# RELIABILITY BASED ELASTO-PLASTIC TOPOLOGY OPTIMIZATION

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**Abstract**
This paper present elasto-plastic topology optimization with reliability constraint. It recalls fundamental concepts from first order reliability analysis and introduces an algorithm for topology optimization of elasto-plastic structures. The presented numerical example shows dependence of the volume fraction on probability of failure.

**Keywords**: *Structural topology optimization, Elastoplastic analysis, Reliability-based optimization*

## 1. Introduction

An important aspect of any optimization process is robustness to variability of structural parameters, either material or loading dependent. The more the structure becomes optimal, the lower the resistance to its parameter changes. One of the possible ways to tackle this issue is to add to the optimization formulation an additional constraint for the probability of failure. The designer will then assure that his or her optimized structure does not go below the assumed safety level. Since the probability of failure of engineering structures must be small (approximately 0.0001), it is possible to obtain a relatively fast estimation of reliability by using first or second order methods i.e. First Order Reliability Method (FORM) or Second Order Reliability Method (SORM). Most often, several iterations (finite element solutions) are enough to obtain convergence. Recent advances in reliability based topology has been presented in paper by Kharmanda et al. 2002 [1], Kim et al. 2009 [2], Kharmanda et al. 2014 [3], Kang and Liu 2018 [4], Chun et al. 2019 [5]. The authors of this paper have also several papers in this topic for more than a decade (Logo et al. 2009 [8], Movahedi Rad et al. 2009 [9], Logo et al. 2011 [10]).
In the present paper iterative topology optimization algorithm together with elasto-plastic material formulation and reliability approach will be shortly described.
Analysis of elasto-plastic structure will be presented on numerical example. All aspect of numerical analysis, finite element formulation, topology optimization as well as reliability analysis library are performed by our own software implemented in MATLAB and C++.

## 1. Theoretical background on reliability analysis

Topological optimization allows to minimize the mass of the structure while maintaining certain mechanical properties. By minimizing the mass, we also reduce the durability of the structure, in terms of random events. In order to maintain control over the reliability in the optimization process, it is necessary to assess also the safety level, together with mechanical properties. These can be: loads, material parameters or shape parameters. Vector of random parameters $\mathbf{x}$ belongs to probabilistic space $\Omega$ (fig.1). Some realizations of random variables $\bar{\mathbf{x}} = \{x_1 \dots x_N\}$ may lead to a topology that does not meet the constraints Such structures are in a state of failure. Constraints therefore divide the probabilistic space into two domains: the safe domain $\Omega_s$ and the failure domain $\Omega_f$ (fig.2). To check in which state given random realization is, limit state function $g(\mathbf{x})$ is introduced. Similarly to the constraints in deterministic optimization there is a convention that a negative value of function $g(\mathbf{x})$ means the $\mathbf{x}$ is in failure domain while positive value means safe domain of $\mathbf{x}$.

$$g(\mathbf{x} \in \Omega_s) > 0 \text{ safe domain}$$

$$g(\mathbf{x} \in \Omega_f) < 0 \text{ failer domain}$$

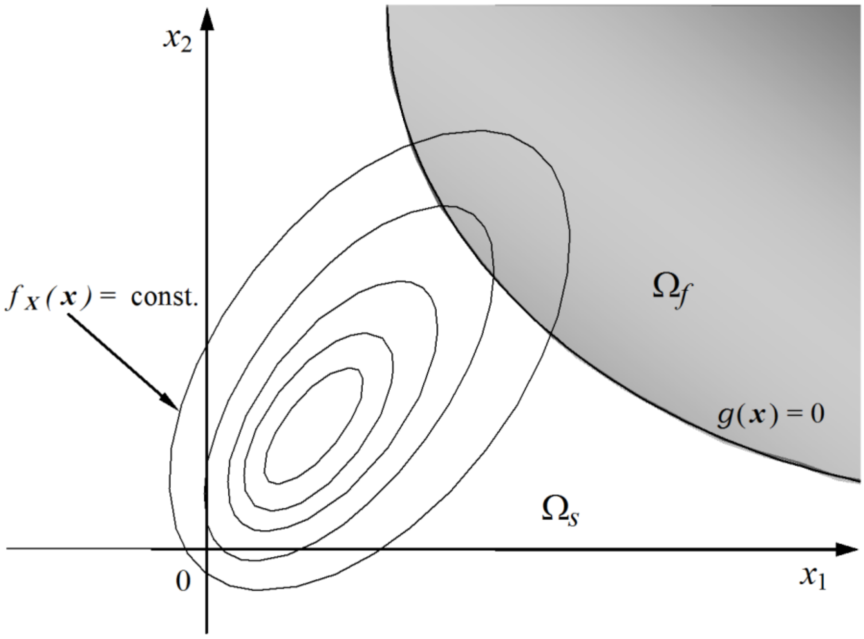



Figure 1. Probabilistic design space.

Equation $g(\mathbf{x}) = 0$ determines limit state surface which separates safe and failure domains (fig 1). Probability of failure can be computed as integration of joint probability distribution over failure domain. It is expressed by following formula:

$$P_f[\mathbf{x} \in \Omega_f] = P[g(\mathbf{x}) < 0] = \int_{g(\mathbf{x})<0} f_X(\mathbf{x})dx \tag{1}$$

Direct computation of probability of failure using above formula is very difficult or even impossible due to an implicit form of a function $g(\mathbf{x})$, which in the case of topology optimization, is based on finite element computations. There are several methods to determine the probability of failure $P_f$. Most versatile one: Monte Carlo (MC) method, allows to asses $P_f$ for any function continuous or discontinuous. Most serious drawback of MC method is numerical complexity. Therefore, combining this method with very complex topological optimization is not an option. Another method: Importance Sampling is less complex, but still it takes an unacceptably long time to determine the probability of failure. Most promising method seems to be First Order Probability Method. (FORM). The foundations for modern methods of reliability analysis were developed by Hasofer and Lind [6]. In their work the concept of the so-called design point or most probable point was introduced.

Design point is the is the realization of random variables with the most probable failure scenario. In other words, it is a point on the limit surface in which the joint probability density function reaches its maximum value (fig 2). Therefore, the neighborhood of this point has greatest contribution in determining the integral (1). Hence, in the case of a small probability of failure, the approximation of the limit surface by the plane will result an acceptable estimation of the probability of failure. In topological optimization, as in the case of engineering structures, the probability of failure should be very small, usually around 0.001. What makes the FORM method a good choice for coupling with topological optimization.

Rackwitz and Fiesler [7], were the authors of the first iterative gradient based algorithm for searching the design point. At the beginning, to facilitate computations, probabilistic design space is transformed to dimensionless standard normal space $\mathbf{x} \to \mathbf{u}$. The design point is the closest point of limit state surface to the center of the coordinate system of standard normal space. So its determination is in fact an optimization task formulated as follows:

$$\text{Find} \qquad \min\|\mathbf{u}^2\| = \mathbf{u}\cdot\mathbf{u}^T, \tag{2}$$

$$\text{constrains} \qquad g(\mathbf{u}) = 0. \tag{3}$$

The iterative Rackwitz-Fiesler formula has the following form:

$$\mathbf{u}^{(n+1)} = \frac{1}{\|\nabla g(\mathbf{u}^{(n)})\|^2}\left(\nabla g\left(\mathbf{u}^{(n)}\right)^T\mathbf{u}^{(n)} - g\left(\mathbf{u}^{(n)}\right)\right)\nabla g(\mathbf{u}^{(n)}). \tag{4}$$

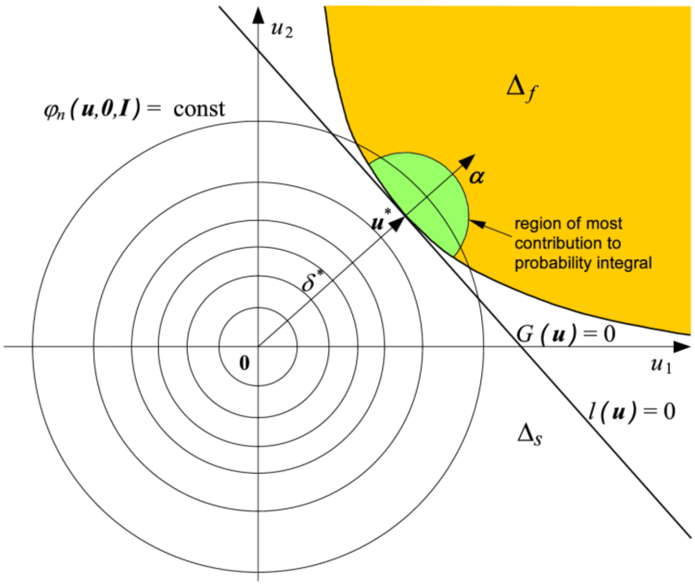



Figure 2. First Order Reliability Method. Design point concept.

Having a design point $\mathbf{u}^*$ it is easy to use a linear estimation of the probability of failure. We assume that approximate limit surface is a hyperplane and is tangent to the limit surface at the design point (fig 2). Thus, the hyperplane equation has the form:

$$l(\mathbf{u}) = -\alpha\mathbf{u} + \beta, \tag{5}$$

where $\alpha$ is a unit vector with the direction opposite to the gradient $\nabla g(\mathbf{u})$ at the design point $\mathbf{u}^*$

$$\alpha = \frac{\nabla g(\mathbf{u})}{\|\nabla g(\mathbf{u})\|}\bigg|_{\mathbf{u}=\mathbf{u}^*}, \tag{6}$$

and $\beta$ is an Hasofer-Lind's reliability index. The linear estimation of the probability of failure has the form:

$$P_{FORM} = \Phi(-\beta), \tag{7}$$

$$\beta_{FORM} = \text{sign}\left(g(\mathbf{0})\right)\|\mathbf{u}^*\|. \tag{8}$$

## 3. Topology optimization approach

In the case of topology optimization random variables vector $\mathbf{x}$ can be composed of loads or material constants. Shape of the structure is result of topology optimization therefore random nature of shape parameter is not taken into consideration. Now we present complete topology optimization with reliability constrains in the form of algorithm and flowchart (see next page):

**Algorithm 2.** *Topology optimization of elastoplastic structures with reliability constrains.*

***Step 1***. Initialize *design variables* to a vector of ones $\boldsymbol{\rho}_{\boldsymbol{e}}^{(0)} = \{1, 1, \dots, 1\}$ and *erased element list* to an empty list $\mathcal{L} = \{\}$. Initialize random variables vector $\mathbf{x} = \{x_1 \dots x_N\}$.
Until the load capacity or displacement condition is exceeded repeat ***Steps 2 to 7***.

***Step 2***. At every $k$-th iteration solve nonlinear equilibrium equations for the elastoplastic problem

$$\mathbf{K}\left(\boldsymbol{\rho}_{\boldsymbol{e}}^{(k)}\right)\mathbf{u}\left(\boldsymbol{\rho}_{\boldsymbol{e}}^{(k)}\right) - \mathbf{f} = \mathbf{0}.$$

***Step 3***. Determine the stress intensity vector calculated as the average of equivalent von Mises stresses evaluated at each Gauss point, then normalize the obtained values dividing them by the yield limit

$$\bar{\sigma}_i = \frac{1}{N_g\,\sigma_0} \sum_{g=1}^{N_g} \sigma_{i,g}^{g}, \qquad i = 1,2,\dots,N .$$

***Step 4***. Apply a design filter to avoid the checkerboard phenomenon

$$\{\bar{\sigma}_i\}_{\text{filter}} = \frac{1}{\rho_i^{(k)} \sum_{j=1}^{N} H_j} \sum_{j=1}^{N} H_j \rho_j^{(k)} \bar{\sigma}_j$$

where $H_j$ is the convolution operator $H_j = r_{\min} - \|\boldsymbol{x}_i - \boldsymbol{x}_j\|$ and $j$ belongs to the set of elements for which the distance from the center of the $i$-th element is smaller than the filter radius $r_{\min}$.

***Step 5***. Perform FORM to assess probability of failure:

$$P_{FORM} = \Phi(-\beta)$$

If structure is not reliable ($P_{FORM} > P_{Limit}$) then stop.

***Step 6***. Select $n$ finite elements with the smallest stress intensities $\bar{\sigma}_e < \bar{\sigma}_{min} + \bar{\sigma}_t$ (usually $\bar{\sigma}_t = 0.005$) and add the list of the newly selected elements $\ell$ to the list of previously erased elements, $\mathcal{L}^{(k)} = \{\mathcal{L}^{(k-1)};\ \ell\}$.

***Step 7***. Using the current list of erased elements $\mathcal{L}$ update corresponding design variables applying the following iterative formula:

$$\rho_l^{(k)} = \max_{l \notin \mathcal{L}}\left(\rho_{\min}, [\{\bar{\sigma}_l\}_{\text{filter}}]^p\, \rho_l^{(k-1)}\right).$$

## 4. Numerical example

Reliability assessment in topological optimization will be illustrated on simple benchmark example presented in Fig 3. The examples deals with a simple cantilever with a force applied at the free end. Regular rectangular mesh composed with Lagrange four-node finite elements was used in the example. Mesh dimension is 40x20 elements. Aluminum was chosen as the material with following parameters: Young's modulus $E = 71$ GPa, Poisson's ratio $v = 0.11$, yield stress $\sigma_0 = 260$MPa, thicknes $h = 0.22$units.

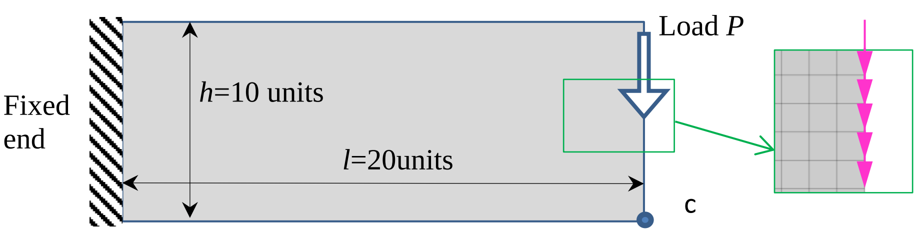
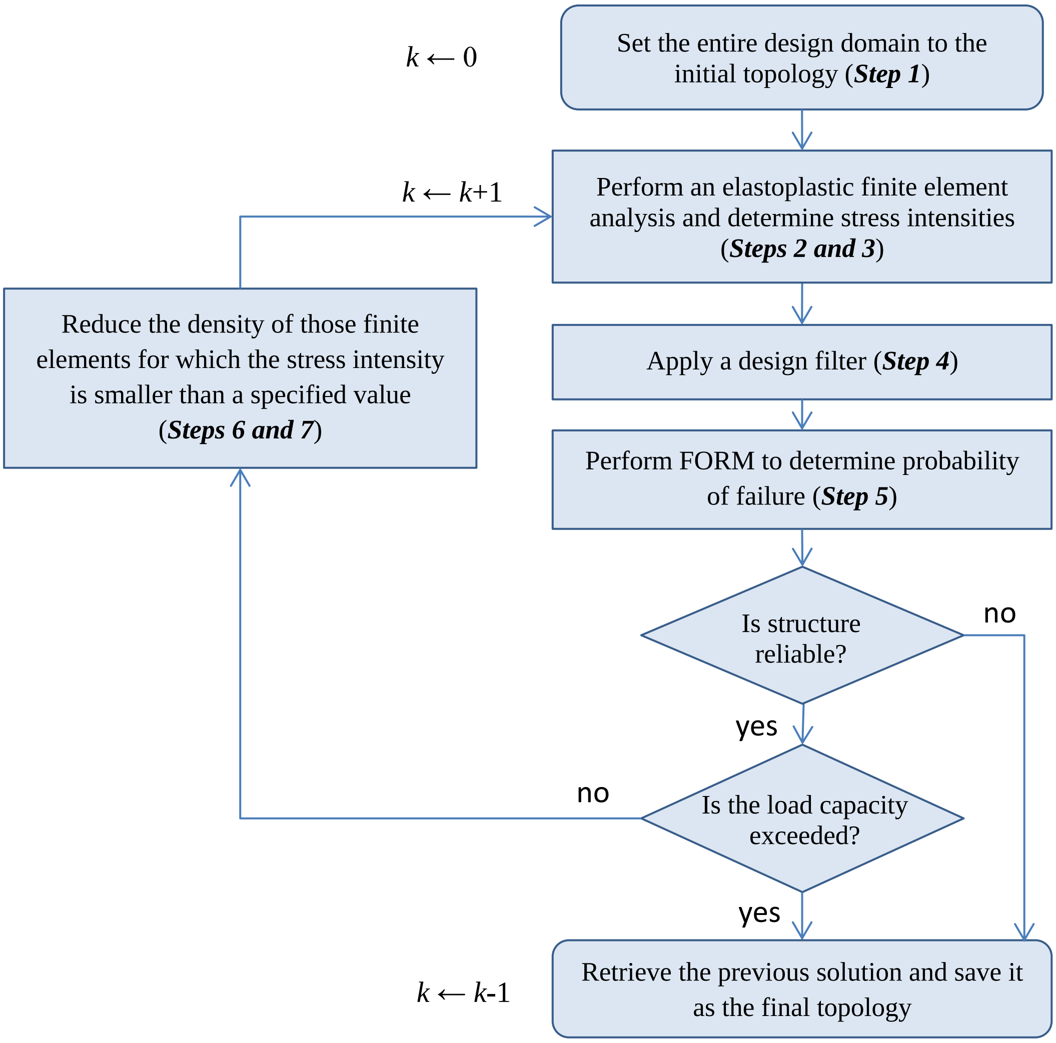



Figure 3. Design domain for cantilever under investigation.




Flowchart of the proposed method for topology optimization under reliability constraint.

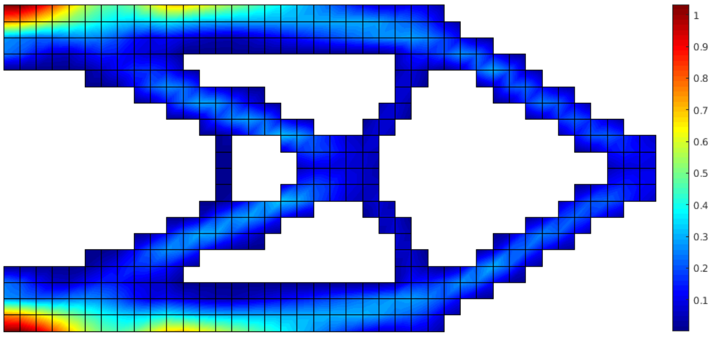



Figure 4. Optimal topology with reliability constraints.

Probabilistic nature is represented by three random variables Young's modulus, yield stress and Poisson's ratio. Distribution parameters of random variables displayed in Table 1.

| variable | mean value | standard deviation |
|---|---|---|
| Young's modulus | 71GPa | 5% |
| Yield stress | 260MPa | 5% |
| Poisson's ratio | 0.11 | 10% |

Table 1. Probabilistic properties of material parameters.

Limit state function reflects displacement condition. Unsafe state means the displacement in point c (see fig 4) $u_c$ exceeds permissible value 0.1m ($l$/200), so the limit state function is specified by the formula:

$$g(\mathbf{u}) = 0.1 - u_c. \tag{9}$$

In the Figure 5 dependence of the volume fraction on probability of failure has been presented. iteration

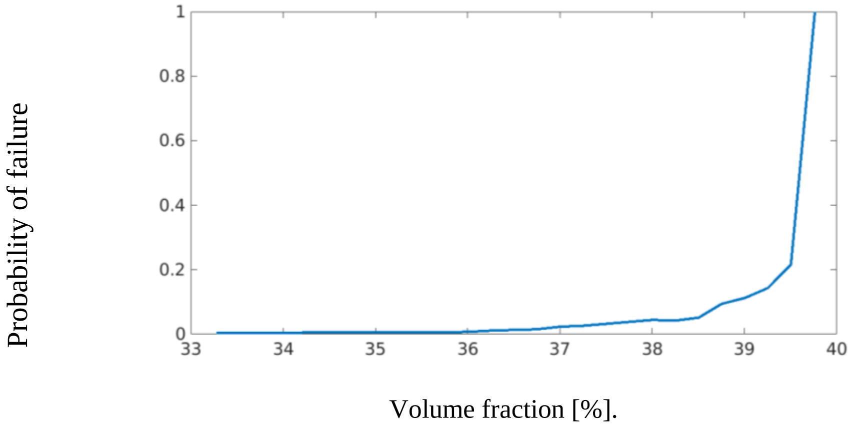



Figure 5. Probability of failure as a function of volume fraction.

## 3. Conclusions

In the present paper topology optimization with reliability constraint has been formulated. The final topology of a cantilever with reliability constraint is shown Fig.4. Additionally, in Fig 5. it can be observed that above certain level of volume fraction the probability of failure rapidly grows.

## Acknowledgments

The present study was supported by the National Research, Development and Innovation Office (Grant K 119440) and by the joint grant of the Hungarian and the Polish Academy of Sciences.

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