore the integration over ξ in (38) can be restricted to the open interval $]0, \infty[$.

We assume that the bulk loads $\mathbf{f}$ vanish and either $\mathbf{u}$ or $\boldsymbol{\sigma}\,\mathbf{n}$ vanishes at every point of $\partial\Omega$. The Hamiltonian equations are

$$\mathbf{v} = \frac{\partial H}{\partial \mathbf{p}} \equiv \frac{1}{\rho}\,\mathbf{p}(t, x, \xi), \tag{41}$$

$$\dot{\boldsymbol{\zeta}} = \frac{\partial H}{\partial \mathbf{q}} \equiv \mathsf{N}(\xi, x)^{-1}\,\mathbf{q}(t, x; \xi), \tag{42}$$

$$\rho\,\dot{\mathbf{v}} \equiv \dot{\mathbf{p}} = -\mathrm{D}_{\mathbf{u}} H \equiv \operatorname{div} \boldsymbol{\sigma}, \tag{43}$$

$$\mathsf{N}(\xi, x)\,\ddot{\boldsymbol{\zeta}} \equiv \dot{\mathbf{q}} = -\mathrm{D}_{\boldsymbol{\zeta}} H = -\xi\,\mathsf{N}(\xi, x)\,\mathbf{w}(t, x, \xi) \tag{44}$$

with $\mathbf{w} \equiv \xi\,\boldsymbol{\zeta} + \mathbf{e}$. The symbol $\mathrm{D}_{\mathbf{u}}$ denotes the Gâteaux derivative with respect to $\mathbf{u} \in \mathscr{L}^2(\Omega; \mathbb{R}^d)$, while $\mathrm{D}_{\boldsymbol{\zeta}}$ denotes the Gâteaux derivative with respect to $\boldsymbol{\zeta} \in \mathscr{L}_m$. If $\mathsf{N}(\xi, x)$ is invertible $(m \times \lambda)$-almost everywhere, then the last equation is equivalent to

$$\ddot{\boldsymbol{\zeta}} + \xi^2\,\boldsymbol{\zeta} = -\xi\,\mathbf{e}. \tag{45}$$

The first two equations reproduce the definition of the generalized momenta and are easy to verify. We shall prove the third and fourth line. Noting that

$$\int\limits_{[0,\infty[} \xi\,\mathsf{N}(\xi, x)\,\boldsymbol{\zeta}(t, x; \xi)\,m(\mathrm{d}\xi, x) = \tilde{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \mathsf{G}_0\,\mathbf{e},$$

one arrives at the formulae

$$\mathrm{D}_{\mathbf{u}}\,H[\delta\mathbf{u}] = \int\limits_{\Omega} (\tilde{\sigma}_{kl} + [\mathsf{G}_0\,\mathbf{e}]_{kl}\,)\ \delta u_{k,l}\,\lambda(\mathrm{d}x) = -\int\limits_{\Omega} \sigma_{kl,l}\ \delta u_k\ \lambda(\mathrm{d}x)$$

and

$$\mathrm{D}_{\boldsymbol{\zeta}} H[\delta\boldsymbol{\zeta}] = \int\limits_{\Omega} \Bigg[\int\limits_{[0,\infty[} \xi^2\,\langle \mathsf{N}(\xi, x)\,\boldsymbol{\zeta}(t, x; \xi), \delta\boldsymbol{\zeta}(t, x; \xi)\rangle m(\mathrm{d}\xi, x) + \Bigg\langle \mathbf{e}, \int\limits_{[0,\infty[} \xi\,\mathsf{N}(\xi, x)\,\delta\boldsymbol{\zeta}(t, x; \xi)\,m(\mathrm{d}\xi, x) \Bigg\rangle \Bigg]\,\lambda(\mathrm{d}x).$$

The third equation is thus equivalent to the equation of motion (3), while the fourth one is equivalent to (23).

The Hamiltonian represents an elastic medium, defined by the Hamiltonian

$$H_0 = \frac{1}{2} \int\limits_{\Omega} \left[\frac{\mathbf{p}^2}{\rho} + \langle \mathbf{e}, \mathsf{G}_0\,\mathbf{e}\rangle \right] \lambda(\mathrm{d}x),$$

which interacts with a one-parameter family of oscillators (23) driven by the strain rate of the elastic medium.

4 Lagrangian formulation of linear dynamic viscoelasticity

The Lagrangian $L(\mathbf{u}, \boldsymbol{\zeta}, \mathbf{v}, \dot{\boldsymbol{\zeta}})$ can now be constructed by applying the Legendre transformation $\mathbf{p} = \rho\,\mathbf{v}$, $\mathbf{q} = \mathsf{N}\,\dot{\boldsymbol{\zeta}}$,

$$L(\mathbf{u}, \boldsymbol{\zeta}, \mathbf{v}, \dot{\boldsymbol{\zeta}}) = \int\limits_{\Omega} \left[\mathbf{p}^\top\,\mathbf{v} + \int\limits_{]0,\infty[} \langle \mathbf{q}, \dot{\boldsymbol{\zeta}}\rangle\,\mathrm{d}\mu(\xi) - h \right] \lambda(\mathrm{d}x) \tag{46}$$

to the Hamiltonian (38, 39), which yields

$$L(\mathbf{u}, \boldsymbol{\zeta}, \mathbf{v}, \dot{\boldsymbol{\zeta}}) = \int\limits_{\Omega} \left[\frac{\rho\, \mathbf{v}(t,x)^2}{2} + \frac{1}{2} \int\limits_{]0,\infty[} \langle \dot{\boldsymbol{\zeta}}(t,x;\xi), \mathsf{N}(\xi,x)\, \dot{\boldsymbol{\zeta}}(t,x;\xi) \rangle\, m(\mathrm{d}\xi, x) \right] \lambda(\mathrm{d}x)$$
$$-\frac{1}{2} \int\limits_{\Omega} \langle \mathbf{e}(t,x), \mathsf{G}_0(x)\, \mathbf{e}(t,x) \rangle\, \lambda(\mathrm{d}x)$$
$$-\frac{1}{2} \int\limits_{\Omega} \int\limits_{]0,\infty[} \xi^2\, \langle \boldsymbol{\zeta}(t,x;\xi), \mathsf{N}(\xi,x)\, \boldsymbol{\zeta}(t,x;\xi) \rangle m(\mathrm{d}\xi, x)\, \lambda(\mathrm{d}x)$$
$$- \int\limits_{\Omega} \left\langle \int\limits_{]0,\infty[} \xi\, \mathsf{N}(\xi,x)\, \boldsymbol{\zeta}(t,x;\xi), \mathbf{e}(t,x) \right\rangle \lambda(\mathrm{d}x). \tag{47}$$

The viscoelastic boundary value problem can now be expressed in terms of a Hamiltonian action principle

$$\delta \int\limits_{t_0}^{t_1} L(\mathbf{u}, \boldsymbol{\zeta}, \mathbf{v}, \dot{\boldsymbol{\zeta}})\, \mathrm{d}t = 0 \tag{48}$$

with $\delta\mathbf{u}(t_0) = \delta\mathbf{u}(t_1) = 0$, $\delta\boldsymbol{\zeta}(t_0) = \delta\boldsymbol{\zeta}(t_1) = 0$.

5 A Poisson structure on the phase space $\mathscr{P}$

For the study of integrals of motion it is convenient to formulate the equations of motion in terms of a Poisson bracket on a space of functionals of the conjugate fields. For simplicity we shall restrict the Poisson structure to a class $\mathscr{I}$ of functionals.

The space of auxiliary variables $\boldsymbol{\zeta}$ is the completion

$$\Sigma = \mathscr{L}^2(\mathbb{R}_+;\, \mathit{\Sigma};\, m)$$

of the space of $\mathit{\Sigma}$-valued functions $\boldsymbol{\zeta} :]0,\infty[\, \to \mathit{\Sigma}$ with finite norm

$$\|\mathbf{v}\|_m^2 := \int\limits_{]0,\infty[} |\mathbf{v}(\xi)|^2\, m(\mathrm{d}\xi). \tag{49}$$

The phase space is $\mathscr{P} := \mathbb{R}^d \times \mathbb{R}^d \times \Sigma \times \Sigma$. We shall define a Poisson structure on the set $\mathscr{I}$ of functionals $F : \mathscr{P} \to \mathbb{R}$ of the form

$$F(\mathbf{u}, \mathbf{p}, \mathbf{q}(\cdot), \boldsymbol{\zeta}(\cdot)) = \int\limits_{\Omega} f_0(x, \mathbf{u}(x), \mathbf{p}(x))\, \lambda(\mathrm{d}x) + \int\limits_{\Omega} \int\limits_{]0,\infty[} F_0(x, \xi, \boldsymbol{\zeta}(x,\xi), \mathbf{q}(x,\xi))\, m(\mathrm{d}\xi, x)\, \lambda(\mathrm{d}x) \tag{50}$$

(λ denotes the Lebesgue measure on $\mathbb{R}^d$), where f_0, F_0 are Carathéodory functions twice differentiable with respect to the field variables $\mathbf{u}, \mathbf{p}, \boldsymbol{\zeta}, \mathbf{q}$. The functional $F \in \mathscr{I}$ can thus be identified with the pair (f_0, F_0) of functions. We shall express the relation (50) in short by $F = \Phi(f_0, F_0)$. Note that $H \in \mathscr{I}$.

The functional derivatives of $F \in \mathscr{I}$ can be calculated explicitly:

$$\mathrm{D}_{\boldsymbol{\zeta}} F[\overline{\boldsymbol{\zeta}}] = \int\limits_{\Omega} \int\limits_{]0,\infty[} \frac{\partial F_0}{\partial \boldsymbol{\zeta}}(x, \xi, \boldsymbol{\zeta}(x,\xi), \mathbf{q}(x,\xi))\, \overline{\boldsymbol{\zeta}}(x,\xi)\, m(\mathrm{d}\xi, x)\, \lambda(\mathrm{d}x).$$

In general the functional derivatives of F with respect to the fields are Gâteaux derivatives in $\Lambda := \mathscr{L}^2\left(\Omega, \mathbb{R}^d; \lambda\right) \times \mathscr{L}^2\left(\Omega, \mathbb{R}^d; \lambda\right) \times \mathscr{L}^2\left(\Omega \times \mathbb{R}_+, \mathbb{R}^d; \lambda \otimes m\right) \times \mathscr{L}^2\left(\Omega \times \mathbb{R}_+, \mathbb{R}^d; \lambda \otimes m\right)$. If the derivatives of

f_0, F_0 with respect to the field variables are bounded, then the functional derivatives of F are also Fréchet derivatives in Λ.

We shall define the Poisson structure on $\mathscr{I}$ by the following canonical Poisson bracket:

$$\{F, G\}_1(\mathbf{u}, \mathbf{p}, \boldsymbol{\zeta}, \mathbf{q}) := \int_\Omega \left[\sum_{k=1}^{d} \left[\frac{\partial f_0}{\partial u_k} \frac{\partial g_0}{\partial p_k} - \frac{\partial f_0}{\partial p_k} \frac{\partial g_0}{\partial u_k} \right] \right] \lambda(\mathrm{d}x)$$
$$+ \int_\Omega \left[\int_{[0,\infty[} \sum_{k=1}^{d} \left[\mathrm{D}_{\zeta_k} F_0 \, \mathrm{D}_{q_k} G_0 - \mathrm{D}_{q_k} F_0 \, \mathrm{D}_{\zeta_k} G_0 \right] m(\mathrm{d}\xi, x) \right] \lambda(\mathrm{d}x)$$
$$=: Q_0(f_0, g_0) + Q_1(F_0, G_0), \tag{51}$$

where $F = \Phi(f_0, F_0)$ and $G = \Phi(g_0, G_0)$. Before performing the integration in the second line the integrands are evaluated by substituting the fields $\mathbf{u}(x)$, $\mathbf{p}(x)$, $\boldsymbol{\zeta}(x, \xi)$, and $\mathbf{q}(x, \xi)$.

If the Hamiltonian flow $\mathbf{u}(t, x, ID)$, $\mathbf{p}(t, x, ID)$, $\boldsymbol{\zeta}(t, x, \xi, ID)$, $\mathbf{q}(t, x, \xi, ID)$, (where ID stands for the initial data at $t = 0$), is substituted into a functional $F \in \mathscr{I}$, the functional becomes a function of time and its time derivative can be expressed in terms of the Poisson bracket with H:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\}_1$$

by the chain rule of differentiation and equations (41–44)

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \int_\Omega \left[\mathrm{D}_{\mathbf{u}}\, f_0(t, x, \mathbf{u}(t, x), \mathbf{p}(t, x))\, \dot{\mathbf{u}}(t, x) + \mathrm{D}_{\mathbf{p}}\, f_0(t, x, \mathbf{u}(t, x), \mathbf{p}(t, x))\, \dot{\mathbf{p}}(t, x) \right] \lambda(\mathrm{d}x)$$
$$+ \int_\Omega \int_{]0,\infty[} \Big[\mathrm{D}_{\boldsymbol{\zeta}}\, F_0(x, \xi, \boldsymbol{\zeta}(t, x, \xi), \mathbf{q}(t, x, \xi)) \cdot \dot{\boldsymbol{\zeta}}(t, x, \xi)$$
$$+ \mathrm{D}_{\mathbf{q}}\, F_0(x, \xi, \boldsymbol{\zeta}(t, x, \xi), \mathbf{q}(t, x, \xi))\, \dot{\mathbf{q}}(t, x, \xi) \Big]\, m(\mathrm{d}\xi, x)\, \lambda(\mathrm{d}x),$$

where $\mathrm{D}_x\, f := \partial f / \partial x$.

We now proceed to prove that the Poisson bracket $\{\cdot, \cdot\}_1$ satisfies the Jacobi identity

$$T(F, G, H) := \{\{F, G\}_1, H\}_1 + \{\{H, F\}_1, G\}_1 + \{\{G, H\}_1, F\}_1 = 0 \tag{52}$$

for arbitrary $F, G, H \in \mathscr{I}$.

We begin by recalling the Jacobi identity for finite-dimensional systems

Lemma 1 *The bilinear form*

$$\{f, g\} := \sum_{k=1}^{D} \left[\frac{\partial f}{\partial X_k} \frac{\partial g}{\partial Y_k} - \frac{\partial f}{\partial Y_k} \frac{\partial g}{\partial X_k} \right], \tag{53}$$

defined on $\mathscr{C}^2\left(\mathbb{R}^D \times \mathbb{R}^D, \mathbb{R}\right)$*, satisfies the Jacobi identity.*

Proof Each term of the expression

$$T_0(f, g, h) := \{\{h, f\}, g\} + \{\{g, h\}, f\} + \{\{f, g\}, h\} \tag{54}$$

is a product of the second-order derivative of one of the functions by first-order derivatives of the other two functions with respect to the corresponding variables X, Y with the corresponding indices. The second-order derivatives of g appear in those terms of (54) which contain g in the inner bracket.

It is sufficient to show that the terms involving a fixed type of second-order derivatives of g cancel. The terms involving the second-order derivatives of f, h differ by relabeling of the functions and therefore they cancel analogously. We begin by showing that the sum of the terms involving the second-order derivative of g with respect to X_k, X_l for a fixed pair k, l vanishes:

$$-\frac{\partial f}{\partial Y_k}\,\frac{\partial^2 g}{\partial X_k\,\partial X_l}\,\frac{\partial h}{\partial Y_l}+\frac{\partial h}{\partial Y_k}\,\frac{\partial^2 g}{\partial X_k\,\partial X_l}\,\frac{\partial f}{\partial Y_l}=0.$$

The terms involving $\partial^2 g/\partial Y_k\,\partial Y_l$ cancel in a similar way.

The terms involving $\partial^2 g/\partial X_k\,\partial Y_l$ add up to

$$\frac{\partial f}{\partial X_k}\,\frac{\partial^2 g}{\partial Y_k\,\partial X_l}\,\frac{\partial h}{\partial Y_l}+\frac{\partial f}{\partial Y_k}\,\frac{\partial^2 g}{\partial X_k\,\partial Y_l}\,\frac{\partial h}{\partial X_l}-\frac{\partial f}{\partial Y_l}\,\frac{\partial^2 g}{\partial Y_k\,\partial X_l}\,\frac{\partial h}{\partial X_k}-\frac{\partial f}{\partial Y_k}\,\frac{\partial^2 g}{\partial X_k\,\partial Y_l}\,\frac{\partial h}{\partial X_l}=0.$$

This terminates the proof. □

Theorem 6.9 in [31] implies the following proposition:

Proposition 1 *If the Radon measure m on a domain $\mathscr{D}$ in a finite-dimensional space $\mathbb{R}^n$ is positive and finite, then there is a sequence of measures m_n on $\mathscr{D}$ of the form*

$$m_n=\sum_{k=1}^{N_n} c_k^n\,\delta_{r_k^n}$$

with $N_n < \infty$ such that, for every continuous and bounded function $f : \mathscr{D} \to \mathbb{R}$,

$$\sum_{k=1}^{N_n} c_k^n\, f(r_k^n) \equiv \int\limits_{\mathscr{D}} f(r)\, m_n(\mathrm{d}r) \to \int\limits_{\mathscr{D}} f(r)\, m(\mathrm{d}r) \quad \text{for } n \to \infty, \tag{55}$$

where δ_a denotes the Dirac measure with support at a:

$$\delta_a(U)=\begin{cases}1, & a\in U,\\ 0, & a\notin U.\end{cases}$$

In the language of measure theory, the measures m_n converge narrowly to the measure m.

Theorem 3 *If the set Ω is bounded and $m(]0,\infty[) < \infty$, then the bilinear form $\{\cdot,\cdot\}_1$ on $\mathscr{I}$ satisfies the Jacobi identity.*

Proof We shall apply discretization to approximate $T(F, G, H)$ by expressions of the form $T_0(f, g, h)$ [Eq. (54)].

$$T(F,G,H)=Q_0(Q_0(f_0,g_0),h_0)+Q_1(Q_1(F_0,G_0),H_0). \tag{56}$$

The first term on the right-hand side of (56) involves a single integral with respect to λ_Ω, where λ_Ω is defined as the restriction of the Lebesgue measure λ to Ω. The second term involves a double integral, which can also be converted to an integral with respect to the product measure $\lambda_\Omega \otimes m$, by the Fubini theorem. By Proposition 1 there is a sequence of positive measures m_n with support on a finite set of points $\xi_k^n \geq 0$, $k = 1,\ldots,N_n$, converging narrowly to m, and a sequence of measures λ_n with support on a finite set of points $x_k^n \in \mathbb{R}^d$, $k = 1,\ldots,N_n$, converging narrowly to λ_Ω. By the Fubini theorem, the sequence $\lambda_n \otimes m_n$ converges narrowly to $\lambda_\Omega \otimes m$. Substituting the approximating measures λ_n for λ_Ω and m_n for m in $T(F, G, H)$ results in the expression (54) for the Poisson bracket (53). The X and Y coordinates in the approximating Poisson bracket (53) are $\mathbf{u}_k := \mathbf{u}(x_k)$, $\mathbf{p}_k := \mathbf{p}(x_k)$, $\boldsymbol{\zeta}_{kl} := \boldsymbol{\zeta}(x_k, \xi_l)$, $\mathbf{q}_{kl} := \mathbf{q}(x_k, \xi_l)$, with $k, l = 1,\ldots N_n$, while $f(X) = \sum_k f_0(x_k, \mathbf{u}_k, \mathbf{p}_k) + \sum_k \sum_l F_0(x_k, \xi_l, \boldsymbol{\zeta}_{kl}, \mathbf{q}_{kl})$. By Lemma 1 the approximating expressions (54) vanish, hence their limit $T(F, G, H)$ also vanishes for arbitrary $F, G, H \in \mathscr{I}$, which completes the proof. □

Corollary 1 *The bilinear form $\{\cdot,\cdot\}_1$ is a Poisson bracket on $\mathscr{P}$.*

6 Examples of scalar viscoelastic relaxation moduli

6.1 Introduction

We shall illustrate the variety of viscoelastic models covered by the above theory.

We shall consider the initial-value problem (IVP)

$$\left.\begin{array}{l}\rho\, u_{,tt} = \sigma_{,x}, \\ \sigma = G * u_{,tx}\end{array}\right\} x \in \mathbb{R}, \quad t > 0, \tag{57}$$

$$u(t,x) = 0, x \in \mathbb{R}, \quad t < 0, \tag{58}$$

$$u^{(k)}(0+, x) = u_k(x), x \in \mathbb{R}, \quad k = 0, 1, \tag{59}$$

with ρ a positive function of $x \in \mathbb{R}$.

The spectral measure m will be calculated for two examples of one-dimensional hereditary viscoelastic models. Note that the theory does not apply to the Newtonian viscosity ($G(t) = c\,\delta(t)$) because $\delta(t)$ is not the Fourier transform of a finite Radon measure.

6.2 Hyperbolic models, nonsingular relaxation kernels

We shall consider the following relaxation modulus

$$G(s) = b\,\theta(s) + \nu\,\frac{(s+a)^{\alpha-1}}{\Gamma(\alpha)}, \quad s \in \mathbb{R} \tag{60}$$

with the constants $0 < \alpha < 1$, $\nu, b > 0$, $a \geq 0$. The second term is a locally integrable completely monotone function, hence it is causal positive definite. For $a = 0$ it is unbounded at $s \to 0$. The relaxed modulus $G_\infty := \lim_{s\to\infty} G(s) = b$ is finite, but the instantaneous modulus

$$G_0 := \lim_{s\to 0+} G(s) = \begin{cases} b + \nu\, a^{\alpha-1}/\Gamma(\alpha), & a > 0, \\ \infty, & a = 0, \end{cases} \tag{61}$$

can be infinite.

For $a > 0$ the singularity in the relaxation modulus disappears and the propagation speed $c(\omega) \leq c_\infty < \infty$, $c(\omega) := \omega/k(\omega)$, where $k(\omega)$ denotes the wavenumber.

The Laplace transform of the relaxation modulus can be expressed in terms of the incomplete gamma function [1]:

$$\tilde{G}(p) = \frac{b}{p} + \nu\, \mathrm{e}^{pa}\, p^{-\alpha}\, \frac{\Gamma(\alpha, ap)}{\Gamma(\alpha)}, \tag{62}$$

hence the measure m in the Fourier integral representation of G is now

$$\begin{aligned} m(\mathrm{d}\xi) &= \frac{1}{\pi} \lim_{p\to -\mathrm{i}\xi+0} \mathrm{Re}\, \tilde{G}(p) = b\,\delta(\xi) + \frac{\nu}{\pi}\, \mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\xi a}\,(-\mathrm{i}\xi)^{-\alpha}\, \frac{\Gamma(\alpha, -\mathrm{i}a\xi)}{\Gamma(\alpha)}\right] \mathrm{d}\xi \\ &= b\,\delta(\xi) + \nu\, f(\xi)\,\xi^{-\alpha} \mathrm{d}\xi, \end{aligned} \tag{63}$$

where

$$f(\xi) := \frac{1}{\pi}\, \mathrm{Re}\left[\cos(\pi\alpha/2 - |\xi|a)\,|\xi|^{-\alpha}(a\,\xi)^\alpha\ \mathrm{Re}\,\gamma^*(\alpha, -\mathrm{i}a\xi)\right] \tag{64}$$

and γ^* is the univalent nonsingular function defined in [1]. Figures 1 and 2 show that f is non-negative as expected.

It is easy to see from Figs. 1 and 2 that $f(\xi) \sim \text{const}$ for $\xi \to 0$. The tail behavior of m is determined by $f(\xi)/\xi^\alpha = \mathrm{o}[1/\xi]$ as $\xi \to \infty$, by a standard asymptotic argument (see, e.g., [16]).

In the limit $a \to 0$ the limit $G_0 = \lim_{t\to 0+} G(t)$ becomes infinite and $m(\mathrm{d}\xi)$ degenerates to an infinite measure $b\,\delta(\xi) + \frac{\nu}{\pi}\cos(\pi\alpha/2)|\xi|^{-\alpha}\,\mathrm{d}\xi$. Consequently the theory does not apply. The dispersion relation in the limit case yields an unbounded propagation speed

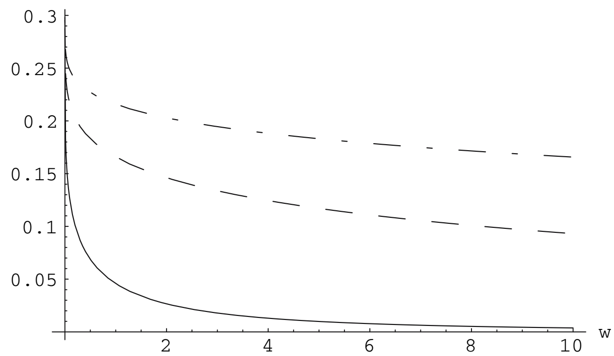

Fig. 1 The function $f(w)$ for $\alpha = 0.2$ and $a = 0.4$ (*solid line*), 0.01 (*dashed*) and 0.001 (*dot dashed*)

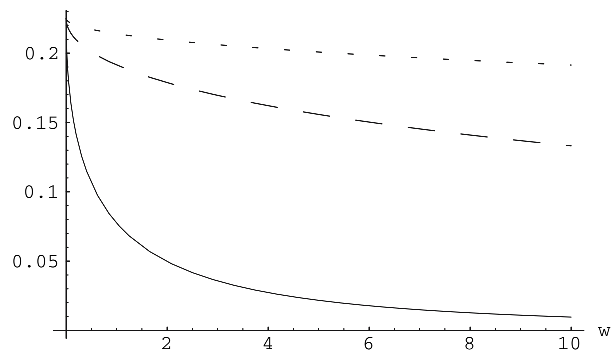

Fig. 2 The function $f(w)$ for $\alpha = 0.5$ and $a = 0.4$ (*solid line*), 0.01 (*dashed*) and 0.001 (*dotted*)

$$c(\omega)^2 = \frac{b}{\rho} + \frac{\nu}{\rho}|\omega|^{1-\alpha}\,\mathrm{e}^{-\mathrm{i}\pi(1-\alpha)/2}, \tag{65}$$

indicating that disturbances propagate with infinite speed.

6.3 Hyperbolic models with singular relaxation kernels and smooth solutions

If the relaxation kernel G' has an integrable singularity at 0, then the solutions of the IVP (57–59) are infinitely differentiable at the wavefronts and therefore singularities do not propagate.

An example of this class is $G(t) = G_1\, E_\alpha\,(-(t/\tau)^\alpha) + G_\infty\,\theta(t)$, where $G_1 = G_0 - G_\infty$, $\tau > 0, 0 < \alpha < 1$ and E_α is the Mittag–Leffler function [1]. Since $E'_\alpha(0)$ is finite, $G'(t) \sim a\, t^{\alpha-1}$ for $t \to 0$.

Since

$$\frac{(\tau\, p)^{\alpha-1}}{1 + (\tau p)^\alpha} = \int_0^\infty \mathrm{e}^{-pt}\, E_\alpha\left(-(t/\tau)^\alpha\right)\, \mathrm{d}t \tag{66}$$

[37],

$$m(\mathrm{d}\xi) = G_\infty\,\delta(\xi) + G_1\,\frac{\cos\left((1-\alpha)\pi/2\right)}{\pi}\,\frac{|\xi|^{-\alpha-1}}{1 + |\xi|^{2\alpha}}\,\mathrm{d}\xi. \tag{67}$$

The total mass $m(]0, \infty[\,)$ is finite and the theory developed in the previous sections applies.

7 Hamiltonian and Lagrangian formulation of poroelasticity

Poroelasticity, as developed by Biot [8], and poroacoustics [3], for a multiphase porous medium consisting of N phases, can be expressed in the form

$$\mathbf{K} * \ddot{\mathbf{u}} = \nabla^\top \mathsf{G} * (\nabla \dot{\mathbf{u}}), \tag{68}$$

where $\mathbf{u} : \mathbb{R} \to V := \mathbb{R}^{Nd}$; $\mathbf{K} : \mathbb{R} \to S^N$; $\mathsf{G} : \mathbb{R} \to \Sigma$; $\operatorname{supp} \mathbf{K} \subset [0, \infty[$; $\operatorname{supp} \mathsf{G} \subset [0, \infty[$; $\Omega \subset \mathbb{R}^d$; S and S^N denote the spaces of symmetric tensors of second rank on $\mathbb{R}^d$ and V, respectively; $S_N = S \oplus \cdots \oplus S$ (N copies of S) and Σ is the space of symmetric operators on S_N.

We shall ignore the constraint following from the fact that the phases do not overlap on the microscopic scale and the associated volume fraction variables. We shall only focus on the memory effects specific to poroelasticity and poroacoustics.

The left-hand side of Eq. (68) can be expressed in the form $\mathbf{K}' * \dot{\mathbf{u}}$, where $\mathbf{K}'$ denotes the distributional derivative of $\mathbf{K}$. The material response functions $\mathbf{K}$, G may additionally depend on $x \in \Omega$.

We shall assume that $\mathbf{K}' = \delta' \mathbf{K}_0 + \mathbf{K}_1$, $\mathbf{K}_0 \in S^N$, $\mathbf{K}_0 > 0$, $\mathbf{K}_1 \in \mathscr{L}^1_{\text{loc}} \left(\mathbb{R}_+; S^N\right)$, and the functions $\mathbf{K}_1$ and G are causal positive definite:

$$\int_{-\infty}^{\infty} \mathbf{y}^\top \mathbf{K}_1 * \mathbf{y} \, \mathrm{d}t \geq 0, \tag{69}$$

$$\int_{-\infty}^{\infty} \langle \mathbf{z}, \mathsf{G} * \mathbf{z} \rangle \, \mathrm{d}t \geq 0 \tag{70}$$

for arbitrary $\mathbf{y} \in \mathscr{L}^2(\mathbb{R}; V)$ and $\mathbf{z} \in \mathscr{L}^2(\mathbb{R}; S_N)$ with compact support.

The energy balance assumes the form

$$\int_{\Omega} \left[\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \dot{\mathbf{u}}^\top \mathbf{K}_0 \, \dot{\mathbf{u}} + \dot{\mathbf{u}}^\top \mathbf{K}_1 * \dot{\mathbf{u}} + \frac{1}{2} \langle \nabla \dot{\mathbf{u}}, \mathsf{G} * \nabla \dot{\mathbf{u}} \rangle \right] \lambda(\mathrm{d}x) = \int_{\partial\Omega} \mathbf{g}^\top \, \dot{\mathbf{u}} \, a(\mathrm{d}x) + \int_{\Omega} \mathbf{f}^\top \, \dot{\mathbf{u}} \, \lambda(\mathrm{d}x), \tag{71}$$

where $\mathbf{f}$, $\mathbf{g}$ denote the external volume loads and external tractions.

Invoking the Bochner theorem and Lemma A1,

$$\mathbf{K}_1(s) = \int_{[0,\infty[} \cos(\xi s) \, \mathbf{M}(\xi) \, \mu(\mathrm{d}\xi), \tag{72}$$

$$\mathsf{G}(s) = \int_{[0,\infty[} \cos(\xi s) \, \mathsf{N}(\xi) \, \nu(\mathrm{d}\xi), \tag{73}$$

where $\mu, \nu : \mathbb{R} \to \mathbb{R}$ are positive Borel measures on $[0, \infty[$, $\mathbf{M} : [0, \infty[\to S$ is non-negative almost everywhere with respect to the measure μ and $\mathsf{N} : [0, \infty[\to \Sigma$ is non-negative almost everywhere with respect to the measure ν.

Let $\mathbf{u}_n$ denote the displacement of the nth phase,

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{u}_N \end{bmatrix}, \quad \mathbf{e} := \begin{bmatrix} \mathbf{e}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{e}_N \end{bmatrix} \tag{74}$$

and $\mathbf{e}_n := \left[\nabla \mathbf{u}_n + (\nabla \mathbf{u}_n)^\top\right]/2$, $1 \leq n \leq N$,

$$\boldsymbol{\kappa}(t, x; \xi) := \left[\int_0^{\infty} \cos(\xi s) \, \dot{\mathbf{u}}(t - s, x) \, \mathrm{d}s - \mathbf{u}(t, x) \right] / \xi, \tag{75}$$

$$\boldsymbol{\zeta}(t, x; \xi) := \left[\int_0^{\infty} \cos(\xi s) \, \dot{\mathbf{e}}(t - s, x) \, \mathrm{d}s - \mathbf{e}(t, x) \right] / \xi. \tag{76}$$

The kinetic and stored energy densities are defined in terms of the auxiliary variables $\boldsymbol{\kappa}$ and $\boldsymbol{\zeta}$:

$$W_\mathrm{k} := \frac{1}{2} \int\limits_{]0,\infty[} [\xi\,\boldsymbol{\kappa}(\xi) + \mathbf{u}]^\top\, \mathbf{M}(\xi)\, [\xi\,\boldsymbol{\kappa}(\xi)\ + \mathbf{u}]\,\mu(\mathrm{d}\xi), \tag{77}$$

$$W_\mathrm{s} := \frac{1}{2} \int\limits_{]0,\infty[} \langle [\xi\,\boldsymbol{\zeta}(\xi) + \mathbf{e}], \mathsf{N}(\xi)\, [\xi\,\boldsymbol{\zeta}(\xi) + \mathbf{e}]\rangle_{\mathbb{C}}\, \nu(\mathrm{d}\xi) \tag{78}$$

where the arguments t, x have been suppressed for brevity. The terms on the left-hand side can now be replaced by the rate of an energy so that the energy balance is converted to an energy conservation equation. Using the generalized coordinates $\boldsymbol{\kappa}$, $\boldsymbol{\zeta}$ and the generalized momenta $\mathbf{p} = \mathsf{K}_0^{-1}\,\dot{\mathbf{u}}$, $\mathbf{q}_\kappa(t, x; \xi) = \mathbf{M}(t, x; \xi)\,\dot{\boldsymbol{\kappa}}$, and $\mathbf{q}_\zeta(t, x; \xi) = \mathsf{N}(t, x; \xi)\,\dot{\boldsymbol{\zeta}}$, the energy conservation equation can be expressed in the form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Omega} \left[\frac{1}{2}\mathbf{p}^\top\, \mathbf{K}_0^{-1}\,\mathbf{p} + W_\mathrm{k}(t,x) + W_\mathrm{s}(t,x) + \frac{1}{2}\int\limits_{]0,\infty[} \mathbf{q}_\kappa(\xi)^\top\,\mathbf{M}(\xi)\,\mathbf{q}_\kappa(\xi)\,\mu(\mathrm{d}\xi) \right. \\ &\qquad \left. + \frac{1}{2}\int\limits_{]0,\infty[} \mathbf{q}_\zeta(\xi)^\top\,\mathsf{N}(\xi)\,\mathbf{q}_\zeta(\xi)\,\nu(\mathrm{d}\xi) \right] \lambda(\mathrm{d}x) \\ &= \int\limits_{\partial\Omega} \mathbf{g}^\top\,\dot{\mathbf{u}}\,\mathrm{d}a + \int\limits_{\Omega} \mathbf{f}^\top\,\dot{\mathbf{u}}\,\lambda(\mathrm{d}x). \end{aligned} \tag{79}$$

For any boundary conditions that make the right-hand side vanish the left-hand side is the time derivative of a Hamiltonian H. The Hamiltonian equations of motion are similar to the equations of linear elasticity. The additional equation for $\dot{\boldsymbol{\kappa}} = \mathrm{D}_{\mathbf{q}_\kappa} H$ recalls the definition of the generalized momentum $\mathbf{q}_\kappa$, while $\dot{\mathbf{q}}_\kappa = -\mathrm{D}_{\boldsymbol{\kappa}}\,H$ is equivalent to

$$\ddot{\boldsymbol{\kappa}} + \xi^2\,\boldsymbol{\kappa} = -\xi\,\boldsymbol{\kappa}. \tag{80}$$

Finally, the equation $\dot{\mathbf{p}} = -\partial H/\partial \mathbf{u}$ is equivalent to the conservation of momentum

$$\mathbf{K}_0\,\ddot{\mathbf{u}} = \dot{\mathbf{p}} = -\mathrm{D}_{\mathbf{u}}\,H = \operatorname{div}\boldsymbol{\sigma} - \int\limits_{]0,\infty[} \mathbf{M}(\xi)\,(\xi\,\boldsymbol{\kappa}(\xi) + \mathbf{u})\,\mu(\mathrm{d}\xi) = \operatorname{div}\boldsymbol{\sigma} - \mathbf{K}_1 * \dot{\mathbf{u}}. \tag{81}$$

8 Conclusions

A conserved energy can be associated with general hereditary viscoelastic materials. The dynamics of such materials can be expressed in Hamiltonian form. The conserved energy consists of the kinetic energy and the stored energy. The latter is the sum of an elastic energy and the energy of a continuum of oscillators driven by the strain rate. The dissipated energy thus resides in the oscillators and is accounted for by a non-negative quadratic functional. The Hamiltonian is the sum of the elastic Hamiltonian, the oscillator Hamiltonians, and a linear interaction term.

The oscillator representation of dissipation in viscoelasticity obtained here resembles some recent models of an oscillator interacting with a heat bath [18,10]. Such models yield the Langevin equation or a Fokker–Planck equation for the oscillator. In our case each particle of the elastic subsystem is connected by springs to its neighbors and to a continuum of oscillators (Fig. 3). We conjecture that thermal effects can be included by randomizing the initial data for the auxiliary fields.

The stored energy can be expressed as a quadratic functional of the auxiliary fields representing the oscillators. The integral can be viewed as extending over the imaginary axis iξ in the complex plane. Different energy concepts can be obtained by varying the complex-plane spectral representation of the material response functions, as pointed out in [40]. Real viscoelastic materials generally have locally integrable completely monotone (LICM) relaxation moduli. By a theorem in [19], LICM functions are positive definite and therefore

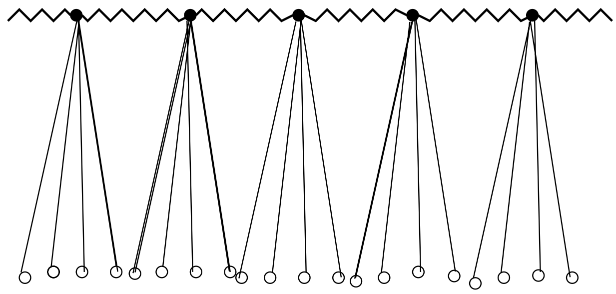

Fig. 3 A schematic representation of a viscoelastic medium

have the spectral representation in terms of the Fourier transform of a positive Radon measure. By the Bernstein theorem [44], an LICM function has an alternative spectral representation in terms of the Laplace transform of a positive Radon measure. A Laplace transform can be viewed as an integral over the negative real axis. The associated energy is monotonely decreasing if the system is closed [24].

The theory can be extended to single-integral physically nonlinear viscoelastic constitutive equations of the single-integral type [22,25] by replacing the strain $\mathbf{e}$ by a nonlinear function $\mathbf{f}(\mathbf{e}, x)$ in Eq. (35). Eulerian formulation requires, however, a transformation of the Poisson brackets to a noncanonical Lie–Poisson form [36].

The theory is in sharp contrast with earlier approaches to accommodate dissipative memory effects in pseudo-Hamiltonian theory by including them in a symmetric dissipative bracket, which additionally accounts for nonconservation of the Hamiltonian [7]. In this case the Hamiltonian is the pure field Hamiltonian corresponding to the elastic part of the energy. The dissipative bracket accounts for the motion of the system out of symplectic leaves (cf. [35]), while it is restricted to a single symplectic leave with respect to the Poisson structure defined in Sect. 5.

The Hamiltonian formulation can be used in the quantum-mechanical description of the interaction of phonon field (the elastic energy component) with matter (the oscillators).

A Factorization of the relaxation spectral measure

We shall prove that the tensor-valued Radon measure M in Theorem 1 can be factored into a positive (real-valued) Radon measure m and a positive-semidefinite tensor-valued function:

Lemma A1 (i) *The measure M in (8) has a Radon–Nikodym derivative N with respect to a positive Radon measure m. The function N is defined everywhere except for a set $E \subset [0, \infty[$ of zero measure m. $\mathsf{N}(\xi)$ is symmetric, positive semidefinite and bounded for m-almost all ξ;*
(ii) *$m(E) = m(-E)$ for every Borel subset $E \subset \mathbb{R}$ and $\mathsf{N}(\xi) = \mathsf{N}(-\xi)$ for m-almost all $\xi \in \mathbb{R}$.*

Proof Let E be a measurable subset of the real line $\mathbb{R}$.

Ad (i):

By the Riesz theorem M can be considered as a Borel measure.

Equation (9) implies that the distribution $\mathsf{M}(\xi)$ is pointwise self-adjoint, or, more precisely, the measure M is self-adjoint on $S_{\mathbb{C}}$ for every Borel subset E of $\overline{\mathbb{R}_+}$:

$$\mathsf{M}(E)^{\dagger} = \mathsf{M}(E).$$

In view of the symmetry of the operators $\mathsf{G}(t)$ and (9) the tensor $\mathsf{M}(E)$ is also symmetric on $S_{\mathbb{C}}$:

$$\mathsf{M}(E)^{\top} = \mathsf{M}(E)$$

for every Borel $E \subset \overline{\mathbb{R}_+}$. Consequently $\mathsf{B} := \mathsf{M}(E)$ is also real and positive semidefinite and therefore

$$|\mathbf{v}^{\top}\,\mathsf{B}\,\mathbf{w}|^2 \leq \left(\mathbf{v}^{\top}\,\mathsf{B}\,\mathbf{v}\right)\,\left(\mathbf{w}^{\top}\,\mathsf{B}\,\mathbf{w}\right) \leq |\mathbf{v}|^2\,|\mathbf{w}|^2\,m(E)^2,$$

where $m(E)$ denotes the trace of B. m is the sum of a finite number of Borel measures; hence it is a Borel measure. m is a positive measure because the tensor $\mathsf{M}(E)$ is real for every Borel E. On account of (7) $\mathsf{M}(E)$ is positive semidefinite for every Borel E.

Since

$$|\mathbf{v}^\top \mathsf{M}(E)\mathbf{w}| \le |\mathbf{v}|\,|\mathbf{w}|\,m(E), \tag{82}$$

the components of M are continuous with respect to the measure m and by the Radon–Nikodym theorem the measure M has a density N, integrable with respect to m.

The density N inherits the symmetry properties of M and is m-almost everywhere real, symmetric, and positive semidefinite. On account of (82)

$$|\mathsf{N}(\xi)| := \sup_{|\mathbf{v}|=1} \langle \mathbf{v}, \mathsf{N}(\xi)\,\mathbf{v}\rangle \le 1$$

for m-almost all $\xi \in [0, \infty[$.

Ad (ii): Eqs. (10) and (9) imply that $M(E) = M(-E)$ for every Borel subset E of $\mathbb{R}$, hence (ii) follows by the definition of m and N. □

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