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ON A PROBLEM OF NIRENBERG CONCERNING EXPANDING MAPS IN HILBERT SPACE

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ABSTRACT. Let $\mathbf{H}$ be a Hilbert space and $f\colon \mathbf{H} \to \mathbf{H}$ a continuous map which is expanding (i.e., $\|f(\mathbf{x}) - f(\mathbf{y})\| \geq \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbf{H}$) and such that $f(\mathbf{H})$ has nonempty interior. Are these conditions sufficient to ensure that f is onto? This question was stated by Nirenberg in 1974. In this paper we give a partial negative answer to this problem; namely, we present an example of a map $F\colon \mathbf{H} \to \mathbf{H}$ which is not onto, continuous, $F(\mathbf{H})$ has nonempty interior, and for every $\mathbf{x}, \mathbf{y} \in \mathbf{H}$ there is $n_0 \in \mathbb{N}$ (depending on $\mathbf{x}$ and $\mathbf{y}$) such that for every $n \geq n_0$

$$\|F^n(\mathbf{x}) - F^n(\mathbf{y})\| \geq c^{n-m}\|\mathbf{x} - \mathbf{y}\|$$

where F^n is the nth iterate of the map F, c is a constant greater than 2, and m is an integer depending on $\mathbf{x}$ and $\mathbf{y}$. Our example satisfies $\|F(\mathbf{x})\| = c\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbf{H}$.

We show that no map with the above properties exists in the finite-dimensional case.

1. INTRODUCTION

In 1974 Nirenberg [9] stated the following problem:

(P_1) Let $\mathbf{H}$ be a Hilbert space and let $f\colon \mathbf{H} \to \mathbf{H}$ be a continuous map that is expanding and whose range contains an open set. Does f map $\mathbf{H}$ onto $\mathbf{H}$?

This question could be generalized to the case (in this paper called (P_2)) when the spaces considered are Banach spaces $\mathbf{X}$, $\mathbf{Y}$.

There are several partial positive answers to (P_1) and (P_2) in the following cases:

(a) $\mathbf{X}$ is finite dimensional [1, 2],

(b) $f = I - C$ where C is compact or a contraction or more generally a k-set-contraction [6, 10],

(c) f strongly monotone, i.e., there exists $s > 0$ such that [3, 7]

$$\mathrm{Re}\langle f(\mathbf{x}) - f(\mathbf{y}), \mathbf{x} - \mathbf{y}\rangle \geq s\|\mathbf{x} - \mathbf{y}\|^2 \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbf{X}.$$

In [4] Chang and Shujie proved the surjectivity of the map $f\colon \mathbf{X} \to \mathbf{Y}$ ($\mathbf{X}$, $\mathbf{Y}$ Banach spaces) under the additional assumptions that $\mathbf{Y}$ is reflexive, f is

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Fréchet-differentiable, and

$$\limsup_{\mathbf{x}\to\mathbf{x}_0} \|f'(\mathbf{x}) - f'(\mathbf{x}_0)\| < 1 \quad \text{for all } \mathbf{x}_0 \in \mathbf{X}.$$

Seven years ago Morel and Steinlein [8] gave a beautiful counterexample to (P_2) in the case when f acts in the Banach space $L^1(\mathbb{N})$.

In this paper we suggest a negative answer to (P_1); namely, we present an example of a map $F: \mathbf{H} \to \mathbf{H}$ which is not onto, continuous, $F(\mathbf{H})$ has nonempty interior, and for every $\mathbf{x}, \mathbf{y} \in \mathbf{H}$ there is $n_0 \in \mathbb{N}$ (depending on $\mathbf{x}$ and $\mathbf{y}$) such that for every $n \geq n_0$

$$\|F^n(\mathbf{x}) - F^n(\mathbf{y})\| \geq c^{n-m}\|\mathbf{x} - \mathbf{y}\|,$$

where F^n is the nth iterate of F, c is a constant greater than 2, and m is an integer depending on $\mathbf{x}$ and $\mathbf{y}$. This condition means that the distance between any two trajectories of the discrete dynamical system $F: \mathbf{H} \to \mathbf{H}$ tends to infinity in an exponential way.

2. The example

We start by constructing a map $f: L^2(\mathbb{N}) \to L^2(\mathbb{N})$ with the following properties:

(a) f is continuous,
(b) $B(0, 1) \subset f(L^2(\mathbb{N}))$ where $B(0, 1)$ is the unit ball in $L^2(\mathbb{N})$,
(c) $f(L^2(\mathbb{N})) \neq L^2(\mathbb{N})$,
(d) f is an injection.

Then we define a map F by $F(\mathbf{x}) := cf(\mathbf{x})$. Taking into account the properties of f we show that F satisfies the required assumptions.

To define f we first introduce a continuous function $\psi: R^+ \to R^+$ such that

$\psi(t) := t$ for all t so that $t \leq 1$ and $2 \leq t$,
$\alpha t < \psi(t) < t$ for $1 < t < 2$,
ψ is C^1,

where α is a fixed number which satisfies $0 < \alpha < 1$.

Now for every $\mathbf{x} \in L^2(\mathbb{N})$ let $n_\mathbf{x}$ denote the minimal natural number such that

$$\left(\sum_{i=1}^{n_\mathbf{x}} x_i^2\right)^{1/2} \leq \psi(\|\mathbf{x}\|) \leq \left(\sum_{i=1}^{n_\mathbf{x}+1} x_i^2\right)^{1/2}.$$

(We allow $n_\mathbf{x} = 0$ and then the left side of the above inequality is 0.) We set

$$f(\mathbf{x}) := \begin{cases} \mathbf{x} & \text{for all } \mathbf{x} \text{ such that } \|\mathbf{x}\| \leq 1 \text{ or } 2 \leq \|\mathbf{x}\|, \\ (x_1, x_2, \ldots, x_{n_\mathbf{x}}, \alpha_\mathbf{x} x_{n_\mathbf{x}+1}, \sqrt{1-\alpha_\mathbf{x}^2}\, x_{n_\mathbf{x}+1}, x_{n_\mathbf{x}+2}, x_{n_\mathbf{x}+3}, \ldots) & \\ \qquad\qquad \text{for } 1 < \|\mathbf{x}\| < 2, \end{cases}$$

where $\alpha_\mathbf{x}$ satisfies

$$\left(\sum_{i=1}^{n_\mathbf{x}} x_i^2 + \alpha_\mathbf{x}^2 x_{n_\mathbf{x}+1}^2\right)^{1/2} = \psi(\|\mathbf{x}\|). \tag{1}$$

(Of course $0 \leq \alpha_\mathbf{x} < 1$; if $x_{n_\mathbf{x}+1} = 0$ then $\alpha_\mathbf{x} := 0$.)

The continuity of f and properties (b) and (c) are easy to prove. So we must only prove (d).

Before passing to the proof we make the obvious observation that

$$\|f(\mathbf{x})\| = \|\mathbf{x}\| \quad \text{for every } \mathbf{x} \in L^2(\mathbb{N}). \tag{2}$$

Taking into account this observation we show (d).

Lemma. *Let* $\mathbf{x}, \mathbf{y} \in L^2(\mathbb{N})$ *and* $f(\mathbf{x}) = f(\mathbf{y})$. *Then* $\mathbf{x} = \mathbf{y}$.

Proof. By definition of f and (2) it is sufficient to consider the case when $1 < \|\mathbf{x}\| < 2$ and $1 < \|\mathbf{y}\| < 2$. By (2) we see immediately that $\psi(\|\mathbf{x}\|) = \psi(\|\mathbf{y}\|)$, and from (1) and the fact that $f(\mathbf{x}) = f(\mathbf{y})$ it follows that $n_{\mathbf{x}} = n_{\mathbf{y}}$ and, consequently, $x_i = y_i$ for both $i = 1, 2, \dots, n_{\mathbf{x}}$ and $i = n_{\mathbf{x}} + 2, n_{\mathbf{x}} + 3, \dots$. Since $\|\mathbf{x}\| = \|\mathbf{y}\|$ we conclude that $|x_{n_{\mathbf{x}}+1}| = |y_{n_{\mathbf{x}}+1}|$ and since

$$\alpha_{\mathbf{x}} x_{n_{\mathbf{x}}+1} = \alpha_{\mathbf{y}} y_{n_{\mathbf{x}}+1}, \qquad \sqrt{1-\alpha_{\mathbf{x}}^2}\, x_{n_{\mathbf{x}}+1} = \sqrt{1-\alpha_{\mathbf{y}}^2}\, y_{n_{\mathbf{x}}+1}$$

where $\alpha_{\mathbf{x}} \geq 0$, we see that $x_{n_{\mathbf{x}}+1} = y_{n_{\mathbf{x}}+1}$, which finishes the proof.

Now we define $F(\mathbf{x}) := cf(\mathbf{x})$, $c > 2$. We show the following

Theorem. *The map* F *has the following properties*:

(a_1) F *is continuous,*

(b_1) $F(L^2(\mathbb{N}))$ *has nonempty interior,*

(c_1) F *is not onto,*

(d_1) *for arbitrary* $\mathbf{x}, \mathbf{y} \in H$ *there is* $n_0 \in \mathbb{N}$ *(depending on* $\mathbf{x}$ *and* $\mathbf{y}$*) such that for every* $n \geq n_0$

$$\|F^n(\mathbf{x}) - F^n(\mathbf{y})\| \geq c^{n-m}\|\mathbf{x} - \mathbf{y}\| \tag{3}$$

where F^n *is the* n*th iterate of* F, c *is a constant greater than* 2, *and* m *is an integer depending on* $\mathbf{x}$ *and* $\mathbf{y}$.

Proof. Properties (a_1), (b_1), (c_1) are easy to prove. We show (d_1).

By definition of f and (2), for every $\mathbf{x} \in L^2(\mathbb{N})$

$$\|F^n(\mathbf{x})\| = c^n\|\mathbf{x}\|, \tag{4}$$

and there is some integer p depending on $\mathbf{x}$ (we choose the smallest one) such that

$$F^n(\mathbf{x}) = c^{n-p}F^p(\mathbf{x}) \quad \text{for } n \geq p. \tag{5}$$

Now consider the expression $\|F^n(\mathbf{x}) - F^n(\mathbf{y})\|$. By (5),

$$\begin{aligned}\|F^n(\mathbf{x}) - F^n(\mathbf{y})\| &= \|c^{n-p}F^p(\mathbf{x}) - c^{n-k}F^k(\mathbf{y})\| \\ &= c^{n-p}\|F^p(\mathbf{x}) - c^{p-k}F^k(\mathbf{y})\|\end{aligned}$$

(k corresponds to $\mathbf{y}$ according to (5)), and since

$$c^{p-k}F^k(\mathbf{y}) = F^p(\mathbf{y})$$

(without loss of generality we can assume that $p \geq k$) we have

$$\|F^p(\mathbf{x}) - c^{p-k}F^k(\mathbf{y})\| = \|F^p(\mathbf{x}) - F^p(\mathbf{y})\| > 0 \quad \text{for } \mathbf{x} \neq \mathbf{y},$$

because f, and hence F, is an injection. Finally, since $c > 2$ there is n_0 such that for every $n \geq n_0$

$$\|F^n(\mathbf{x}) - F^n(\mathbf{y})\| \geq c^{n-p}\|\mathbf{x} - \mathbf{y}\|$$

and $m := \max\{k, p\} = p$. Thus, the proof of (d_1) is finished.

Proposition. *There is no map* F_1 *with properties* (a_1), (b_1), (c_1), (d_1), *and* (e_1) $\|F_1(\mathbf{x})\| = c\|\mathbf{x}\|$ *in the finite-dimensional case.*

Proof. Assume that $F_1: \mathbb{R}^n \to \mathbb{R}^n$ is such a map. Then, by (c_1) and (e_1) there is $0 \neq \mathbf{x}_0 \notin F_1(\mathbb{R}^n)$. From (e_1) it follows that F_1 maps spheres (centered at 0) into spheres, in particular it maps the sphere $\mathscr{S}$ with radius $\|\mathbf{x}_0\|/c$ into the sphere with radius $\|\mathbf{x}_0\|$. By (a_1) and (d_1) $F_1|_{\mathscr{S}}$ is continuous injection and because each sphere in a finite-dimensional space is compact, $F_1|_{\mathscr{S}}$ is a homeomorphism onto a compact proper subset of the other sphere. But this contradicts the well-known theorem stating that the necessary condition for a compact set in $\mathbb{R}^n$ to be homeomorphic to a sphere in $\mathbb{R}^n$ is that its complement has exactly two connected components [5].

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References

1. L. E. J. Brouwer, *Bewels der Invarianz des n-dimensionalen Gebiets*, Math. Ann. **71** (1912), 305–313.
2. ——, *Zur Invarianz des n-dimensionalen Gebiets*, Math. Ann. **72** (1912), 55–56.
3. F. E. Browder, *The solvability of non-linear functional equations*, Duke Math. J. **30** (1963), 557–566.
4. K. C. Chang and L. Shujie, *A remark on expanding maps*, Proc. Amer. Math. Soc. **85** (1982), 583–586.
5. K. Kuratowski, *Topology*, vol. II, Academic Press and PWN, New York and Warsaw, 1968.
6. J. Leray, *Topologie des espaces abstraits de M. Banach*, C. R. Acad. Sci. Paris Sér. I Math. **200** (1935), 1083–1085.
7. G. J. Minty, *Montone (nonlinear) operators in Hilbert space*, Duke Math. J. **29** (1962), 341–346.
8. J. M. Morel and H. Steinlein, *On a problem of Nirenberg concerning expanding maps*, J. Funct. Anal. **59** (1984), 145–150.
9. L. Nirenberg, *Topics in nonlinear functional analysis*, Lecture Notes, Courant Inst. of Math. Sci., New York Univ., New York, 1974.
10. R. D. Nussbaum, *Degree theory for local condensing maps*, J. Math. Anal. Appl. **37** (1972), 741–766.

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