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Topology optimization of elasto-plastic structures under reliability constraints: A first order approach

Piotr Tauzowski [a], Bartlomiej Blachowski [a], János Lógó [b]

[a] Institute of Fundamental Technological Research, Polish Academy of Sciences, Poland
[b] Budapest University of Technology and Economics, Hungary

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ABSTRACT

The objective of this study is to propose a relatively simple and efficient method for reliability based topology optimization for structures made of elasto-plastic material. The process of determining the optimal topology of elasto-perfectly plastic structures is associated with the removal of material from the structure. Such a process leads to weakening of structural strength and stiffness causing at the same time increase the likelihood of structural failure. An important aspect of engineering design is to track this probability during the optimization process and not allow the structure safety to exceed a certain level specified by the designer. The purpose of this work is to combine the previously developed yield-limited topology optimization method with reliability analysis using first order approach. Effectiveness of the proposed methodology is demonstrated on benchmark problems proposed by Rozvany and Maute, and the elasto-plastic topology design of L-shape structure which is frequently used in different approaches for stress constrained topology optimization.

1. Introduction

Reliability analysis and reliability-based optimal design, especially first order methods (FORM), belong to the best known and used techniques in structural analysis and design. They have number of decades history in research. However, topology optimization and reliability analysis/design is relatively new, but rapidly developed research field. As it is known in the 150 years of history of topology optimization [41], the elasto-plastic layout problems were the fundamental issues only in the first 90 years. Later, when the continuum based structural topology problems dominated the field, linearly/non-linearly elastic or stress limited elastic problems were investigated, plasticity is almost totally neglected. In recent years the elasto-plastic continuum based topology problems with the achievement of reliability analysis could come into focus of the interest of researchers. This paper is an extended version of our CIVIL-COMP-OPTI 2019 conference presentation from this lately mentioned field. At first the probabilistic and reliability based topology optimization papers are overviewed.

Until the end of last century almost one could not find any publication on topology optimization considering uncertainties. Exceptions to this most often use multiple load or reliability constraints. Uncertainty is typically limited to the loading, although some works have considered extensions to support conditions and material properties. The probabilistic works can be classified into three types of group.

The first type is where the structure is assumed built precisely as designed (no uncertainty in geometry) and the load is uncertain. The loading uncertainties can handle the magnitude, line of action or the point of application of the loads.

The second type is where the load is deterministic and the nodal locations that are used to define the geometry of the structure are uncertain. The load uncertainty problem for a finite number of load patterns with discrete probabilities has been analyzed previously using a slightly expanded form of the optimization problem. The nodal uncertainty problem is significantly more complex because it involves randomness in the inverse of the stiffness matrix. To make the results analytically tractable, it is assumed that the uncertainties in the nodal locations are small relative to the length scale of the structural elements. An important application of such an uncertainty model is in representing fabrication errors.

The third type of group is rather a computational strategy what is proposed for robust structural topology optimization in the presence of uncertainties with known second order statistics. The strategy combines deterministic topology optimization techniques with a perturbation method for the quantification of uncertainties associated with structural stiffness, such as uncertain material properties and/or structure geometry. The use of perturbation transforms

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the problem of topology optimization under uncertainty to an augmented deterministic topology optimization problem. This in turn leads to significant computational savings when compared with simulation-based (Monte Carlo-based) optimization algorithms.

As a consequence of the achievements of the great number of deterministic topological investigations about two decades ago something has started in the field of probability based topology optimization. Researchers used the advantages that reliability based design optimization (RBDO) and stochastic optimization have significant results in all fields of engineering and mathematics during the last 60 years. Books, edited volumes, proceedings were published during the last thirty years containing the general reliability theorems, numerical procedures -e.g. Frangopol et al. [19,20], Jendo and Dolinski [25], Calafiore and Dabbene [11], Melchers [47]. The mathematical background of the probabilistic optimization methods were presented among others by Kall [27], Prekopa [56] and Marti [43]. Aoues and Chateauneuf [1] provide benchmark study of selected numerical methods of RBDO (the single loop approach, the two-level approach and the decoupled approach). Very efficient procedures were developed in the two main fields of RBDO: namely approximate reliability methods (see eg. Kaymaz [31]) and advance simulation methods (see eg. Papadrakakis and Lagaros [54], Schüller et al. [61], Vietor and Van den Akker [72], Vietor and Marti [73], Beer and Liebscher [5].) The overview and comparison of the reliability procedures can be followed by the review article of Valdebenito and Schüeller [71], Beck and de Santana Gomes [4]. Ben-Tal and Neminovski [6,7] works provided a strong foundation of the probability based topology optimization almost a decade earlier than the topic become "popular".

A number of methods have been developed to achieve the optimization goal among which heuristic (Blachowski and Gutkowski, 2008) [8] and gradient based (Blachowski and Gutkowski, 2010) [9] methods could be good examples for solution techniques. Although the clear concept, the solution of topology optimization problems poses significant technical challenges. Problems are typically large-scale and discrete, and often exhibit some numerical difficulty associated with underlying mechanics (such as instability of members, checkerboard patterns in continua). For these reasons, the majority of topology optimization research has focused on deterministic design problems, neglecting uncertainty that arises in most engineering applications.

As it was mentioned earlier until the end of last century almost one could not find any publication on topology optimization considering uncertainties. Fortunately, this trend has changed and a great number of works were published during the last twelve years. Generally the publications are classified according the object of the topology optimization (trusses, continuum type structures, grids, plates...), the type of numerical procedures (e.g. homogenization method, the solid isotropic material with penalization (SIMP) approach, the evolutionary structural optimization (ESO) method, the level set-based topology optimization method, meta-heuristic methods...), the formation of the uncertainties (loading, geometry, stiffness, production tolerance). A common thing among many of these different approaches is to reformulate the uncertainties and create an alternate deterministic formulation. The uncertain objective function or the uncertain constraints can be replaced by their statistical averages. An alternative is to minimize the influence of stochastic variability on the mean design by including higher order statistics such as variance. These approaches are commonly referred to as robust design optimization (RDO). The description of the reliable and robust tool for structural shape optimization is the object of the paper of Sienz and Hinton [63] in 1997. Reliability based design optimization (RBDO) looks to constrain or minimize (or maximize) a measure of the probability of failure such as reliability index. Also other methods use the upper bound theorems of the stochastic optimization to provide an equivalent deterministic expression which bounds the probabilistic ones.

Truss optimization by Liu and Moses [36] and stochastic optimization works of Marti and Stöckl [44,45] provide early information about this topic. The paper of Duan et al. [16] is among the very first publications in the field of uncertainty based topology optimization. This work presents an entropy-based topological optimization method for truss structures by the use of iteration technique. Kharmanda et al. [32] present how to integrate reliability analysis into topology optimization. Robust optimization formulation without statistical information and its application to micro-electro-mechanical systems (MEMS) devices is discussed by Han and Kwak [22]. In the robust optimal design procedure, a deterministic optimization for performance improvement is followed by a sensitivity analysis with respect to uncertainties such as MEMS fabrication errors and changes of material properties. During the process of the deterministic optimization and sensitivity analysis, dominant performances and critical uncertain variables are identified to define the gradient index. Maute and Frangopol [46], Kim et al. [33] also present a methodology for the design of micro-electro-mechanical systems (MEMS) by topology optimization accounting for stochastic loading and boundary conditions as well as material properties. This methodology combines the advances in material-based topology optimization for compliant mechanisms undergoing large displacements and design optimization under uncertainties using first order reliability analysis methods. Mogami et al. [50] discrete frame elements in their topology optimization work. Their paper concerns a reliability-based topology optimization method for frame structures that considers uncertainties in applied loads and nonstructural mass at the early conceptual design stage. The effects that multiple criteria, namely, stiffness and eigenfrequency, have upon system reliability are evaluated by regarding them as a series system, where mode reliabilities can be evaluated using first-order reliability methods.

The methodology of the robust design with consideration of highly nonlinear structural behaviour is discussed by Schumacher and Olschinka [62].

By the use of Mindlin plate elements with the von Karman strain–displacement relation a reliability-based topology optimization method is presented by Jung and Cho [26]. The classical topology optimization program is generalized as plate volume minimization problem having probabilistic displacement constraints by the use of the performance measure approach.

Califore and Dabbene [11] apply two standard theorems for finding optimal solutions for uncertain convex optimization problems to truss topology optimization under uncertainty on the load pattern and/or on the material characteristics. In these approaches the optimal design should minimize the expected value of the objective function with respect to uncertainty (average approach), while in the second one it should minimize the worst-case objective (worst-case or min–max approach). Both approaches are shown to lead to exact and numerically efficient solution schemes when the uncertainty enters the data in simple form.

Kang and Luo [30] present a non-probabilistic reliability-based topology optimization method for the design of continuum structures undergoing large deformations. The variation of the structural system is treated with the multi-ellipsoid convex model, which is a realistic description of the parameters being inherently uncertain-but-bounded or lacking sufficient probabilistic data.

The level-set based topology optimization algorithm of Chen and Chen [13] is suitable to consider geometric uncertainties, as well. Their robust optimization method consists of two parts: First, the geometric uncertainty is quantitatively modeled by combing level set equation with a random normal boundary velocity field. Second, a partial differential equation-based approach is employed to overcome the deficiency of conventional level set model which cannot explicitly maintain the point correspondences between the current and the perturbed boundaries. With the explicit point correspondences, shape sensitivity defined on different perturbed designs can be mapped back to the current design.

Dunning et al. [17,18] introduce an efficient and accurate approach to robust structural topology optimization. The objective is to minimize expected compliance with uncertainty in loading magnitude and applied direction, where uncertainties are assumed normally distributed and statistically independent. This approach is analogous to a multiple load case problem where load cases and weights are derived analytically to accurately and efficiently compute expected compliance and sensitivities. Illustrative examples using a level-set-based topology optimization method are then used to demonstrate the proposed approach.

Topology optimization with uncertainty in the magnitude and locations of the applied loads and with small uncertainty in the locations of the structural nodes is the object of the paper of Guest and Igusa [21]. Their method is based on the assumption that the loading uncertainties are taken into consideration as "safety factors" of the deterministic load cases in the load combination. This technique is extended for nonlinear effects of global instability [24] and material property uncertainties [2], to put more control on the variability of the final design via including variance of the compliance [3].

Sigmund [64] considered manufacturing errors by modelling structures as uniformly too thin or thick and optimizing for the worst case performance under these conditions. This idea was also applied to the robust design of photonic crystal waveguides by Wang et al. [74].

A single-loop algorithm for system reliability-based topology optimization is presented by Silva et al. [67], Nguyen et al. [53]. The proposed single-loop algorithm accounts for the statistical dependence between the limit-states by using the matrix-based system reliability method to compute the system failure probability and its parameter sensitivities. The reliability-based topology optimization method presented by Yoo et al. [78] is based on bidirectional evolutionary structural optimization which uses response surface methods.

The publications of Lógó et al. [40,39], Lógó [37,38], Pintér et al. [55] provide an appropriate tool for continuum type topology optimization procedure using a first order approximation for compliance in the presence of uncertainty in applied loads. The robustness also considered in Csébfalvi and Lógó [15], Rashki et al. [58]. In this lastly mentioned paper a simulation-based method is described for reliability based design optimization. The method provides multi-level of solutions by performing only one simulation run.

The last five years brings a real expansions of the achievements in the field of reliability based topology optimization (RBTO). New procedures, more accurate and robust solutions have elaborated. Kanakasabai and Dhingra [28] introduce an approach for RBTO in which the computational effort involved in solving the RBTO problem is equivalent to that of solving a deterministic topology optimization problem. The methodology presented is built upon the bidirectional evolutionary structural optimization (BESO) method used for solving the deterministic optimization problem.

The hybrid probabilistic and interval model is elaborated by Xia et al. [76] To improve the computational efficiency, a hybrid perturbation random moment method (HPRMM) to estimate the objective function and a hybrid perturbation inverse mapping method (HPIMM) to evaluate the component reliability is proposed. Based on HPRMM and HPIMM, the nested loop optimization is converted into an efficient single-loop process.

A topology description function approach and a first order reliability method are employed for topology optimization and reliability calculation by J. Liu et al. [34]. K. Liu et al.[35] have elaborated novel segmental multi-point linearization (SML) method for a more accurate estimation of the gradient of failure probability. This type of application increases the robustness of the optimal solution. A hybrid sequential approximate programming (HSAP) method is developed by Meng et al. [49] to calculate the optimum efficiently by developing a distance-checking criterion and a convex approximate method. Since the distance-checking criterion identifies the feasibility of the probabilistic constraint effectively, the proposed method combines the efficiency of the sequential approximate programming method and the accuracy of SORM. The convex approximate method is also constructed using the sensitivity and function value of the probabilistic constraint.

To avoid the oscillatory and non-convergent properties of the iterative single loop solution procedure for RBDO in case of nonlinear performance function, a chaotic single loop approach is proposed by almost the previously cited same team (Meng et al. [48]) to achieve the convergence control of original iterative algorithm. Additionally, an oscillation-checking method is constructed to detect the oscillation of iterative process of the design variables.

A spatially varying geometric uncertainties due to manufacturing errors are modeled with a random threshold model by Kang and Liu [29]. The projection threshold is represented by a transformation of a Gaussian random field, which is then discretized by means of the expansion optimal linear estimation. The structural response and their sensitivities are evaluated with the polynomial chaos expansion, and the accuracy of the proposed method is verified by Monte Carlo simulations.

Canelas, Carrasco and López [12] present a RBDO, which is based on the approximation of the safe region in the random space by a polytope-like region. This polytope is in its turn transformed into quite a simple region by using generalized spherical coordinates. The failure probability can be estimated by considering simple quadrature rule.

Chun et al. [14] propose a method to incorporate constraints on the first-passage probability into reliability-based optimization of structural design or topology. For efficient evaluations of first-passage probability during the optimization, the failure event is described as a series system event consisting of instantaneous failure events defined at discrete time points. The probability of the series system event is then computed by use of a system reliability analysis method termed as the sequential compounding method. The adjoint sensitivity formulation is derived for calculating the parameter sensitivity of the first-passage probability to facilitate the use of efficient gradient-based optimization algorithms.

Moustapha, and Sudret [51] present a generalization of the existing surrogate-assisted and simulation-based RBDO techniques using a unified framework that includes three independent blocks, namely adaptive surrogate modelling, reliability analysis, and optimization.

Da Silva, Beck and Sigmund [65] elaborate a robust design approach, based on eroded, intermediate and dilated projections, to handle uniform manufacturing uncertainties in stress-constrained topology optimization. In addition, a simple scheme is proposed to increase accuracy of stress evaluation at jagged edges, based on limiting sharpness of the projections to intentionally allow a thin layer of intermediate material between solid and void phases.

A non-probabilistic reliability-based topology optimization framework for compliant mechanisms with interval uncertainties is introduced by Wang et al. [75]. As a result of the combination of the SIMP (solid isotropic material with penalization) model and the set-theoretical interval method, the uncertainty quantification analysis is conducted to obtain mathematical approximations and boundary laws of considered mean compliance. By normalization treatment of the limit-state function, a quantified measure of the non-probabilistic reliability is defined.

Luo et al. [42] present an effective procedure for solution of stress constrained topology optimization problems under load and material uncertainties. Their method is based on the performance measure approach (PMA) and the method of moving asymptotes (MMA) proposed by Svanberg [68]. To overcome the stress singularity phenomenon caused by the combined stress and reliability constraints, a reduction strategy on target reliability index is proposed and utilized together with the ε-relaxation approach. The proposed algorithm is able to solve RBDO under stress constraints in an iterative manner based on the double-loop strategy. In this strategy outer loop was responsible for updating design parameters, while the inner loop took care about reliability assessment. The effectiveness of the proposed double-loop approach has been demonstrated on design of an L-shape structure frequently used as a benchmark problem in stress constrained topology optimization. Alternative formulation based on sequential optimization has been proposed by dos Santos et al. [60]. This formulation allowed for decoupling the problem into 2 steps related to deterministic optimization and reliability analysis, respectively. In particular, the deterministic optimization was addressed with an efficient methodology based on the topological derivative concept combined with a level-set method. As an illustrative examples they used T-bracket structure corresponding to engineering design of transmission towers. Next interesting paper on stress-constrained topology optimization under uncertainties has been proposed by da Silva et al. [66]. Reliability-based topology optimization of elasto-plastic structures handle uniform manufacturing uncertainties in topology optimization. The proposed methodology has been validated on three practical examples, namely: L-shaped problem, optimal design for an eye-bar belonging to an eye-bar-chain of a suspended bridge and the portal frame problem. Recently, the problem of uncertainty-oriented topology optimization with local stress constraints has been studied by Xia et al. [77]. The paper described an effective procedure for interval parametric structures to achieve optimal material configurations under consideration of local stiffness and strength failure. The numerical examples concerned planar L-shaped structure, 3D block structure and aeronautical joint. The ε-relaxed stress criterion and global stress aggregation approach are involved to circumvent the stress singularity and multi-constrained problems. Combined the orthogonal polynomial expansion with the set allocation theorem, an interval dimension-by-dimension method is proposed to determine feasible bounds of structural responses under unknown-but-bounded load and material uncertainties.

Recent representative examples of topology optimization could be papers by Tauzowski et al. [70,69] where a new direction of topology optimization procedure-functor oriented [69] - is created and the element size effect is considered [70]. In addition to it the Authors of this paper successfully extended this newly introduced formulation into plastic topology design [10]. Despite chosen optimization type a critical aspect of the obtained optimal solution is its dependence on a character of applied loading. Often deterministic loading are applied in the structural optimization, which can bring to the optimal solution sensitive to small variation in the applied load. It can even result in failure of the structure subjected to loading slightly different than those assumed during deterministic optimization. For that reason recently more attention is paid to the design optimization under uncertainty and in particular the so-called reliability-based topology optimization (RBTO). One can see from the wide range of overview of the publications of reliability based topology optimization in the last three decades the fully plastic application is missing or marginally covered (eq. stress limited, but elastic design).

The objective of this study is to propose a relatively simple and efficient method for reliability based topology optimization for structures made of elasto-plastic material. To the best Authors' knowledge the only paper which treated similar topic was paper by Kaymaz and Marti [31]. However, contrary to the proposed method their approach was devoted to layout optimization. The remaining part of the paper is organized as follows: in the second section background information about reliability analysis and reliability-based design optimization are recalled. Also some fundamental equation for elasto-plasticity theory are recalled along with return mapping algorithm. Then, proposed method for reliability-based topology optimization of elasto-plastic structures is introduced. Finally, two numerical examples are presented.

2. Problem statement

We are looking for minimum-weight structure made of elasto-plastic continuum. The structure is subjected to random loading and its reliability is assumed to be greater than prescribed value. Schematically such a problem has been shown in Fig. 1. To the best Author's knowledge such a problem has not been solve before. In our study, a reliability-based optimization problem can be formulated as follows: The above nested optimization problem can be formulated as follows:

$$\begin{aligned} &\text{find } \boldsymbol{\rho} \\ &\text{minimize } V(\boldsymbol{\rho}) = \sum_{e=1}^{N_e} \rho_e V_e, \\ &\text{subject to } r_i(\boldsymbol{\rho}, \mathbf{q}) = 0, \qquad i = 1,2,\ldots,N_{dof} \\ &\qquad \Pr\left(q_j(\boldsymbol{\rho},\boldsymbol{\theta}) \leqslant 0\right) \geqslant R_j^t,\ j = 1,2,\ldots,N_c \\ &\qquad \rho_e^L \leqslant \rho_e \leqslant \rho_e^U, \qquad e = 1,2,\ldots,N_e \end{aligned} \tag{1}$$

where the objective function is the mass of the system, the design vector $\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots, \rho_{N_e})^T$ represents density of individual finite element, the random parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_{N_r})^T$ it can be loads or material parameter, N_e and N_r denote the number of design variables and random parameters, respectively, r_i is the i-th component of residual vector represents equilibrium constraint. This constraint is meet by the iterative Newton–Raphson algorithm – solution of elasto-plastic problem. g_j is the performance function of the j-th design constraint, for $j = 1, \ldots, N_c, N_c$ is the number of constraints, R_j^t is the target reliability level for j-th probabilistic constraint and ρ_e^L and ρ_e^U are, respectively, the lower and upper bounds on ρ_e, for $e = 1, \ldots, N_e$.

$$\Pr\left(q_j(\boldsymbol{\rho},\boldsymbol{\theta}) \leqslant 0\right) = \int_{q_j(\boldsymbol{\rho},\boldsymbol{\theta})\leqslant 0} f_{\boldsymbol{\theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta} \tag{2}$$

where $f_{\boldsymbol{\theta}}(\boldsymbol{\theta})$ is probability density function of $\boldsymbol{\theta}$.

3. Methodology

The problem, which is the topic of the present paper consists of two main issues: topological analysis of elasto-plastic structures and reliability analysis. In this chapter, we will briefly describe a methodology applied in solving these issues, starting with remainder of the fundamental information for reliability analysis and elasto-plasticity.

3.1. Reliability analysis

Some parameters of the designed structure are in fact random. These include, for example, wind loads. In the case of mass production, they can be material parameters or shape parameters. Considering the random nature of these parameters in our project, these

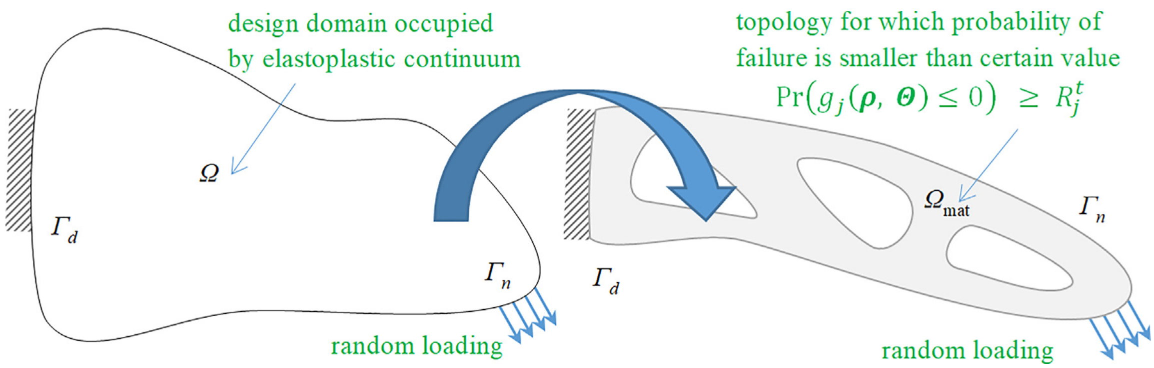


Fig. 1. Reliability-based topology optimization of elastoplastic structure.

parameters are represented by random variables and stored into vector θ. The vector of random variables belongs to the probabilistic space Ω (Fig. 2). The random nature of the design parameters means that the parameters are scattered. It may cause that some values of these parameters lead to dangerous or even destructive states. That is why the design space is divided into two half-spaces: a safe area Ω_s and a failure area Ω_f. This division is made by introducing a function called the performance function $g(\theta)$, which in the failure zone takes negative values and in the safe area positive ones (Fig. 2). The boundary surface, which is determined by the equation $g(\theta) = 0$ is called limit state surface and separates the safe area from the failure area. Failure probability is the integral over the failure area from the joint probability density function. This is expressed by the following formula:

$$P_f\left[\theta \in \Omega_f\right] = P_f[g(\mathbf{x}) < 0] = \int_{g(\theta)<0} f_\theta(\mathbf{x})d\theta, \quad (3)$$

where $f_\Theta(\theta)$ is probability density function of Θ.

3.2. Numerical techniques for reliability assessment

Analytical determination of the probability of failure directly from the formula (2) is usually not possible due to the fact that the definition of the limit state function is based on numerical solutions such as the finite element method. This implicit form of the limit state function forced the emergence of a number of methods that allow numerical estimation of the probability of failure. One of the best known and easiest methods often used as a reference method is the Monte Carlo method. It consists in a random selection a certain number N of implementations of random variables $\{\theta_1, \ldots, \theta_N\}$ based on their probability distributions. Then the value of the limit function $g(\mathbf{x})$ is calculated for each $\theta \in \Theta$. The probability of failure is the percentage of those points for which the value of the limit state function $g(\mathbf{x}) < 0$. Designers want the structure to be safe. The probability of failure should then be very low. In civil engineering, it is assumed that the probability of failure of a building structure should be in the order of 0.0001. To determine such a probability of failure, we need to randomly generate at least $N = 100000$ implementations of random variable. The duration of calculations for such a number of limit state function values based on finite element analysis may not be acceptable.

In order to be able to determine very low probabilities of failure, a number of other methods for their determination were developed shortly. Among them stands out the First Order Reliability Method (FORM). This method is based on a linear approximation of the limit state surface (Fig. 3). The idea of this method is based on the observation that the neighborhood of the Most Probable Point (MPP) point have the largest contribution in the probability of failure integral (3). It has been depicted on the Fig. 3 in green color. In such a small area, a linear approximation of the limit state surface usually gives a sufficient accuracy of the probability of failure estimation for low probabilities encountered in engineering. The original random variables are transformed from original probabilistic space $\Omega(\mathbf{x})$into a standard normal space $\Delta(\mathbf{u})$, $\mathbf{u} = T(\mathbf{x})$. For correlated random variables, Rosenblatt transformation or approximate Nataf [52] transformation should be used. There are two first order approaches for evaluation reliability constraints, namely *Reliability Index Approach* (RIA):

$$\begin{array}{ll} \text{minimize} & ||\mathbf{u}||, \\ \text{subject to} & g(\mathbf{u}) = 0, \end{array} \quad (4)$$

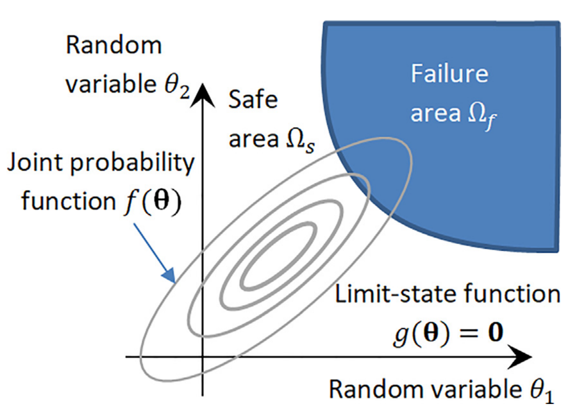


Fig. 2. Division of the probabilistic space into safe and failure area.

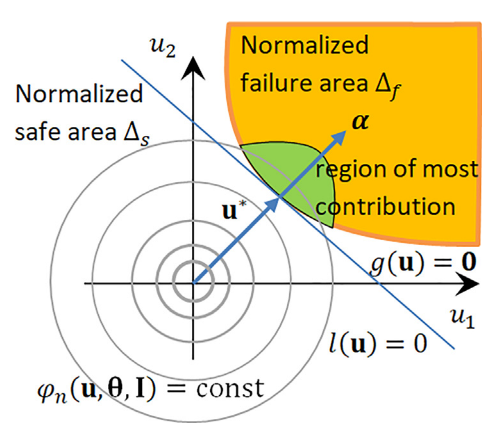


Fig. 3. Graphical representation of First Order Reliability Method.

and *Performance Measure Approach* (PMA):

$$\begin{array}{ll} \text{maximize} & g(\mathbf{u}), \\ \text{subject to} & ||\mathbf{u}|| = \beta^t, \end{array} \tag{5}$$

where β^t is target reliability index the structure should never exceed. Detailed comparison of both methodologies the reader can find in the paper by Youn et al. [79]. Both above mentioned problems are cases of nonlinear programming. General purpose numerical procedures such as sequential quadratic programming (*SQP*) can therefore be applied. However, in both cases it is much better to derive iterative formulas dedicated to these issues, characterized by greater simplicity and efficiency than general procedures. One of the simplest iterative formula for RIA was derived by Hasofer and Lind [23] and upgraded by Rackwitz and Fiesler [57]. This formula at the k-th iteration can be presented in the following way:

$$\mathbf{u}^{(k+1)} = \frac{1}{||\nabla g(\mathbf{u}^{(k)})||^2}\left(\nabla g(\mathbf{u}^{(k)})^T\mathbf{u}^{(k)} - g(\mathbf{u}^{(k)})\right)\nabla g(\mathbf{u}^{(k)}) \tag{6}$$

where $\nabla g(\mathbf{u}^{(k)})$ denotes derivative of the performance function with respect to random variables. Similar iterative formulas can be derived for the PMA approach, which was done by Youn et al. [79]. They also compared both methods in terms of efficiency. Most examples support the use of PMA, however RIA has also shown certain advantage in some cases. Much depends on the nature of the limit state function, on the convexity, so in our case of topology optimization of elastic–plastic problems we will test both approaches. Solving problems (4) or (5) most probable point (MPP) $\mathbf{u}^*$ is determined (see Fig. 3). Hasofer-Lind's [23] reliability index β_{FORM} and Probability of failure P_{FORM} have the forms respectively:

$$\beta_{\text{FORM}} = \text{sgn}(g(\mathbf{0}))||\mathbf{u}^*||, \tag{7}$$

$$P_{\text{FORM}} = \Phi(-\beta_{\text{FORM}}), \tag{8}$$

where Φ denotes cumulative distribution function for standard normal distribution. However, using the Rackwitz - Fiesler procedure is not always the best solution. At the initial stage of topology optimization, the probability of failure is very small, the procedure in the figure above processes very small numbers, which may be the reason for numerical instability. We have experienced this in our own calculations as well as it is confirmed in the above mentioned paper [79]. As a remedy for these inconveniences, an approach called Performance Measure Approach (PMA) is also implemented in our system, and in particular its variant called Hybrid Measure Approach (HMA), which also allows to solve nonconvex problems. The iterative procedure is described by the following formulas:

$$\mathbf{n}(\mathbf{u}^{(k+1)}) = \frac{\nabla g(\mathbf{u}^{(k)})}{||\nabla g(\mathbf{u}^{(k)})||}, \tag{9}$$

$$\mathbf{u}^{(k+1)} = \begin{cases} \beta^t \mathbf{n}(\mathbf{u}^{(k)}) & k \leqslant 2 \\ \frac{\mathbf{n}(\mathbf{u}^{(k)}) + \mathbf{n}(\mathbf{u}^{(k-1)}) + \mathbf{n}(\mathbf{u}^{(k-2)})}{||\mathbf{n}(\mathbf{u}^{(k)}) + \mathbf{n}(\mathbf{u}^{(k-1)}) + \mathbf{n}(\mathbf{u}^{(k-2)})||} & k > 2 \end{cases} \tag{10}$$

In the RIA approach, the reliability index is determined, while in the HMA algorithm we look for the minimum value of the performance function at an safety distance β^t. If this minimum value $g_{min} = g(\mathbf{u}_3) > 0$ is positive, it means that constraint is not active. Otherwise, if a fragment of the failure area was located closer than safe distance $g_{min} = g(\mathbf{u}_3) < 0$, it means that reliability constraint is active. To summarize, the HMA algorithm allows you to check the reliability constraint without determining the probability of failure. In addition, as it can be seen in the figure 3, the algorithm works in a region that is approximately distant by β^t from the system origin, which means that it does not process very small values of probability of failure helps to improve the algorithm stability. It is worth noting that both of the above algorithms are gradient based. The finite difference method was used to determine the gradients. Since we operate in standard, normal space on dimensionless random variables, there is no difficulties in determining the perturbation value and a constant value of 0.0001 is sufficient.

3.3. Reliability-based topology optimization

The optimal design is the one for which we find the minimum value of a certain function of the design parameters (in topological optimization, most often density) while fulfilling certain constraints. A reliable design is one for which we control the probability of failure. Combining these two approaches we obtain reliability based optimization. This means adding a new design constraint. Therefore, a reliable design will be less optimal because it must meet additional constraints related to safety $f(\mathbf{x}^{\text{opt}}_{\text{DET}}) < f(\mathbf{x}^{\text{opt}}_{\text{RBDO}})$ (Fig. 4). In an optimal design there is a risk of exceeding the limits caused by fluctuations in certain design parameters. To prevent this difficulty, we formulate the reliability based optimization problem as follows:

$$\begin{array}{ll} \min\limits_{\boldsymbol{\rho}} & V(\boldsymbol{\rho}) \\ \text{s.t.} & \mathbf{r}(\boldsymbol{\rho}, \mathbf{q}) = \mathbf{0} \\ & P_f(\boldsymbol{\rho}, \mathbf{x}) < P_t \end{array} \tag{11}$$

where $\boldsymbol{\rho}$ denotes design variables (density of individual finite elements), vector $\mathbf{r}$ represents residua of the equilibrium equations and P_f is a probability of failure. The above general formulation of reliability optimization problem shows that two issues are combined: minimizing the objective function and determining the probability of failure. Because in the first-order approach (FORM) determining the probability of failure is in fact also an optimization problem, the task of reliability optimization is a combination of two optimization problems. Several methods have been developed to solve such a nested formulation:

- **Double loop**, Optimization loop minimizing the objective function is the first loop. Inside optimization loop reliability analysis loop is called to check the reliability constraint - second loop. It is nested approach because reliability loop is nested inside opti-

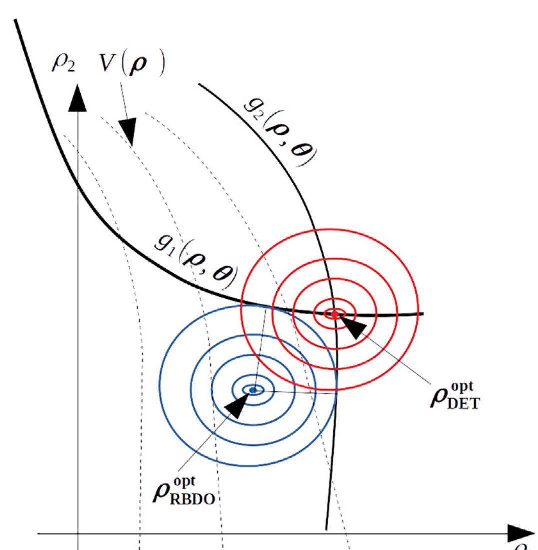


Fig. 4. Deterministic vs reliability-based design optimization.

mization loop. Double loop approach is the most commonly used and hence the best tested approach, the advantage of this method is its stability. Reliability estimation for each set of design variables allows for precisely meeting of reliability constraints. Among the advantages, it can be also mention the modularity. Reliability may be estimated by external software called in optimization loop. However this modularity can be limited by the need to determine reliability index gradients if it is not delivered by an external reliability software. Among the disadvantages of this approach are the numerical complexity caused by the need to estimate the reliability indicator for each value of decision variables.

- **Single loop**, A single loop, in this version of the algorithm, one iterative procedure updates both the search for the minimum objective function as well as the search for the design point $\mathbf{u}^*$ (Fig. 3) on which the failure probability FORM estimation is based. This approach allows to avoid a nested reliability task by replacing reliability constraints with deterministic ones. Hence the basic advantage of this method (and unfortunately the only one) is the chance for high numerical efficiency resulting from the lack of the need to determine the reliability index for each value of decision variables. Unfortunately, this method has a number of disadvantages. The main disadvantage is the possibility of decrease in numerical efficiency as well as accuracy depending on various factors such as the selection of the starting point. This approach is not modular. It is difficult to outsource any part of the algorithm to external programs. The entire procedure must be originally programmed.
- **Decoupled**, Optimization and reliability analysis are performed separately. Both are repeated sequentially until some convergence criterion is meet. Decoupled approach is implemented in two independent procedures that perform optimization and reliability tasks. The coupling consists in the fact that the data for the optimization procedure are based on the results of reliability analysis. The process is repeated iteratively. Among the greatest advantages of this approach are modularity. Procedures are connected only by data. So you can use completely independent procedures, or even two different, programs to perform reliability calculations and optimization. This makes the implementation of this method the easiest of these three approaches. However, disadvantages include numerical efficiency, especially in the context of topological optimization. To solve the problem by the "decoupled" method it will be necessary to call the optimization procedure several times. It can significantly increase the calculation time especially for large tasks.

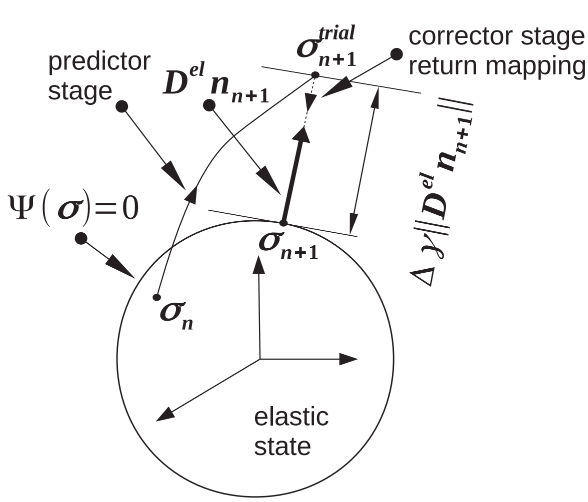


Fig. 5. Predictor and corrector stage of the elastoplastic analysis.

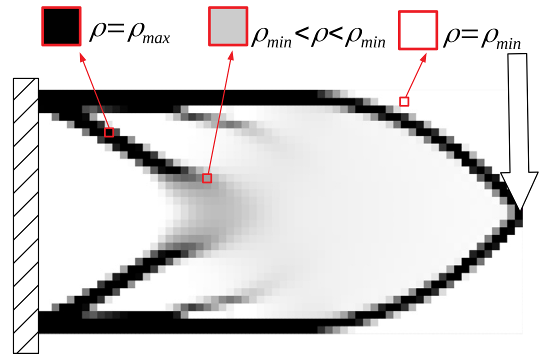


Fig. 6. Design variable of topology optimization as density.

3.4. Elasto-plastic analysis

In this section, fundamental relationships governing the elasto-plastic problem will be briefly reminded such as the elastic law, associative flow rule, yield surface. It is worth noting that issues related to plasticity, stress will never exceed the yield point. Hence, in the optimization formulation there is no need to impose constraint on stresses because they are met by definition. However, an important limitation that we must meet is the load capacity of the structure.

3.4.1. Fundamental relations

The formulation of the elasto-plastic problem is based on the classical Prandtl-Reuss assumption about the additivity of elastic $\boldsymbol{\varepsilon}^{el}$ and plastic $\boldsymbol{\varepsilon}^{pl}$ strains:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{pl} + \boldsymbol{\varepsilon}^{el}. \tag{12}$$

Cauchy stress tensor can be expressed as follows:

$$\boldsymbol{\sigma} = \mathbf{D}^{el} : \boldsymbol{\varepsilon}^{el}. \tag{13}$$

The von Mises yield condition was chosen, having the following form:

$$\Psi = \sqrt{3J_2} - \sigma_0, \tag{14}$$

where Ψ is yield surface, σ_0 is the yield stress, J_2 is second stress invariant and $\mathbf{s}(\boldsymbol{\sigma})$is stress tensor deviatoric part. Standard associative flow rule was assumed in the form:

$$\dot{\boldsymbol{\varepsilon}}^{pl} = \dot{\gamma}\mathbf{n} = \dot{\gamma}\frac{\partial \Psi}{\partial \boldsymbol{\sigma}}, \tag{15}$$

where $\dot{\gamma}$ is the plastic flow multiplier and $\mathbf{n}$ is Prandtl-Reuss flow vector which can be expressed as follow:

$$\mathbf{n} = \frac{\partial \Psi}{\partial \boldsymbol{\sigma}} = \sqrt{\frac{3}{2}}\frac{\mathbf{s}}{||\mathbf{s}||}, \tag{16}$$

where $\mathbf{s}$ is deviatoric part of the stress.

3.4.2. Predictor–corrector solution strategy

Predictor–corrector solution strategy is most frequently used solution methodology for nonlinear equations system. Predictor and corrector are two stages of computation performed on each iteration until the convergence condition is met. In elasto-plastic analysis predictor stage n-th displacement increment $\Delta\mathbf{q}$ is evaluated according to formula:

$$\Delta\mathbf{q} = -\left(\mathbf{K}_T^{(n)}\right)^{-1}\mathbf{r}^{(n)}, \tag{17}$$

where $\mathbf{K}_T^{(n)}$ is tangent matrix and $\mathbf{r}^{(n)}$ is residual vector. Finite element Eq. (17) can be solved by Newton–Raphson iteration

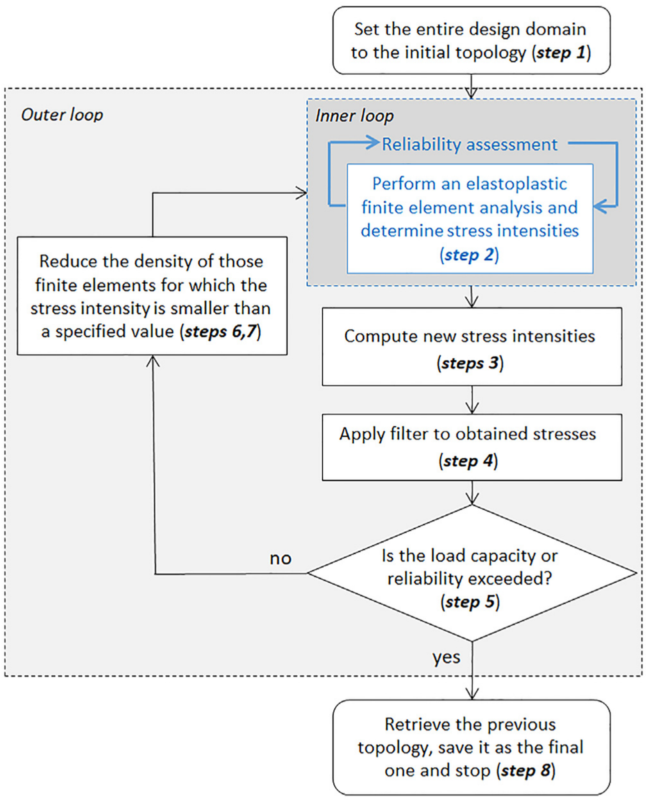


Fig. 7. Flowchart of the proposed optimization algorithm.

procedure. Global tangent matrix is computed and assembled in according to classical formula:

$$\mathbf{K}_T^{(n)} = \bigcup_{e=1}^{Ne} \int_{Ve} \mathbf{B}^T \mathbf{D}^{el-pl} \mathbf{B}\, dv, \tag{18}$$

$$\mathbf{r}^{(n+1)} = \mathbf{f} - \bigcup_{e=1}^{Ne} \int_{Ve} \mathbf{B}^T \boldsymbol{\sigma}^{(n+1)}\, dv, \tag{19}$$

where $\mathbf{f}$ is load vector, $\mathbf{B}$ is strain derivatives matrix. It is worth noting here that the derivatives of displacements with respect to global coordinates require the determination of the Jacobian matrix at each Gaussian point. In topological optimization, regular meshes containing identical finite elements, in term of shape, are most often used. Therefore, it is worth organizing the algorithm so that $\mathbf{B}$ matrices are calculated once. Similarly, the element tangent matrix $\mathbf{K}_T^{(e)}$ for non-plasticized areas, in which an elastic stiffness matrix is identical for each element in regular finite element mesh. This approach significantly speeds up computations. Having $\Delta\mathbf{q}$, prediction of the stress and strain are evaluated according to equations below:

$$\Delta\varepsilon = \mathbf{B}\Delta\mathbf{q}, \tag{20}$$

$$\boldsymbol{\varepsilon}_{trail}^{(n+1)} = \boldsymbol{\varepsilon}^{(n)} + \Delta\varepsilon, \tag{21}$$

$$\boldsymbol{\sigma}_{trail}^{(n+1)} = \mathbf{D}^{el} \boldsymbol{\varepsilon}_{trail}^{(n+1)}, \tag{22}$$

The purpose of the next stage called the corrector is to modify the values of mechanical quantities including stresses so that the condition that stresses cannot exceed the surface of plastic flow is met. In addition, the associated flow law implies that the stress vector in the plastic state is perpendicular to the plastic flow surface. The algorithm for this correction in plasticity is called return mapping and requires the use of the Newton–Raphson procedure at each Gauss point to evaluate plastic flow multiplier $\Delta\gamma$. In the particular case of three-axis stress state and perfect plasticity, the stress correction procedure does not require iteration and $\Delta\gamma$ can be evaluated from single formula. Geometrical interpretation of return mapping algorithm is depicted in Fig. 5. Stresses and strains at the n-th iteration can be computed from the following formulas:

$$\boldsymbol{\varepsilon}^{(n+1)} = \boldsymbol{\varepsilon}_{trail}^{(n+1)} - \Delta\gamma \mathbf{n}^{(n+1)}, \tag{23}$$

$$\boldsymbol{\sigma}^{(n+1)} = \boldsymbol{\sigma}_{trail}^{(n+1)} - \Delta\gamma \mathbf{D}^{el} \mathbf{n}^{(n+1)}. \tag{24}$$

4. Computational procedure

In the present section all information presented so far will be gathered into one procedure which allow to determine minimum volume structure which is able to carry given probabilistic loads with prescribed level of reliability. For this purpose we combine the yield limited topology optimization method described in detail in the previous Authors' paper (Blachowski et al. [10]) and reliability assessment using first order method described in Section 3.

As a result the proposed topology optimization with reliability assessment leads to nested optimization problem. In the first level optimization is conduced towards minimization of the structural volume (Fig. 6). Then, at each iteration reliability index is determined and compared with its prescribed desired value. To perform this comparison optimization of the second level is performed. In this way reliability index is determined using FORM. If the structure at the given iteration step still satisfies reliability condition the redundant volume is removed based on stress intensity criterion, i.e. least stressed element is removed. The procedure is repeated until either current design violated reliability condition or external loading causes yield flow of the optimized structure (Fig. 7).

As illustrations of the algorithm described in p.4 three topologies for selected iterations of outer loop illustrating the operation of our algorithm are presented (Fig. 8). The first row depicts the topology at iteration 11. It illustrates the beginning of the topology optimization process. In this figure, the material being removed from the vertices, i.e. the least-stressed areas of the project design. The next iteration, namely 175 is shown in the second row of the table it illustrate the formation of empty spaces within the structure. The final iteration is 350, and it is shown in last row of the table illustrating the final result. An internal loop is made at each iteration to check that the reliability constraints are met. Outer loop operates on design variables changing density of finite element while inner loop, related to reliability constraints checking working on random variables (in our examples horizontal force F_{random}). Constraints can be checked using PMA or RIA algorithms, depending on the user's choice.

5. Numerical examples

5.1. Two-bar benchmark problem

To verify correctness of the proposed algorithm a benchmark example was chosen. The example was taken from the seminal paper by Rozvany and Maute [59], where analytical solution for the considered problem has been found. Graphical representation of the task is presented in Fig. 9. Analytical relation between force direction angle θ and inclination α is expressed by the formula

$$\alpha = \arctan\sqrt{\left(\sqrt{\tan^4\theta + 8\tan^2\theta} - \tan^2\theta\right)/4}, \tag{25}$$

which is depicted in Fig. 10. Several examples for different angles of acting force were performed for linear analysis. Bar angles in obtained topologies shows close consistency with analytical

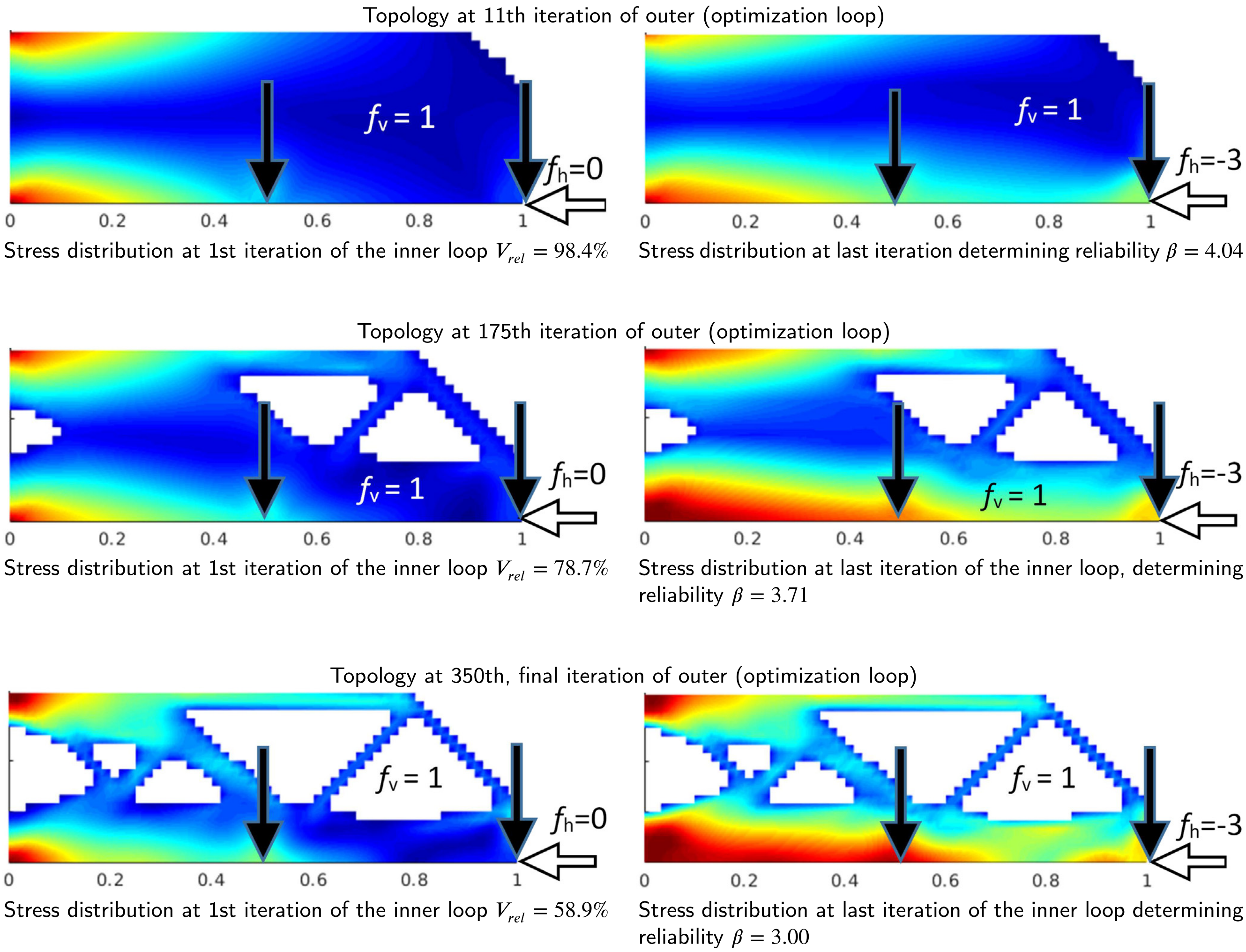


Fig. 8. Convergence of the iterative process for reliability-based topology optimization.

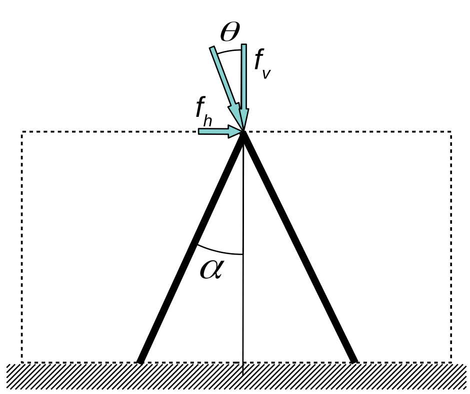


Fig. 9. Benchmark problem of Rozvany and Maute [59] for reliability-based optimization.

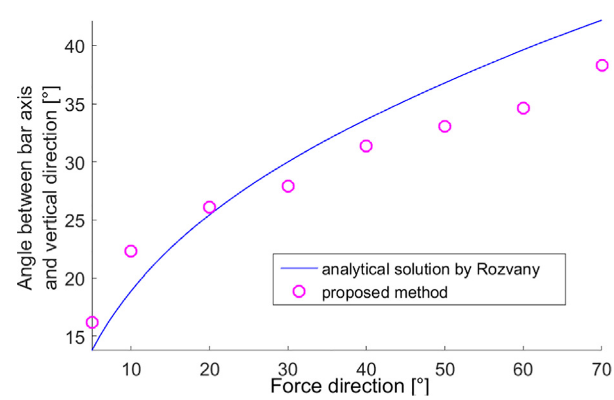


Fig. 10. Relation between force direction and bars direction.

solution [59]. Optimal topologies for these solutions are presented in Figs. 11 and 12.

5.2. L-shape structure in plane stress

The second numerical example concerns an L-shape structure widely used as a benchmark in the literature devoted to stress-constrained topology optimization. In this benchmark some difficulties arise and are related to stress concentration in re-entrant corner. The schematic L-shape bracket is presented in Fig. 13. The L-shape bracket will be solve for two different loading conditions. The first case will correspond to low stress intensity i.e. elastic case, while the second one will correspond to high-stress intensity. This second case will require elastoplastic analysis in order to find the optimal topology. There is some preliminary explanation regarding the load of the structure. The elastoplastic structure has some limited load capacity. It is therefore necessary to ensure that load is lower than maximal. Thus, the actual load applied to the structure is expressed by the following formula: $\mathbf{f}_{\text{real}} = \phi\gamma\mathbf{f}$. In this formula γ is the minimal load capacity factor calculated for the load vectors $\left(\mathbf{f}_i = \mathbf{f}_d \pm 3\sigma_i \mathbf{f}_i^p\right), i = 1, \ldots, N_r$, where $\mathbf{f}_d$- deterministic part of load, $\mathbf{f}_i^p$- probabilistic load related with ith random variable, N_r is the number of independent random variables, ϕ is the reduction factor determining the load level in relation to the maximum load the user wants to apply ($\phi < 1$). In following numerical examples two load levels are applied: low stress intensity load, where $\phi = 0.1$ and high stress intensity load $\phi = 0.8$, which means such a load value that causes plastic zones in the structure. (see Figs. 14–17).

Algorithm 1. Topology optimization of elastoplastic structures with reliability constraints

Step 1 Initialize design variables to a vector of ones $\rho_e^{(0)} = \{1, 1, \ldots, 1\}$and erased element list to an empty list $\mathscr{L} = \{\}$. Initialize random variables vector $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_{N_r}\}$.

Step 2 (inner loop) Asses reliability constraints using:

Performance Measure Approach (PMA)	or Reliability Index Approach (RIA)
min $g(\mathbf{u})$	min $\|\|\mathbf{u}\|\|$
subject to: $\|\|\mathbf{u}\|\| = \beta^t$	subject to: $g(\mathbf{u}) = 0$

At every k-th iteration of inner loop solve nonlinear equations of the elasto-plastic problem

$$\mathbf{r}\left(\boldsymbol{\rho}_e^{(k)}, \mathbf{u}^{(k)}\right) = 0.$$

Step 3 Determine the stress intensity calculated as the average of equivalent von Mises stresses evaluated at each Gauss point, then normalize the obtained values dividing them by the yield limit

$$\overline{\sigma}_i = \frac{1}{N_g \sigma_0} \sum_{g=1}^{N_g} \sigma_{i,g}^g, \qquad i = 1, 2, \ldots, N.$$

Step 4 Apply a design filter to avoid the checkerboard phenomenon and reduce mesh dependence of the results.

Step 5 If load capacity or reliability $P_{\text{FORM}} > P_{\text{limit}}$ exceed permissible limit then go to Step 8.

Step 6 Select n finite elements with the smallest stress intensities $\overline{\sigma}_e < \overline{\sigma}_{e_{min}} + \overline{\sigma}_t$ (usually $\overline{\sigma}_t = 0.005$) and add the list of the newly selected elements l to the list of previously erased elements,

$$\mathscr{L}^{(k)} = \left\{\mathscr{L}^{(k-1)}; l\right\}.$$

Step 7 Using the current list of erased elements L update corresponding design variables applying the following iterative formula and go to Step 2.:

$$\boldsymbol{\rho}^{(k)} = \max_{l \notin \mathscr{L}}\left(\boldsymbol{\rho}_{min}\left[\{\overline{\sigma}_l\}_{\text{filter}}\right]^p \boldsymbol{\rho}_l^{(k-1)}\right).$$

Step 8 Retrieve the topology from the previous iteration, save it as the final one and stop.

5.2.1. Low stress intensity - elastic case

In the case of low stress intensity the following material constants are assumed: $E = 1$ and $\nu = 0.3$. Vertical load is deterministic $f_\nu = 1$, while horizontal load is probabilistic with standard normal distribution $f_h = N(0, 0.4)$. Performance function is computed according to the following formula:

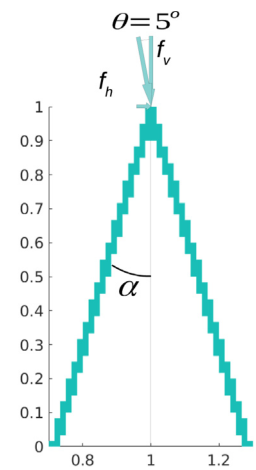


Fig. 11. Optimal topology for force angle $\theta = 5°$. Obtained bar angle $\alpha = 16.2°$, while theoretical value (Eq. 25) $\alpha_r = 13.8°$.

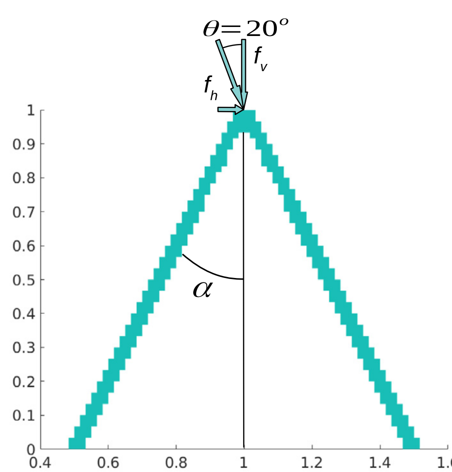


Fig. 12. Optimal topology for force angle $\theta = 20°$. Obtained bar angle $\alpha = 26.1°$, while theoretical value (Eq. 25) $\alpha_r = 25.4°$.

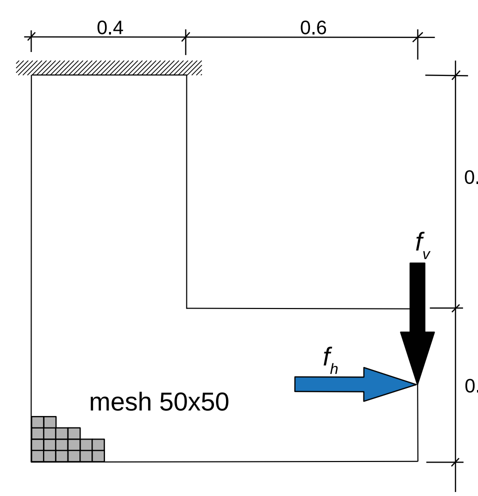


Fig. 13. L-shape structure with deterministic and probabilistic loads.

$$g(x) = |q_h| < 3q_{\text{init}}, \tag{26}$$

where q_h describes current horizontal displacement at the force and q_{init} is horizontal displacement computed for initial design space (at

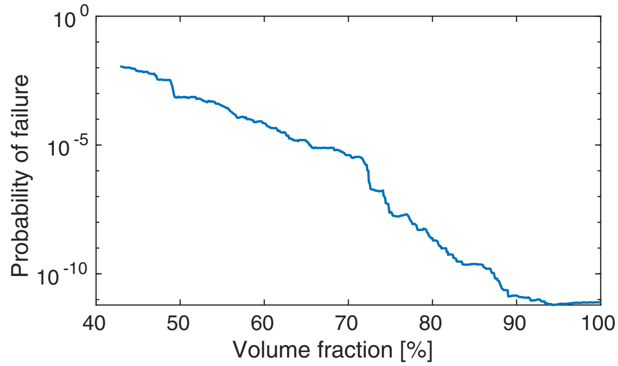


Fig. 14. Dependence of the probability of failure on volume fraction.

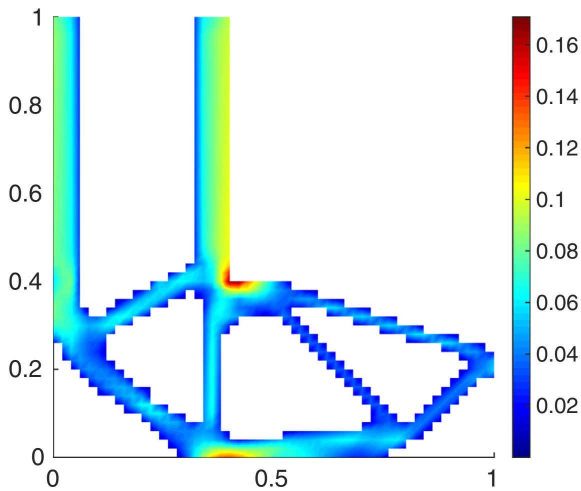


Fig. 15. Deterministic optimal topology in the case of low stress intensity (purely elastic range).

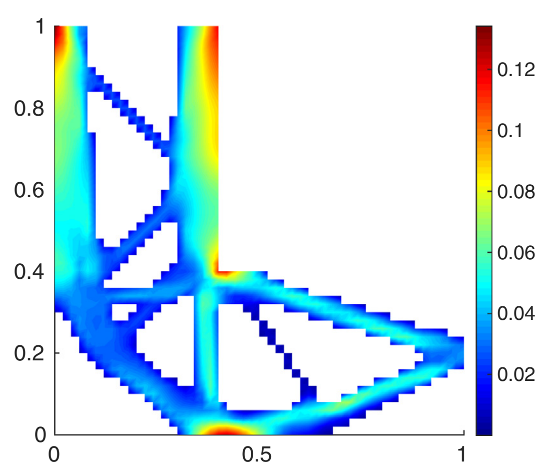


Fig. 16. Reliability-based optimal topology in the case of low stress intensity.

the beginning of topology optimization process). The target value of the probability of failure is $P_t = 0.015$. For deterministic load only the probability of failure $P_f = 0.36$, while the corresponding $V_{rel} = 35\%$. For comparison purpose these values has been obtain using the reliability based optimization and both deterministic and probabilistic loads, the obtained in this case probability of failure and volume are $P_f = 0.014$ and $V_{rel} = 52\%$, respectively.

5.2.2. High stress intensity - elasto-plastic case.

To demonstrate ability of the proposed methodology in the case of elasto-plastic range the same example as in the previous subsection was computed with yield stress $\sigma_0 = 1$. Horizontal probabilistic force with standard normal distribution taken the following values $f_h = N(0, 0.2)$. Target probability of failure was increase to value $P_t = 0.03$.

5.3. Cantilever structure in plane stress

Another example demonstrating the effectiveness of the presented algorithm is the topological optimization of the elastically plastic bracket. The task diagram is presented at Fig. 18. The beam is loaded with two vertical deterministic forces $f_v = 1$ in the middle of the span and at the free end of cantilever. and finally a horizontal force of random nature was imposed. Force has a Gaussian distribution $N(0, 1.2)$. The horizontal displacement of the point of application of random force is the basis for the definition of the performance function (as in previous examples). The performance function is given by the formula (27):

$$g(x) = |q_h| < 2q_{init}, \tag{27}$$

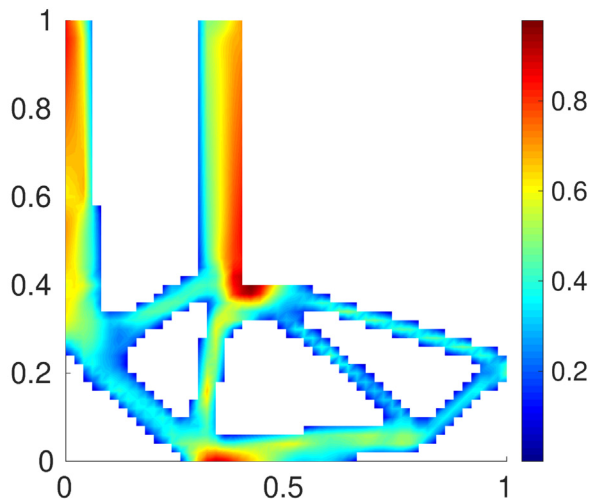
where q_{init} is vertical displacement of initial design domain, at the beginning of optimization process, q_h is current horizontal displacement at the point of random force application. The safety level was assumed assuming an acceptable reliability index $\beta_T = 3$. The same performance function and safety level was applied for low as well as high stress intensity case. The deterministic solution is more optimal, but less reliable. Taking into account the reliability constraints will provide a solution with the desired safety level at the expense of more weight. This relation is illustrated in Fig. 19.

5.3.1. Low stress intensity - elastic case

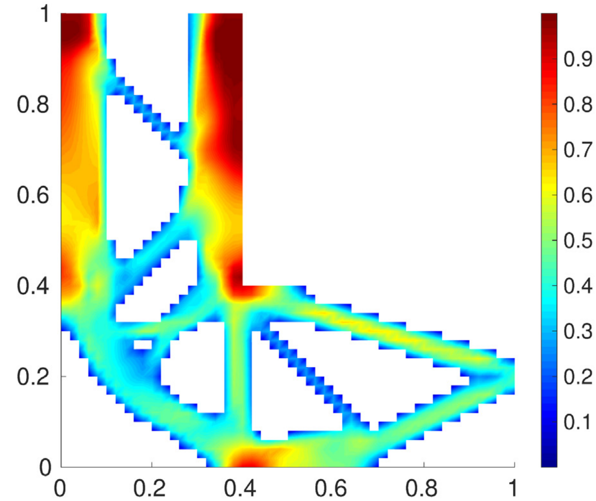
In the case of low stress intensity the following material constants are assumed: $E = 1$ and $\nu = 0.3$. Vertical load is deterministic $f_v = 1$, while horizontal load is probabilistic with standard normal distribution $f_h = N(0, 1.2)$. The resulting topologies are presented in Fig. 20. For a deterministic elastic solution (fig. 20 a)), the failure probability was determined as $\beta = 2.52$. Therefore, optimal structure has a greater probability of failure than the assumed safety level $\beta^t = 3$. This is the motivation to improve the safety of the structure and perform optimization with reliability constraints.

5.3.2. High stress intensity - elasto-plastic case

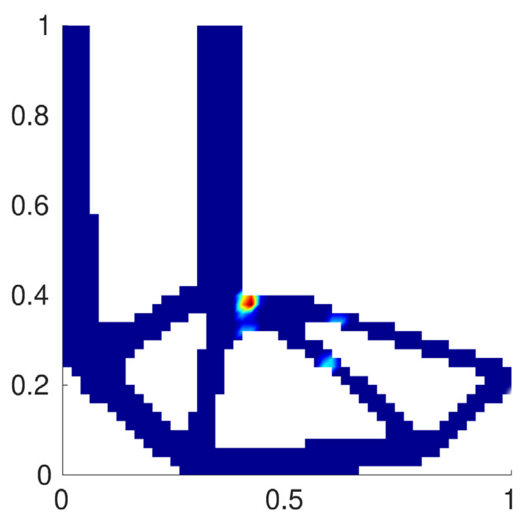
The same material data are applied as in the low stress intensity example. A few words of commentary require a lack of failure probability value for deterministic topology at high stress intensity (Fig. 20 c)). It should be remembered that elasto-plastic structures have some limited load capacity. The equilibrium problem for loads exceeding the load capacity cannot be solved. This is a natural limitation that must be taken into account when using plastic models. If topological optimization for such structures ends with active constraint of load capacity, it means that perturbation increasing the load value is not possible because then the load capacity of the structure is exceeded. Therefore, it is impossible to determine the probability of failure regardless of the performance function because the random variability of the load will surely exceed the load capacity of the structure. Without the failure probability determined, the structure may be considered insufficiently reliable due to any random values. Of course, it is possible to calibrate numerical examples by slightly offsetting the load from the capacity value so that the probability of failure can be computed for all cases. We considered that, it is important to pay attention to the need to take into account the load capacity of elastic–plastic structures.



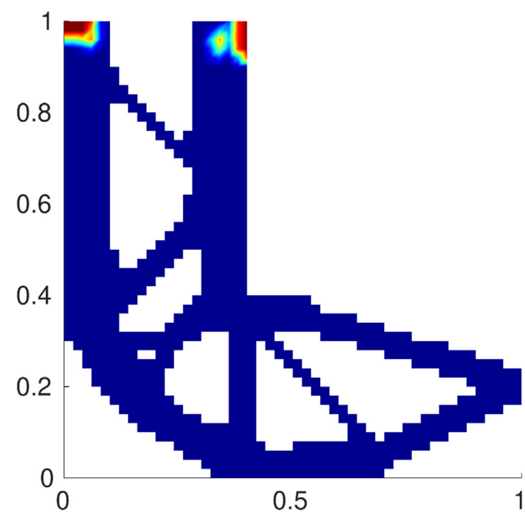
a) Deterministic optimal topology in the case of high stress intensity



b) Reliability-based optimal topology in the case of high stress intensity



c) Plastic zones under deterministic load only



d) Plastic zones under combined action of deterministic and probabilistic loads

Fig. 17. High stress intensity results.

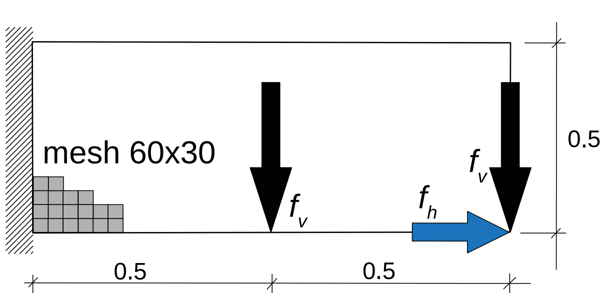


Fig. 18. Cantilever structure with deterministic f_v and probabilistic f_h loads.

6. Conclusions

In the present paper a novel methodology for reliability-based topology optimization of elasto-plastic structures has been proposed. The methodology consists of two nested optimization problems called outer and inner loop. Outer loop is based on a heuristic

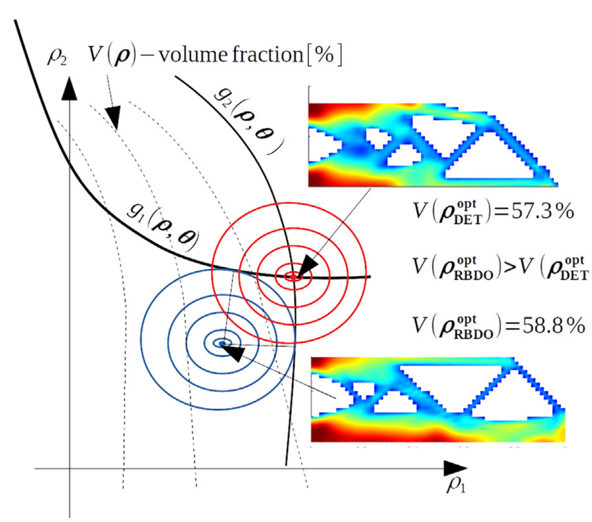
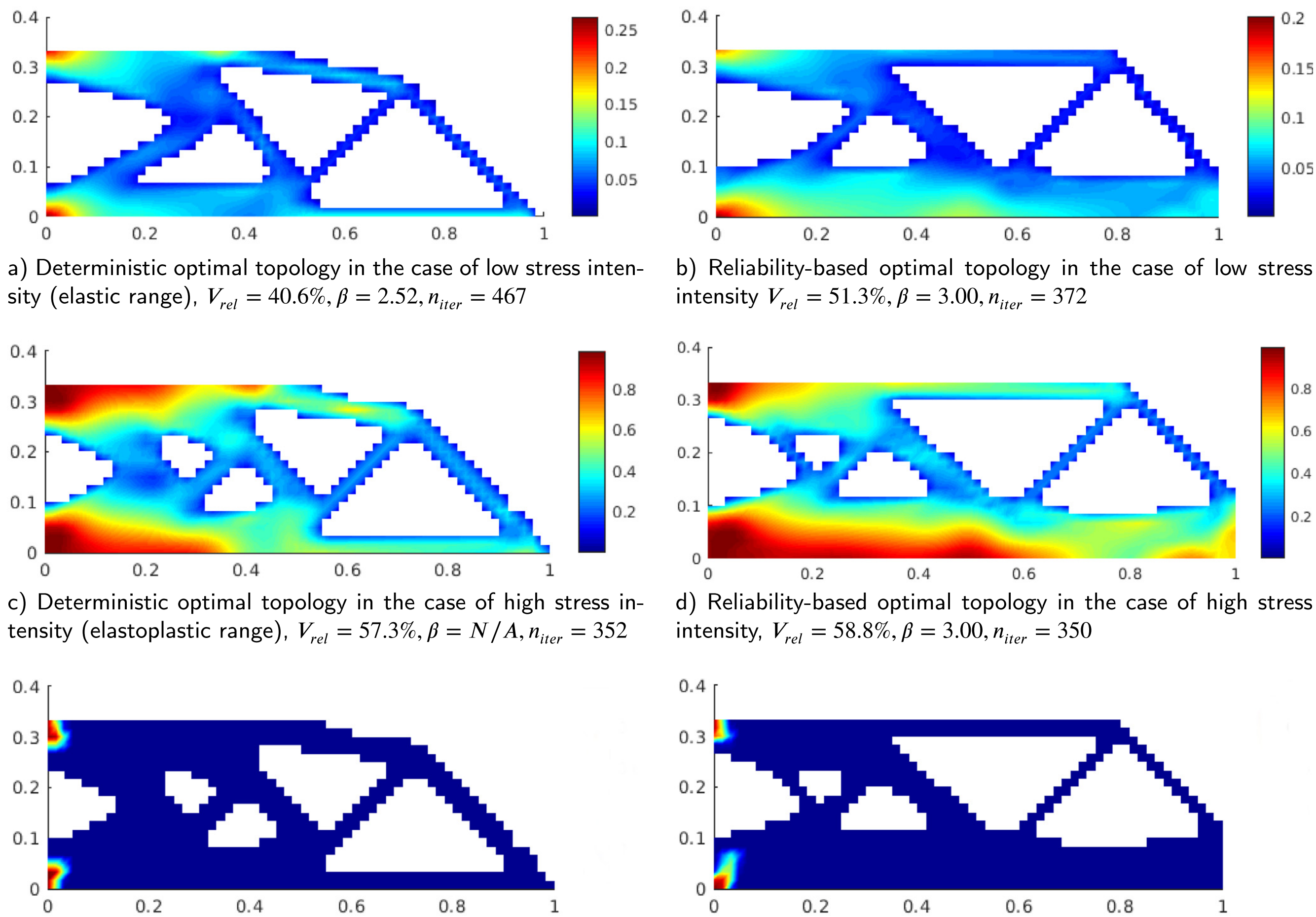


Fig. 19. Deterministic vs reliability based solution.



a) Deterministic optimal topology in the case of low stress intensity (elastic range), $V_{rel} = 40.6\%, \beta = 2.52, n_{iter} = 467$

b) Reliability-based optimal topology in the case of low stress intensity $V_{rel} = 51.3\%, \beta = 3.00, n_{iter} = 372$

c) Deterministic optimal topology in the case of high stress intensity (elastoplastic range), $V_{rel} = 57.3\%, \beta = N/A, n_{iter} = 352$

d) Reliability-based optimal topology in the case of high stress intensity, $V_{rel} = 58.8\%, \beta = 3.00, n_{iter} = 350$

e) Plastic zones under deterministic load only

f) Plastic zones under combined action of deterministic and probabilistic loads

Fig. 20. Comparison between deterministic and reliability-based solution for cantilever structure.

algorithm derived on the basis of the fully stress design concept. This loop allows for reducing volume of the optimized structure. The inner loop utilizes the well-known FORM method which is responsible for calculation of the reliability index. To demonstrate the effectiveness of the proposed methodology three numerical examples have been used. The first one is a two-bar benchmark problem for which analytical solution is known, the second one is an L-shape bracket frequently used for validation of different methods of stress-constrained topology optimization and finally the third one is cantilever structure subjected to two forces, one of which is deterministic and the other probabilistic one. Based on the calculated examples one can conclude that the results obtained by proposed methodology are in good agreement with analytical solution in the case of two bar structure and in the case of L-shape bracket provide optimal topologies comparable with those obtain by other methods.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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