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Disclination kinematics

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A MATHEMATICAL theory of moving disclinations is developed. Kinematics is derived for a continuous distribution of disclinations and dislocations as well as for moving discrete disclination and dislocation lines. The concept of the plastic velocity is used to give the theory a symmetrical form. The new concepts of disclination and dislocation loop currents are introduced. The relation between the disclination theory and the incompatibility theory is given.

Praca poświęcona jest matematycznej teorii ruchomych dysklinacji. Przedstawiona jest kinematyka ciągłego rozkładu dysklinacji i dyslokacji oraz ruchomych pojedynczych linii dysklinacji i dyslokacji. Wykorzystuje się pojęcie prędkości plastycznej, co nadaje teorii symetryczną formę. Wprowadza się pojęcie prądu pętli dysklinacji i dyslokacji. Przeprowadza się porównanie teorii dysklinacji i teorii niezgodności.

Работа посвящена математической теории подвижных дисклинаций. Представлена кинематика непрерывного распределения дисклинаций и дислокаций, а также подвижных единичных линий дисклинаций и дислокаций. Используется понятие пластической скорости, что придает теории симметричную форму. Вводится понятие тока петли дисклинации и дислокации. Проводится сравнение теории дисклинаций с теорией несовместности.

1. Introduction

THIS ARTICLE develops disclination kinematics. We introduce kinematical quantities which appear in the theory of moving disclinations and dislocations in a linearly elastic, continuous medium. We refer to the results of dislocation dynamics [1, 2, 3, 4, 5, 6, 7, 8, 29], the theory of stationary disclinations [9, 10, 11, 12, 13, 14, 15], the dynamic theory of incompatibility [16], the theory of disclinations in the Cosserat-continuum [17] and the four-dimensional theory of disclinations [18, 19, 20, 21].

Disclination dynamics will be published independently [31].

By "defects" we shall mean the combination of dislocations and disclinations. We first give the compatibility conditions for the classical dynamic elasticity theory in terms of the basic total fields: the strain, bend-twist, and velocity. However, the constitutive equations relate the stress only to the basic elastic fields, which do not necessarily satisfy the compatibility conditions. The difference between the total and elastic fields gives the plastic or stress-free fields. These do not satisfy the compatibility conditions either, and the deviation from compatibility logically provides the definitions for the defect densities and their currents. The Burgers and Frank vectors are defined as in the static theory.

Section 5 treats the moving discrete defect line. A procedure is developed to find the basic plastic fields. The analysis is facilitated by introducing the concepts of defect loop densities and their currents in terms of which the basic plastic fields are conveniently expressed. These expressions are all surface integrals. The defect densities and their currents then follow by a straightforward derivation from the equations in Sect. 4 as line integrals along the defect line.

In Sect. 6 we show how the incompatibility tensor and its current are related to disclination kinematics.

Throughout the development of this paper we find that many concepts or quantities from dislocation kinematics generalize into pairs in disclination kinematics. For example, dislocations generalize to defects consisting of dislocations and disclinations, and dislocation current generalizes to defect current. We have found it useful to introduce the new concept of "basic fields" consisting of the strain, bend-twist, linear and rotational velocity. Then the distortion and velocity of dislocation kinematics generalize to the basic fields of disclination kinematics. These ideas are summarized in Tables 1 and 2.

Table 1. Defect densities and currents in disclination kinematics

Quantity	Dislocation		Disclination	
	Density	Current	Density	Current
Defects	α_{pl}	J_{kl}	θ_{pq}	S_{kq}
Loops	β^{*}_{kl}	v^{*}_{l}	ϕ^{*}_{kq}	ψ^{*}_{q}

Table 2. Basic fields in various stages of defect theory

	Dislocation	Disclination
Statics	β_{kl}	$e_{kl},\ \varkappa_{mq}$
Dynamics	$\beta_{kl},\ v_l$	$e_{kl},\ \varkappa_{mq},\ v_l,\ w_q$

2. The plastic strain problem

2.1. The plastic velocity

Before stating the general problem of the dynamic plastic strain, we make a digression here to introduce and motivate a new concept, the plastic velocity v_l^P, due to KOSSECKA [28].

For this purpose, we anticipate some of the material covered in Sect. 3.2. Kossecka defined a defect by a displacement field u_l^T, which has a jump discontinuity on the defect surface $S(t)$. It is then found that the spatial derivative of the displacement, i.e. the total distortion β_{kl}^T, has a singularity on this surface. The singular part of the total distortion is found to be the plastic distortion β_{kl}^P. For a discrete dislocation line it is given explicitly by the well-known Eq. (3.25) in Sect. 3.2, where the dislocation is the boundary of $S(t)$. It is well known for the static case that the elastic fields of a dislocation in an infinite body do not depend on the specific location of the defect surface, but only that its boundary coincides with the dislocation line [14].

Kossecka found similarly that the time derivative of the displacement, i.e. the total velocity v_l^T, has a singularity on the defect surface $S(t)$ and called the singular part the plastic velocity v_l^P. For a discrete dislocation line it is given explicitly by Eq. (3.26) in Sect. 3.2.

Furthermore in the dynamic case it is found that the dislocation density and current depend only on the particular location of the dislocation line and not the defect surface (viz. (3.28)–(3.29)). In other words, the defect surface $S(t)$ can be chosen arbitrarily except for its boundary which must coincide with the dislocation line.

2.2. The elastic and plastic fields

For a given displacement $u_l^T(\mathbf{r}, t)$, which is a function of space and time, the total distortion and velocity are defined by

$$\beta_{kl}^T = u_{l,k}^T, \tag{2.1}$$

$$v_l^T = \dot{u}_l^T. \tag{2.2}$$

Here the comma denotes the spatial derivative and the dot the time derivative.

In general when defects are present, in the elastic medium, the total deformation is not completely elastic, but a part of it is plastic. Therefore, we generalize the results Kossecka found for the special case discussed in Sect. 2.1, i.e. we assume that the total distortion and velocity consist of an elastic and plastic part

$$\beta_{kl}^T = \beta_{kl} + \beta_{kl}^P, \tag{2.3}$$

$$v_l^T = v_l + v_l^P. \tag{2.4}$$

The plastic distortion β_{kl}^P is a well-known concept in the dislocation theory. The existence of a plastic velocity v_l^P has not been so widely recognized. However, at this point it can be argued that in an elasto-plastic deformation a part of the relative motion between neighbouring points of the medium may be of a plastic nature. Therefore, the plastic velocity is introduced for completeness. As it is already apparent from Eqs. (2.3) and (2.4) and as we shall see further on, this quantity lends a certain amount of symmetry to the dynamic equations. In this sense its introduction is in the same spirit as other fundamental theories of physics where symmetry considerations have often motivated the full development of the theory.

The strain is the symmetric part of the distortion. Hence, we have for the symmetric part of Eqs. (2.1) and (2.3) the following equations:

$$e_{kl}^T = u_{(l,k)}^T, \tag{2.5}$$

$$e_{kl}^T = e_{kl} + e_{kl}^P. \tag{2.6}$$

Here the parentheses on the subscripts indicate that the symmetric part is to be taken with respect to k and l.

3. Review of dislocation kinematics in the linearly elastic continuous medium

3.1. Continuous distribution of dislocations

For a continuous simply-connected body that undergoes an elasto-plastic deformation without breaking, the total deformation satisfies the classical conditions of compatibility. These conditions are a consequence of the fact that a total displacement function u_l^T can be defined for every point of the continuous body at any time. Hence, in the mathematical sense they are integrability conditions. These conditions can be formulated in a variety of ways. We shall show three forms for them in this paper, one below for application to dislocations, a second in Sect. 4.1 for disclinations, and a third in Sect. 6.1 for the incompatibility. In the present section the compatibility conditions are formulated in terms of the *total distortion* β_{kl}^T and the *total velocity* v_l^T, which could be called the ***basic total fields*** for dislocation dynamics. These quantities are in general defined by

(3.1) $$\beta_{kl}^T = u_{l,k}^T,$$

(3.2) $$v_l^T = \dot{u}_l^T,$$

for a given displacement. Conversely, if Eqs. (3.1) and (3.2) are regarded as partial differential equations for u_l^T, then it can be shown that the necessary and sufficient conditions to assure the existence of a continuous single-valued solution for u_l^T throughout the body is that the relations

(3.3) $$\varepsilon_{pmk}\beta_{kl,m}^T = 0,$$

(3.4) $$v_{l,k}^T - \dot{\beta}_{kl}^T = 0,$$

are satisfied everywhere in the body. These relations are the compatibility conditions expressed in terms of the total distortion and velocity for the dynamic case.

Though dislocations and their currents have traditionally been defined entirely in terms of kinematical concepts, their physical significance does not become evident until their relation to dynamics is considered. The relation between kinematics and dynamics is established by the constitutive equations which, in the case of the linear elasticity theory, are the Hooke's law and the equation of motion. They relate the stress σ_{ij} to elastic fields. In classical or compatible elasticity the total and the elastic fields are one and the same, i.e. the deformation is purely elastic. However, when dislocations and their currents are present the total fields exceed the elastic ones by amounts which are stress-free or plastic:

(3.5) $$\beta_{kl}^T = \beta_{kl} + \beta_{kl}^P,$$

(3.6) $$v_l^T = v_l + v_l^P.$$

These relations correspond to Eqs. (1.5) and (2.5) in ref. [28]. In general the elastic or plastic fields do not satisfy compatibility conditions of the type in Eqs. (3.3) and (3.4) separately, i.e. they are not derivable from displacement funtcions. The plastic fields may be introduced arbitrarily into the body in which case they determine the defects and their currents, whereas the elastic fields are subject to the constitutive equations. If the plastic distortion is given, then the *dislocation density* tensor is defined by

(3.7) $$\alpha_{pl} \equiv -\varepsilon_{pmk}\beta_{kl,m}^P.$$

If in addition the plastic velocity is also given, then the *dislocation current* tensor is defined by

$$J_{kl} \equiv -v^P_{l,k} + \dot{\beta}^P_{kl}. \tag{3.8}$$

In other words, the dislocation density and current can be regarded as the deviation of the plastic deformation from compatibility. The definition (3.7) was first given by Kröner [23]. Kosevich [3] and Teodosiu [8] called J_{kl} the "dislocation flux" and gave an expression like Eq. (3.8), but without the term involving the plastic velocity v^P_l. Mura [4] has introduced a different tensor, V_{mnl}, which he called the "dislocation-flux tensor". It is related to our result by the relation

$$J_{kl} = -\varepsilon_{kmn} V_{mnl}. \tag{3.9}$$

We shall not need this tensor in our subsequent work, but Eq. (3.9) will allow us to show the connection with Mura's work. The continuity equations for dislocations

$$\alpha_{pl,p} = 0, \tag{3.10}$$

$$\dot{\alpha}_{pl} + \varepsilon_{pmk} J_{kl,m} = 0, \tag{3.11}$$

follow directly from Eqs. (3.7) and (3.8). Relation (3.10) is by now well known and implies that dislocations cannot end inside the body. Relation (3.11) was first given by Holländer [1] and shows that the dislocation density can only change by a dislocation current, i.e. by the motion of dislocations. Several other authors have also given this equation, but in different forms, which have been compared with each other by Günther [5]. This is Eq. (3.9) of Kossecka [7], except that in the present work we have reversed the sign in the definition of J_{kl} from that of Ref. [7], in keeping with the older literature.

The field equations for dislocations

$$\varepsilon_{pmk} \beta_{kl,m} = \alpha_{pl}, \tag{3.12}$$

$$v_{l,k} - \dot{\beta}_{kl} = J_{kl}, \tag{3.13}$$

follow from Eqs. (3.3) to (3.9). The fundamental meaning of these equations is that the dislocation density and current are the sources of elastic distortion and velocity. Equation (3.12) is well-known from Kröner's work [23]. Equation (3.13) has also been given by Holländer [1], Kosevich [2, 3], Teodosiu [8], and Amari [6]. In the four-dimensional non-Riemannian formulation it has been given by Simmons [29] with his Eq. (3.14). It corresponds to Eq. (3.7) in Ref. [7].

We define a *Burgers circuit* as any closed curve λ inside the body. For generality the position of this curve could be a function of time $\lambda(t)$. The *Burgers vector* associated with this Burgers circuit is then defined for a given plastic distortion by the closed line integral

$$b_l(t) \equiv -\oint_{\lambda(t)} \beta^P_{kl}(\mathbf{r}, t)\, dL_k. \tag{3.14}$$

The minus sign in this definition is introduced to conform to sign convention *A* [24] or FS/RH [25] for the Burgers vector. It is clear that the Burgers vector is a function of time, both because the integrand β^P_{kl} explicitly depends on time and because the Burgers circuit

$\lambda(t)$ may move with time. By Stokes' theorem (see Appendix A in ref. [14]) and Eq. (3.7) we can also write

(3.15) $$b_l(t) = \int_{\sigma(t)} \alpha_{pl}(\mathbf{r}, t)\, dS_p,$$

where the Burgers surface $\sigma(t)$ is any open surface inside the body bounded by $\lambda(t)$. It follows from Eqs. (3.5), (3.1) and (3.14) that

(3.16) $$\oint_{\lambda(t)} \beta_{kl}(\mathbf{r}, t)\, dL_k = b_l(t).$$

We see that the above relations for the Burgers vector are unchanged from the stationary case, i.e. Eqs. (3.7), (3.8) and (3.11) of Ref. [14]. The rate of change of the Burgers vector is

(3.17) $$\dot{b}_l(t) = -\oint_{\lambda(t)} [J_{kl}(\mathbf{r}, t) - \varepsilon_{pmk}\alpha_{pl}(\mathbf{r}, t) v'_m(\mathbf{r}, t)]\, dL_k.$$

This relation can be derived from Eqs. (3.14) or (3.16) by applying (A3) (see Appendix), using Eqs. (3.7) and (3.8) or (3.12) and (3.13) and noting that certain terms vanish on integrating around a closed circuit. Here v'_m is the velocity of the Burgers circuit $\lambda(t)$. The first term in Eq. (3.17) represents the loss in the Burgers vector due to the dislocation current out of the Burgers circuit, and the second term represents the gain in the Burgers vector due to the new dislocations that are encircled by the Burgers circuit due to its motion. From Eq. (3.15) and (A2) we have, alternatively,

(3.18) $$\dot{b}_l = \int_{(\sigma)t} [(\dot{\alpha}_{pl} + (\alpha_{pl} v'_m)_{,m})\, dS_p - \alpha_{pl} v'_{m,p}\, dS_m].$$

By Eqs. (3.10), (3.11) and Stokes' theorem this relation is equivalent to Eq. (3.17).

Consider now the case where $\sigma(t)$ is a closed surface. Then we have for Eq. (3.15)

(3.19) $$b_l = \oint_{\sigma(t)} \alpha_{pl}\, dS_p = \int_{V(t)} \alpha_{pl,p}\, dV = 0,$$

by the divergence theorem (Appendix A in [14]) and Eq. (3.10). Of course the same result can also be obtained from Eq. (3.14) since when the surface σ closes, the boundary line λ shrinks to zero. Similarly, the rate of change of the Burgers vector, Eq. (3.17), vanishes for a closed surface σ, and this can also be shown from Eq. (3.18) by using the divergence theorem and Eq. (3.10). The above result shows that the continuity equation for α_{pl} implies conservation of the Burgers vector.

Finally, for later use we note that the plastic strain is the symmetric part of the plastic distortion:

(3.20) $$e^P_{kl} = \beta^P_{(kl)},$$

i.e. the same relationship holds for the total strain and distortion, cf. Eqs. (2.1) and (2.5).

3.2. The discrete dislocation line

The discrete dislocation line $L(t)$ which may change its position with time t is defined as the boundary of a surface $S(t)$, where the material below S has been plastically displaced with respect to the material above S by a constant amount given by the Burgers vector b_l.

Hence the difference between the displacement just below and above $S(t)$ is given by

$$[u_l(\mathbf{r}, t)] = -b_l. \tag{3.21}$$

Now, if we can find the corresponding plastic distortion and velocity, then the equations in Sect. 3.1. can be used to find all the desired results for kinematics. We use the straightforward procedure that was developed before [14] to find these quantities. Let us first assume that $S(t)$ is closed, enclosing a volume $V(t)$ which is smoothly changing with respect to time. Then

$$u_l^T(\mathbf{r}, t) = \int_{V(t)} \delta(\mathbf{R}) b_l dV' \tag{3.22}$$

represents a displacement that is constant and equal to b_l inside $V(t)$ and that has the jump, Eq. (3.21), at the surface $S(t)$. By Eq. (3.1) this displacement leads to the following distortion

$$\beta_{kl}^T(\mathbf{r}, t) = \int_{V(t)} \delta_{,k}(\mathbf{R}) b_l dV' = -\int_{V(t)} \delta_{,k'}(\mathbf{R}) b_l dV' = -\oint_{S(t)} \delta(\mathbf{R}) b_l dS_k', \tag{3.23}$$

where we have used the divergence theorem (see Appendix A in Ref. [14]). By Eq. (3.2) and (A1) we find that Eq. (3.22) leads to the following velocity:

$$v_l^T(\mathbf{r}, t) = \oint_{S(t)} \delta(\mathbf{R}) b_l v_k'(\mathbf{r}', t) dS_k', \tag{3.24}$$

where v_k' is the velocity of the surface $S(t)$. These results reveal that the distortion and velocity are concentrated at the surface $S(t)$. We now simply generalize these results to the open surface $S(t)$ used in the definition of the dislocation loop above, and furthermore assume they are the basic plastic fields.

In other words, we postulate that the plastic distortion and velocity are given by

$$\beta_{kl}^P(\mathbf{r}, t) = -\int_{S(t)} \delta(\mathbf{R}) b_l dS_k', \tag{3.25}$$

$$v_l^P(\mathbf{r}, t) = \int_{S(t)} \delta(\mathbf{R}) b_l v_k'(\mathbf{r}', t) dS_k', \tag{3.26}$$

for a moving discrete dislocation line. It can be shown that these expressions give the jump condition (3.21) [28]. These relations correspond to Eqs. (5.17) of ref. [7], and (1.6) and (2.6) of Ref. [28]. By the methods of Appendix B of Ref. [14], it is easily shown that (3.25) is consistent with (3.14), just as for stationary dislocations [14]. Relation (3.26) further clarifies the meaning of the plastic velocity. For a discrete loop its direction is the same as the direction of the Burgers vector and its magnitude is determined by the component of the surface velocity v_k' normal to the surface dS_k'. If $S(t)$ changes in time only by the motion of its boundary, then $v_k' dS_k' = 0$, then the plastic velocity vanishes. This has usually been assumed in the literature. We note from Eqs. (3.25) and (3.26) that

$$v_l^P = -v_k' \beta_{kl}^P, \tag{3.27}$$

i.e. the plastic velocity and distortion are not completely independent for a discrete dislocation line. This contrasts with a continuous distribution where v_l^P and β_{kl}^P may be prescribed in a completely arbitrary manner. The reason for this is that for a discrete dislocation line we have introduced a restriction, namely the constancy of the Burgers vector of the dislocation line. This means that the strength of the dislocation remains constant as it moves. On the other hand, for a continuous distribution the strength of the dislocation density may be varied independently of its motion. We next find the dislocation density from Eqs. (3.7) and (3.25),

$$\text{(3.28)} \quad \alpha_{pl}(\mathbf{r}, t) = \int_{S(t)} \varepsilon_{pmk}\delta_{,m}(\mathbf{R}) b_l dS'_k = -\int_{S(t)} \varepsilon_{pmk}\delta_{,m'}(\mathbf{R}) b_l dS'_k = \oint_{L(t)} \delta(\mathbf{R}) b_l dL'_p,$$

where we have used Stokes' theorem (see Appendix A in Ref. [14]). The dislocation current is found from Eqs. (3.8), (3.25) and (3.26),

$$\text{(3.29)} \quad J_{kl}(\mathbf{r}, t) = -\int_{S(t)} \delta_{,k}(\mathbf{R}) b_l v'_p(\mathbf{r}', t) dS'_p + \oint_{L(t)} \varepsilon_{pmk}\delta(\mathbf{R}) b_l v'_m(\mathbf{r}', t) dL'_p$$
$$- \int_{S(t)} \delta_{,k'}(\mathbf{R}) b_l v'_p(\mathbf{r}', t) dS'_p = \oint_{L(t)} \varepsilon_{pmk}\delta(\mathbf{R}) b_l v'_m(\mathbf{r}', t) dL'_p,$$

where we have used (A2). In this relation v'_m represents the velocity of the dislocation line $L(t)$. The expressions (3.28) and (3.29) have also been given by TEODOSIU [8]. The above relation is the combination of Eqs. (2.28) and (2.23) in Ref. [7]. We note that for a discrete dislocation line

$$\text{(3.30)} \quad J_{kl} = \varepsilon_{pmk}\alpha_{pl} v'_m.$$

This relation was also given by TEODOSIU [8]. It is easy to verify that Eqs. (3.28), (3.29) satisfy the continuity conditions (3.10), (3.11). Also it is easily shown by the methods of Appendix B of ref. [14] that Eq. (3.28) is consistent with Eq. (3.15).

3.3. The dislocation loop current

It is well known that the plastic distortion β^P_{kl} can be identified with the dislocation loop density tensor [14]. This means that a distribution of dislocations can always be represented by a continuous distribution of infinitesimal dislocation loops.

We wish to show in this section that a similar type of interpretation can also be given to the plastic velocity v^P_l. First we note from Eq. (3.8) that we can write

$$\text{(3.31)} \quad \int_V \dot{\beta}^P_{kl} dV = \oint_S v^P_l dS_k + \int_V J_{kl} dV,$$

by the divergence theorem where V is an arbitrary volume enclosed by the surface S.

We now investigate the last term in this equation. For a discrete dislocation line it is given by

$$\text{(3.32)} \quad \int_V J_{kl}(\mathbf{r}, t) dV = \oint_{L(t)} \varepsilon_{pmk} b_l v'_m(\mathbf{r}', t) dL'_p,$$

from Eq. (3.29) where the line integration is only over that part of the curve $L(t)$ which lies inside V. An infinitesimal loop will in general fall completely inside V, so that it is indeed a closed line integral.

If the loop moves as a rigid unit, i.e. when $v'_m(\mathbf{r}', t)$ is constant with respect to $\mathbf{r}'$ around the line $L(t)$, the line integral vanishes because the integrand is then a constant with respect to the integration. Note that this result remains valid even when $v'_m(\mathbf{r}', t)$ changes with respect to time. Hence Eq. (3.32) vanishes if the dislocation loop retains its original character, i.e. it does not change its size, shape, and orientation. The same result is assumed to hold for a continuous distribution of moving infinitesimal dislocation loops. Now the left hand side of Eq. (3.31) gives the rate of change of dislocation loop density inside V. Hence the right hand side must represent the dislocation loop current crossing the boundary S of V.

So the plastic velocity v_l^P can be identified with the density of the *dislocation loop current*.

If the above restrictions on Eq. (3.32) are relaxed, i.e. if the dislocation loop is allowed to change its size, shape, or orientation, then Eq. (3.32) represents the dislocation loop creation inside V. Hence the dislocation current J_{kl} can also be interpreted as a source function for dislocation loop creation. In this case v_l^P still represents the dislocation loop current, even though Eq. (3.32) does not vanish. Equation (3.31) then shows in general that the rate of change of dislocation loop density inside the volume V is due to the dislocation loop current crossing its surface S plus the creation of dislocation loop density inside V.

4. Continuous distribution of moving defects

As mentioned in the Introduction, we shall denote the combination of dislocations and disclinations by the word *defects* in the remainder of this paper.

1. The compatibility equations

In this section we shall formualte the compatibility conditions for the total deformation in terms of the *basic total fields* by which we mean the *strain* e_{kl}^T, *bend-twist* $\varkappa_{mq}^T$, *linear velocity* v_l^T, and *rotational velocity* w_q^T. For a given total displacement u_l^T these quantities are defined by

(4.1) $$e_{kl}^T = u_{(l,k)}^T,$$

(4.2) $$\varkappa_{mq}^T = \omega_{q,m}^T = 1/2\varepsilon_{klq}u_{l,km}^T,$$

(4.3) $$v_l^T = \dot{u}_l^T,$$

(4.4) $$w_q^T = \dot{\omega}_q^T = 1/2\varepsilon_{klq}\dot{u}_{l,k}^T.$$

On the other hand, if Eqs. (4.1) to (4.4) are regarded as partial differential equations for u_l^T, then it is easy to show that the necessary and sufficient conditions to assure the existence of a continuous single-valued solution for u_l^T is that the relations

(4.5) $$\varepsilon_{pmk}(e_{kl,m}^T + \varepsilon_{klq}\varkappa_{mq}^T) = 0,$$

(4.6) $$\varepsilon_{pmk}\varkappa_{kq,m}^T = 0,$$

(4.7) $$v_{l,k}^T - \dot{e}_{kl}^T - \varepsilon_{klq}w_q^T = 0,$$

(4.8) $$w_{q,k}^T - \dot{\varkappa}_{kq}^T = 0,$$

are satisfied everywhere in the body. These are the compatibility conditions in terms of the basic total fields for the dynamic case.

4.2. The defect density and current tensors

If defects and their currents are present then the basic total fields exceed the elastic ones by amounts which are stress-free or plastic:

$$e_{kl}^T = e_{kl} + e_{kl}^P, \tag{4.9}$$

$$\varkappa_{kq}^T = \varkappa_{kq} + \varkappa_{kq}^P, \tag{4.10}$$

$$v_l^T = v_l + v_l^P, \tag{4.11}$$

$$w_q^T = w_q + w_q^P. \tag{4.12}$$

This occurs in a general elasto-plastic deformation. In general the basic elastic or plastic fields do not satisfy the compatibility conditions separately, i.e. they are not derivable from a displacement function. The basic plastic fields may then be introduced arbitrarily into the body, in which case they determine the defect density and current in the body. The *dislocation density* α_{pl} and the *disclination density* θ_{pq} are then defined by

$$\alpha_{pl} \equiv -\varepsilon_{pmk}(e_{kl,m}^P + \varepsilon_{klq}\varkappa_{mq}^P), \tag{4.13}$$

$$\theta_{pq} \equiv -\varepsilon_{pmk}\varkappa_{kq,m}^P, \tag{4.14}$$

while the *dislocation current* J_{kl} and the *disclination current* S_{kq} are defined by

$$J_{kl} \equiv -v_{l,k}^P + \dot{e}_{kl}^P + \varepsilon_{klq} w_q^P, \tag{4.15}$$

$$S_{kq} \equiv -w_{q,k}^P + \dot{\varkappa}_{kq}^P. \tag{4.16}$$

In other words, the defect densities and currents measure the deviation of the basic plastic fields from compatibility. The definitions (4.13) and (4.14) have been given before [13, 14]. KLUGE'S [21] formulation in terms of the Cosserat-continuum gave relations similar to Eqs. (4.15) and (4.16), but without the plastic velocity term v_l^P. Moreover, the author misinterpreted S_{kl} as the foreign atom current. Also note that the relation (4.15) can easily be derived from Eq. (3.8), while Eq. (4.16) can be written down in analogy to Eq. (3.8). Relations (4.13) to (4.16) lead directly to the continuity equations for the defects

$$\alpha_{pl,p} + \varepsilon_{lpq}\theta_{pq} = 0, \tag{4.17}$$

$$\theta_{pq,p} = 0, \tag{4.18}$$

$$\dot{\alpha}_{pl} + \varepsilon_{pmk}(J_{kl,m} + \varepsilon_{klq}S_{mq}) = 0, \tag{4.19}$$

$$\dot{\theta}_{pq} + \varepsilon_{pmk}S_{kq,m} = 0. \tag{4.20}$$

SCHAEFER [10] and ANTHONY et al. [26] first gave the relation (4.17) which has been interpreted to mean that dislocations can end on disclinations [13, 14]. Relation (4.18) shows that disclinations cannot end inside the body. Relation (4.19) was first given in the four-dimensional non-Riemannian formulation by SIMMONS [29], his Eq. (3.24). Relations (4.19)–(4.20) have also been given by KLUGE [20, 21], SCHAEFER [17], and GÜNTHER [18, 19]. They require that the change in dislociaton and disclination density can only be achieved by their currents. From Eqs. (4.5) to (4.16) follow the field equations for defects,

$$\varepsilon_{pmk}(e_{kl,m} + \varepsilon_{klq}\varkappa_{mq}) = \alpha_{pl}, \tag{4.21}$$

$$\varepsilon_{pmk}\varkappa_{kq,m} = \theta_{pq}, \tag{4.22}$$

$$v_{l,k} - \dot{e}_{kl} - \varepsilon_{klq} w_q = J_{kl}, \tag{4.23}$$

$$w_{q,k} - \dot{\varkappa}_{kq} = S_{kq}. \tag{4.24}$$

The fundamental meaning of these equations is that the defect densities and currents are the sources of the basic elastic fields. Relations (4.21) and (4.22) have been given before [13, 14] and the relations (4.23), (4.24) have been given by KLUGE [20, 21] and SCHAEFER [17] for the Cosserat-continuum.

Instead of the dislocation density it is sometimes convenient to use the *contortion* which is defined by [23, 13]

$$K_{lp} \equiv \varepsilon_{pmk} e^{P}_{kl,m} - \varkappa^{P}_{lp}. \tag{4.25}$$

Comparing with Eq. (4.13) we see that the contortion is related to the dislocation density by

$$K_{lp} = 1/2\,\delta_{lp}\alpha - \alpha_{pl}, \tag{4.26}$$

$$\alpha_{pl} = \delta_{pl} K - K_{lp} = \varepsilon_{pmk}\varepsilon_{klq} K_{mq}. \tag{4.27}$$

In terms of the contortion the continuity equations (4.17) and (4.19) become

$$K_{,l} - K_{lp,p} + \varepsilon_{lpq}\theta_{pq} = 0, \tag{4.28}$$

$$\varepsilon_{pmk}[J_{k,m} + \varepsilon_{klq}(S_{mq} + \dot{K}_{mq})] = 0. \tag{4.29}$$

The field equation for the contortion is

$$-\varepsilon_{pmk} e_{kl,m} + \varkappa_{lp} = K_{lp}. \tag{4.30}$$

These relations in terms of the contortion will be useful in Sect. 6 relating the present work to the incompatibility tensor.

4.3. The characteristic vectors

We define a *Burgers circuit* as any closed curve $\lambda(t)$, inside the body, whose position can change continuously with time. The *characteristic vectors* associated with this Burgers circuit are the *total Burgers vector* B_l and the *Frank vector* Ω_q which are defined by

$$B_l(t) \equiv - \oint_{\lambda(t)} [e^{P}_{kl}(\mathbf{r}, t) - \varepsilon_{lqr}\varkappa^{P}_{kq}(\mathbf{r}, t) x_r]\, dL_k, \tag{4.31}$$

$$\Omega_q(t) \equiv - \oint_{\lambda(t)} \varkappa^{P}_{kq}(\mathbf{r}, t)\, dL_k. \tag{4.32}$$

By Stokes' theorem (Appendix A in ref. [14]) and Eqs. (4.13) and (4.14) these relations can also be put in terms of the defect densities,

$$B_l(t) = \int_{\sigma(t)} [\alpha_{pl}(\mathbf{r}, t) - \varepsilon_{lqr}\theta_{pq}(\mathbf{r}, t) x_r]\, dS_p, \tag{4.33}$$

$$\Omega_q(t) = \int_{\sigma(t)} \theta_{pq}(\mathbf{r}, t)\, dS_p, \tag{4.34}$$

where the Burgers surface $\sigma(t)$ is any open surface bounded by $\lambda(t)$. From Eqs. (4.9) and (4.10), (4.1) and (4.2), (4.31) and (4.32), we also find

$$\oint_{\lambda(t)} [e_{kl}(\mathbf{r}, t) - \varepsilon_{lqr}\varkappa_{kq}(\mathbf{r}, t) x_r]\, dL_k = B_l(t), \tag{4.35}$$

$$\oint_{\lambda(t)} \varkappa_{kq}(\mathbf{r}, t)\, dL_k = \Omega_q(t). \tag{4.36}$$

The relations (4.31) to (4.36) for the characteristic vectors are unchanged from those for the stationary case, i.e. Eqs. (4.5) and (4.6), (4.7) and (4.8) and (4.20) and (4.21) of Ref. [14]. The rates of change of the characteristic vectors are

$$\dot{B}_l = -\oint_{\lambda(t)} [J_{kl} - \varepsilon_{pmk}\alpha_{pl}v'_m - \varepsilon_{lqr}(S_{kq} - \varepsilon_{pmk}\theta_{pq}v'_m)x_r]\,dL_k, \tag{4.37}$$

$$\dot{\Omega}_q = -\oint_{\lambda(t)} [S_{kq} - \varepsilon_{pmk}\theta_{pq}v'_m]\,dL_k. \tag{4.38}$$

These relations can be derived from Eqs. (4.31) and (4.32) or (4.35) and (4.36) by applying (A3) and using Eqs. (4.13) to (4.16) or (4.21) to (4.24). Here v'_m is the velocity of the Burgers circuit $\lambda(t)$.

Consider now the case where $\sigma(t)$ is a closed surface. Then we have for Eqs. (4.33) and (4.34)

$$B_l = \oint_{\sigma(t)} (\alpha_{pl} - \varepsilon_{lqr}\theta_{pq}x_r)\,dS_p = \int_{V(t)} (\alpha_{pl,p} - \varepsilon_{lqp}\theta_{pq} - \varepsilon_{lqr}\theta_{pq,p}x_r)\,dV = 0, \tag{4.39}$$

$$\Omega_q = \oint_{\sigma(t)} \theta_{pq}\,dS_p = \int_{V(t)} \theta_{pq,p}\,dV = 0, \tag{4.40}$$

where we have used the divergence theorem and Eqs (4.17) and (4.18). This shows that the continuity equations for α_{pl} and θ_{pq} imply the conservation of the characteristic vectors. Of course the same conclusion, Eqs. (4.39) and (4.40) can also be reached from the fact that the characteristic vectors can be written as line integrals. Similarly, we can conclude from Eqs. (4.37) and (4.38) that the rate of change of the characteristic vectors vanishes for a closed surface σ.

5. The moving discrete defect line

5.1. The basic elastic fields

The discrete defect line $L(t)$ which may change its position with time is defined as the boundary of a surface $S(t)$, where the material below S has been plastically displaced with respect to the material above S by an amount which represents a rigid motion.

Hence the difference between the displacement just below and above $S(t)$ is given by

$$[u_l(\mathbf{r}, t)] = -b_l - \varepsilon_{lqr}\Omega_q(x_r - x_r^0), \tag{5.1}$$

where b_l represents a rigid translation and the second term a superposed constant rotation of amount Ω_q around an axis through the point x_r^0. The constant b_l will be called the Burgers vector for the discrete dislocation line contained in the defect line, and is to be distinguished from the general Burgers vector defined by Eq. (4.31). The constant Ω_q will be identified with the Frank vector, Eq. (4.32). The rotation term in Eq. (5.1) is conventionally associated with the discrete disclination line. However, it is clear that the same discrete defect line can be described by different values of x_r^0 and b_l [14].

Now our problem is how to embody the relation (5.1) into definitions for the basic plastic fields. Then the equations in Sect. 4 can be applied to find all the desired results. We use a straightforward procedure that has been developed earlier [14]. Let us first assume that $S(t)$ is closed, enclosing a volume $V(t)$, which is smoothly changing with respect to time. Then the expression

$$u_l^T(\mathbf{r}, t) = \int_{V(t)} \delta(\mathbf{R}) \{b_l + \varepsilon_{lqr} \Omega_q (x'_r - x_r^0)\} dV' \tag{5.2}$$

represents a displacement that is the same as Eq. (5.1) inside $V(t)$ and vanishes outside $V(t)$. Thus it has the required jump across $S(t)$. Equation (5.2) can be regarded as describing a grain of volume $V(t)$ whose orientation with respect to the surrounding material is given by the rigid motion of Eq. (5.1), and whose boundary $S(t)$ is migrating. We shall assume that the axis of rotation remains constant, i.e. x_r^0 is not a function of time. By Eqs. (4.1) to (4.4) the relation (5.2) then leads to the following basic total fields:

$$e_{kl}^T(\mathbf{r}, t) = -\oint_{S(t)} \delta(\mathbf{R}) \{b_l + \varepsilon_{lqr} \Omega_q (x'_r - x_r^0)\} dS'_{k\,(kl)}, \tag{5.3}$$

$$\varkappa_{mq}^T(\mathbf{r}, t) = -1/2\, \varepsilon_{klq} \oint_{S(t)} \delta_{,m}(\mathbf{R}) \{b_l + \varepsilon_{lpr} \Omega_p (x'_r - x_r^0)\} dS'_k - \oint_{S(t)} \delta(\mathbf{R}) \Omega_q dS'_m, \tag{5.4}$$

$$v_l^T(\mathbf{r}, t) = \oint_{S(t)} \delta(\mathbf{R}) \{b_l + \varepsilon_{lqr} \Omega_q (x'_r - x_r^0)\} v'_k(\mathbf{r}', t) dS'_k, \tag{5.5}$$

$$w_q^T(\mathbf{r}, t) = 1/2 \varepsilon_{klq} \oint_{S(t)} \delta_{,k}(\mathbf{R}) \{b_l + \varepsilon_{lpr} \Omega_p (x'_r - x_r^0)\} v'_s(\mathbf{r}', t) dS'_s, \tag{5.6}$$

where we have used the divergence theorem and (A1). Here v'_k is the velocity of the surface $S(t)$. These results show that the basic total fields are concentrated at the surface $S(t)$. The next step is to assume that these relations hold in the same form for the open surface $S(t)$ used in the definition of the defect loop, and that they represent the basic plastic fields.

To facilitate writing down the final results it is convenient at this point to introduce four new quantities that can be related to defect loops and their currents, distributed over the open surface $S(t)$. They are the *dislocation loop density* $\beta_{kl}^\star$, the *disclination loop density* $\phi_{kq}^\star$, the *dislocation loop current* $v_l^\star$, and the *disclination loop current* $\psi_q^\star$ (Table 1). These quantities are defined by

$$\beta_{kl}^\star(\mathbf{r}, t) \equiv -\int_{S(t)} \delta(\mathbf{R}) \{b_l + \varepsilon_{lqr} \Omega_q (x'_r - x_r^0)\} dS'_k, \tag{5.7}$$

$$\phi_{kq}^\star(\mathbf{r}, t) \equiv -\int_{S(t)} \delta(\mathbf{R}) \Omega_q dS'_k, \tag{5.8}$$

$$v_l^\star(\mathbf{r}, t) \equiv \int_{S(t)} \delta(\mathbf{R}) \{b_l + \varepsilon_{lqr} \Omega_q (x'_r - x_r^0)\} v'_k(\mathbf{r}', t) dS'_k, \tag{5.9}$$

$$\psi_q^\star(\mathbf{r}, t) \equiv \int_{S(t)} \delta(\mathbf{R}) \Omega_q v'_k(\mathbf{r}', t) dS'_k. \tag{5.10}$$

Then we postulate that the basic plastic fields are given by

$$e_{kl}^P = \beta_{(kl)}^\star, \tag{5.11}$$

$$\varkappa_{mq}^P = 1/2 \varepsilon_{klq} \beta_{kl,m}^\star + \phi_{mq}^\star, \tag{5.12}$$

$$v_l^P = v_l^\star, \tag{5.13}$$

$$w_q^P = 1/2 \varepsilon_{klq} \dot{\beta}_{kl}^\star + \psi_q^\star. \tag{5.14}$$

These are the results we sought in this section. We note here that the transition from Eqs. (5.3)–(5.6) to Eqs. (5.11)–(5.14) involves a certain arbitrariness in that the latter equations may contain line integrals along $L(t)$ that vanish in the former. Thus, instead of Eq. (5.14) we could alternatively have defined $w_q^P = 1/2\ \varepsilon_{klq} v^\star_{l,k}$, by comparison with Eq. (5.6). But this definition was found to be inconsistent with the relation (4.15) and therefore discarded. Relations (5.7) and (5.8) have been given before by MURA [12] and the relations (5.11) and (5.12) in Ref. [14]. The relations (5.9) and (5.10) as well as (5.13) and (5.14) form logical extensions of Eqs. (5.7) and (5.8) as well as Eqs. (5.11) and (5.12). We note that the theory of line defects can be constructed entirely without the quantities $v_l^\star$ and $\psi_q^\star$ if we assume that $S(t)$ changes in time only by the motion of its boundary, i.e. when $v_k' dS_k' = 0$. In this case v_l^P also vanishes, but not w_q^P. Nevertheless it is comfortable to deal with an arbitrarily moving $S(t)$, for then the theory is more symmetric.

The total Burgers vector can be found from Eq. (4.31), Eqs. (5.11) and (5.12) as well as (5.7) and (5.8)

$$B_l = b_l - \varepsilon_{lqr}\Omega_q x_r^0, \tag{5.15}$$

where we have integrated using the method of Appendix B in [14]. Hence the total Burgers vector is also a constant for the discrete defect line. In a similar way we can show that Eqs. (5.12) and (5.8) are consistent with Eq. (4.32).

5.2. The defect density and current tensors

From Eqs. (4.13) to (4.16) and Eqs. (5.11) to (5.14) we find the following relationships:

$$\alpha_{pl} = -\varepsilon_{pmk}(\beta^\star_{kl,m} + \varepsilon_{klq}\phi^\star_{mq}), \tag{5.16}$$

$$\theta_{pq} = -\varepsilon_{pmk}\phi^\star_{kq,m}, \tag{5.17}$$

$$J_{kl} = -v^\star_{l,k} + \dot{\beta}^\star_{kl} + \varepsilon_{klq}\psi^\star_q, \tag{5.18}$$

$$S_{kq} = -\psi^\star_{q,k} + \dot{\phi}^\star_{kq}. \tag{5.19}$$

These are the fundamental relations between the defect densities and their currents and the corresponding loop densities and loop currents. From Eqs. (5.7) to (5.10) these relations can then be written as the following line integrals:

$$\alpha_{pl}(\mathbf{r}, t) = \oint_{L(t)} \delta(\mathbf{R})\{b_l + \varepsilon_{lqr}\Omega_q(x_r' - x_r^0)\}\, dL_p', \tag{5.20}$$

$$\theta_{pq}(\mathbf{r}, t) = \oint_{L(t)} \delta(\mathbf{R})\Omega_q\, dL_p', \tag{5.21}$$

$$J_{kl}(\mathbf{r}, t) = \oint_{L(t)} \varepsilon_{pmk}\delta(\mathbf{R})\{b_l + \varepsilon_{lqr}\Omega_q(x_r' - x_r^0)\} v_m'(\mathbf{r}', t)\, dL_p', \tag{5.22}$$

$$S_{kq}(\mathbf{r}, t) = \oint_{L(t)} \varepsilon_{pmk}\delta(\mathbf{R})\Omega_q v_m'(\mathbf{r}', t)\, dL_p', \tag{5.23}$$

where we have used Stokes' theorem and (A2). Now v_m' is the velocity of the defect line $L(t)$. These relations represent the defect densities and their currents for a moving discrete defect line.

We shall make a few comments here about these relations. First from Eqs. (5.20) to (5.21) we find that Eq. (4.33) also leads to Eq. (5.15), and that Eq. (5.21) is consistent with Eq. (4.34), using the methods of Appendix B in Ref. [14]. Second, Eqs. (5.20) to (5.23) satisfy the continuity Eqs. (4.17) to (4.20) by (A3). Third, we defined the discrete disclination line as the rotational component of Eq. (5.1) which corresponds to the terms containing Ω_q in Eqs. (5.20) to (5.23). Hence the discrete disclination line contributes to the dislocation density in Eq. (5.20) and the dislocation current in Eq. (5.22). ANTHONY [11] and GÜNTHER [18] therefore felt it necessary to split these quantities into two components, one associated with the discrete dislocation line, and the other associated with the discrete disclination line. We do not think this distinction is necessary [14]. Fourth, we note that for a discrete defect line the relations (5.20) to (5.23) show that

$$J_{kl} = \varepsilon_{pmk}\alpha_{pl}v'_m, \tag{5.24}$$

$$S_{kq} = \varepsilon_{pmk}\theta_{pq}v'_m, \tag{5.25}$$

$$\alpha_{pl} = \varepsilon_{lqr}\theta_{pq}(x_r - x_r^0) + \oint_{L(t)} \delta(\mathbf{R})b_l dL'_p, \tag{5.26}$$

$$J_{kl} = \varepsilon_{lqr}S_{kq}(x_r - x_r^0) + \oint_{L(t)} \varepsilon_{pmk}\delta(\mathbf{R})b_l v'_m dL'_p. \tag{5.27}$$

Relation (5.24) to (5.25) have also been given by GÜNTHER [18].

6. Relation to the incompatibility tensor and its current

6.1. The compatibility equations

In this section we shall formulate the compatibility conditions for the total deformation in terms of the total *strain* e_{kl}^T and *velocity* v_l^T. For a given total displacement u_l^T these quantities are defined by Eqs. (2.5) and (2.2)

$$e_{kl}^T = u_{(l,k)}^T, \tag{6.1}$$

$$v_l^T = \dot{u}_l^T. \tag{6.2}$$

On the other hand, if Eq. (6.1) to (6.2) are regarded as partial differential equations for u_l^T, then it is easy to show that the necessary and sufficient conditions to assure the existence of a continuous single-valued solution for u_l^T is that the relations

$$-\varepsilon_{pmk}\varepsilon_{qnl}e_{kl,mn}^T = 0, \tag{6.3}$$

$$-v_{l,kn}^T + \dot{e}_{kl,n}^T + \dot{e}_{ln,k}^T - \dot{e}_{nk,l}^T = 0 \tag{6.4}$$

are satisfied everywhere in the body. These are the compatibility conditions for the total strain and velocity in the dynamic case. Note that Eq. (6.3) can also be obtained by eliminating $\varkappa_{mq}^T$ from Eqs. (4.5) and (4.6), as well as Eq. (6.4) by eliminating $w_{q;}^T$ from Eq. (4.7) and (4.8), using Eq. (4.5) to eliminate $\varkappa_{mq}^T$.

6.2. The incompatibility tensor and its current

If defects and their currents are present, then we have by Eqs. (2.6) and (2.4)

$$e^T_{kl} = e_{kl} + e^P_{kl}, \tag{6.5}$$

$$v^T_l = v_l + v^P_l, \tag{6.6}$$

for an elasto-plastic deformation where the elastic or plastic components do not satisfy the compatibility conditions separately. The plastic strain and velocity may be arbitrarily prescribed without specifying the exact nature of the defects and their currents. In that case it is still possible to define the *incompatibility tensor* η_{pq} and the *incompatibility current* F_{nkl} as follows:

$$\eta_{pq} = \varepsilon_{pmk}\,\varepsilon_{qnl}\,e^P_{kl,mn}, \tag{6.7}$$

$$F_{nkl} = v^P_{l,kn} - \dot{e}^P_{kl,n} - \dot{e}^P_{ln,k} + \dot{e}^P_{nk,l}. \tag{6.8}$$

In other words, the incompatibility tensor and its current measure the deviation of the plastic strain and velocity from compatibility. The fundamental meaning of these equations is that an arbitrary plastic strain and velocity lead to incompatibility and its current, which are just other words for defects and their currents. The definition (6.7) has been given by KRÖNER [23]. The definition (6.8) however differs from the incompatibility current given by KOSSECKA [16].

Relations (6.7) and (6.8) lead directly to the continuity equations

$$\eta_{pq,p} = 0, \tag{6.9}$$

$$\dot{\eta}_{pq} + 1/2\,\varepsilon_{pmk}\,\varepsilon_{qnl}\,F_{nkl,m} = 0. \tag{6.10}$$

KRÖNER [23] has given the relation (6.9), which implies that there are no sources or sinks for incompatibility. Relation (6.10) shows that the incompatibility can only change by its current. From Eqs. (6.1) to (6.8) follow the field equations for the incompatibility:

$$-\varepsilon_{pmk}\,\varepsilon_{qnl}\,e_{kl,mn} = \eta_{pq}, \tag{6.11}$$

$$-v_{l,kn} + \dot{e}_{kl,n} + \dot{e}_{ln,k} - \dot{e}_{nk,l} = F_{nkl}. \tag{6.12}$$

The fundamental meaning of these equations is that if defects are present with a distribution of incompatibility η_{pq} and its current F_{nkl}, then elastic strain and velocity is produced according to these laws in order to insure the continuity of matter. Consequently these equations are the mathematical formulation of the statement that incompatibility and its current are the sources of elastic strain and velocity.

6.3. Relation to defect densities and their currents

From the definitions of the incompatibility and its current (6.7) and (6.8) and the definitions of the defect densities and their currents (4.13) to (4.16), it follows that

$$\eta_{pq} = -(\varepsilon_{qnl}\,\alpha_{pl,n} + \theta_{pq})_{(pq)}, \tag{6.13}$$

$$F_{nkl} = -J_{(kl),n} - J_{(ln),k} + J_{(nk),l}. \tag{6.14}$$

These are the fundamental relations between the incompatibility and the defects, and their currents, respectively. For the subsequent analysis, however, it is convenient to work in terms of the contortion rather than the dislocation density. So, by Eqs. (4.25), (4.14) and (6.7), we find

$$\eta_{pq} = \varepsilon_{qnl} K_{lp,n} - \theta_{qp}, \tag{6.15}$$

and by Eqs. (4.25), (4.15), (4.16) and (6,8) we find

$$F_{nkl} = -J_{kl,n} - \varepsilon_{klq}(S_{nq} + \dot{K}_{nq}). \tag{6.16}$$

7. Summary

We derived the compatibility equations for a deformation in terms of the basic total fields, i.e. the strain, bend-twist, linear and rotational velocity. The basic plastic fields in general violate these compatibility equations and thus motivate the definitions of the defect densities and their currents. We introduced the characteristic vectors, i.e. the total Burgers vector and the Frank vector. These can be given as surface integrals over the defect densities, whereas their time derivatives are given as line integrals over the defect densities and their currents.

We defined the moving discrete defect line in terms of a surface across which there is an appropriate displacement jump. A straightforward procedure motivated the definitions of the basic plastic fields consistent with this jump condition. In this process it was convenient to define also the defect loop densities and loop currents. Once the basic plastic fields were found the results for continuous distributions could be specialized to the discrete case. In this way we found the defect densities and their currents as line integrals along the defect line. We then noted some special relations between the defect densities and their currents for the discrete case.

We conclude the paper with an Appendix showing identities for the time derivative of volume, surface and line integrals. These are very useful in calculation for the moving discrete defect line.

8. Appendix A. Kinematics of volume, surface and line integrals

For the case of moving discrete defect lines it is often necessary to find the time derivative of certain volume, surface or line integrals whose position may be changing with time. When the integrand is a general field $f(\mathbf{r}, t, \mathbf{r}')$, which is a function of position $\mathbf{r}$ and time t, and the variable of integration $\mathbf{r}'$, these derivatives are given by

$$\frac{\partial}{\partial t}\int\limits_{V(t)} f(\mathbf{r}, t, \mathbf{r}')dV' = \int\limits_{V(t)}\left(\frac{\partial f}{\partial t} + f_{,i'}v_i' + fv_{i,i'}\right)dV' = \oint\limits_{S(t)} fv_i' dS_i' + \int\limits_{V(t)}\frac{\partial f}{\partial t}dV', \tag{A.1}$$

$$\begin{aligned}\frac{\partial}{\partial t}\int\limits_{S(t)} f(\mathbf{r}, t, \mathbf{r}')dS_i' &= \int\limits_{S(t)}\left[\left(\frac{\partial f}{\partial t} + f_{,j'}v_j' + fv_{j,j'}'\right)dS_i' - fv_{j,i'}'dS_j'\right] \\ &= \oint\limits_{L(t)} \varepsilon_{ijk} fv_j' dL_k' + \int\limits_{S(t)}\left(\frac{\partial f}{\partial t}dS_i' + f_{,i'}v_j' dS_j'\right),\end{aligned} \tag{A.2}$$

$$\text{(A.3)}\qquad \frac{\partial}{\partial t}\int\limits_{L(t)} f(\mathbf{r}, t, \mathbf{r}')\,dL'_i = \int\limits_{L(t)}\left[\left(\frac{\partial f}{\partial t}+f_{,j'}v'_j\right)dL'_i+fv'_{i,j'}dL'_j\right],$$

where

$$\text{(A.4)}\qquad v'_i = \dot{x}'_i$$

is the velocity of the volume, surface or line, respectively. The second equalities in Eqs. (A1) and (A2) follow from the divergence theorem and Stokes' theorem (Appendix A in Ref. [14]).

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