

On the elastic orthotropy

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EXPRESSIONS for Kelvin moduli of orthotropic linearly elastic bodies in terms of Young's moduli and Poisson's ratios are presented. The check of the orthotropy criterion is posed as a variational problem.

1. Introduction

MOST ELASTIC bodies which are considered in physics and in engineering either are, or can be, with acceptable accuracy, considered as orthotropic. Most of metal crystals (due to the symmetries of the lattice) and composites (because of the production techniques reasons) show these properties rigorously, while most of the polycrystalline textured materials, biological tissues, rock structures etc. can be, without significant errors, also considered as orthotropic. Moreover, all the second rank tensors exhibit the orthotropic symmetries, thus one can expect that, according to the generalized Curie law [11], the fourth order tensor-valued functions of e.g. one second rank tensor and arbitrary number of scalars, or of several coaxial second rank tensors etc., would preserve at least this symmetry. Reduction of the description of elastic properties of the anisotropic bodies to the orthotropic ones gives rise to a significant gain in the effectiveness and comprehensibility of the description, reducing the number of independent elastic constants from 18 to 9⁽¹⁾ and the characteristic axes (eigenvectors of the proper elastic states) from 18 to 9 (to three, in fact).

In recent years some papers presenting the invariant properties, structure and transformation rules of plane fourth rank two-dimensional compliance and stiffness tensors (plane Hooke's tensors – according to [12]) were published [16, 17, 12, 4]. These papers exhausted the problem to some extent. Such a complete study of general three-dimensional case seems to be too complex to be useful, some helpful interesting facts, however, especially in the cases of higher symmetries, still can be established [15]. In the present paper the authors will touch only some aspects of the problem for the case of general orthotropy, with no claim for the completeness and generality of the considerations.

For the sake of brevity in this paper we shall use the term “*at least orthotropic*” for any class of symmetry having three mutually orthogonal planes of symmetry, i.e. we shall understand materials of higher symmetries, such as cubic, tetragonal transversely isotropic and isotropic as particular cases of orthotropic materials.

⁽¹⁾ The problem: 18 or 21 constants? will be discussed later in this paper, compare also [9].

We shall discuss here the properties of elastic compliance and elastic stiffness tensors, but almost all the results remain valid for any fourth order tensors of the same symmetries:

$$(1) \quad C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}.$$

Throughout the present paper we shall use both the tensors and their representations and the matrices, which may not represent any tensors, thus to avoid any confusion we shall denote tensorial quantities using **boldface** symbols, while for matrices we shall use *sanserif* characters.

2. Modified Voigt matrices, Kelvin moduli, Young's moduli and Poisson's ratios

It is widely used to represent Hooke's symmetric operators mapping the set of second order symmetric strain tensors onto the set of the same type stress tensors (or *vice versa*) using 6×6 matrices. Thus instead of

$$(2) \quad \varepsilon_{ij} = C_{ijkl}\sigma_{kl}$$

one writes

$$(3) \quad \varepsilon_K = C_{KL}\sigma_L,$$

where⁽²⁾

$$(4) \quad \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \sqrt{2}\varepsilon_{23} \\ \sqrt{2}\varepsilon_{31} \\ \sqrt{2}\varepsilon_{12} \end{bmatrix}, \quad \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\sigma_{23} \\ \sqrt{2}\sigma_{31} \\ \sqrt{2}\sigma_{12} \end{bmatrix}.$$

Note please, that this notation differs from the classical Voigt one by the factor $\sqrt{2}$ (in the case when $i \neq j$). This modification makes it possible to operate with the corresponding matrices of representations using "tensorial" rules (cf. [6, 2]). The following correspondence between the representations of the compliance

⁽²⁾ I.e. we take, in fact, the projections of the tensors onto the six-dimensional subspace defined by the following orthonormal basis \mathbf{t}_i : $\mathbf{t}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1$, $\mathbf{t}_2 = \mathbf{e}_2 \otimes \mathbf{e}_2$, $\mathbf{t}_3 = \mathbf{e}_3 \otimes \mathbf{e}_3$, $\mathbf{t}_4 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2)$, $\mathbf{t}_5 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1)$, $\mathbf{t}_6 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$.

tensor takes place:

$$(5) \quad \begin{aligned} C_{11} &= C_{1111}, & C_{22} &= C_{2222}, & C_{33} &= C_{3333}, \\ C_{44} &= 2C_{2323}, & C_{55} &= 2C_{1313}, & C_{66} &= 2C_{1212}, \\ C_{12} &= C_{1122}, & C_{13} &= C_{1133}, & C_{32} &= C_{3322}, \\ C_{45} &= C_{2313}, & C_{46} &= C_{2312}, & C_{56} &= C_{1312}, \\ C_{14} &= \sqrt{2}C_{1123}, & C_{15} &= \sqrt{2}C_{1113}, & C_{16} &= \sqrt{2}C_{1112}, \\ C_{24} &= \sqrt{2}C_{2223}, & C_{25} &= \sqrt{2}C_{2213}, & C_{26} &= \sqrt{2}C_{2212}, \\ C_{34} &= \sqrt{2}C_{3323}, & C_{35} &= \sqrt{2}C_{3313}, & C_{36} &= \sqrt{2}C_{3312}. \end{aligned}$$

For general considerations, as well as for the sake of illustrativeness, another representation, which can be traced back to Lord Kelvin and was intensively developed during last decade by J. RYCHLEWSKI [9] and some other authors (for references see [13, 8]), can be very fruitful. The elastic compliance tensor, as any fourth order tensor preserving symmetries (1) can be represented in the following form:

$$(6) \quad \mathbf{C} = \frac{1}{\lambda_1} \boldsymbol{\kappa}_1 \otimes \boldsymbol{\kappa}_1 + \dots + \frac{1}{\lambda_6} \boldsymbol{\kappa}_6 \otimes \boldsymbol{\kappa}_6,$$

where $\boldsymbol{\kappa}_I$ ($I = 1$ to 6) are the second order tensors, mutually orthogonal in the sense of the scalar product in the corresponding linear space: $\boldsymbol{\kappa}_I \cdot \boldsymbol{\kappa}_J = \delta_{IJ}$; J. Rychlewski proposed to call them *proper elastic states*, while the non-negative scalars λ_I he proposed to call the *Kelvin moduli* [9]. Using (6) one can write the (reversed) Hooke's law in the following form:

$$(7) \quad \boldsymbol{\varepsilon} = \frac{1}{\lambda_1} (\boldsymbol{\kappa}_1 \cdot \boldsymbol{\sigma}) \boldsymbol{\kappa}_1 + \dots + \frac{1}{\lambda_6} (\boldsymbol{\kappa}_6 \cdot \boldsymbol{\sigma}) \boldsymbol{\kappa}_6.$$

In the case of orthotropy (or higher symmetry), a triplet of proper elastic states represent three pure shears in the mutually orthogonal planes; in appropriate basis their representations take the following form:

$$(8) \quad \boldsymbol{\kappa}_4 \propto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad \boldsymbol{\kappa}_5 \propto \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\kappa}_6 \propto \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is not difficult to notice that the other three proper elastic states have *in this basis* diagonal representations. Moreover, in view of mutual orthogonality, the triads of diagonal terms form rows of some orthogonal 3×3 matrix. Among

many possible representations, the following one seems to be convenient:

$$(9) \quad \begin{aligned} \kappa_1 &\propto \begin{bmatrix} \cos \theta \cos \varphi & 0 & 0 \\ 0 & \cos \theta \sin \varphi & 0 \\ 0 & 0 & \sin \theta \end{bmatrix}, \\ \kappa_2 &\propto \begin{bmatrix} -\cos \gamma \sin \varphi & 0 & 0 \\ +\sin \gamma \sin \theta \cos \varphi & \cos \gamma \cos \varphi & 0 \\ 0 & +\sin \gamma \sin \theta \sin \varphi & 0 \\ 0 & 0 & -\sin \gamma \cos \theta \end{bmatrix}, \\ \kappa_3 &\propto \begin{bmatrix} \sin \gamma \sin \varphi & 0 & 0 \\ +\cos \gamma \sin \theta \cos \varphi & -\sin \gamma \cos \varphi & 0 \\ 0 & +\cos \gamma \sin \theta \sin \varphi & 0 \\ 0 & 0 & -\cos \gamma \cos \theta \end{bmatrix}. \end{aligned}$$

Parameters φ , θ , γ can assume any values, they do not represent any angles in "physical" space⁽³⁾.

Assuming particular values for these parameters and taking some Kelvin moduli equal to each other, one obtains all the particular cases of higher symmetries:

- isotropy

$$\sin \theta = \frac{1}{\sqrt{3}}, \quad \cos \varphi = \frac{1}{\sqrt{2}}, \quad \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6,$$

- cubic symmetry

$$\sin \theta = \frac{1}{\sqrt{3}}, \quad \cos \varphi = \frac{1}{\sqrt{2}}, \quad \lambda_2 = \lambda_3, \quad \lambda_4 = \lambda_5 = \lambda_6,$$

- transversal isotropy in $\{x_1, x_2\}$ -plane

$$\sin \gamma = 0, \quad \cos \varphi = \frac{1}{\sqrt{2}}, \quad \lambda_2 = \lambda_6, \quad \lambda_4 = \lambda_5,$$

- tetragonal symmetry⁽⁴⁾ (about the x_3 -axis)

$$\sin \gamma = 0, \quad \cos \varphi = \frac{1}{\sqrt{2}}, \quad \lambda_4 = \lambda_5.$$

⁽³⁾ The presented *form* of the representation of the orthogonal matrix corresponds nevertheless to the "rotational" interpretation of the matrix Q as a representation of the tensor $Q = n \otimes n + (I - n \otimes n) \cos \gamma + n \cdot e \sin \gamma$ (cf. [3]).

⁽⁴⁾ We mean here the same case as that considered e.g. in [14], we are not considering here the other case pointed out in [7], which in fact can be reduced to the previous one; for details see [4].

Let us assume that the preferred orthotropy axes are known. One can notice at once that for $i \geq 4$ Kelvin moduli λ_i are equal to the corresponding doubled Kirchhoff moduli:

$$(10) \quad \lambda_4 = 2G_{23}, \quad \lambda_5 = 2G_{13}, \quad \lambda_6 = 2G_{12}.$$

For the determination of the other three Kelvin moduli we shall consider uniaxial tension (compression) along each of the preferred axes x_i . Denoting:

$$(11) \quad Q_{ij} \equiv (\kappa_i)_{jj} \quad (\text{no summation!})$$

one obtains, for the case of tension along x_1 , the following components of ϵ

$$(12) \quad \epsilon_{ii} = \frac{1}{\lambda_1} Q_{1i} \kappa_1 \cdot \sigma + \frac{1}{\lambda_2} Q_{2i} \kappa_2 \cdot \sigma + \frac{1}{\lambda_3} Q_{3i} \kappa_3 \cdot \sigma = \sigma_{11} \sum_{j=1}^3 \frac{Q_{ji} Q_{j1}}{\lambda_j}$$

(no summation! Compare (7)).

Denoting

$$(13) \quad E_i \equiv \frac{\sigma_{ii}}{\epsilon_{ii}}, \quad \nu_{ij} \equiv -\frac{\epsilon_{jj}}{\epsilon_{ii}} \quad (\text{no summation!})$$

one can write

$$(14) \quad \begin{aligned} \frac{1}{E_1} &= \frac{Q_{11}^2}{\lambda_1} + \frac{Q_{21}^2}{\lambda_2} + \frac{Q_{31}^2}{\lambda_3}, \\ -\frac{\nu_{12}}{E_1} &= \frac{Q_{12} Q_{11}}{\lambda_1} + \frac{Q_{22} Q_{21}}{\lambda_2} + \frac{Q_{32} Q_{31}}{\lambda_3}, \\ -\frac{\nu_{13}}{E_1} &= \frac{Q_{13} Q_{11}}{\lambda_1} + \frac{Q_{23} Q_{21}}{\lambda_2} + \frac{Q_{33} Q_{31}}{\lambda_3}. \end{aligned}$$

Similar relations are valid for the uniaxial tension along the other two axes. Thus, introducing new symbols

$$(15) \quad \nu_{ii} = -1, \quad A_{ij} = -\frac{\nu_{ij}}{E_i} \quad (\text{no summation!})$$

we obtain the following relation

$$(16) \quad \frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \quad (\text{no summation!})$$

and we are able to write

$$(17) \quad A = Q^T \hat{L}^{-1} Q,$$

where \hat{L} is the diagonal matrix of the first three Kelvin moduli λ_i . Thus we have reduced the problem of expressing the Kelvin moduli in terms of Young's moduli and Poisson's ratios to the eigenvalue problem for a symmetric 3×3 matrix A .

Among many known ways of solving this problem, a particular one, giving rise to relatively simple analytic formulae can be pointed out.

Note that, due to orthogonality of Q , the invariants of the matrices A and \hat{L}^{-1} are equal to each other; similar relation (such as (17)) holds true for the normalised deviatoric parts of these tensors, therefore also their corresponding invariants are equal to each other.

Thus, denoting

$$(18) \quad A' \equiv A - \frac{1}{3} \text{tr} A \mathbf{I}, \quad z_i \equiv \frac{1}{\sqrt{\text{tr}(A'^2)}} \left(\frac{1}{\lambda_i} - \frac{1}{3} \text{tr} A \right), \quad b \equiv \sqrt{54} \frac{\det A'}{\left(\sqrt{\text{tr}(A'^2)} \right)^3},$$

where \mathbf{I} is the unit matrix, one can express the invariants of A' by its eigenvalues z_i writing:

$$(19) \quad \begin{aligned} z_1 + z_2 + z_3 &= 0, \\ z_1^2 + z_2^2 + z_3^2 &= 1, \\ z_1 z_2 z_3 &= \frac{b}{\sqrt{54}}. \end{aligned}$$

Solutions of the system (19) are equal to the real parts of the three branches of the following complex expression:⁽⁵⁾

$$(20) \quad z_{1,2,3} = -\frac{1}{\sqrt{6}} \left[(b + i\sqrt{1-b^2})^{1/3} + (b - i\sqrt{1-b^2})^{1/3} \right].$$

Consequently, one obtains

$$(21) \quad \frac{1}{\lambda_i} = z_i \sqrt{\text{tr} A^2 - \frac{1}{3} (\text{tr} A)^2} + \frac{1}{3} \text{tr} A,$$

while the *compliance distributors* Q_{ij} ⁽⁶⁾ can be easily obtained as the normalised solutions of the following linear system:

$$(22) \quad A_{ij} Q_{Kj} = \frac{1}{\lambda_K} Q_{Ki} \quad (\text{no summation over capital index } K!).$$

3. Orthotropy criterion

Throughout the previous section it was assumed, that the distinguished axes of orthotropy are known, as they often really *are* in physics and in the engineering applications. This cannot be considered as a rule, however. Such practical

⁽⁵⁾ Note that always $|b| \leq 1$, thus substituting $b = \cos 3\alpha$ and using the Moivre formulae, one can easily verify that expressions (20) satisfy Eqs. (19), cf. [1].

⁽⁶⁾ They are simultaneously the *stiffness distributors*.

situations can be pointed out, when not only the axes are not known, but there is even no information if the material is at least orthotropic, or not.

If the material is orthotropic and the basis of unit vectors along the orthotropy axes is adopted, then the matrix of the six-dimensional representation of the elastic compliance tensor (cf. [7]) has the following form:

$$(23) \quad \mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}, \quad (C_{ij} = C_{ji}).$$

If the compliance tensor is given in an *arbitrary basis*, then the answer for the question if the material is at least orthotropic or not, is not immediate. Even if one solves (numerically) the six-dimensional eigenvalue problem, finds the Kelvin moduli and then calculates the proper elastic states and finds out, that the three of them are pure shears, there is still no certainty, that we have to do with the case of orthotropy. There exist such five orthogonal (in the sense of scalar product of second rank tensors) pure shears among which there are no three of them acting in mutually orthogonal planes in the “physical” space [5]. To find out the answer, one has to analyze the proper axes of almost all the proper elastic states.

Another possible approach to finding the answer to the orthotropy question can be as follows:

1. Take a general three-parameter expression for the rotation tensor \mathbf{R} in 3-dimensional “physical” space.
2. For a known representation (in a chosen basis) of elastic compliance tensor, find a general form of the representation in a rotated basis

$$C'_{ijkl} = R_{ip}R_{jq}R_{kr}R_{ls}C_{pqrs}.$$

3. Find out if there exist such values of the parameters (e.g. $\{\varphi_1, \varphi_2, \varphi_3\}$), defining rotation \mathbf{R} , which lead to vanishing of all these components of \mathbf{C} which vanish for the case of the representation (23) in the “proper” orthotropy basis.

This way can be fairly laborious as regards the computer programming, and it forces us to leave quite convenient and familiar six-dimensional space and to return to the space of fourth rank tensors.

The present authors in their recent paper written jointly with J. RYCHLEWSKI [4] have shown – in the case of plane elasticity – an effective criterion of orthotropy, expressed by the relation between the components of the compliance tensor in an arbitrary basis. It is not clear if such a criterion of reasonable complexity can be formulated for the 3-dimensional case. In the present paper the authors are

going to propose a way, which, being in fact equivalent to the one presented above, makes it possible to remain all the time in the six-dimensional space, to verify the intermediate results (which can solely be useful for certain other considerations) and to use standard computer procedures.

The one who wants to rotate the bases in a six-dimensional space should first establish the set of allowed orthogonal matrices, having their counterparts in the sets of rotations in three-dimensional space. The complete set of rotation matrices in six-dimensional space depends on 15 parameters⁽⁷⁾. The set of rotations in 3-dimensional space is 3-parametric; this fact has some important implications: first – the 21 independent parameters describing general compliance tensor can be chosen in such a way, that 18 of them can be considered as material constants, the remaining three fixing the orientation of the axes of proper elastic states in a 3-dimensional space (the same considerations reduce the number of material constants for an orthotropic material from 12 to 9), and second – the set of the physically meaningful rotations in a six-dimensional space must be also three-parametric.

We shall find a general representation of the rotation tensor in a six-dimensional space, generated by the rotation in the “physical” 3-dimensional space. It is not difficult to verify (compare [4]), that the rotation $R_{(1)}$ by the angle φ_1 around the axis x_1 generates the following orthogonal 6×6 matrix $R_{(1)}$ describing the same rotation in a six-dimensional space:

$$(24) \quad R_{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2 \varphi_1 & \sin^2 \varphi_1 & -\frac{1}{\sqrt{2}} \sin 2\varphi_1 & 0 & 0 \\ 0 & \sin^2 \varphi_1 & \cos^2 \varphi_1 & \frac{1}{\sqrt{2}} \sin 2\varphi_1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} \sin 2\varphi_1 & -\frac{1}{\sqrt{2}} \sin 2\varphi_1 & \cos 2\varphi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos \varphi_1 & \sin \varphi_1 \\ 0 & 0 & 0 & 0 & -\sin \varphi_1 & \cos \varphi_1 \end{bmatrix}.$$

The corresponding expressions for the rotations around the other axes are straightforward. Any rotation R can be achieved as the superposition of the rotation around three non-coplanar, for example orthogonal, axes, thus a general expression for rotation matrix in a six-dimensional space can be obtained as a product of matrices describing rotations around the coordinate axes x_i by the angles φ_i ,

$$(25) \quad R = R_{(1)}R_{(2)}R_{(3)}.$$

⁽⁷⁾ Every matrix of the type AA^T is symmetric, thus the orthogonality condition $RR^T = 1$ imposes only 21 constraints on the 36 components of the matrix R .

The complete expression for such \mathbf{R} is quoted in the Appendix⁽⁸⁾.

Let us denote for convenience the "vector" composed of the elements of the k -th column of the matrix \mathbf{R} by $r_{(K)}$:

$$(26) \quad r_{(K)} \equiv [R_{1K}, R_{2K}, R_{3K}, R_{4K}, R_{5K}, R_{6K}].$$

It is not difficult to notice, that the "vectors" r_1, r_2, r_3 represent in a six-dimensional space three pure shears in mutually orthogonal planes; indeed, denoting by $k_{(i)}$ the six-dimensional representation of the tensors κ_i listed in (8), one obtains:

$$(27) \quad r_{(i)} = \mathbf{R}k_{(i)} \quad (i = 4, 5, 6).$$

Thus:

The elastic compliance 6×6 matrix \mathbf{C} describes at least an orthotropic material if and only if there exists such a rotation \mathbf{R} in 3-dimensional space, that the components of the last three columns of its six-dimensional representation \mathbf{R} represent the three proper elastic states.

If some symmetric tensor σ , (certain vector in a six-dimensional space) represents a proper state, then obviously it is *energetically orthogonal* [10] to any symmetric tensor (six-dimensional vector) from its orthogonal complement; the reverse is equally true, i.e.:

σ is a proper state if and only if for any \mathbf{a} , $\mathbf{a} \cdot \sigma = 0$ implies $\mathbf{a} \cdot \mathbf{C} \cdot \sigma = 0$.

Hence:

The elastic compliance matrix \mathbf{C} describes at least an orthotropic material if and only if there exists such an orthogonal 6×6 matrix \mathbf{R} , describing the rotation in three-dimensional space and consisting of such columns $r_{(K)}$, that $r_{(K)}^T \mathbf{C} r_{(J)} = 0$ ⁽⁹⁾ for $K = 1, 2, 3, 4, 5, 6$, $J = 4, 5, 6$, $K \neq J$.

Let us define, for a given compliance matrix \mathbf{C} , the following function F of the three parameters $\{\varphi_1, \varphi_2, \varphi_3\}$ determining the "physical" rotation in six-dimensional space:

$$(28) \quad F(\varphi_1, \varphi_2, \varphi_3) = \sum_{K=1}^6 \sum_{\substack{J=4 \\ J \neq K}}^6 |r_{(K)}^T \mathbf{C} r_{(J)}|.$$

⁽⁸⁾ It is evident that the set of such matrices is a subgroup of the group of orthogonal 6×6 matrices. Some interesting properties of the matrices of this subgroup, connected with the preservation of the tensorial invariants of the six-dimensional "vectors", can be pointed out. For example, sum of the first three terms in the case of 1-st, 2-nd and 3-rd column is equal to 1, while for the other columns it is equal to zero, and the same applies to the rows (the columns of the transposed matrix).

⁽⁹⁾ By $r_{(K)}^T$ we denote the row matrix having the same components as the column matrix $r_{(K)}$, thus $r_{(K)}^T \mathbf{C} r_{(J)}$ is a number.

The orthotropy condition can then be expressed as follows

$$(29) \quad \begin{aligned} \min_{\varphi_1, \varphi_2, \varphi_3} F(\varphi_1, \varphi_2, \varphi_3) &= 0, \\ -\pi/2 &\leq \varphi_1 \leq \pi/2, \\ -\pi/2 &\leq \varphi_2 \leq \pi/2, \\ -\pi/2 &\leq \varphi_3 \leq \pi/2. \end{aligned}$$

The ratio of the minimal value of the function $F(\varphi_1, \varphi_2, \varphi_3)$ to the norm of the elastic compliance tensor

$$(30) \quad \chi = \frac{\min_{\varphi_1, \varphi_2, \varphi_3} F(\varphi_1, \varphi_2, \varphi_3)}{\|\mathbf{C}\|}$$

can be considered as a good measure of obliqueness (departure from orthotropy) of the elastic properties.

Observe that χ is an invariant of the compliance tensor \mathbf{C} . Indeed: all the matrices of its six-dimensional representations are mutually connected by the relation: $\mathbf{C}' = \mathbf{R}^T \mathbf{C} \mathbf{R}$, while for any orthogonal 6×6 matrix \mathbf{R} , generated by rotation in a 3-dimensional space and any "vector" $r_{(J)}$, the product $\mathbf{R} r_{(J)} \equiv r'_{(J)}$ is again a "vector" of the same type, its "tensorial" invariants being preserved, thus:

$$(31) \quad \sum_{K=1}^6 \sum_{\substack{J=4 \\ J \neq K}}^6 |r_{(K)}^T \mathbf{C}' r_{(J)}| = \sum_{K=1}^6 \sum_{\substack{J=4 \\ J \neq K}}^6 |r'_{(K)}^T \mathbf{C} r'_{(J)}| = F(\varphi'_1, \varphi'_2, \varphi'_3).$$

This means that changing the representation of the compliance tensor \mathbf{C} (and consequently the components of the matrix \mathbf{C}), one merely renames variables in (29), what, evidently, can not affect the result of the minimization procedure. This completes the proof.

An entirely different approach to the problem under consideration can be also proposed: any compliance 6×6 matrix \mathbf{C} can be represented as

$$(32) \quad \mathbf{C} = \mathbf{P}^T \mathbf{L}^{-1} \mathbf{P},$$

where \mathbf{P} denotes a 6×6 orthogonal matrix and \mathbf{L} is the diagonal matrix of Kelvin moduli. For example, in the case of orthotropy one can take

$$(33) \quad \mathbf{P} = \mathbf{B} \mathbf{R}, \quad \text{where} \quad \mathbf{B} \equiv \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},$$

\mathbf{Q} and \mathbf{R} denote the same matrices as those in (11) and (25). Thus one can reduce the considerations on the obliqueness of the elastic properties to the problem of invariants of the matrix \mathbf{P} ; we shall not pursue this line in the present paper however.

This completes for the time being our considerations on the orthotropy condition.

Appendix

For reference, a general form of a 6×6 orthogonal three-parameter matrix describing the rotation in the three-dimensional space should be quoted. We shall omit here the elementary calculations and expose only the final result of multiplying matrices $R_{(1)}$, $R_{(2)}$ and $R_{(3)}$, where the first of them has already been shown (Eq.(24)), and the other two can be easily obtained by renaming the variables and reshuffling the columns and rows. For better comprehension we shall split the matrix R into four submatrices which will be specified separately:

$$R = \left[\begin{array}{c|c} R^{(1)} & R^{(2)} \\ \hline R^{(3)} & R^{(4)} \end{array} \right],$$

where

$$R^{(1)} = \left[\begin{array}{ccc} \cos^2 \varphi_2 \cos^2 \varphi_3 & \cos^2 \varphi_2 \sin^2 \varphi_3 & \sin^2 \varphi_2 \\ \sin^2 \varphi_1 \sin^2 \varphi_2 \cos^2 \varphi_3 & \sin^2 \varphi_1 \sin^2 \varphi_2 \sin^2 \varphi_3 & \\ + \cos^2 \varphi_1 \sin^2 \varphi_3 & + \cos^2 \varphi_1 \cos^2 \varphi_3 & \sin^2 \varphi_1 \cos^2 \varphi_2 \\ -\frac{1}{2} \sin 2\varphi_1 \sin \varphi_2 \sin 2\varphi_3 & +\frac{1}{2} \sin 2\varphi_1 \sin \varphi_2 \sin 2\varphi_3 & \\ \cos^2 \varphi_1 \sin^2 \varphi_2 \cos^2 \varphi_3 & \cos^2 \varphi_1 \sin^2 \varphi_2 \sin^2 \varphi_3 & \\ + \sin^2 \varphi_1 \sin^2 \varphi_3 & + \sin^2 \varphi_1 \cos^2 \varphi_3 & \cos^2 \varphi_1 \cos^2 \varphi_2 \\ +\frac{1}{2} \sin 2\varphi_1 \sin \varphi_2 \sin 2\varphi_3 & -\frac{1}{2} \sin 2\varphi_1 \sin \varphi_2 \sin 2\varphi_3 & \end{array} \right],$$

$$R^{(2)} = \left[\begin{array}{ccc} \frac{\sin 2\varphi_2}{\sqrt{2}} \sin \varphi_3 & -\frac{\sin 2\varphi_2}{\sqrt{2}} \cos \varphi_3 & -\cos^2 \varphi_2 \frac{\sin 2\varphi_3}{\sqrt{2}} \\ -\frac{\sin 2\varphi_1}{\sqrt{2}} \cos \varphi_2 \cos \varphi_3 & -\frac{\sin 2\varphi_1}{\sqrt{2}} \cos \varphi_2 \sin \varphi_3 & -\sin^2 \varphi_1 \sin^2 \varphi_2 \frac{\sin 2\varphi_3}{\sqrt{2}} \\ -\sin^2 \varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \sin \varphi_3 & +\sin^2 \varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \cos \varphi_3 & +\cos^2 \varphi_1 \frac{\sin 2\varphi_3}{\sqrt{2}} \\ \frac{\sin 2\varphi_1}{\sqrt{2}} \cos \varphi_2 \cos \varphi_3 & \frac{\sin 2\varphi_1}{\sqrt{2}} \cos \varphi_2 \sin \varphi_3 & -\frac{\sin 2\varphi_1}{\sqrt{2}} \sin \varphi_2 \cos 2\varphi_3 \\ -\cos^2 \varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \sin \varphi_3 & +\cos^2 \varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \cos \varphi_3 & -\cos^2 \varphi_1 \sin^2 \varphi_2 \frac{\sin 2\varphi_3}{\sqrt{2}} \\ & & +\sin^2 \varphi_1 \frac{\sin 2\varphi_3}{\sqrt{2}} \\ & & +\frac{\sin 2\varphi_1}{\sqrt{2}} \sin \varphi_2 \cos 2\varphi_3 \end{array} \right],$$

$$R^{(3)} = \begin{bmatrix} -\frac{\sin 2\varphi_1}{\sqrt{2}} \sin^2 \varphi_2 \cos^2 \varphi_3 & -\frac{\sin 2\varphi_1}{\sqrt{2}} \sin^2 \varphi_2 \sin^2 \varphi_3 & -\frac{\sin 2\varphi_1}{\sqrt{2}} \cos^2 \varphi_2 \\ +\frac{\sin 2\varphi_1}{\sqrt{2}} \sin^2 \varphi_3 & +\frac{\sin 2\varphi_1}{\sqrt{2}} \cos^2 \varphi_3 & \\ +\cos 2\varphi_1 \sin \varphi_2 \frac{\sin 2\varphi_3}{\sqrt{2}} & -\cos 2\varphi_1 \sin \varphi_2 \frac{\sin 2\varphi_3}{\sqrt{2}} & \\ \cos \varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \cos^2 \varphi_3 & \cos \varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \sin^2 \varphi_3 & -\cos \varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \\ +\sin \varphi_1 \cos \varphi_2 \frac{\sin 2\varphi_3}{\sqrt{2}} & -\sin \varphi_1 \cos \varphi_2 \frac{\sin 2\varphi_3}{\sqrt{2}} & \\ -\sin \varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \cos^2 \varphi_3 & -\sin \varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \sin^2 \varphi_3 & \sin \varphi_1 \frac{\sin 2\varphi_2}{\sqrt{2}} \\ +\cos \varphi_1 \cos \varphi_2 \frac{\sin 2\varphi_3}{\sqrt{2}} & -\cos \varphi_1 \cos \varphi_2 \frac{\sin 2\varphi_3}{\sqrt{2}} & \end{bmatrix},$$

$$R^{(4)} = \begin{bmatrix} \cos 2\varphi_1 \cos \varphi_2 \cos \varphi_3 & \cos 2\varphi_1 \cos \varphi_2 \sin \varphi_3 & \frac{1}{2} \sin 2\varphi_1 \sin^2 \varphi_2 \sin 2\varphi_3 \\ +\frac{1}{2} \sin 2\varphi_1 \sin 2\varphi_2 \sin \varphi_3 & -\frac{1}{2} \sin 2\varphi_1 \sin 2\varphi_2 \cos \varphi_3 & +\frac{1}{2} \sin 2\varphi_1 \sin 2\varphi_3 \\ & & +\cos 2\varphi_1 \sin \varphi_2 \cos 2\varphi_3 \\ -\sin \varphi_1 \sin \varphi_2 \cos \varphi_3 & -\sin \varphi_1 \sin \varphi_2 \sin \varphi_3 & -\frac{1}{2} \sin 2\varphi_2 \sin 2\varphi_3 \\ -\cos \varphi_1 \cos 2\varphi_2 \sin \varphi_3 & +\cos \varphi_1 \cos 2\varphi_2 \cos \varphi_3 & +\sin \varphi_1 \cos \varphi_2 \cos \varphi_2 \\ -\cos \varphi_1 \sin \varphi_2 \cos \varphi_3 & -\cos \varphi_1 \sin \varphi_2 \sin \varphi_3 & \frac{1}{2} \sin \varphi_1 \sin 2\varphi_2 \sin 2\varphi_3 \\ +\sin \varphi_1 \cos 2\varphi_2 \sin \varphi_3 & -\sin \varphi_1 \cos 2\varphi_2 \cos \varphi_3 & +\cos \varphi_1 \cos \varphi_2 \cos 2\varphi_3 \end{bmatrix}.$$

As it has already been pointed out, summation of the elements of the rows and columns of $R^{(1)}$ gives 1, while the sums over the columns of $R^{(2)}$ and rows of $R^{(3)}$ vanish.

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