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A qualitative approach to Hooke's tensors. Part II

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A STRAIGHTFORWARD AND COMPLETE DESCRIPTION of all possible invariant linear decompositions of the space of Hooke's tensors has been given in Part I, [1]. In this Part II we demonstrate various eleborations and consequences of these decompositions. This gives a qualitative decription of the anisotropy of Hooke's tensors. In particular, we demonstrate examples A through G, not only important but also astonishing. When reference is made to the formulae in [1], we shall add "Part I" to the number. The notions and notations are the same (see Appendices 1, 2 in [1]).

1. Introduction

THIS PART IS A DIRECT CONTINUATION of [1], but it has a different character. The purpose now is to investigate the qualitative consequences of the invariant decompositions presented in the previous part. The main and most interesting results are grouped in seven examples A-G, Sec. 7. We intend to demonstrate on these examples that materials of totally different structure can, under certain types of action, react quite similarly or even identically. We hope that these examples will contribute to a novel kind of thinking on the isotropy of properties of condensed matter.

2. Energy decompositions of Hooke's tensors

2.1. Let us decompose the quadratic form (energy, work, stress intensity and so on)

$$\omega \cdot \mathbf{H} \cdot \omega = \omega_{\mathcal{P}} \cdot \mathbf{H} \cdot \omega_{\mathcal{P}} + 2\omega_{\mathcal{D}} \cdot \mathbf{H} \cdot \omega_{\mathcal{P}} + \omega_{\mathcal{D}} \cdot \mathbf{H} \cdot \omega_{\mathcal{D}}, \tag{2.1}$$

where $\omega_{\mathcal{P}} \equiv \mathbb{I}_{\mathcal{P}} \cdot \omega$, $\omega_{\mathcal{D}} \equiv \mathbb{I}_{\mathcal{D}} \cdot \omega$.

EXAMPLE 1. Let $\mathbf{C}$ be a compliance tensor of an elastic material, and σ an acting stress. Then (see (1.4), Part I):

• $\sigma_{\mathcal{P}} \cdot \mathbf{C} \cdot \sigma_{\mathcal{P}} = \sigma_{\mathcal{P}} \cdot (\mathbf{C} \cdot \sigma_{\mathcal{P}})_{\mathcal{P}}$ is doubled work of the hydrostatic part of stress $\sigma_{\mathcal{P}}$ on the resultant deformation,

• $\sigma_{\mathcal{P}} \cdot \mathbf{C} \cdot \sigma_{\mathcal{D}} = \sigma_{\mathcal{P}} \cdot (\mathbf{C} \cdot \sigma_{\mathcal{D}})_{\mathcal{P}}$ is doubled work of the hydrostatic part on the deformation caused by the deviatoric part of stress and *vice versa*,

• $\sigma_{\mathcal{D}} \cdot \mathbf{C} \cdot \sigma_{\mathcal{D}} = \sigma_{\mathcal{D}} \cdot (\mathbf{C} \cdot \sigma_{\mathcal{D}})_{\mathcal{D}}$ is doubled work of the deviatoric part on the resultant deformation.

Quite similarly, we can decompose energy of deformation $\boldsymbol{\varepsilon} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}$. Let us note that, for example, $\varepsilon_{\mathcal{P}} \cdot \mathbf{S} \cdot \varepsilon_{\mathcal{P}} \neq \sigma_{\mathcal{P}} \cdot \mathbf{C} \cdot \sigma_{\mathcal{P}}$.

2.2. There is a correspondence between the used decomposition of quadratic form (2.1) and the unique decomposition of its Hooke's tensor

$$\mathbf{H} = \mathbf{H}^{\mathcal{P}} + \mathbf{H}^{\mathcal{PD}} + \mathbf{H}^{\mathcal{D}}, \tag{2.2}$$

where

$$\mathbf{H}^{\mathcal{P}} \equiv \mathbb{I}_{\mathcal{P}} \circ \mathbf{H} \circ \mathbf{I}_{\mathcal{P}}, \tag{2.3}$$

$$\mathbf{H}^{\mathcal{PD}} \equiv \mathbb{I}_{\mathcal{P}} \circ \mathbf{H} \circ \mathbb{I}_{\mathcal{D}} + \mathbb{I}_{\mathcal{D}} \circ \mathbf{H} \circ \mathbb{I}_{\mathcal{P}}, \tag{2.4}$$

$$\mathbf{H}^{\mathcal{D}} \equiv \mathbb{I}_{\mathcal{D}} \circ \mathbf{H} \circ \mathbb{I}_{\mathcal{D}}. \tag{2.5}$$

This is an immediate result of two identities: $\mathbf{H} = \mathbb{I}_{\mathcal{S}} \circ \mathbf{H} \circ \mathbb{I}_{\mathcal{S}}$ and $\mathbb{I}_{\mathcal{S}} = \mathbb{I}_{\mathcal{P}} + \mathbb{I}_{\mathcal{D}}$ (see (4.15), Part I).

Clearly, linear operators on the space $\mathcal{H}$

$$\mathbf{H} \to \mathbf{H}^{\mathcal{L}}, \qquad \mathcal{L} = \mathcal{P}, \mathcal{PD}, \mathcal{D} \tag{2.6}$$

are invariant orthogonal projectors, i.e.

$$(\mathbf{R} * \mathbf{H}^{\mathcal{L}}) = (\mathbf{R} * \mathbf{H})^{\mathcal{L}}, \qquad (\mathbf{H}^{\mathcal{L}})^{\mathcal{L}} = \mathbf{H}^{\mathcal{L}}, \qquad \mathbf{R} \in \mathcal{O}.$$

They are mutually orthogonal, i.e.

$$\mathbf{H}^{\mathcal{P}} \cdot \mathbf{H}^{\mathcal{PD}} = \mathbf{H}^{\mathcal{P}} \cdot \mathbf{H}^{\mathcal{D}} = \mathbf{H}^{\mathcal{D}} \cdot \mathbf{H}^{\mathcal{PD}} = 0$$

and their sum is an identity operator.

We have obtained a new *invariant orthogonal decomposition of the space of Hooke's tensors*

$$\mathcal{H} = \mathcal{H}^{\mathcal{P}} \dot{+} \mathcal{H}^{\mathcal{PD}} \dot{+} \mathcal{H}^{\mathcal{D}}, \qquad 21 = 1 + 5 + 15 \tag{2.7}$$

which we will call *energy decomposition.* The dimensions will become evident soon.

2.3. It is not difficult to obtain this decomposition in an explicit form. It is simplest to begin with non-orthogonal decomposition (see (7.15), Part I).

$$\mathbf{H} = \mathrm{h}_{\mathcal{P}}\mathbb{I}_{\mathcal{P}} + \mathrm{h}_{\mathcal{D}}\mathbb{I}_{\mathcal{D}} + (\mathbf{1}\otimes\boldsymbol{\omega} + \boldsymbol{\omega}\otimes\mathbf{1}) + \mathfrak{c}\times(\mathbf{1}\otimes\boldsymbol{\varrho} + \boldsymbol{\varrho}\otimes\mathbf{1}) + \mathbf{D}. \tag{2.8}$$

Only the decomposition of the part

$$\mathbf{L} \equiv \mathfrak{c}\times(\mathbf{1}\otimes\boldsymbol{\varrho} + \boldsymbol{\varrho}\otimes\mathbf{1}) \tag{2.9}$$

is not immediately clear. But $\mathbf{1}\cdot\mathbf{L}\cdot\mathbf{1} = \mathbf{0}$ and taking $\mathbb{I}_{\mathcal{D}} = \mathbb{I}_{\mathcal{S}} - \mathbb{I}_{\mathcal{D}}$, we have

$$\begin{aligned} \mathbb{I}_{\mathcal{P}}\circ\mathbf{L}\circ\mathbb{I}_{\mathcal{P}} &= \mathbf{0}, \\ \mathbb{I}_{\mathcal{P}}\circ\mathbf{L}\circ\mathbb{I}_{\mathcal{D}} + \mathbb{I}_{\mathcal{D}}\circ\mathbf{L}\circ\mathbb{I}_{\mathcal{P}} &= \mathbb{I}_{\mathcal{P}}\circ\mathbf{L} + \mathbf{L}\circ\mathbb{I}_{\mathcal{P}} = \frac{2}{3}(\mathbf{1}\otimes\boldsymbol{\varrho} + \boldsymbol{\varrho}\otimes\mathbf{1}), \\ \mathbb{I}_{\mathcal{D}}\circ\mathbf{L}\circ\mathbb{I}_{\mathcal{D}} &= \mathbf{L} - \frac{2}{3}(\mathbf{1}\otimes\boldsymbol{\varrho} + \boldsymbol{\varrho}\otimes\mathbf{1}). \end{aligned} \tag{2.10}$$

Finally, a *complete energy decomposition of Hooke's tensor* has the following unique explicit form:

$$\begin{aligned} \mathbf{H} = h_{\mathcal{P}}\mathbb{I}_{\mathcal{P}} &+ (\mathbf{1}\otimes\boldsymbol{\varphi} + \boldsymbol{\varphi}\otimes\mathbf{1}) \\ &+ \left[h_{\mathcal{D}}\mathbb{I}_{\mathcal{D}} + \left(\mathfrak{c} - \frac{2}{3}\mathfrak{i}\right)\times(\mathbf{1}\otimes\boldsymbol{\psi} + \boldsymbol{\psi}\otimes\mathbf{1}) + \mathbf{D}\right], \end{aligned} \tag{2.11}$$

where

$$\boldsymbol{\varphi} = \frac{1}{3}(3\boldsymbol{\omega} + 2\boldsymbol{\varrho}) = \frac{1}{7}(7\boldsymbol{\alpha} + 2\boldsymbol{\beta}), \qquad \boldsymbol{\psi} = \boldsymbol{\varrho} = \frac{2}{3}(\boldsymbol{\alpha} - \boldsymbol{\beta}). \tag{2.12}$$

This is an orthogonal decomposition. The orthogonality of the second and fourth part follows from ((5.14), Part I). Deviators $\boldsymbol{\varphi}, \boldsymbol{\psi}$ are expressed by Novozhilov's deviators ((7.12), Part I) as follows:

$$\boldsymbol{\varphi} = \frac{1}{3}\boldsymbol{\mu}_{\mathcal{D}}, \qquad \boldsymbol{\psi} = \boldsymbol{\varrho} = \frac{2}{7}(3\boldsymbol{\nu}_{\mathcal{D}} - 2\boldsymbol{\mu}_{\mathcal{D}}). \tag{2.13}$$

2.4. *The complete energy decomposition of the space of Hooke's tensor* which corresponds to (2.11) has the form

$$\mathcal{H} = \mathcal{J}_{\mathcal{P}} \dotplus \mathfrak{D}_i \dotplus (\mathcal{J}_{\mathcal{D}} \dotplus \mathfrak{D}_{\mathfrak{n}} \dotplus \mathsf{D}), \qquad 21 = 1 + 5 + (1 + 5 + 9), \tag{2.14}$$

where $\mathfrak{n} = \mathfrak{c} - \frac{2}{3}\mathfrak{i}$. This decomposition is unique.

The quadratic form (2.1) we started with takes the form

$$\omega \cdot \mathbf{H} \cdot \omega = h_{\mathcal{P}} |\omega_{\mathcal{P}}|^2 + 2 \operatorname{tr} \omega (\varphi \cdot \omega_{\mathcal{D}}) + h_{\mathcal{D}} |\omega_{\mathcal{D}}|^2 + 2\psi (\omega_{\mathcal{D}})^2 + \omega_{\mathcal{D}} \cdot \mathbf{D} \cdot \omega_{\mathcal{D}}. \tag{2.15}$$

EXAMPLE 2. There is an important class of elastic anisotropic materials, in which the deviatoric and spherical parts of stress and deformation are *energy orthogonal* in the sense [2] of, i.e. such as

$$\mathbf{C}^{\mathcal{PD}} = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{S}^{\mathcal{PD}} = \mathbf{0}. \tag{2.16}$$

According to (2.11), (2.13) we have

$$\mathbf{C}^{\mathcal{PD}} = \frac{1}{3} (\mathbf{1} \otimes \mu_{\mathcal{D}} + \mu_{\mathcal{D}} \otimes \mathbf{1}), \tag{2.17}$$

so $\mu_D = 0$. By the definition,

$$\mathbf{C} \cdot \mathbf{1} = c_{\mathcal{P}} \mathbf{1} \quad , \quad \mathbf{S} \cdot \mathbf{1} = s_{\mathcal{P}} \mathbf{1} \quad , \quad c_{\mathcal{P}} s_{\mathcal{P}} = 1. \tag{2.18}$$

Thus, the hydrostatic stress causes a change in volume without any deviatoric deformation, whereas deviatoric stress causes only deviatoric deformation without change in volume. Such materials we called in [3, 4] *volume isotropic*. Their spectral decomposition ((1.9), Part I) takes the form

$$\mathbf{S} = s_{\mathcal{P}} \mathbb{I}_{\mathcal{P}} + \mu_1 \delta_1 \otimes \delta_1 + \cdots + \mu_5 \delta_5 \otimes \delta_5, \tag{2.19}$$

where $\delta_1, \cdots, \delta_5$ are proper deviators, and 6 Kelvin moduli are the bulk modulus s and 5 deviatoric moduli $\mu_1, \cdots, \mu_5$.

3. On true Hooke's tensors

3.1. A non-zero Hooke's tensor $\mathbf{H}$ will be called *a true Hooke's tensor* when its quadratic form is non-negative definite:

$$\omega \cdot \mathbf{H} \cdot \omega \geqslant 0 \qquad \text{for every} \qquad \omega \in \mathcal{S}. \tag{3.1}$$

The true character of a Hooke's tensor can be easily determined when spectral decomposition ((1.9), Part I) is used: Kelvin moduli $h_1 \geqslant \mathbf{0}, \cdots, h_6 \geqslant \mathbf{0}$ are to be non-negative. It would be more difficult, however, to satisfy the necessary and sufficient conditions imposed on the systems $(h_{\mathcal{P}}, h_{\mathcal{D}}, \alpha, \beta, \mathbf{D})$ or on equivalent systems[1] (see (7.4), Part I).

[1] Of course, one can use the classical Sylvester criterion, when $\mathbf{H}$ is taken in a matrix form, but this does not lead to easy formulation.

3.2. The invariant parts of a true Hooke's tensor *do not have to be* true Hooke's tensors. This is, however, the case for the most important part. For basic decomposition

$$\mathbf{H} = \mathbf{H}^{is} \dot{+} \mathbf{H}^{an} , \qquad 21 = 2 + 19 \tag{3.2}$$

we have the following, not at all obvious, theorem.

THEOREM 1. The isotropic part $\mathbf{H}^{is}$ of every true Hooke's tensor $\mathbf{H}$ is a true Hooke's tensor. This means that for (3.1) the following inequalities are the case:

$$h_{\mathcal{P}} \equiv \frac{1}{3}\mathbf{1}\cdot\mathbf{H}\cdot\mathbf{1} \geqslant \mathbf{0}, \tag{3.3}$$

$$h_{\mathcal{D}} \equiv \frac{1}{5}\left(\operatorname{Tr}\mathbf{H} - h_{\mathcal{P}}\right) \geqslant \mathbf{0}, \tag{3.4}$$

with one of them being sharp. If the tensor $\mathbf{H}$ is not isotropic, i.e. $\mathbf{H}^{an} \neq \mathbf{0}$, then

$$h_{\mathcal{P}} \geqslant 0 , \qquad h_{\mathcal{D}} > 0. \tag{3.5}$$

P r o o f. If tensor $\mathbf{H}$ is isotropic, $\mathbf{H}^{an} = \mathbf{0}$, then

$$\omega\cdot\mathbf{H}\cdot\omega = h_{\mathcal{P}}\left|\omega_{\mathcal{P}}\right|^2 + h_{\mathcal{D}}\left|\omega_{\mathcal{D}}\right|^2 \tag{3.6}$$

and condition (3.1) means exactly (3.3) and (3.4).

Let us take a true Hooke's tensor which is not isotropic, i.e. $\mathbf{H}^{an} \neq \mathbf{0}$. For spherical action $\omega = \omega_{\mathcal{P}} = p\,\mathbf{1}$ we have

$$\omega\cdot\mathbf{H}\cdot\omega = h_{\mathcal{P}}\,p^2 \geqslant \mathbf{0} \Longrightarrow h_{\mathcal{P}} \geqslant \mathbf{0}. \tag{3.7}$$

Let us take any deviatoric action $\omega = \omega_{\mathcal{D}} = \delta \in \mathcal{D}$. We have

$$\omega\cdot\mathbf{H}\cdot\omega = \delta\cdot\mathbf{H}\cdot\delta = \delta\cdot\mathbf{H}^{\mathcal{D}}\cdot\delta = \delta\cdot\left(h_{\mathcal{D}}\mathbb{I}_{\mathcal{D}} + \mathbf{K}\right)\cdot\delta \geqslant \mathbf{0}, \tag{3.8}$$

where

$$\mathbf{K} \equiv \left(\mathbf{H}^{an}\right)^{\mathcal{D}} = \mathbb{I}_{\mathcal{D}} \circ \mathbf{H}^{an} \circ \mathbb{I}_{\mathcal{D}} . \tag{3.9}$$

The operator $\delta \rightarrow \mathbf{K}\cdot\delta$ is a symmetric linear operator which transforms the 5-dimensional space of deviators $\mathcal{D}$, with scalar product $\delta_1\cdot\delta_2$, into itself. Thus we have its spectral decomposition

$$\mathbf{K} = k_1\varkappa_1 \otimes \varkappa_1 + \cdots + k_5\varkappa_5 \otimes \varkappa_5, \tag{3.10}$$

where proper deviators constitute the orthonormal basis in $\mathcal{D}$, $\varkappa_i\cdot\varkappa_j = \delta_{ij}$.

Since $\mathbb{I}_{\mathcal{D}}$ is the unity operator on $\mathcal{D}$, $\mathbb{I}_{\mathcal{D}} \cdot \boldsymbol{\delta} \equiv \boldsymbol{\delta}$, then every deviator is its proper element corresponding to eigenvalue 1, so

$$\mathbb{I}_{\mathcal{D}} = \varkappa_1 \otimes \varkappa_1 + \dots + \varkappa_5 \otimes \varkappa_5 . \tag{3.11}$$

Operator $h_{\mathcal{D}}\mathbb{I}_{\mathcal{D}}+\mathbf{K}$ is non-negative definite, so all its eigenvalues are non-negative

$$h_{\mathcal{D}} + k_1 \geqslant 0, \dots, h_{\mathcal{D}} + k_5 \geqslant 0 . \tag{3.12}$$

But

$$\begin{aligned} k_1 + \dots + k_5 = \operatorname{Tr} \mathbf{K} \equiv K_{pqpq} = \mathbb{I}_{\mathcal{D}\,pqab}\, H^{an}_{abcd}\, \mathbb{I}_{cdpq} \\ = H^{an}_{abcd}\, \mathbb{I}_{\mathcal{D}\,abcd} = \mathbf{H}^{an} \cdot \mathbb{I}_{\mathcal{D}} \end{aligned} \tag{3.13}$$

because of $\mathbb{I}_{\mathcal{D}} \circ \mathbb{I}_{\mathcal{D}} = \mathbb{I}_{\mathcal{D}}$. Since the anisotropic part $\mathbf{H}^{\mathrm{an}}$ is orthogonal to any isotropic Hooke's tensor, then

$$k_1 + \dots + k_5 = 0. \tag{3.14}$$

As tensor $\mathbf{H}$ is not isotropic, then $\mathbf{K} \neq \mathbf{0}$, hence its smallest eigenvalue $k_{\min}$ must be negative. We have therefore

$$h_{\mathcal{D}} \geqslant |k_{\min}| > 0. \tag{3.15}$$

COROLLARY 1. A true Hooke's tensor without an isotropic part does not exist. Indeed, for $\mathbf{H} = \mathbf{H}^{an}$, one can always find such a deviator $\boldsymbol{\delta}$ that

$$\boldsymbol{\delta} \cdot \mathbf{H} \cdot \boldsymbol{\delta} = \boldsymbol{\delta} \cdot \mathbf{K} \cdot \boldsymbol{\delta} = k_{\min} |\boldsymbol{\delta}|^2 < 0.$$

COROLLARY 2. The isotropic part of a true Hooke's tensor is its closest true Hooke's tensor (in the sense of distance $|\mathbf{A} - \mathbf{B}|$). Indeed, it is an orthogonal projection of $\mathbf{H}$ on $\mathcal{H}^{is}$.

REMARK. The norm of the isotropic part $\left|\mathbf{H}^{is}\right|$ can differ substantially from the norm of the entire Hooke's tensor

$$|\mathbf{H}|^2 = \left|\mathbf{H}^{is}\right|^2 + |\mathbf{H}^{an}|^2 . \tag{3.16}$$

Similarly, the values of quadratic forms $\boldsymbol{\omega} \cdot \mathbf{H}^{is} \cdot \boldsymbol{\omega}$ (e.g. energy) can substantially differ from one another. The problem of choosing a 'good' isotropic approximation of Hooke's tensor, for example choosing an isotropic elastic material to approximate an anisotropic elastic material, is a problem *per se* (see e.g. [5, 6]). Everything here depends on the purpose of the approximation.

4. Spatial symmetry and invariant decompositions

4.1. The relation between spatial symmetry of Hooke's tensors and their spectral decompositions was examined in detail (see [3, 7, 8] and review [4]). Here we shall only deal with the relation between this symmetry and the **invariant linear** decompositions obtained in Part I.

THEOREM 2. A group of spatial symmetry of Hooke's tensor is the intersection of symmetry groups of its anisotropy deviators, e.g.

$$\mathcal{O}(\mathbf{H}) = \mathcal{O}(\boldsymbol{\alpha}) \cap \mathcal{O}(\boldsymbol{\beta}) \cap \mathcal{O}(\mathbf{D}). \tag{4.1}$$

P r o o f. obvious (see also [9]). Clearly, the pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ can be replaced by any other pair $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ described in ((7.3), Part I).

Therefore, all the possible types of symmetry of Hooke's tensors are types of symmetry of triples: two second-order deviators $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and a fourth-order deviator $\mathbf{D}$. An analysis of the resultant possibilities would lead us in a new way to eight classical groups of symmetry of linear elasticity (see also [9]). We will not do so.

IMPORTANT REMARK. In a general situation, the triple $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{D})$ loses the common elements of symmetry very quickly. In other words, most anisotropic materials do not have any axes or planes of elastic symmetry

4.2. We shall demonstrate how, in the language of anisotropy deviators, appear the axes of symmetry of Hooke's tensors.

Let $\mathbf{R}_{\mathbf{k}}(\varphi)$ be a rotation of our basic Euclidean space by the angle φ around the axis directed by a unit vector $\mathbf{k}$. The straight line will be called a *total symmetry axis* of the tensor $\mathbf{A}$ when

$$\mathbf{R}_{\mathbf{k}}(\varphi) * \mathbf{A} = \mathbf{A} \qquad \text{for every angle } \varphi, \tag{4.2}$$

and its *n-fold symmetry axis* when it is not its axis of total symmetry, but

$$\mathbf{R}_{\mathbf{k}}(\varphi) * \mathbf{A} = \mathbf{A} \qquad \text{for } \varphi = \frac{2\pi}{n}, \tag{4.3}$$

where n is the minimal integer $\geqslant 2$ of such integers for which this equation is satisfied. As the tensor $\mathbf{A}$ we can here take any Euclidean tensor.

It can be demonstrated that the space of second-order deviators can be decomposed with respect to any fixed axis into an orthogonal direct sum

$$\mathcal{D} = \overset{\circ}{\mathcal{D}}_{\mathbf{k}} + \overset{1}{\mathcal{D}}_{\mathbf{k}} + \overset{2}{\mathcal{D}}_{\mathbf{k}}, \qquad 5 = 1 + 2 + 2 \tag{4.4}$$

such that any rotation $\mathbf{R}_{\mathbf{k}}(\varphi)$

– preserves every deviator on the straight line $\overset{\circ}{\mathcal{D}}_{\mathbf{k}}$,

– rotates every deviator in plane $\overset{l}{\mathcal{D}}_{\mathbf{k}}$, $l = 1, 2$, by the angle $l\varphi$[2].

Clearly, every deviator on the straight line $\overset{\circ}{\mathcal{D}}_{\mathbf{k}}$ has the form

$$\boldsymbol{\omega} = a\,(\mathbf{1} - 3\mathbf{k}\otimes\mathbf{k})\,. \tag{4.5}$$

Similarly, an orthogonal decomposition of the space of fourth-order deviators

$$\begin{aligned} \mathsf{D} &= \overset{0}{\mathsf{D}}_{\mathbf{k}} + \overset{1}{\mathsf{D}}_{\mathbf{k}} + \overset{2}{\mathsf{D}}_{\mathbf{k}} + \overset{3}{\mathsf{D}}_{\mathbf{k}} + \overset{4}{\mathsf{D}}_{\mathbf{k}} \\ 9 &= \ 1 \ + \ 2 \ + \ 2 \ + \ 2 \ + \ 2, \end{aligned} \tag{4.6}$$

takes place. In this case, every rotation $\mathbf{R}_{\mathbf{k}}(\varphi)$

– preserves every deviator on the straight line $\overset{0}{\mathsf{D}}_{\mathbf{k}}$,

– rotates every deviator in plane $\overset{l}{\mathsf{D}}_{\mathbf{k}}$, $l = 1, 2, 3, 4$, by the angle $l\varphi$.

Therefore, higher symmetry axes (i.e. 3-fold and 4-fold) of a Hooke's tensor are caused only by the presence of the invariant term $\mathbf{D}$ in its decomposition. Strictly speaking, the following theorem is the case.

THEOREM 3. A Hooke's tensor $\mathbf{H}$ has a higher symmetry axis $\mathbf{k}$ only if its anisotropy deviators have the form

$$\boldsymbol{\alpha} = a\,(\mathbf{1} - 3\mathbf{k}\otimes\mathbf{k})\ , \qquad \boldsymbol{\beta} = b\,(\mathbf{1} - 3\mathbf{k}\otimes\mathbf{k})\ , \qquad \mathbf{D} \neq \mathbf{0}. \tag{4.7}$$

It has more than one higher symmetry axis only if

$$\boldsymbol{\alpha} = \mathbf{0}\ , \qquad \boldsymbol{\beta} = \mathbf{0}\ , \qquad \mathbf{D} \neq \mathbf{0}. \tag{4.8}$$

P r o o f. Let $\mathbf{k}$ be a higher symmetry axis of $\mathbf{H}$. Rotations around $\mathbf{k}$ by the angles $2\pi/3$ or $2\pi/4$ preserve second-order deviator $\boldsymbol{\omega}$ only if $\mathbf{k}$ is its total symmetry axis, so if $\boldsymbol{\omega} \in \overset{0}{\mathsf{D}}_{\mathbf{k}}$. If $\mathbf{D} = \mathbf{0}$ then $\mathbf{k}$ would be a total symmetry axis of $\mathbf{H}$, which we dismissed. Let $\mathbf{k}, \mathbf{l}$ be two different higher symmetry axes of $\mathbf{H}$. Then the formulae (4.7) would have to be the case both for $\mathbf{k}$ and for $\mathbf{l}$, which is only possible when $a = b = 0$.

Let us examine only one, but important example.

EXAMPLE 3. Elasticity tensor $\mathbf{S}$ of a *cubic crystal*. The symmetry group $\mathcal{O}(\mathbf{S})$ is a symmetry group of a cube. We have here three 4-fold axes and four 3-fold axes. This is more than needed, according to the theorem and decomposition

[2] All the elements $\overset{1}{\mathcal{D}}_k, \overset{2}{\mathcal{D}}_k$ are pure shears (see [8]).

(7.3), Part I, for tensor $\mathbf{S}$ to belong to the following important class of Hooke's tensors

$$\mathbf{S} = \mathbf{S}^{is} + \mathbf{D}, \qquad \boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{0}\,. \tag{4.9}$$

The deviator $\mathbf{D}$ is easy to be explicitly define. By denoting by $\mathbf{k}, \mathbf{l}, \mathbf{m}$ the directions of the edges of the cube, we see that tensor

$$\mathbf{D} \backsim 3\mathbb{I}_{\mathcal{S}} - 5\mathbf{K} = 3iI_{\mathcal{P}} + 2\mathbb{I}_{\mathcal{S}} - 5\mathbf{K}, \tag{4.10}$$

where

$$\mathbf{K} \equiv \mathbf{k}\otimes\mathbf{k}\otimes\mathbf{k}\otimes\mathbf{k} + \mathbf{l}\otimes\mathbf{l}\otimes\mathbf{l}\otimes\mathbf{l} + \mathbf{m}\otimes\mathbf{m}\otimes\mathbf{m}\otimes\mathbf{m}, \tag{4.11}$$

is a deviator with the required symmetry of a cube.

A complete invariant decomposition of elasticity tensor of the cubic crystal has, therefore, the form

$$\mathbf{S} = \lambda_{\mathcal{P}}\mathbb{I}_{\mathcal{P}} + \lambda_{\mathcal{D}}\mathbb{I}_{\mathcal{D}} + \delta\mathbf{D}. \tag{4.12}$$

The relation with the given spectral decomposition (1.10), Part I, is obvious

$$\begin{aligned} \mathbb{I}_{\mathcal{P}} &= \mathbf{P_1}\,, & \lambda_{\mathcal{P}} &= h_1\,, \\ \mathbb{I}_{\mathcal{D}} &= \mathbf{P_2} + \mathbf{P_3}\,, & \lambda_{\mathcal{D}} &= \frac{1}{5}\left(2h_2 + 3h_3\right)\,, \\ \mathbf{D} &= -3\mathbf{P_2} + 2\mathbf{P_3}\,, & \delta &= \frac{1}{5}\left(h_3 - h_2\right)\,. \end{aligned} \tag{4.13}$$

The projectors onto proper subspaces defined by (1.12), Part I, are therefore

$$\mathbf{P_1} = \mathbb{I}_{\mathcal{P}}\,, \quad \mathbf{P_2} = \mathbf{K} - \mathbb{I}_{\mathcal{P}}, \quad \mathbf{P_3} = \mathbb{I}_{\mathcal{S}} - \mathbf{K} \tag{4.14}$$

which can be demonstrated by direct projecting: $\boldsymbol{\omega} \to \mathbf{P}_i \cdot \boldsymbol{\omega}\,, \quad i = 1, 2, 3.$

5. Invariant decompositions of plane Hooke's tensors (see also [10])

5.1. Plane tensors of any order q are generated by the Euclidean plane $\mathcal{E}$, $\dim \bigotimes^q \mathcal{E} = 2^q$. Still

$$\mathcal{S} = \operatorname{sym} \mathcal{E} \bigotimes \mathcal{E}\,, \qquad \mathcal{H} = \operatorname{sym} \mathcal{S} \bigotimes \mathcal{S}, \tag{5.1}$$

but now the situation is far simpler since

$$\dim \mathcal{S} = 3, \qquad \dim \mathcal{H} = 6\,. \tag{5.2}$$

5.2. The counterpart of decompositions ((4.12), (4.13), Part I), is

$$\begin{aligned} &\mathcal{S} = \mathcal{P}\dot{+}\mathcal{D}, \quad 3 = 1+2, \\ &\omega = \omega_{\mathcal{P}} + \omega_{\mathcal{D}}, \quad \begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} + \begin{pmatrix} u & v \\ v & -u \end{pmatrix}. \end{aligned} \tag{5.3}$$

The plane deviator $\omega_{\mathcal{D}}$ is always, in the sense of mechanics, a *pure shear* τ in plane $\mathcal{E}$, (see any textbook of solid mechanics or [8]). We have

$$\omega_{\mathcal{P}} = \tau, \qquad \tau = t\,(\mathbf{m}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{m}), \qquad t^2 \equiv \frac{1}{2}\tau\cdot\tau \tag{5.4}$$

where mutually orthogonal unit vectors $(\mathbf{m}, \mathbf{n})$ are the *shear directions.*

5.3. The nonlinear invariant spectral decompositions of the plane Hooke's tensor have the form

$$\mathbf{H} = h_1\omega_1\otimes\omega_1 + h_2\omega_2\otimes\omega_2 + h_3\omega_3\otimes\omega_3 \tag{5.5}$$

where $\omega_k\cdot\omega_l = \delta_{kl}$. These decompositions are presented simply and in enough detail in [4].

5.4. The action of permutation operators remains exactly the same as that described in Part I, but its results are far simpler. *The first basic decomposition of the space $\mathcal{H}$, with respect to internal symmetry* has the form

$$\mathcal{H} = \mathcal{H}_{\mathfrak{s}} \dot{+} \mathcal{H}_{\mathfrak{t}}\,, \qquad 6 = 5+1. \tag{5.6}$$

The dimension of $\mathcal{H}_{\mathbf{s}}$ follows from the fact that the condition of total symmetry $\mathfrak{s}\times\mathbf{H} = \mathbf{H}$ imposes on the six free components H_{1111}, H_{2222}, $H_{1122}, H_{1212}, H_{1112}, H_{2212}$, only one constraint $H_{1122} = H_{1212}$.

5.5. *The second basic decomposition of $\mathcal{H}$, according to the symmetry with respect to the group of rotations and mirror reflections in plane $\mathcal{E}$, has the form*

$$\mathcal{H} = \mathcal{H}^{is} \dot{+} \mathcal{H}^{an}\,, \qquad 6 = 2+4. \tag{5.7}$$

The description of the plane $\mathcal{H}^{is}$ of plane isotropic Hooke's tensors differs from the former one only by the change of coefficient 3^{-1} into 2^{-1} in formulae (4.10), (4.11), Part I

$$\begin{aligned} \mathbb{I}_{\mathcal{S}} &= \mathfrak{c}\times(\mathbf{1}\otimes\mathbf{1})\,, & \mathbb{I}_{\mathcal{S}}\cdot\mathbb{I}_{\mathcal{S}} &= \dim\mathcal{S} = 3, \\ \mathbb{I}_{\mathcal{D}} &= \left(\mathfrak{c} - \frac{1}{2}\mathfrak{i}\right)\times(\mathbf{1}\otimes\mathbf{1})\,, & \mathbb{I}_{\mathcal{D}}\cdot\mathbb{I}_{\mathcal{D}} &= \dim\mathcal{D} = 2, \\ \mathbb{I}_{\mathcal{P}} &= \frac{1}{2}(\mathbf{1}\otimes\mathbf{1})\,, & \mathbb{I}_{\mathcal{P}}\cdot\mathbb{I}_{\mathcal{P}} &= \dim\mathcal{P} = 1. \end{aligned} \tag{5.8}$$

Here and henceforth, the symbol **1** denotes now the *plane* unit $\mathbf{1} \backsim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

5.6. Let us consider the counterparts of two anisotropic parts of the canonical decomposition (6.4), Part I. There are two simple albeit not self-evident facts.

LEMMA 1. Every plane tensor of the form

$$\mathbf{1} \otimes \boldsymbol{\tau} + \boldsymbol{\tau} \otimes \mathbf{1}, \tag{5.9}$$

where $\boldsymbol{\tau}$ is any plane deviator (so any pure shear), is totally symmetric with respect to permutations.

Proof. Taking proper directions of $\boldsymbol{\tau}, \mathbf{a} = (\mathbf{m} + \mathbf{n})/\sqrt{2}$, $\mathbf{b} = (\mathbf{n} - \mathbf{m})/\sqrt{2}$, we have

$$\boldsymbol{\tau} = t\,(\mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b}) \sim t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.10}$$

At the same time, of course,

$$\mathbf{1} = \mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.11}$$

Hence

$$\mathbf{1} \otimes \boldsymbol{\tau} + \boldsymbol{\tau} \otimes \mathbf{1} = t\,(\mathbf{a} \otimes \mathbf{a} \otimes \mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b} \otimes \mathbf{b} \otimes \mathbf{b}) \tag{5.12}$$

and the right-hand side does not react to permutations.

5.7. Therefore, the plane counterpart of the 10-dimensional invariant space $\mathfrak{D}$ will be the plane $\mathfrak{D}$ consisting of all plane tensors of the form (5.9). As $\dim\mathcal{H}_{\mathfrak{s}} = 5$, then the plane counterpart of q-dimensional complement D will be the plane of fourth-order plane deviators D.

COROLLARY. The anisotropic part $\mathbf{H}^{an}$ of plane Hooke's tensor is totally symmetric

$$\mathbf{H}_{\mathfrak{s}} = h_{\mathfrak{s}}\mathbb{I}_{\mathfrak{s}} + \mathbf{H}^{an}, \qquad \mathbf{H}_{\mathfrak{t}} = h_{\mathfrak{t}}\mathbb{I}_{\mathfrak{t}}. \tag{5.13}$$

LEMMA 2. For every non-zero plane fourth-order deviator $\mathbf{D}$, there exists a pure shear $\boldsymbol{\gamma}$ such that

$$\mathsf{D} = \boldsymbol{\gamma} \otimes \boldsymbol{\gamma} - \boldsymbol{\gamma}^{\perp} \otimes \boldsymbol{\gamma}^{\perp} = \mathbf{2}\boldsymbol{\gamma} \otimes \boldsymbol{\gamma} - |\boldsymbol{\gamma}|^2\, \mathbb{I}_{\mathcal{D}} \tag{5.14}$$

where the complementary pure shear $\boldsymbol{\gamma}^{\perp}$ is defined by $\boldsymbol{\gamma}$ through formulae $\boldsymbol{\gamma}^{\perp} \cdot \boldsymbol{\gamma} = \mathbf{0}$, $|\boldsymbol{\gamma}^{\perp}| = |\boldsymbol{\gamma}|$.

P r o o f. Let us apply a spectral decomposition (5.5)

$$\mathbf{D} = d_1\,\omega_1 \otimes \omega_1 + d_2\omega_2 \otimes \omega_2 + d_3\omega_3 \otimes \omega_3. \tag{5.15}$$

From conditions of orthogonality to $\mathcal{H}^{is}$ we have $\mathbf{1}\cdot\mathbf{D} = 0$, $\operatorname{Tr}\mathbf{D} = 0$ which gives

$$d_1\,\mathbf{tr}\omega_1 = d_2\,\mathbf{tr}\omega_2 = d_3\,\mathbf{tr}\omega_3 = 0\,, \qquad d_1 + d_2 + d_3 = 0\,. \tag{5.16}$$

The only non-zero solution $\mathbf{D}$ ordered in such a manner that $d_1 < d_2 < d_3$, is

$$d_3 = -d_1\,, \qquad d_2 = 0\,, \qquad \mathbf{tr}\omega_1 = \mathbf{tr}\omega_2 = 0 \tag{5.17}$$

which gives the first formula (5.14). The second formula follows from the equation

$$\mathbb{I}_{\mathfrak{D}} = \tau_1 \otimes \tau_1 + \tau_2 \otimes \tau_2 \tag{5.18}$$

valid for every orthonormal basis in $\mathfrak{D}$, $\tau_i \cdot \tau_k = \delta_{ik}$.

5.8. Summing up: *The canonical decomposition of the space of plane Hooke's tensors has the form*

$$\mathcal{H} = \mathcal{H}^{is} \dotplus \mathfrak{D} \dotplus \mathsf{D}, \qquad 6 = 2+2+2, \tag{5.19}$$

while every complete invariant decomposition has the form

$$\mathcal{H} = (\mathcal{J}_{\mathfrak{n}_1} + \mathcal{J}_{\mathfrak{n}_2}) \dotplus \mathfrak{D} \dotplus \mathsf{D}\,, \qquad 6 = (1+1)+2+2. \tag{5.20}$$

In other words, every plane Hooke's tensor has the form

$$\mathbf{H} = h_1\mathbb{I}_{\mathfrak{n}_1} + h_2\mathbb{I}_{\mathfrak{n}_2} + (\mathbf{1}\otimes\tau + \tau\otimes\mathbf{1}) + \mathbf{D}. \tag{5.21}$$

Pure shear τ *and fourth-order deviator* $\mathbf{D}$ (5.14) *are uniquely defined, whereas invariants* h_1, h_2 *depend on the choice of permutation operators* $\mathfrak{n}_1, \mathfrak{n}_2$.

Explicit formulae for $(h_1, h_2, \tau, \mathbf{D})$ are easy to obtain.

5.9. *Energy decomposition* of plane Hooke's tensor

$$\mathbf{H} = \mathbf{H}^{\mathcal{P}} + \mathbf{H}^{\mathcal{P}\mathcal{D}} + \mathbf{H}^{\mathcal{D}} \tag{5.22}$$

is unique,

$$\mathbf{H} = h_{\mathcal{P}}\mathbb{I}_{\mathcal{P}} + (\mathbf{1}\otimes\tau + \tau\otimes\mathbf{1}) + (h_{\mathcal{D}}\mathbb{I}_{\mathcal{D}} + \mathbf{D})\,, \tag{5.23}$$

and the quadratic form corresponding to $\mathbf{H}$ can be written in the form

$$\begin{aligned}\omega\cdot\mathbf{H}\cdot\omega = {}& h_{\mathcal{P}}\,|\omega_{\mathcal{P}}|^2 + 2\mathrm{tr}\,\omega\,(\tau\cdot\omega_{\mathcal{D}})^2 \\ & + \left(h_{\mathcal{D}} - |\gamma|^2\right)|\omega_{\mathcal{D}}|^2 + \mathbf{2}\,(\gamma\cdot\omega_{\mathcal{D}})^2.\end{aligned} \tag{5.24}$$

5.10. The rotation $\mathbf{R}$ of plane $\mathcal{E}$ by the angle φ rotates the 3-dimensional space $\mathcal{S}$ around the axis $\mathcal{P}$ by the angle 2φ as it immediately follows from formula (5.4).

Thus the orthogonal basis of pure shears $(\gamma, \gamma^{\perp})$ rotates in the usual manner

$$\mathbf{R} * \gamma = \cos 2\varphi\gamma + \sin 2\varphi\gamma^{\perp}, \qquad \mathbf{R} * \gamma^{\perp} = -\sin 2\varphi\gamma + \cos 2\varphi\gamma^{\perp}; \tag{5.25}$$

hence, by taking $\mathbf{D} = \gamma \otimes \gamma - \gamma^{\perp} \otimes \gamma^{\perp}$ we obtain

$$(\mathbf{R} * \mathbf{D}) \cdot \mathbf{D} = |\mathbf{D}|^2 \cos 4\varphi. \tag{5.26}$$

So, every fourth-order deviator $\mathbf{D}$ rotates, as it should, by the angle 4φ. Therefore the plane Hooke's tensor $\mathbf{H}$ can have three well-known kinds of symmetry:

1. Symmetry of circle (isotropy), if $\tau = \mathbf{0}, \quad \mathbf{D} = 0$,
2. Symmetry of square (tetragonal), if $\tau = \mathbf{0}, \quad \mathbf{D} \neq 0$,
3. Symmetry of rectangle (ortothropy), if $\tau \neq \mathbf{0}, \quad \mathbf{D} \neq 0$,

(see also [10] where it was also pointed out that the plane elastic continuum without any symmetry has orientation 'left' or 'right').

5.11. To every 3-dimensional Hooke's tensor we can assign an infinite number of plane Hooke's tensors that correspond to it.

Let us take, in 3-dimensional space $\mathcal{E}$, a plane $\mathcal{K}$ defined by its unit normal vector $\mathbf{k}$. The orthogonal projection of the vectors $\mathbf{x} \in \mathcal{E}$ onto the plane $\mathcal{K}$ is an operation defined by projector $\mathbf{P}(\mathbf{k})$,

$$\mathbf{P}(\mathbf{k}) = \mathbf{1} - \mathbf{k} \otimes \mathbf{k}, \qquad \mathbf{x} \to \mathbf{P}\mathbf{x} = \mathbf{x} - (\mathbf{k}\mathbf{x})\,\mathbf{k}. \tag{5.27}$$

To this corresponds the linear operation of orthogonal projecting of 3-dimensional tensors onto the plane $\bigotimes^q \mathcal{E} \ni \mathbf{A} \to \mathbf{P} * \mathbf{A} \in \bigotimes^q \mathcal{K}$, defined for simple tensors by the formula

$$\mathbf{P} * (\mathbf{a}_1 \otimes \cdots \otimes \mathbf{a}_q) \equiv \mathbf{P}\mathbf{a}_1 \otimes \cdots \otimes \mathbf{P}\mathbf{a}_q. \tag{5.28}$$

The orthogonal plane projection of the 3-dimensional tensor of q-order

$$\mathbf{A} = \mathbf{A}_{i \cdots j} \mathbf{n}_i \otimes \cdots \otimes \mathbf{n}_j \tag{5.29}$$

onto the plane will therefore be the plane tensor of q-order

$$\mathbf{P}(\mathbf{k}) * \mathbf{A} = \mathbf{A}_{i \cdots j}\, \mathbf{P}(\mathbf{k})\, \mathbf{n}_i \otimes \cdots \otimes \mathbf{P}(\mathbf{k})\, \mathbf{n}_j. \tag{5.30}$$

The relations between the tensor $\mathbf{A}$ and its plane projections $\mathbf{P}(\mathbf{k}) * \mathbf{A}$ are in many situations quite essential.

EXAMPLE 4. Let us take an elastic sample with the compliance tensor $\mathbf{C}$. Let us cut out from this sample a thin plate with the normal vector $\mathbf{k}$. The plane part of the plate's deformation under the plane state of stress is defined by the plane compliance tensor of the plate, being nothing else but a plane projection $\mathbf{P}(\mathbf{k}) * \mathbf{C}$.

6. On description, qualification and design of elastic materials

6.1. Let us begin with remarks on the qualitative description and qualification of elastic materials.

The issue of symmetry is purposefully left out. It is so extensively discussed that one gets a false impression that the main differences in the behavior of elastic materials consist in the differences in their symmetry. *This is not so.* The fascination with the symmetry of physical properties is quite well justified in the physics of crystals, and also due to the simple and economical production technologies of composites, imposing the symmetry of their structure, e.g. ortothropy. Yet the contemporary technologies offer more and more sophisticated opportunities for shaping of the materials with pre-selected properties. But complicated structure leads to a prompt loss of symmetry. It suffices, for instance, to put into a composite three different kinds of fibres, mutually non-orthogonal, and there is no trace of symmetry. More importantly, materials of totally different symmetry can, in certain conditions, behave similarly or even identically.

It is therefore necessary to find manners of description, not connected with symmetry (ortothropy, etc.), and as a consequence, designing of the properties of materials, which would be deeper and more universal than those now used. One can remain particulalrly hopeful about the *invariant descriptions: non-linear of type ((1.9), Part I) and linear of type ((7.3), Part I). As decision variables* of designing of the properties of a material at the point under cosideration can serve here, in particular, spectral variables $(h_1, ..., h_6; \omega_1, ..., \omega_6)$ or the invariant parts of tensors like $(h_{\mathcal{P}}, h_{\mathcal{D}}, \alpha, \beta, \mathbf{D})$. This, however, calls for a deeper insight into the sense of these quantities.

6.2. Even at the stage of formulation of invariant decompositions, we gave introductory examples, pointing out their qualitative sense and possible applications. While illustrating the first basic decomposition (2.12), Part I, we made reference in Example 1 to the classical discussion on the number of parts of an elasticity tensor. While recalling the second basic decomposition (3.1), Part I, we demonstrated, in Theorem 1, the sense of the isotropic part of the elasticity tensor as an independent elasticity tensor. In Theorem 3 we showed the independent meaning of the deviatoric part $\mathbf{D}$ of Hooke's tensor as a true source of the presence of the axis of elastic symmetry of the third or/and fourth order. By introducing the energy decomposition (1.7) we made an immediate reference, in Example 2, to an important class of elastic materials, in which hydrostatic pressure and stress deviator are separated in terms of energy.

7. Some surprising applications: astonishing elastic materials

7.1. Let us now quote **quite different** examples. By using the technique of invariant linear decompositions, we shall point out some *novel types of elastic anisotropy* The very fact of their existence was, at least for me, quite surprising. I shall therefore use a manner of presentation slightly different from the standards of applied mathematics. I shall posit a series of questions, deliberately provocative. Yet each will have an unambiguous answer, proven in the papers quoted below. We shall limit our presentation to examples of linear elastic materials.

7.2. EXAMPLE A. Let us begin with acoustics. The great practical and theoretical role of longitudinal elastic waves is widely known (see, e.g. [6, 11]). An isotropic elastic material is capable of conducting a longitudinal wave in *each* direction and with the same speed. Many courses of the theory of elasticity and acoustics consider this property to be almost synonymous with elastic isotropy. Moreover, proving this is recommended as its experimental verification. The following question should be inportant:

> *Are there any anisotropic materials with the stiffness tensor* $\mathbf{S} = \mathbf{S}^{is} + \mathbf{S}^{an}, \mathbf{S}^{an} \neq 0$ *, capable of conducting a longitudinal acoustic signal in each direction?*

The answer is not less surprising than the question: YES, there are. In paper [12] I proved that these were the materials with the stiffness

$$\mathbf{S} = \mathbf{S}^{is} + \mathsf{t} \times (\mathbf{1} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \mathbf{1}) \tag{7.1}$$

and only these. The proof follows from the invariant decomposition (7.4), Part I. Let us note that the anisotropy of these materials is completely *undetectable* (so to speak, invisible) in experiments with longitudinal waves.

7.3. EXAMPLE B. Let us realize, in an elastic body with compliance $\mathbf{C}$, a stress state of *pure shear* $\sigma = \tau = t\,(\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m})$. This is one of the favorite ways of loading a sample (often realized on thin metal plates or twisted pipes). The size of the change of the originally right angle between shear directions $(\mathbf{m}, \mathbf{n})\,, \mathbf{mn} = 0$, is decided by the parameter

$$G\,(\mathbf{m}, \mathbf{n}) \equiv [4\,(\mathbf{m} \otimes \mathbf{n}) \cdot \mathbf{C} \cdot (\mathbf{m} \otimes \mathbf{n})]^{-1} \tag{7.2}$$

usually called *shear modulus for directions* $(\mathbf{m}, \mathbf{n})$ (see, e.g. [13]). As an isotropic material does not have any pre-distinguished direction, the shear modulus $\mathbf{G}(\mathbf{m}, \mathbf{n})$ will be identical for each pair $(\mathbf{m}, \mathbf{n})$, so it will be an invariant of the stiffness tensor $\mathbf{C}$, called *the Kirchhoff modulus.* Our next off-beat problem is:

> *Are there any* **anisotropic** *elastic materials, for which* $G(\mathbf{m}, \mathbf{n})$ *does not depend on the shear directions ?*

In [14] we showed that there was an infinite number of such materials and that they were defined by the formula

$$\mathbf{C} = \mathbf{C}^{is} + (\mathbf{1} \otimes \boldsymbol{\omega} + \boldsymbol{\omega} \otimes \mathbf{1}) \tag{7.3}$$

(cf. the invariant decomposition (7.15), Part I, in which we need to take $\boldsymbol{\rho} = \mathbf{0}$, $\mathbf{D} = \mathbf{0}$).

7.4. EXAMPLE C. In Example 4 we have demonstrated that the plane part of deformation of a thin plate, cut out from a material with compliance $\mathbf{C}$ and under plane load $\boldsymbol{\sigma}$, is defined by plane Hooke's tensor, being an orthogonal projection of the tensor $\mathbf{C}$ onto the plane of this plate. We formulate another off-beat qualitative question:

> *Is there any such anisotropic material* $\mathbf{C} = \mathbf{C}^{is} + \mathbf{C}^{an}$, $\mathbf{C}^{an} \neq \mathbf{0}$, *that each thin plate cut out from it will be isotropic, i.e.*

$$\mathbf{P}(\mathbf{k}) * \mathbf{C} \in \mathcal{H}^{is}(\mathbf{k}) \quad \textit{for each direction } \mathbf{k}\,? \tag{7.4}$$

Here $\mathcal{H}^{is}(\mathbf{k})$ is a set of plane isotropic Hooke's tensors.

The answer is surprising, again: YES. In [15] I have demonstrated that all such materials are defined by the formula

$$\mathbf{C} = \mathbf{C}^{is} + \mathrm{t} \times (\mathbf{1} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \mathbf{1}) \tag{7.5}$$

(see the invariant decomposition (7.4), Part I, where $\boldsymbol{\alpha} = \mathbf{0}$, $\mathbf{D} = \mathbf{0}$).

7.5. EXAMPLE D. In an isotropic material the Hooke's stress tensor $\boldsymbol{\sigma}$ and the corresponding tensor of small deformations $\boldsymbol{\varepsilon}$ are coaxial, i.e. they take the diagonal form in a common basis, or they are commutative $\boldsymbol{\sigma\varepsilon} = \boldsymbol{\varepsilon\sigma}$. Is this property equivalent to isotropy? Or, in other words:

> *Are there any linear elastic* **anisotropic** *materials which preserve the coaxiality of stress and deformation?*

I was relieved to establish that the answer is NO, at least when there is lack of internal stresses. The proof is given in [16].

7.6. EXAMPLE E. In a similar manner I have proved that only isotropic materials have an invariant elasticity constant ν called *Poisson's ratio*, [16]. Both proofs follow from the invariant decompositions of Hooke's tensors presented in Part I.

7.7. EXAMPLE F. Let us take an example that looks a bit more sophisticated. Let us introduce the main tensor of the theory of elastic waves – *Christoffel's tensor* $\Lambda(\mathbf{n})$

$$\rho\Lambda(\mathbf{n}) \equiv \mathbf{nSn}, \qquad \rho\Lambda_{ij} = S_{pijq}n_p n_q. \tag{7.6}$$

It defines the triple of plane elastic waves that can propagate in the direction $\mathbf{n}$, $\mathbf{n}\cdot\mathbf{n} = 1$. The displacement vectors accompanying these waves are mutually orthogonal, while the phase velocities are v_1, v_2, v_3. It is not difficult to demonstrate that

$$\operatorname{tr}\Lambda(\mathbf{n}) = v_1^2 + v_2^2 + v_3^2. \tag{7.7}$$

Neighbours [17] have demonstrated a long time ago that in cubic crystals, as in an isotropic body, the sum of squares of phase velocities does not depend on the direction of propagation $\mathbf{n}$ (this directly follows, after all, from the formulae in Examples 3). Question:

Is this property the case for other materials?

YES. By using the invariant decomposition (7.3), Part I, one can demonstrate that $v_1^2 + v_2^2 + v_3^2 = \text{const}$ for all materials with the stiffness

$$\mathbf{S} = \mathbf{S}^{is} + \mathrm{m} \times (\mathbf{1}\otimes\boldsymbol{\gamma} + \boldsymbol{\gamma}\otimes\mathbf{1}) + \mathbf{D}, \tag{7.8}$$

where $\mathrm{m} = 5 - 4c$.

7.8. EXAMPLE G. Let us finish with a neat example. As regards the first and simplest property of a solid body, taught at secondary schools (at least in Europe, as to my knowledge), is the elasticity modulus. At the more advanced stages of education this is called the *Young modulus in direction* $\mathbf{n}$ defined by the formula

$$E(\mathbf{n}) = [(\mathbf{n}\otimes\mathbf{n})\cdot\mathbf{C}\cdot(\mathbf{n}\otimes\mathbf{n})]^{-1}. \tag{7.9}$$

It determines the stiffness of a thin fibre (thin bar), cut out from an elastic body, with compliance $\mathbf{C}$, in direction $\mathbf{n}$, under tension $\boldsymbol{\sigma} = s\,\mathbf{n}\otimes\mathbf{n}$.

For an isotropic body $E(\mathbf{n})$ is independent of the direction $\mathbf{n}$, hence E is an invariant – a true elasticity constant, called simply the *Young modulus* of the isotropic material in question. For an anisotropic body one should rather not expect that such a constant exists. More interesting becomes our next off-beat question:

Are there any **anisotropic** *bodies,* $\mathbf{C}^{an} \neq 0$, *having yet the invariant Young modulus, i.e. the bodies with fibres of equal stiffness,*

$$E(\mathbf{n}) = E = \text{const} \qquad \text{for all} \quad \mathbf{n}? \tag{7.10}$$

A closer look at the invariant decomposition (7.4), Part I, and formula (7.9) demonstrarte that the answer is YES! Indeed, the tensor $\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}$ is orthogonal to the part $\mathsf{t} \times (\mathbf{1} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \mathbf{1})$ of the tensor $\mathbf{C}$. Thus, for $E(\mathbf{n}) = \text{const}$ it is sufficient that

$$\mathbf{C} = \mathbf{C}^{is} + \mathsf{t} \times (\mathbf{1} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \mathbf{1})\,. \tag{7.11}$$

A proof of the necessity of this form and a detailed description of this type of anisotropy can be found in [15].

7.9. By using the technique of invariant decompositions of Hooke's tensors we have demonstrated that there are broad classes of **anisotropic** materials of any marked anisotropy, which, in certain conditions, behave just as if they were **isotropic** ones.

This was a deliberate intellectual provocation. Were these thoughts to be elaborated on, a different broader problem could be formulated. This would be *the issue of distinguishability and indistinguishability of the classes of anisotropy in fixed classes of actions.* This immediately leads to another problem of the *choice of anisotropy type* adapted to the prevailing mode of the predicted work of the material being designed. Another group of problems follows from the choice of strategy of experimental *identification* of a Hooke's tensor when there is no preliminary, given *a priori*, information, e.g. information on the structure of the material in question. Finally, as a matter of course, there come several natural ideas of transposing the ideas developed herein onto non-linear elasticity and non-elasticity. Each of these subjects calls for a separate study.

References

1. J. RYCHLEWSKI, *A qualitative approach to Hooke's tensors. Part I*, Arch. Mech., **52**, 4–5, 737–759, 2000.

2. J. RYCHLEWSKI, *Elastic energy decomposition and the limit criteria*, Uspekhy Mekh. (Advances in Mechanics), **7**, 3, 51–80, 1984.

3. J. RYCHLEWSKI, *On Hooke's law*, PMM, **48**, 420–433, 1984, (English translation Prikl. Matem. Mekhan., **48**, 303–314, 1984).

4. J. RYCHLEWSKI, *Unconventional approach to linear elasticity*, Arch. Mech., **47**, 2, 149–171, 1995.

5. J. RYCHLEWSKI, *Zur Abschätzung der Anisotropie*, ZAMM, **65**, 256–258.

6. F. I. FEDOROV, *Theory of elastic waves in crystals*, Nauka, Moscow 1965, (English ed. Plenum Press, New York 1968).

7. S. SUTCLIFFE, *Spectral decomposition of the elasticity tensor*, Trans. ASME J. Appl. Mech., **59**, 762–773, 1992.

8. A. BLINOWSKI, J. RYCHLEWSKI, *Pure shears in the mechanics of materials*, Mat. Mech. of Solids, **4**, 471–503, 1998.

9. S. FORTE, M. VIANELLO, *Symmetry classes for elasticity tensors*, J. of Elasticity, **43**, 81–108, 1996.

10. A. BLINOWSKI, J. OSTROWSKA-MACIEJEWSKA, J. RYCHLEWSKI, *Two-dimensional Hooke's tensors isotropic decomposition, effective symmetry criteria*, **48**, 2, 325–345, 1996.

11. G. BACKUS, *A geometrical picture of anisotropic elastic tensors*, Reviews of Geophysics and Space Physics, **8**, 633–671, 1970.

12. J. RYCHLEWSKI, *Elastic waves under unusual anisotropy*, J. Mech. Phys. Solids, (in print).

13. YU. I. SIROTIN, M. P. SHASKOLSKAYA, *Foundations of Crystal Physics*, Nauka, Moscow 1979 (2d ed.).

14. J. RYCHLEWSKI, *Anisotropic materials with invariant Kirchhoff shear modulus* (in print).

15. J. RYCHLEWSKI, *Anisotropic materials with invariant Young modulus* (in print).

16. J. RYCHLEWSKI, *Immanent properties of elastic isotropy* (in print).

17. J. R. NEIGIBOURS, *An approximation method for determining the elastic constants of single crystals*, J. Acoust. Soc. Am., **26**, 865–869, 1954.

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