Archives of Mechanics • Archiwum Mechaniki Stosowanej • 29, 6, pp. 749–767, Warszawa 1977

Disclination dynamics

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A MATHEMATICAL theory of moving disclinations in a linearly elastic, infinitely extended, homogeneous body is developed. The theory is a generalization of disclination statics to dynamics on the one hand, and of the dynamics of dislocations to disclinations on the other hand. As expected, many quantities in the simpler theories correspond to pairs of quantities in the present theory, revealing a completeness which is lacking in the simpler theories. The boundary value problem for the infinite medium is completely solved. The state quantities, i.e. the elastic strain, bend-twist and velocities, are expressed as closed integrals in terms of the defects in the body, that is as volume integrals for a continuous distribution, and as line integrals for discrete lines. These integrals are given in terms of Green's tensor and an integral of Green's tensor which we have termed Green's potential tensor. The relation between disclination dynamics and the incompatibility theory is given.

Przedstawiona jest matematyczna teoria ruchomych dysklinacji w ciele liniowo-sprężystym, nieskończonym i jednorodnym. Teoria ta jest z jednej strony uogólnieniem statyki dysklinacji na dynamikę, z drugiej strony dynamiki dyslokacji na dynamikę dysklinacji; w tym sensie jest teorią kompletną w porównaniu z wymienionymi teoriami prostszymi. Odpowiednim wielkościom z teorii prostszych odpowiadają w obecnej teorii pary wielkości. Podane są rozwiązania dla ośrodka nieskończonego. Zmienne stanu, jak odkształcenie, skręcenie i prędkości, dane są w postaci całek po defektach; dla rozkładu ciągłego są to całki objętościowe, dla dyskretnych defektów całki po liniach defektów. Całki są splotami z tensorem Greena i tzw. potencjałem Green'a. Rozważa się związki dynamiki dysklinacji z teorią niezgodności.

Представлена математическая теория подвижных дисклинаций в бесконечном и однородном, линейно-упругом теле. Эта теория является с одной стороны обобщением статики дисклинаций на динамику, с другой же стороны динамики дислокаций на динамику дисклинаций; в этом смысле она является полной теорией по сравнению с упомянутыми более простыми теориями. Соответствующим величинам из более простых теорий отвечают в настоящей теории пары величин. Приведены решения для бесконечной среды. Переменные состояния, такие как деформация, кручение и скорости, даны в виде интегралов по дефектам; для непрерывного распределения это объемные интегралы, для дискретных дефектов интегралы по линиям дефектов. Интегралы являются свертками с тензором Грина и т. наз. потенциалом Грина. Рассматриваются связи динамики дисклинаций с теорией несовместности.

1. Introduction

1.1. Background

THIS ARTICLE develops a general theory of moving disclinations and dislocations in a linearly elastic, infinitely extended, homogeneous body.

This theory, together with Disclination Kinematics [31], may be regarded as the culmination of three different lines of work. First, it is an extension of dislocation dynamics, [1, 2, 3, 4, 5, 6, 7, 8, 29], to include disclinations. Second, it is an extension of the general theory of stationary disclinations [9, 10, 11, 12, 13, 14, 15] to include dynamics. Third, it is an extension of the dynamic theory of incompatibility [16] by identifying the defects that give rise to the incompatibility as dislocations and disclinations.

Some general work in disclination dynamics has already been done by other workers. SCHAEFER [17] has formulated a theory in the framework of the Cosserat-continuum and we closely follow his kinematical development. GÜNTHER [18, 19] has developed a four-dimensional non-Riemannian formalism and we shall note points of correspondence at the appropriate places in the present paper. The above two authors extend some earlier work by KLUGE [20, 21] who considered his theory to apply to foreign atoms instead of disclinations.

The results of [31] will be used throughout.

1.2. Outline of paper

In Sect. 2 we derive a general solution of the dynamic plastic strain problem which can be posed without specifying the nature of the defects involved. This problem is a generalization of Eshelby's "transformation problem" [22]. We express the equation of motion in terms of the total displacement and the basic plastic fields by using Hooke's law. The resulting partial differential equation is then solved for the displacement as a closed volume-time integral by using the dynamic Green's tensor function. This solution forms the basis of all subsequent applications to dynamics.

Sect. 3 reviews classical dislocation dynamics to set the stage for the following sections. It shows the basic approach that will be generalized to disclinations and will also serve as a basis for comparison with the later sections.

In Sect. 4 we derive the fields for a continuous distribution of moving defects. By "defects" we shall mean the combination of dislocations and disclinations. The constitutive equations relate the stress only to the basic elastic fields which do not necessarily satisfy the compatibility conditions. The difference between the total and elastic fields gives the plastic or stress-free fields. We derive closed volume-time integrals for the basic elastic fields in terms of the defect densities and their currents. For the strain formula it is necessary to introduce an integral of Green's tensor which we have termed Green's potential tensor.

Sect. 5 treats the moving discrete defect line. We find the displacement as a close surface-time integral. Then the basic elastic fields are derived as closed line-time integrals along the defect line. For the strain it is again necessary to use Green's potential.

In Sect. 6 we show the relations between the incompatibility theory and disclination dynamics.

Throughout the development of this paper we find that many concepts or quantities from dislocation dynamics generalize into pairs in disclination dynamics. The distortion and velocity of dislocation dynamics generalize to the basic fields of disclination dynamics. In a similar way we also find that many terms from disclination statics generalize into pairs in disclination dynamics. The basic fields are enlarged from two to four quantities by the addition of the velocities. The nomenclature that has developed in this field is summarized here:

b_l	Burgers vector,
B_l	total Burgers vector,
C_{ijkl}	elastic constants,
e_{kl}	strain,
F_{nkl}	incompatibility current,
G_{jn}	Green's tensor function,

H_{jn} Green's potential tensor function,
J_{kl} dislocation current,
K_{lp} contortion,
$L(t)$ defect line,
P plastic (superscript),
$\mathbf{r}$ radius vector,
$\mathbf{R}$ relative radius vector,
S_{kq} disclination current,
$S(t)$ defect surface,
t time,
T relative time, total (superscript),
u_l^T total displacement,
v_l linear velocity,
v_l^* dislocation loop current,
w_q rotational velocity,
α_{pl} dislocation density,
β_{kl} distortion,
β_{kl}^* dislocation loop density,
δ_{ln} Kronecker delta,
$\delta(\mathbf{r})$ Dirac delta function,
ε_{klq} permutation symbol,
η_{pq} incompatibility tensor,
θ_{pq} disclination density,
$\varkappa_{mq}$ bend-twist,
$\lambda(t)$ Burgers circuit,
ϱ mass density,
σ_{kj} stress,
$\sigma(t)$ Burgers surface,
ϕ_{kq}^* disclination loop density,
ψ_{kq}^* disclination loop current,
ω_q rotation,
Ω_q Frank vector.

This paper basically addresses itself to solving boundary value problems. The important subject of the forces on and the energy of the defects is not treated here. The solution in the static case seems tractable; the dynamic solution is, however, still open. Moreover, we shall not treat applications to special problems or geometries in the present paper. These would be quite straightforward and could be useful in analyzing experimental data.

2. The dynamic plastic strain problem

2.1. The constitutive equation

In this section we state the dynamic plastic strain problem which can be posed without specifying the nature of the defects involved.

Given an infinitely extended homogeneous anisotropic body with the plastic strain e_{kl}^P and the plastic velocity v_l^P prescribed as functions of space and time, we are to find the resulting total displacement u_m^T as a function of space and time.

This problem is a generalization of ESHELBY's [22] "transformation problem" to an anisotropic medium with a dynamic and inhomogeneous stress-free strain and velocity. We remark here that for our purpose the concept of "stress-free" is identical with "plastic".

To formulate the problem mathematically, we shall also need the constitutive equations, i.e. Hooke's law and the equation of motion. Hooke's law relates the stress σ_{ij} to the elastic strain e_{kl} as follows:

$$\sigma_{ij} = C_{ijkl} e_{kl}, \tag{2.1}$$

where the C_{ijkl} are the anisotropic elastic constants. We use the Einstein summation convention over repeated indices. Since both σ_{ij} and e_{kl} are symmetric, the elastic constants satisfy the symmetry conditions

$$C_{ijkl} = C_{jikl} = C_{ijlk}. \tag{2.2}$$

Note that Hooke's law does not involve the plastic or stress free strain e^P_{kl}. This is because the plastic strain is not a state quantity, whereas constitutive equations must relate state quantities. In terms of the discrete dislocation line discussed in Sect. 2.1 of [31], it means that the stress in Hooke's law does not depend on the location of the defect surface $S(t)$.

We now *assume* that the equation of motion relates the stress divergence to the *elastic* acceleration

$$\sigma_{ij,i} = \varrho \dot{v}_j, \tag{2.3}$$

where ϱ is the mass density. Note that this equation does not involve the plastic velocity v^P_i. This is because we wish to regard the plastic velocity as not being a state quantity. Hence it should not appear in a constitutive equation. In terms of the discrete dislocation line discussed in Sect. 2.1 of [31], this means that the stress divergence in the equation of motion does not depend on the location of the defect surface $S(t)$. In other words, the stress and elastic strain are not affected by the position and motion of the defect surface. The defect surface is simply regarded as an artificial device that is useful for the development of the theory. In the case of continuous distributions, it means that the elastic fields are completely determined by the dislocation density and current.

We wish to point out, however, that another approach is also possible, namely that the equation of motion relates the stress divergence to the *total* acceleration, $\sigma_{ij,i} = \varrho \dot{v}^T_j$, as would be suggested by Newton's law of motion. In this case it would be possible to have elastic fields without dislocation density and current, due to the plastic velocity. We have not investigated this approach.

Next it is convenient to combine the relations (2.2)–(2.6) of [31] and (2.1)–(2.3)into the following expression:

$$C_{ijkl} u^T_{l,ki} - \varrho \ddot{u}^T_j = C_{ijkl} e^P_{kl,i} - \varrho \dot{v}^P_j. \tag{2.4}$$

This is the set of partial differential equations we wish to solve for u^T_l when the plastic fields e^P_{kl} and v^P_j are given.

2.2. Definition and application of Green's tensor

To solve Eq. (2.4) for u^T it is useful to introduce the dynamic Green's tensor function $G_{jn}(\mathbf{r}, t)$, which represents the displacement in the x_j direction at the field point $\mathbf{r}$ and time t arising from a unit impulse in the x_n direction applied at the origin of space and time. Thus G_{jn} is defined for an infinitely extended body by

$$C_{ijkl} G_{jn,ik}(\mathbf{r}, t) + \delta_{ln} \delta(\mathbf{r}) \delta(t) = \varrho \ddot{G}_{ln}(\mathbf{r}, t) \tag{2.5}$$

together with the boundary condition that G_{jn} vanish at infinity in space and time. Here δ_{ln} is the Kronecker delta, while $\delta(\mathbf{r})$ and $\delta(t)$ are Dirac delta functions. For convenience we further define the relative radius vector and time:

$$\mathbf{R} = \mathbf{r} - \mathbf{r}', \tag{2.6}$$

$$T = t - t'. \tag{2.7}$$

Then we can conveniently derive the solution of Eq. (2.4) as follows:

$$\begin{aligned} u_n^T(\mathbf{r}, t) &= \int \delta_{ln}\,\delta(\mathbf{R})\,\delta(T)\,u_l^T(\mathbf{r}', t')\,dV'dt' \\ &= -\int [C_{ijkl}G_{jn,ik}(\mathbf{R}, T) - \varrho\ddot{G}_{ln}(\mathbf{R}, T)]\,u_l^T(\mathbf{r}', t')\,dV'dt' \\ &= -\int [C_{ijkl}G_{jn}(\mathbf{R}, T)\,u^T_{l,k'i'}(\mathbf{r}', t') - \varrho G_{ln}(\mathbf{R}, T)\,\ddot{u}_l^T(\mathbf{r}', t')]\,dV'dt' \\ &= -\int [C_{ijkl}G_{jn}(\mathbf{R}, T)\,e^P_{kl,i'}(\mathbf{r}', t') - \varrho G_{ln}(\mathbf{R}, T)\,\dot{v}_l^P(\mathbf{r}', t')]\,dV'dt' \\ &= -\int [C_{ijkl}G_{jn,i}(\mathbf{R}, T)\,e^P_{kl}(\mathbf{r}', t') - \varrho \dot{G}_{ln}(\mathbf{R}, T)\,v_l^P(\mathbf{r}', t')]\,dV'dt'. \end{aligned} \tag{2.8}$$

In these expressions the integrations are taken over all space and time. The first equality in the derivation follows from the well-known properties of the Kronecker delta and the Dirac delta function, the second equality from Eq. (2.5), the third by partial integrations with respect to space and time where we assume that the integrated parts vanish at infinity, the fourth from Eq. (2.4), and the fifth by additional partial integrations.

We remark that the plastic fields must satisfy certain conditions for the integral in Eq. (2.8) to be finite: it is clear that it is sufficient for e^P_{kl} and v_l^P to be finite in space and time, though these restrictions may not be necessary.

Whether the results listed under *kinematics* [31] hold regardless of the behavior of the fields at infinity, those under *dynamics* have to satisfy some restrictions as the above in order to keep the integrals finite and to be able to perform the necessary partial integrations in the various derivations.

Equation (2.8) gives the total displacement as a function of space and time in terms of a volume-time integral and applies to any defect which can be described by the given plastic strain and velocity. It forms the basis for all subsequent applications to dynamics. A similar result was derived by MURA [4] but without the term containing the plastic velocity v_l^P.

2.3. Compatible plastic strain and velocity

When no defects are present the plastic fields are compatible, i.e. they can be derived from a plastic displacement:

$$e^P_{kl} = u^P_{(l,k)}, \tag{2.9}$$

$$v_l^P = \dot{u}_l^P. \tag{2.10}$$

We then find for the total displacement

$$\begin{aligned} u_n^T(\mathbf{r}, t) &= -[C_{ijkl}G_{jn,i}(\mathbf{R}, T)\,u^P_{l,k'}(\mathbf{r}', t') - \varrho\dot{G}_{ln}(\mathbf{R}, T)\,\dot{u}_l^P(\mathbf{r}', t')]\,dV'dt' \\ &= -\int [C_{ijkl}G_{jn,ik}(\mathbf{R}, T) - \varrho\ddot{G}_{ln}(\mathbf{R}, T)]\,u_l^P(\mathbf{r}', t')\,dV'dt' \\ &= \int \delta_{ln}\,\delta(\mathbf{R})\,\delta(T)\,u_l^P(\mathbf{r}', t')\,dV'dt' = u_n^P(\mathbf{r}, t). \end{aligned} \tag{2.11}$$

Here the first equality follows from Eqs. (2.8) to (2.10), the second by partial integrations, and the third by Eq. (2.5). It follows therefore that for a compatible plastic deformation the elastic displacement vanishes:

$$u_n = u_n^T - u_n^P = 0. \tag{2.12}$$

Hence, in this case all elastic fields vanish.

3. Review of dislocation dynamics

3.1. Continuous distribution of dislocations

In this section we derive the basic elastic fields for continuous dislocation dynamics, namely the distortion and velocity, as volume-time integrals over the dislocation density and its current.

From Eqs. (2.8), (2.2) and (3.20), of [31], we find that the total displacement for a moving dislocation distribution can be written as

$$u_n^T(\mathbf{r}, t) = -\int [C_{ijkl} G_{jn,i}(\mathbf{R}, T)\beta_{kl}^P(\mathbf{r}', t') - \varrho \dot{G}_{ln}(\mathbf{R}, T) v_l^P(\mathbf{r}', t')] dV' dt'. \tag{3.1}$$

From this relation and Eq. (3.1) of [31] we find the total distortion as follows:

$$\begin{aligned}
\beta_{mn}^T(\mathbf{r}, t) &= -\int [C_{ijkl} G_{jn,im}(\mathbf{R}, T)\beta_{kl}^P(\mathbf{r}', t') - \varrho \dot{G}_{ln,m}(\mathbf{R}, T) v_l^P(\mathbf{r}', t')] dV' dt' \\
&= -\int [C_{ijkl} G_{jn,i}(\mathbf{R}, T)\beta_{kl,m'}^P(\mathbf{r}', t') - \varrho \dot{G}_{ln}(\mathbf{R}, T) v_{l,m'}^P(\mathbf{r}', t')] dV' dt' \\
&= \int \{C_{ijkl} G_{jn,i}(\mathbf{R}, T) [\varepsilon_{pmk}\alpha_{pl}(\mathbf{r}', t') - \beta_{ml,k'}^P(\mathbf{r}', t')] - \varrho \dot{G}_{ln}(\mathbf{R}, T)[J_{ml}(\mathbf{r}', t') \\
&\quad - \dot{\beta}_{ml}^P(\mathbf{r}', t')]\} dV' dt' = \int [C_{ijkl} G_{jn,i}(\mathbf{R}, T)\varepsilon_{pmk}\alpha_{pl}(\mathbf{r}', t') \\
&\quad - \varrho \dot{G}_{ln}(\mathbf{R}, T) J_{ml}(\mathbf{r}', t')] dV' dt' + \beta_{mn}^P(\mathbf{r}, t).
\end{aligned} \tag{3.2}$$

Here the first equality follows simply by differentiating Eq. (3.1) under the integral sign where Green's tensor G_{jn} is the only function depending on $\mathbf{r}$, the second equality follows by partial integrations, the third from (3.7) and (3.8) of [31], and the fourth by partial integration and Eq. (2.5). From Eq. (3.5) of [31] we then find the elastic distortion for a moving dislocation distribution to be

$$\beta_{mn}(\mathbf{r}, t) = \int [\varepsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R}, T)\alpha_{pl}(\mathbf{r}', t') - \varrho \dot{G}_{ln}(\mathbf{R}, T) J_{ml}(\mathbf{r}', t')] dV' dt'. \tag{3.3}$$

This relation is also given by Eq. (2.30) of Ref. [7]. It was first obtained by MURA [4] with
√ of [31] the replacement (3.9) for J_{ml}. It is interesting to note that Eq. (3.3) can also be derived unchanged when the plastic velocity terms in from (2.8) and (3.8) of [31] are suppressed. This shows that v_l^P is not essential for the development of the theory, but we feel that its introduction helps the interpretation.

We next find the total velocity from Eqs. (3.2) of [31] and (3.1)

$$\begin{aligned}
v_n^T(\mathbf{r}, t) &= -\int [C_{ijkl} \dot{G}_{jn,i}(\mathbf{R}, T)\beta_{kl}^P(\mathbf{r}', t') - \varrho \ddot{G}_{ln}(\mathbf{R}, T) v_l^P(\mathbf{r}', t')] dV' dt' \\
&= -\int \{C_{ijkl} G_{jn,i}(\mathbf{R}, T)[J_{kl}(\mathbf{r}', t') + v_{l,k'}^P(\mathbf{r}', t')] - \varrho \ddot{G}_{ln}(\mathbf{R}, T) v_l^P(\mathbf{r}', t')\} dV' dt' \\
&= -\int C_{ijkl} G_{jn,i}(\mathbf{R}, T) J_{kl}(\mathbf{r}', t') dV' dt' + v_n^P(\mathbf{r}, t).
\end{aligned} \tag{3.4}$$

Here the first equality follows because the Green's tensor is the only function which depends on t under the integral sign, the second equality by a partial integration and Eq. (3.8) of [31], and the third by a partial integration and Eq. (2.5) From Eq. (3.6) of [31] we then find the elastic velocity for a moving dislocation distribution to be

$$v_n(\mathbf{r}, t) = -\int C_{ijkl} G_{jn,i}(\mathbf{R}, T) J_{kl}(\mathbf{r}', t')\, dV' dt'. \tag{3.5}$$

This relation corresponds to Eq. (3.6) in Ref. [7]. It was first obtained by MURA [4] with the replacement (3.9)√for J_{kl}. Since Mura assumed $v_n^P = 0$, he made no distinction between v_n and $\dot{u}_n^T$, c.f. Eqs. (3.2) and (3.6) of [31]. √of [31]

We note here that the dislocation density α_{pl} and current J_{kl} are state quantities because they can be measured in the present state of the body. Therefore, Eqs. (3.3) and (3.5) show that the elastic distortion β_{mn} and velocity v_n are also state quantities because they can be expressed entirely as volume-time integrals in terms of other state quantities, α_{pl} and J_{kl}. On the other hand, the plastic distortion β_{mn}^P and velocity v_n^P may not be state quantities because we may have to know the prior history of the body to measure them.

3.2. The discrete dislocation line

In this section we find the basic elastic fields for discrete dislocation dynamics as line-time integrals over the dislocation line. First we substitute Eqs. (3.25) and (3.26) of [31] into Eq. (3.1) to find the displacement for a discrete moving dislocation line,

$$u_n^T(\mathbf{r}, t) = \int \int_{S(t')} [C_{ijkl} G_{jn,i}(\mathbf{R}, T) + \varrho G_{ln}(\mathbf{R}, T) v_k'(\mathbf{r}', t')] b_l\, dS_k' dt', \tag{3.6}$$

where we have performed the integration over all space. In this expression the first integral sign refers to the integration over the complete range of time $t'(-\infty, \infty)$, and the second to the integration over the surface $S(t)$ defined in Sect. 3.2 of [31]. A similar relation has been given by MURA [4], but without the term involving the surface velocity v_k'. In other words, Mura assumed that the surface $S(t')$ changes in time only by the motion of its boundary, an alternative we discussed in Sect. 3.2 of [31]. The relation (3.6) corresponds to Eq. (5.13) in Ref. [7] and Eq. (3.5) in Ref. [28].

Next we find the total distortion from Eqs. (3.1) of [31] and (3.6),

$$\beta_{mn}^T = \int \int_{S(t')} [C_{ijkl} G_{jn,im} + \varrho \dot{G}_{ln,m} v_k'] b_l\, dS_k' dt'. \tag{3.7}$$

Now from Eq. (A2) of the Appendix in [31] it follows that

$$\frac{\partial}{\partial t'} \int_{S(t')} \dot{G}_{ln}\, dS_m' = \oint_{L(t')} \varepsilon_{pmk} \dot{G}_{ln} v_k'\, dL_p' - \int_{S(t')} (\ddot{G}_{ln}\, dS_m' + \dot{G}_{ln,m} v_k'\, dS_k'), \tag{3.8}$$

where we have used the relations

$$\begin{aligned} \frac{\partial}{\partial t'} \dot{G}_{ln}(\mathbf{R}, T) &= -\ddot{G}_{ln}(\mathbf{R}, T), \\ \dot{G}_{ln,m'}(\mathbf{R}, T) &= -\dot{G}_{ln,m}(\mathbf{R}, T), \end{aligned} \tag{3.9}$$

which follow from Eq. (2.6) and (2.7). The result (3.8) can be used to do a partial integration with respect to t' on the second term in Eq. (3.7), while Stokes' theorem can be applied to the first term. Hence

$$\begin{aligned}(3.10)\qquad \beta^T_{mn} &= \int\oint_{L(t')} \varepsilon_{pmk} C_{ijkl} G_{jn,i} b_l \, dL'_p \, dt' + \int\int_{S(t')} C_{ijkl} G_{jn,ik} b_l \, dS'_m \, dt' \\ &\quad + \int\oint_{L(t')} \varepsilon_{pmk} \varrho \dot{G}_{ln} v_k b_l \, dL'_p \, dt' - \int\int_{S(t')} \varrho \ddot{G}_{ln} b_l \, dS'_m \, dt' \\ &= \int\oint_{L(t')} \varepsilon_{pmk} [C_{ijkl} G_{jn,i} + \varrho \dot{G}_{ln} v'_k] b_l \, dL'_p \, dt' + \beta^P_{mn},\end{aligned}$$

where the last expression follows from Eqs. (2.5) and (3.25) of [31]. From Eq. (3.5) of [31] we then obtain the elastic distortion for a moving discrete dislocation line

$$(3.11)\qquad \beta_{mn}(\mathbf{r}, t) = \int\oint_{L(t')} \varepsilon_{pmk} [C_{ijkl} G_{jn,i}(\mathbf{R}, T) + \varrho \dot{G}_{ln}(\mathbf{R}, T) v'_k(\mathbf{r}', t')] b_l \, dL'_p \, dt',$$

where now v'_k is the velocity of the dislocation line $L(t')$. This relation could of course also be obtained more directly by a volume integration from Eq. (3.3), (3.28) and (3.29) of [31]. It was first obtained by MURA [4], and it corresponds to Eq. (2.27) in Ref. [7].

Finally we find the total velocity from Eqs. (3.2) of [31] and (3.6),

$$(3.12)\qquad v^T_n = \int\int_{S(t')} [C_{ijkl} \dot{G}_{jn,i} + \varrho \ddot{G}_{ln} v'_k] b_l \, dS'_k \, dt'.$$

This time we have by Eq. (A2) of [31]:

$$(3.13)\qquad \frac{\partial}{\partial t'} \int_{S(t')} G_{jn,i} \, dS'_k = - \oint_{L(t')} \varepsilon_{pmk} G_{jn,i} v'_m \, dL'_p - \int_{S(t')} (\dot{G}_{jn,i} \, dS'_k + G_{jn,ik} v'_p \, dS'_p).$$

Thus by partial integration over t', Eq. (3.12) becomes

$$(3.14)\qquad v^T_n = - \int\oint_{L(t')} \varepsilon_{pmk} C_{ijkl} G_{jn,i} b_l v'_m \, dL'_p \, dt - \int\int_{S(t')} [C_{ijkl} G_{jn,ik} - \varrho \ddot{G}_{ln}] b_l v'_p \, dS'_p \, dt.$$

The second line in this expression equals v^P_n by Eqs. (2.5) and (3.26) of [31]. Therefore we find from Eq. (3.6) of [31] the elastic velocity for a moving discrete dislocation line to be

$$(3.15)\qquad v_n(\mathbf{r}, t) = - \int\oint_{L(t')} \varepsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R}, T) b_l v'_m(\mathbf{r}, t) \, dL'_p \, dt.$$

This relation could of course also have been obtained directly from Eqs. (3.5) and (3.29) of [31]. It was also first obtained by Mura [4] and corresponds to Eq. (3.5) in Ref. [7].

We see that the state quantities β_{mn} and v_n can be written as line integrals along the discrete moving dislocation, i.e. they are expressed entirely in terms of integrals over the only regions of the body where the defect is localized, and the position of the surface $S(t)$ is immaterial. The defect is localized on the line $L(t)$, and therefore any state quantity associated with the dislocation must be a line integral along the dislocation line.

4. Continuous distribution of moving defects

This section contains the main results of the present paper for a continuous distribution of defects, namely closed volume-time integrals of the basic elastic fields in terms of the defect densities and currents.

First, we find a useful expression for the total distortion:

$$(4.1)\qquad u^T_{n,m} = -\int (C_{ijkl}\, G_{jn,im}\, e^P_{kl} - \varrho \dot{G}_{ln,m} v^P_l)\, dV'dt' = -\int (C_{ijkl}\, G_{jn,i}\, e^P_{kl,m'} - \varrho \dot{G}_{ln} v^P_{l,m'})\, dV'dt'$$

$$= \int [C_{ijkl}\, G_{jn,i}(\varepsilon_{pmk}\,\alpha_{pl} - e^P_{ml,k'} - \varepsilon_{pmk}\,\varkappa^P_{lp}) - \varrho \dot{G}_{ln}(J_{ml} - \dot{e}^P_{ml} - \varepsilon_{pml}\, w^P_p)]\, dV'dt'$$

$$= \int (\varepsilon_{pmk}\, C_{ijkl}\, G_{jn,i}\,\alpha_{pl} - \varrho \dot{G}_{ln} J_{ml})\, dV'dt' - \int \varepsilon_{pmk}(C_{ijkl}\, G_{jn,i}\,\varkappa^P_{lp} - \varrho \dot{G}_{kn}\, w^P_p)\, dV'dt' + e^P_{mn}.$$

Here the first equality follows by differentiating Eq. (2.8), the second by partial integrations, the third from Eqs. (4.13) and (4.15), and the fourth by partial integration and Eq. (2.5). √ of [31]
From this relation we shall proceed to derive some of the basic elastic fields. However, it is convenient first to introduce a new quantity, the Green's potential tensor, which is defined in the next section.

4.1. Green's potential tensor

To find the desired expression for the elastic strain it is useful to introduce the dynamic Green's potential tensor function H_{jn}. It is defined in terms of Green's tensor G_{jn} as follows:

$$(4.2)\qquad H_{jn}(\mathbf{r}, t) = \int (4\pi R)^{-1} G_{jn}(\mathbf{r}', t)\, dV'.$$

An explicit expression for Green's potential in the isotropic case has been given by KOSSECKA [16]. The motivation for the name comes from the fact that Green's potential satisfies Poisson's equation

$$(4.3)\qquad H_{jn,ss} = -G_{jn},$$

with Green's tensor as the source function. From Eq. (2.5) we deduce that the Green's potential also satisfies the equation

$$(4.4)\qquad C_{ijkl}\, H_{jn,ik}(\mathbf{r}, t) + (4\pi r)^{-1}\,\delta_{ln}\,\delta(t) = \varrho \ddot{H}_{ln}(\mathbf{r}, t).$$

Green's potential is closely related to the incompatibility source tensor introduced by SIMMONS and BULLOUGH [27] to solve the so-called incompatibility problem, i.e. to find the elastic strain as a closed volume integral over the incompatibility tensor. The incompatibility source tensor was also found useful to express the elastic strain in terms of the defect densities in the static case [14]. However, we could not find a generalization of the incompatibility source tensor to dynamics. For that reason we have introduced the Green's potential as an alternative method.

4.2. The elastic strain

The elastic strain is obtained from Eqs. (4.9), (4.1) of [31], and (4.1) as follows:

$$(4.5)$$

$$e_{mn} = \int (\varepsilon_{pmk}\, C_{ijkl}\, G_{jn,i}\,\alpha_{pi} - \varrho \dot{G}_{ln} J_{ml})\, dV'dt'_{(mn)} - \int \varepsilon_{pmk}(C_{ijkl}\, G_{jn,i}\,\varkappa^P_{lp} - \varrho \dot{G}_{kn}\, w^P_p)\, dV'dt'_{(mn)},$$

where the symbol (mn) implies symmetrization. We now wish to express this equation also in terms of the disclination density and current by using Green's potential. By Eq. (4.3) we rewrite the second line in Eq. (4.5) as follows:

$$\int \varepsilon_{pmk}(C_{ijkl} H_{jn,iss} \varkappa^P_{lp} - \varrho \dot{H}_{kn,ss} w^P_p)\, dV' dt'_{(mn)} = \int \varepsilon_{pmk}(C_{ijkl} H_{jn,is} \varkappa^P_{lp,s'}$$

$$- \varrho \dot{H}_{kn,s} w^P_{p,s'})\, dV' dt'_{(mn)} = -\int \varepsilon_{pmk}[C_{ijkl} H_{jn,is}(\varepsilon_{qsl}\theta_{qp} - \varkappa^P_{sp,l'})$$

$$- \varrho \dot{H}_{kn,s}(S_{sp} - \dot{\varkappa}^P_{sp})]\, dV' dt'_{(mn)} = -\int \varepsilon_{pmk}(\varepsilon_{qsl} C_{ijkl} H_{jn,is}\theta_{qp} - \varrho \dot{H}_{kn,s} S_{sp})\, dV' dt'_{(mn)}$$

$$+ \int \varepsilon_{pmk}(C_{ijkl} H_{jn,il} - \varrho \ddot{H}_{kn})\, \varkappa^P_{sp,s'}\, dV' dt'_{(mn)}.$$

Here the first equality follows by partial integrations, the second from Eq. (4.14) and (4.16), of [31], and the third by partial integration. By Eq. (4.4) the second line in the last expression above vanishes. Hence we find for Eq. (4.5)

$$(4.6) \quad e_{mn}(\mathbf{r}, t) = \int \{[\varepsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R}, T)\alpha_{pl}(\mathbf{r}', t') - \varrho G_{ln}(\mathbf{R}, T) J_{ml}(\mathbf{r}', t')]$$

$$- \varepsilon_{pmk}[\varepsilon_{qsl} C_{ijkl} H_{jn,is}(\mathbf{R}, T)\theta_{qp}(\mathbf{r}', t') - \varrho \dot{H}_{kn,s}(\mathbf{R}, T) S_{sp}(\mathbf{r}', t')]\}\, dV' dt'_{(mn)}.$$

This is the elastic strain due to a continuous distribution of moving defects and their currents.

4.3. The elastic bend-twist

To find the elastic bend-twist we start with the derivative of the total distortion from Eq. (4.1):

$$(4.7) \quad u^T_{n,ms} = \int (\varepsilon_{pmk} C_{ijkl} G_{jn,is}\alpha_{pl} - \varrho \dot{G}_{ln,s} J_{ml})\, dV' dt' -$$

$$- \int \varepsilon_{pmk}(C_{ijkl} G_{jn,is} \varkappa^P_{lp} - \varrho \dot{G}_{kn,s} w^P_p)\, dV' dt' + e^P_{mn,s}.$$

By partial integrations the second line in this expression becomes

$$- \int \varepsilon_{pmk}(C_{ijkl} G_{jn,i} \varkappa^P_{lp,s'} - \varrho \dot{G}_{kn} w^P_{p,s'})\, dV' dt'$$

$$= \int \varepsilon_{pmk}[C_{ijkl} G_{jn,i}(\varepsilon_{qsl}\theta_{qp} - \varkappa^P_{sp,l'}) - \varrho \dot{G}_{kn}(S_{sp} - \dot{\varkappa}^P_{sp})]\, dV' dt'$$

$$= \int \varepsilon_{pmk}(\varepsilon_{qsl} C_{ijkl} G_{jn,i}\theta_{qp} - \varrho \dot{G}_{kn} S_{sp})\, dV' dt' + \varepsilon_{pmn}\varkappa^P_{sp}.$$

Here the first equality follows from Eqs. (4.14) and (4.16) of [31], and the second by partial integration and Eq. (2.5). Hence,

$$(4.8) \quad u^T_{n,ms} = \int (\varepsilon_{pmk} C_{ijkl} G_{jn,is}\alpha_{pl} - \varrho \dot{G}_{ln,s} J_{ml})\, dV' dt'$$

$$+ \int \varepsilon_{pmk}(\varepsilon_{qsl} C_{ijkl} G_{jn,i}\theta_{qp} - \varrho \dot{G}_{kn} S_{sp})\, dV' dt' + e^P_{mn,s} + \varepsilon_{pmn}\varkappa^P_{sp}.$$

Now from Eqs. (4.2) and (4.10) of [31] we have

$$(4.9) \quad \varkappa_{st} = 1/2\, \varepsilon_{tmn} u^T_{n,ms} - \varkappa^P_{st}.$$

Thus the elastic bend-twist for a continuous distribution of defects and their currents is

$$(4.10)\quad \varkappa_{st} = 1/2\int \varepsilon_{tmn}[(\varepsilon_{pmk} C_{ijkl} G_{jn,is}\alpha_{pl} - \varrho\dot{G}_{ln,s}J_{ml}) + \varepsilon_{pmk}(\varepsilon_{qsl} C_{ijkl} G_{jn,i}\theta_{qp} - \varrho\dot{G}_{kn}S_{sp})]\,dV'dt'.$$

4.4. The elastic velocities

We next wish to find expressions for the elastic velocities. From Eq. (2.8) we find the total linear velocity

$$(4.11)\quad \dot{u}_n^T = -\int (C_{ijkl}\dot{G}_{jn,i}e_{kl}^P - \varrho\ddot{G}_{ln}v_l^P)\,dV'dt'$$
$$= -\int [C_{ijkl}G_{jn,i}(J_{kl}+v_{l,k'}^P) - \varrho\dot{G}_{ln}v_l^P]\,dV'dt' = -\int C_{ijkl}G_{jn,i}J_{kl}\,dV'dt' + v_n^P.$$

Here the second equality follows by a partial integration and Eq. (4.15) of [31], and the third by partial integration and Eq. (2.5). From Eqs. (4.3) and (4.11) of [31] we then find the linear elastic velocity for a moving distribution of defects to be

$$(4.12)\quad v_n(\mathbf{r}, t) = -\int C_{ijkl}G_{jn,i}(\mathbf{R}, T)J_{kl}(\mathbf{r}', t')\,dV'dt',$$

which is identical with the corresponding equation for dislocations only, Eq. (3.5).

To find the elastic rotational velocity, we first take the time derivative of Eq. (4.1)

$$(4.13)\quad \dot{u}_{n,m}^T = \int (\varepsilon_{pmk}C_{ijkl}\dot{G}_{jn,i}\alpha_{pl} - \varrho\ddot{G}_{ln}J_{ml})\,dV'dt'$$
$$-\int \varepsilon_{pmk}(C_{ijkl}\dot{G}_{jn,i}\varkappa_{lp}^P - \varrho\ddot{G}_{kn}w_p^P)\,dV'dt' + \dot{e}_{mn}^P.$$

Now the second line in this expression can be rewritten by partial integration

$$-\int \varepsilon_{pmk}(C_{ijkl}G_{jn,i}\dot{\varkappa}_{lp}^P - \varrho\ddot{G}_{kn}w_p^P)\,dV'dt' = -\int \varepsilon_{pmk}[C_{ijkl}G_{jn,i}(S_{lp}+w_{p,l'}^P) - \varrho\ddot{G}_{kn}w_p^P]\,dV'dt'$$
$$= -\int \varepsilon_{pmk}C_{ijkl}G_{jn,i}S_{lp}\,dV'dt' + \varepsilon_{pmn}w_p^P.$$

Here the first equality follows from Eq. (4.16) of [31], and the second from partial integration and Eq. (2.5). Hence we have for Eq. (4.13)

$$(4.14)\quad \dot{u}_{n,m}^T = \int (\varepsilon_{pmk}C_{ijkl}\dot{G}_{jn,i}\alpha_{pl} - \varrho\ddot{G}_{ln}J_{ml})\,dV'dt'$$
$$-\int \varepsilon_{pmk}C_{ijkl}G_{jn,i}S_{lp}\,dV'dt' + \dot{e}_{mn}^P + \varepsilon_{pmn}w_p^P.$$

Now from Eqs. (4.4) and (4.12) of [31] we have

$$(4.15)\quad w_t = 1/2\,\varepsilon_{tmn}\dot{u}_{n,m}^T - w_t^P.$$

Therefore we find the rotational velocity for a distribution of moving defects to be

$$(4.16)\quad w_t = 1/2\int \varepsilon_{tmn}(\varepsilon_{pmk}C_{ijkl}\dot{G}_{jn,i}\alpha_{pl} - \varrho\ddot{G}_{ln}J_{ml} - \varepsilon_{pmk}C_{ijkl}G_{jn,i}S_{lp})\,dV'dt'.$$

It is possible to reduce this expression to a somewhat simpler form by partial integrations and Eq. (4.19) of [31],

$$(4.17)\qquad w_t = 1/2\int \varepsilon_{tmn}[C_{ijkl}G_{jn,i}(J_{ml,k'}-J_{kl,m'})-\varrho\ddot{G}_{ln}J_{ml}]dV'dt' = -1/2\,\varepsilon_{tmn}\left[\int C_{ijkl}G_{jn,im}J_{kl}dV'dt'+J_{mn}\right],$$

where we have used Eq. (2.5). This expression could alternatively have been obtained directly from the linear elastic velocity, Eq. (4.12), since by Eq. (4.23) of [31]

$$(4.18)\qquad w_t = 1/2\,\varepsilon_{tmn}(v_{n,m}-J_{mn}).$$

In conclusion, Sect. 4 has extended the results of dislocation dynamics of Sect. 3.1 to the more general *disclination* dynamics, whereas on the other hand it has extended the results of disclination statics (Sect. 4 of Ref. [14]) to the more general disclination *dynamics*. The central results obtained are closed integral expressions for the basic elastic fields, the strain (4.6), bend-twist (4.10), linear velocity (4.12), and rotational velocity (4.16) and (4.17). The basic elastic fields are state quantities because they are given entirely as integrals over the defect densities and currents. These expressions can form the basis for applications to particular cases. For example, the case of the moving discrete defect line will be discussed in Sect. 5.

5. The moving discrete defect line

This section contains the main results of the present paper for a moving discrete defect line, namely closed line-time integrals for the basic elastic fields.

First we find an expression for the total displacement due to a moving finite defect loop from Eqs. (2.8) and (5.11), (5.13), (5.7), and (5.9) of [31]:

$$(5.1)\qquad u_n^T(\mathbf{r},t) = \int\int_{S(t')} [C_{ijkl}G_{jn,i}(\mathbf{R},T)+\varrho\dot{G}_{ln}(\mathbf{R},T)v_k'(\mathbf{r}',t)]\{b_l+\varepsilon_{lqr}\Omega_q(x_r'-x_r^0)\}\,dS_k'\,dt',$$

where we have done the integration over all space. In this expression the first integral sign refers to the integration over the complete range of time $t'(-\infty,\infty)$, the second to the integration over the surface $S(t)$ defined in Sect. 5 of [31] and v_k' is the velocity of the surface $S(t')$.

Next we find the total distortion by differentiating:

$$(5.2)\qquad u_{n,m}^T = \int\int_{S(t')} [C_{ijkl}G_{jn,im}+\varrho\dot{G}_{ln,m}v_k']\{b_l+\varepsilon_{lqr}\Omega_q(x_r'-x_r^0)\}\,dS_k'dt'.$$

Now by Eq. (A2) of [31] we have

$$(5.3)\qquad \frac{\partial}{\partial t'}\int_{S(t')} \varrho\dot{G}_{ln}\{b_l+\varepsilon_{lqr}\Omega_q(x_r'-x_r^0)\}\,dS_m' = \oint_{L(t')} \varepsilon_{pmk}\varrho\dot{G}_{ln}\{b_l+\varepsilon_{lqr}\Omega_q(x_r'-x_r^0)\}v_k'\,dL_p' - \int_{S(t')} \varrho\ddot{G}_{ln}\{b_l+\varepsilon_{lqr}\Omega_q(x_r'-x_r^0)\}\,dS_m' - \int_{S(t')} \varrho\dot{G}_{ln,m}\{b_l+\varepsilon_{lqr}\Omega_q(x_r'-x_r^0)\}v_k'\,dS_k' + \int_{S(t')} \varrho\dot{G}_{ln}\varepsilon_{lqm}\Omega_q v_k'\,dS_k',$$

where we have used Eqs. (3.9). This result can be used to do partial integration with respect to t' in Eq. (5.2). Applying also Stokes' theorem we find

$$(5.4)\quad u^T_{n,m} = \int\oint_{L(t')} \varepsilon_{pmk} C_{ijkl} G_{jn,i} \{b_l + \varepsilon_{lqr} \Omega_q (x'_r - x^0_r)\} dL'_p dt'$$

$$+ \int\int_{S(t')} C_{ijkl} G_{jn,ik} \{b_l + \varepsilon_{lqr} \Omega_q (x'_r - x^0_r)\} dS'_m dt' + \int\int_{S(t')} C_{ijkl} G_{jn,i} \varepsilon_{lqm} \Omega_q dS'_k dt'$$

$$+ \int\oint_{L(t')} \varepsilon_{pmk} \varrho \dot{G}_{ln} \{b_l + \varepsilon_{lqr} \Omega_q (x'_r - x^0_r)\} v'_k dL'_p dt' - \int\int_{S(t')} \varrho \ddot{G}_{ln} \{b_l + \varepsilon_{lqr} \Omega_q (x'_r - x^0_r)\} dS'_m dt'$$

$$+ \int\int_{S(t')} \varrho \dot{G}_{ln} \varepsilon_{lqm} \Omega_q v'_k dS'_k dt' = \int\oint_{L(t')} \varepsilon_{pmk} (C_{ijkl} G_{jn,i} + \varrho \dot{G}_{ln} v'_k) \{b_l + \varepsilon_{lqr} \Omega_q (x'_r - x^0_r)\} dL'_p dt'$$

$$+ \int\int_{S(t')} \varepsilon_{pmk} (C_{ijkl} G_{jn,i} + \varrho \dot{G}_{kn} v'_l) \Omega_p dS'_l dt' + \beta^*_{mn},$$

where the last equality follows from Eqs. (2.5) and (5.7) of [31]. From this relation we shall proceed to derive some of the basic elastic fields.

5.1. The elastic strain

The elastic strain is obtained from (4.9), (4.1), (5.11) of [31], and (5.4) as follows:

$$(5.5)\quad e_{mn} = \int\oint_{L(t')} \varepsilon_{pmk} (C_{ijkl} G_{jn,i} + \varrho \dot{G}_{ln} v'_k) \{b_l + \varepsilon_{lqr} \Omega_q (x'_r - x^0_r)\} dL'_p dt'_{(mn)}$$

$$+ \int\int_{S(t')} \varepsilon_{pmk} (C_{ijkl} G_{jn,i} + \varrho \dot{G}_{kn} v'_l) \Omega_p dS'_l dt'_{(mn)}.$$

We now wish to convert the surface integral also to a line integral, and for this we shall use Green's potential. Furthermore we note that by Eq. (A2) of [31]

$$(5.6)\quad \frac{\partial}{\partial t'} \int_{S(t')} A dS'_s = \oint_{L(t')} \varepsilon_{qsl} A v'_l dL'_q - \int_{S(t')} (\dot{A}\, dS'_s + A_{,s} v'_l dS'_l)$$

for any tensor A that depends on $\mathbf{R}$ and T. Thus by Eq. (4.3) the surface integral in Eq. (5.5) becomes

$$- \int\int_{S(t')} \varepsilon_{pmk} (C_{ijkl} H_{jn,iss} + \varrho \dot{H}_{kn,ss} v'_l) \Omega_p dS'_l dt'_{(mn)} = - \int\oint_{L(t')} \varepsilon_{pmk} \varepsilon_{qsl} C_{ijkl} H_{jn,is} \Omega_p dL'_q dt'_{(mn)}$$

$$- \int\int_{S(t')} \varepsilon_{pmk} C_{ijkl} H_{jn,ils} \Omega_p dS'_s dt'_{(mn)} - \int\oint_{L(t')} \varepsilon_{pmk} \varepsilon_{qsl} \varrho \dot{H}_{kn,s} \Omega_p v'_l dL'_q dt'_{(mn)}$$

$$+ \int\int_{S(t')} \varepsilon_{pmk} \varrho \ddot{H}_{kn,s} \Omega_p dS'_s\, dt'_{(mn)} = - \int\oint_{L(t')} \varepsilon_{pmk} \varepsilon_{qsl} (C_{ijkl} H_{jn,is}$$

$$+ \varrho \dot{H}_{kn,s} v'_l) \Omega_p dL'_q dt'_{(mn)} - \int\int_{S(t')} \varepsilon_{pmk} (C_{ijkl} H_{jn,il} - \varrho \ddot{H}_{kn})_{,s} \Omega_p dS'_s dt'_{(mn)}.$$

Here the first equality follows by Stokes' theorem and Eq. (5.6) with $A = \dot{H}_{kn,s}$, and the second by a rearrangement of terms. By Eq. (4.4) the second line in the last expression vanishes. Hence we find from Eq. (5.5) the elastic strain due to a moving discrete defect line

$$(5.7)\quad e_{mn} = \int\limits_{L(t')}\oint \varepsilon_{pmk}(C_{ijkl}G_{jn,i}+\varrho\dot{G}_{ln}v'_k)\{b_l+\varepsilon_{lqr}\Omega_q(x'_r-x^0_r)\}dL'_p dt'_{(mn)}$$

$$-\int\limits_{L(t')}\oint \varepsilon_{pmk}\varepsilon_{qsl}(C_{ijkl}H_{jn,is}+\varrho\dot{H}_{kn,s}v'_l)\Omega_p dL'_q dt'_{(mn)},$$

where now v'_k is the velocity of the defect line $L(t')$. This relation could of course also be obtained more directly by a volume integration from Eqs. (4.6) and (5.20) to (5.23) of [31]. This is the basic relation we sought in this section.

5.2. The elastic bend-twist

To find the elastic bend-twist we start with the derivative of the total distortion from Eq. (5.4):

$$(5.8)\quad u^T_{n,ms} = \int\limits_{L(t')}\oint \varepsilon_{pmk}(C_{ijkl}G_{jn,is}+\varrho\dot{G}_{ln,s}v'_k)\{b_l+\varepsilon_{lqr}\Omega_q(x'_r-x^0_r)\}dL'_p dt'$$

$$+\int\limits_{S(t')}\int \varepsilon_{pmk}(C_{ijkl}G_{jn,is}+\varrho\dot{G}_{kn,s}v'_l)\Omega_p dS'_l dt'+\beta^*_{mn,s}.$$

By Stokes' theorem and Eq. (5.6) with $A=\dot{G}_{kn}$ the above surface integral becomes

$$\int\limits_{L(t')}\oint \varepsilon_{pmk}\varepsilon_{qsl}C_{ijkl}G_{jn,i}\Omega_p dL'_q dt'+\int\limits_{S(t')}\int \varepsilon_{pmk}C_{ijkl}G_{jn,il}\Omega_p dS'_s dt'$$

$$+\int\limits_{L(t')}\oint \varepsilon_{pmk}\varepsilon_{qsl}\varrho\dot{G}_{kn}\Omega_p v'_l dL'_q dt'-\int\limits_{S(t)}\int \varepsilon_{pmk}\ddot{G}_{kn}\Omega_p dS'_s dt'$$

$$=\int\limits_{L(t')}\oint \varepsilon_{pmk}\varepsilon_{qsl}(C_{ijkl}G_{jn,i}+\varrho\dot{G}_{kn}v'_l)\Omega_p dL'_q dt'+\varepsilon_{pmn}\phi^*_{sp},$$

where we have used Eqs. (2.5) and (5.8) of [31]. Hence we have for Eq. (5.8)

$$(5.9)\quad u^T_{n,ms} = \int\limits_{L(t')}\oint \varepsilon_{pmk}(C_{ijkl}G_{jn,is}+\varrho\dot{G}_{ln,s}v'_k)\{b_l+\varepsilon_{lqr}\Omega_q(x'_r-x^0_r)\}dL'_p dt'$$

$$+\int\limits_{L(t')}\oint \varepsilon_{pmk}\varepsilon_{qsl}(C_{ijkl}G_{jn,i}+\varrho\dot{G}_{kn}v'_l)\Omega_p dL'_q dt'+\beta^*_{mn,s}+\varepsilon_{pmk}\phi^*_{sp}.$$

Now we find from Eqs. (4.9) and (5.12) of [31] that

$$(5.10)\quad \varkappa_{st} = 1/2\,\varepsilon_{tmn}(u^T_{n,ms}-\beta^*_{mn,s})-\phi^*_{st}.$$

Thus the elastic bend-twist of a moving discrete defect line is

$$(5.11)\quad \varkappa_{st} = 1/2\int\limits_{L(t')}\oint \varepsilon_{tmn}\varepsilon_{pmk}(C_{ijkl}G_{jn,is}+\varrho\dot{G}_{ln,s}v'_k)\{b_l+\varepsilon_{lqr}\Omega_q(x'_r-x^0_r)\}dL'_p dt'$$

$$+1/2\int\limits_{L(t')}\oint \varepsilon_{tmn}\varepsilon_{pmk}\varepsilon_{qsl}(C_{ijkl}G_{jn,i}+\varrho\dot{G}_{kn}v'_l)\Omega_p dL'_q dt'.$$

This relation could also have been obtained directly by substituting expressions (5.20) to (5.23) of [31] into Eq. (4.10).

5.3. The elastic velocities

We next wish to find expressions for the elastic velocities. From Eq. (5.1) we find the total linear velocity

$$(5.12)\qquad \dot{u}_n^T = \iint_{S(t')} (C_{ijkl}\dot{G}_{jn,i} + \varrho \ddot{G}_{ln} v_k')\{b_l + \varepsilon_{lqr}\Omega_q(x_r' - x_r^0)\}\, dS_k'\, dt'.$$

Now by Eq. (A2) of [31] we have

$$(5.13)\qquad \frac{\partial}{\partial t'} \int_{S(t')} C_{ijkl} G_{jn,i}\{b_l + \varepsilon_{lqr}\Omega_q(x_r' - x_r^0)\}\, dS_k'$$

$$= -\oint_{L(t')} \varepsilon_{pmk} C_{ijlk} G_{jn,i}\{b_l + \varepsilon_{lqr}\Omega_q(x_r' - x_r^0)\} v_m\, dL_p' - \int_{S(t')} C_{ijkl}\dot{G}_{jn,i}\{b_l + \varepsilon_{lqr}\Omega_q(x_r' - x_r^0)\, dS_k'$$

$$- \int_{S(t')} C_{ijkl} G_{jn,ik}\{b_l + \varepsilon_{lqr}\Omega_q(x_r' - x_r^0)\} v_p'\, dS_p'.$$

Thus by a partial integration over t' Eq. (5.12) becomes

$$(5.14)\qquad \dot{u}_r^T = -\int\oint_{L(t')} \varepsilon_{pmk} C_{ijkl} G_{jn,i}\{b_l + \varepsilon_{lqr}\Omega_q(x_r' - x_r^0)\} v_m'\, dL_p'\, dt'$$

$$-\iint_{S(t')} (C_{ijkl} G_{jn,ik} - \varrho \ddot{G}_{ln})\{b_l + \varepsilon_{lqr}\Omega_q(x_r' - x_r^0)\} v_p\, dS_p'\, dt'.$$

The second line in this expression equals v_n^* by Eqs. (2.5) and (5.9) of [31]. Therefore we find from Eqs. (4.11), (4.3), and (5.13) of [31] the linear elastic velocity for a moving discrete defect line

$$(5.15)\qquad v_n(\mathbf{r}, t) = -\int\oint_{L(t')} \varepsilon_{pmk} C_{ijkl} G_{jn,i}(\mathbf{R}, T)\{b_l + \varepsilon_{lqr}\Omega_q(x_r' - x_r^0)\} v_m'(\mathbf{r}', t')\, dL_p'\, dt'.$$

Again the same result could have been found directly from Eqs. (4.12) and (5.22) of [31]. Note that it is not identical to the result (3.15) for dislocations only.

To find the elastic rotational velocity, we first take the time derivative of Eq. (5.4)

$$(5.16)\qquad \dot{u}_{n,m}^T = \int\oint_{L(t')} \varepsilon_{pmk}(C_{ijkl} G_{jn,i} + \varrho \dot{G}_{ln} v_k')\{b_l + \varepsilon_{lqr}\Omega_q(x_r' - x_r^0)\}\, dL_p\, dt'$$

$$+ \iint_{S(t')} \varepsilon_{pmk}(C_{ijkl}\dot{G}_{jn,i} + \varrho \ddot{G}_{kn} v_l')\Omega_p\, dS_l'\, dt' + \dot{\beta}_{mn}^*.$$

Now by Eq. (A2) of [31]

$$(5.17)\qquad \frac{\partial}{\partial t'} \int_{S(t')} G_{jn,i}\, dS_l' = = \oint_{L(t')} \varepsilon_{qsl} G_{jn,i} v_s'\, dL_q' - \int_{S(t')} (\dot{G}_{jn,i}\, dS_l' + G_{jn,il} v_q'\, dS_q').$$

Therefore the surface integral in Eq. (5.16) becomes

$$-\int\oint_{L(t')} \varepsilon_{pmk}\varepsilon_{qsl} C_{ijkl} G_{jn,i}\Omega_p v_s'\, dL_q'\, dt',$$

$$-\iint_{S(t')} \varepsilon_{pmk}(C_{ijkl} G_{jn,il} - \varrho \ddot{G}_{kn})\Omega_p v_q'\, dS_q'\, dt'.$$

The second line in this expression equals $\varepsilon_{pmn}\psi_p^*$ by Eqs. (2.5) and (5.10) of [31]. Therefore we have for Eq. (5.16)

$$(5.18)\qquad \dot{u}^T_{n,m} = \int\oint_{L(t')} \varepsilon_{pmk}(C_{ijkl}\dot{G}_{jn,i}+\varrho\ddot{G}_{ln}v'_k)\{b_l+\varepsilon_{lqr}\Omega_q(x'_r-x^0_r)\}\,dL'_p\,dt' - \int\oint_{L(t')} \varepsilon_{pmk}\varepsilon_{qsl}C_{ijkl}G_{jn,i}\Omega_p v'_s\,dL'_q\,dt' + \dot{\beta}^*_{mn} + \varepsilon_{pmn}\dot{\psi}^*_p.$$

Now from Eqs. (4.15) and (5.14) of [31] it follows that

$$(5.19)\qquad w_t = 1/2\,\varepsilon_{tmn}(\dot{u}^T_{n,m}-\dot{\beta}^*_{mn})-\dot{\psi}^*_t.$$

Therefore the rotational velocity for a moving discrete defect line is

$$(5.20)\qquad w_t = 1/2\int\oint_{L(t')} \varepsilon_{tmn}\varepsilon_{pmk}(C_{ijkl}\dot{G}_{jn,i}+\varrho\ddot{G}_{ln}v'_k)\{b_l+\varepsilon_{lqn}\Omega_q(x'_r-x^0_r)\}\,dL'_p\,dt' - 1/2\int\oint_{L(t')} \varepsilon_{tmn}\varepsilon_{pmk}\varepsilon_{qsl}C_{ijkl}G_{jn,i}\Omega_p v'_s\,dL'_q\,dt'.$$

The same result could have been found directly from Eqs. (4.16) and (5.20) to (5.23) of [31]. By Eqs. (2.3), (5.22) and (A3) of [31] this relation can be reduced to the relation

$$(5.21)\qquad w_t = -1/2\,\varepsilon_{tmn}\left[\int\oint_{L(t')} \varepsilon_{psk}C_{ijkl}G_{jn,im}\{b_l+\varepsilon_{lqr}\Omega_q(x'_r-x^0_r)\}v'_s\,dL'_p\,dt' + J_{mn}\right],$$

that can also be obtained directly from Eqs. (4.17) and (5.22) of [31] or (4.18) and (5.15).

We see that the basic elastic fields (i.e. the strain e_{mn}, the bend-twist $\varkappa_{st}$, the linear and rotational velocities v_n and w_t) can be written as line integrals along the discrete moving defect line $L(t)$, i.e. they are expressed entirely in terms of integrals over the only regions of the body where the defect is localized, and the position of the surface $S(t)$ is immaterial. Hence they are state quantities, because in this case the defect is localized on the line $L(t)$.

Section 5 has extended the results of a moving discrete dislocation line of Sect. 3.2 to a moving discrete *defect* line, whereas it has extended the results of a stationary discrete defect line (Sect. 5 of Ref. [14]) to a *moving* discrete defect line. The central results obtained are closed line integrals for the basic elastic fields, the strain (5.7), the bend-twist (5.11), the linear velocity (5.15), and the rotational velocity (5.20) and (5.21). These expressions are in a form that is directly applicable to special geometries.

6. Relation to the incompatibility problem

6.1. The elastic fields

In this section we find the elastic strain and velocity for a given incompatibility and its current. First we find an expression for the total distortion,

$$(6.1)\qquad u^T_{n,m} = -\oint(C_{ijkl}G_{jn,im}e^P_{kl}-\varrho\dot{G}_{ln,m}v^P_l)\,dV'dt' = \int(C_{ijkl}H_{jn,is}e^P_{kl,m's'} - \varrho\dot{H}_{ln,s}v^P_{l,m's'})\,dV'dt' = \int[C_{ijkl}H_{jn,is}(\varepsilon_{pmk}\varepsilon_{qsl}\eta_{pq}+e^P_{ml,k's'}+e^P_{ks,m'l'}-e^P_{ms,k'l'}) - \varrho\dot{H}_{ln,s}(F_{sml}+\dot{e}^P_{ml,s'}+\dot{e}^P_{ls,m'}-\dot{e}^P_{sm,l'})]\,dV'dt' = \int(\varepsilon_{pmk}\varepsilon_{qsl}C_{ijkl}H_{jn,is}\eta_{pq}$$

$$-\varrho \dot{H}_{ln,s} F_{sml}) dV'dt' + \int (C_{ijkl} H_{jn,ik} - \ddot{H}_{ln})(e^P_{ml,s's'} + e^P_{ls,m's'} - e^P_{sm,l's'}) dV'dt'$$
$$= \int (\varepsilon_{pmk}\varepsilon_{qsl} C_{ijkl} H_{jn,is}\eta_{pq} - \varrho \dot{H}_{ln,s} F_{sml}) dV'dt' + e^P_{mn} - \int (2\pi R)^{-1} e^P_{s[n,m']s'} dV'.$$

Here the first equality follows from Eq. (2.8), the second from Eq. (4.3) and partial integrations, the third from Eqs. (6.7) and (6.8) of [31], the fourth by partial integrations, and the fifth by Eq. (4.3) and (4.4). So by Eqs. (6.5) and (6.1) of [31] we find

$$(6.2)\qquad e_{mn}(\mathbf{r}, t) = \int [\varepsilon_{pmk}\varepsilon_{qsl} C_{ijkl} H_{jn,is}(\mathbf{R}, T)\eta_{pq}(\mathbf{r}', t') - \varrho \dot{H}_{ln,s}(\mathbf{R}, T) F_{sml}(\mathbf{r}', t')] dV'dt'_{(mn)}.$$

This is the elastic strain for a given distribution of incompatibility η_{pq} and its current F_{sml}.

Next we find the total velocity,

$$(6.3)\qquad \dot{u}^T_n = -\int (C_{ijkl}\dot{G}_{jn,i} e^P_{kl} - \varrho \ddot{G}_{ln} v^P_l) dV'dt'$$
$$= \int [C_{ijkl} G_{jn}(F_{ikl} - v^P_{l,k'j'} + \dot{e}^P_{li,k'} - \dot{e}^P_{ik,l'}) + \varrho \ddot{G}_{ln} v^P_l)] dV'dt' = \int C_{ijkl} G_{jn} F_{ikl} dV'dt' + v^P_n.$$

Here the first equality follows from Eq. (2.8), the second from partial integration and Eq. (6.8) of [31] and the third from partial integrations, Eq. (2.5), and a cancellation. Hence by Eqs. (6.6) and (6.2) of [31] we find

$$(6.4)\qquad v_n(\mathbf{r}, t) = \int C_{ijkl} G_{jn}(\mathbf{R}, T) F_{ikl}(\mathbf{r}', t') dV'dt'.$$

This is the linear elastic velocity for a given incompatibility current.

Again, the elastic strain (6.2) and velocity (6.4) are state quantities because these expressions are integrals taken entirely over the incompatibility η_{pq} and its current F_{nkl}, which are also state quantities.

6.2. Consistency with defect theory

In this section we wish to show that Eqs. (6.2) and (6.4) are consistent with their counterparts in the defect theory, Eqs. (4.6) and (4.12). First consider the elastic strain, Eq. (6.2):

$$(6.5)\qquad e_{mn} = \int [\varepsilon_{pmk} C_{ijkl} H_{jn,is}(K_{lp,s'} - K_{sp,l'} - \varepsilon_{qsl}\theta_{qp}) + \varrho \dot{H}_{kn,s}(J_{mk,s'}$$
$$+ \varepsilon_{pmk}(S_{sp} + \dot{K}_{sp}))] dV'dt'_{(mn)} = \int [\varepsilon_{pmk} C_{ijkl}(H_{jn,iss} K_{lp} - \varepsilon_{qsl} H_{jn,is}\theta_{qp})$$
$$+ \varrho(\dot{H}_{kn,ss} J_{mk} + \varepsilon_{pmk}\dot{H}_{kn,s} S_{sp}) - \varepsilon_{pmk}(C_{ijkl} H_{jn,il} - \varrho \ddot{H}_{kn}) K_{sp,s'}] dV'dt'_{(mn)}$$
$$= \int [\varepsilon_{pmk} C_{ijkl}(G_{jn,i}\alpha_{pl} - \varepsilon_{qsl} H_{jn,is}\theta_{qp}) - \varrho(\dot{G}_{kn} J_{mk} - \varepsilon_{pmk}\dot{H}_{kn,s} S_{sp})] dV'dt'_{(mn)}.$$

Here the first equality follows from Eqs. (6.15) and (6.16) of [31], the second by partial integrations and rearrangements, and the third from Eq. (4.26) of [31], Eqs. (4.3) and (4.4). This relation is identical to Eq. (4.6), q.e.d.

Next we find the velocity from Eqs. (6.4) and (6.14) of [31],

$$(6.6)\qquad v_n = -\int C_{ijkl} G_{jn} J_{kl,i'} dV'dt' = -\int C_{ijkl} G_{jn,i} J_{kl} dV'dt',$$

by partial integration. This relation is identical to Eq. (4.12), q.e.d.

These results then show that the dynamic incompatibility theory is completely consistent with disclination dynamics.

7. Summary

We started this paper with the general solution of the dynamic plastic strain problem which is a generalization of Eshelby's transformation problem and very similar to Mura's plastic strain problem. It formed the basis for all dynamic defect fields. We then reviewed dislocation dynamics, including the continuous distribution and the discrete line. This introductory material formed the point of departure for the general theory of disclination dynamics. The latter was renamed defect dynamics because it is a theory that combines disclination and dislocation dynamics.

We derived closed volume-time integrals for the basic elastic fields in terms of the defect densities and their currents. These integrals contain kernels with the dynamic Green's tensor. For the elastic strain we also used as kernel a newly introduced quantity, the dynamic Green's potential tensor. These integral expressions for the basic elastic fields will form the basis for applications to special cases.

We derived the basic elastic fields for a moving discrete defect line as closed line-time integrals along the defect line. These integrals also contained Green's tensor as kernels and, in particular, for the elastic strain we also had to use the Green's potential as a kernel. These integral expressions for the basic elastic fields are in a form that is directly applicable to special geometries.

Finally we compared disclination dynamics with the dynamic incompatibility problem. We identified the relations between the defect densities and their currents and the incompatibility tensor and its current. We showed that the dynamics of the two theories was consistent.

So we have presented a general theory of defect (disclination) dynamics for a linearly elastic, infinitely extended, homogeneous body. The major shortcoming of the present treatment might be the use of the linear theory. This means that in a real solid the resulting fields close to discrete defects will deviate considerably from our formulas, but these fields will become more realistic the further away we are from a defect. However, without the linear assumption we certainly could not have pushed the theory as far as we did. This is the price we paid for a fairly complete analytic treatment which we think might have its usefulness.

Within its clearly prescribed limitations the present theory is completely self-consistent. Aside from its possible intrinsic usefulness, it can be used as the starting point for further generalizations, such as nonlinear effects, couple-stresses, a finite body, or inhomogeneities.

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Received May 19, 1976.