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Nonlinear differential equations with fractional damping with applications to the 1dof and 2dof pendulum

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Summary. Existence, uniqueness and dissipativity is established for a class of nonlinear dynamical systems including systems with fractional damping. The problem is reduced to a system of fractional-order differential equations for numerical integration. The method is applied to a nonlinear pendulum with fractional damping as well as to a nonlinear pendulum suspended on an extensible string. An example of such a fractional damping is a pendulum with the bob swinging in a viscous fluid and subject to the Stokes force (proportional to the velocity of the bob) and the Basset-Boussinesq force (proportional to the Caputo derivative of order 1/2 of the angular velocity). An existence and uniqueness theorem is proved and dissipativity is studied for a class of discrete mechanical systems subject to fractional-type damping. Some particularities of fractional damping are exhibited, including non-monotonic decay of elastic energy. The 2:1 resonance is compared with nonresonant behavior.

Notation

D^α	Caputo derivative
$u' = \mathrm{D}u$	ordinary derivative; $f * g(t) = \int_0^t f(t-s)g(s)\mathrm{d}s$
ϕ	libration angle
ξ	relative spring extension in an extensible pendulum

1 Introduction

Fractional models of viscous damping play an important role in engineering, seismic wave attenuation and polymer rheology. Fractional derivatives represent the singularity of a hereditary viscous kernel, which is its most important aspect, responsible for qualitative differences between singular and regular memory models. Most studies of fractional viscoelasticity are based on analytic methods, which are applicable to linear models [1], [2], [3].

Nonlinearity is another important aspect of engineering structures and rheology. Nonlinearity can be studied jointly with kernel singularity by numerical methods. In this paper we focus on simple discrete dynamical systems. In numerical modeling distributed dynamical systems are typically reduced to discrete systems with many degrees of freedom by the application of the Galerkin method and the focus shifts to the propagation of the oscillations. Furthermore, nonlinear mechanical systems with fractional damping include well established physical effects such as a pendulum swinging in a viscous fluid and subject to the Stokes force and the Basset-Boussinesq force [4].

In addition to a variety of examples and comparisons, we shall also discuss general conditions for well-posedness and dissipativity of the initial-value problems representing discrete mechanical systems. By choosing a mechanical system with two degrees of freedom for numerical analysis we are able to demonstrate interaction of fractional damping with resonances, nonlinear interactions and viscous coupling.

Last but not least, the paper also serves as a demonstration of a method of integrating systems of ordinary differential equations with fractional derivatives using a recently developed predictor-corrector algorithm [5].

For accuracy test the elastic energy is numerically determined and compared with the accumulated energy dissipation. The energy of a mechanical system with fractional damping – like in the case of a viscoelastic medium [6]–[8] – is not uniquely defined by the equations of motion. In view of the numerical algorithm applied we have chosen as the energy of the pendulum the energy of the Hamiltonian system obtained by deleting the fractional damping terms from the equations of motion. This energy does not decay monotonically due to the inertial and elastic effects implicit in fractional damping [9]. In the case of fractional damping of order $0 < \alpha \leq 1$ it is however possible to construct a monotonically decaying energy [10]. Such an energy is a history functional, or, equivalently, it depends on internal variables which cannot be determined by the numerical algorithm adopted in this paper. It can however be determined if an algorithm based on an idea of Yuan and Agrawal [11], [12] is used.

We shall consider the generic initial-value problem (IVP)

$$Au'' + \sum_{k=1}^{N} B_k \mathrm{D}^{\alpha_k} u = f(t, u), \tag{1}$$

$$u(0) = c_0; \quad \mathrm{D}u(0) = c_1 \tag{2}$$

for u: $[0, T] \to \mathbb{R}^d$, where $\mathrm{D} = \mathrm{d}/\mathrm{d}t$, $\mathrm{D}^{\alpha_k} u$ denotes the Caputo fractional derivative of order α_k ([13] and Eq. (9) below), $0 < \alpha_k < 2, A, B_k$ represent $d \times d$ matrices and f is a function defined on a subset of $[0, T] \times \mathbb{R}^d$ with values in $\mathbb{R}^d$, Lipschitz-continuous with respect to the second argument. The choice of the Caputo fractional derivative is consistent with the IVP formulation in terms of initial values of u and u' [24].

In particular, Eq. (1) can represent a nonlinear damped pendulum. A distributed system described in terms of a system of partial differential equations with fractional time derivatives can also be reduced to Eq. (1) by applying the Galerkin method in a suitable basis of functions of the spatial coordinates, such as orthogonal polynomials [14] or finite element shape functions [15].

2 Existence and uniqueness

Consider a class of initial-value problems

$$u'' + K_1 * u'' + Cu' + K_2 * u' + Gu = f, \tag{3}$$

$$u(0) = u_0; \qquad u'(0) = v_0, \tag{4}$$

where K_1, K_2 are matrix-valued kernels. Let χ_α denote the homogeneous distribution on the real line

$$\chi_\alpha(t) := \begin{cases} t^{\alpha-1}/\Gamma(\alpha) & t > 0 \\ 0 & t < 0 \end{cases} \tag{5}$$

[16]. Fractional damping models correspond to the following special case:

$$\begin{aligned} K_1 &= \sum_{k=1}^{n} \chi_{2-\alpha_k} B_k, \qquad 1 < \alpha_k < 2,\ k = 1, \ldots, n, \\ K_2 &= \sum_{l=1}^{m} \chi_{1-\beta_l} E_l, \qquad 0 < \beta_l < 1,\ l = 1, \ldots, m \end{aligned} \tag{6}$$

equivalent to

$$u'' + \sum_{k=1}^{n} B_k \mathrm{D}^{\alpha_k} u + Cu' + \sum_{l=1}^{m} E_l \mathrm{D}^{\beta_l} u + Gu = f, \tag{7}$$

$$u(0) = u_0; \qquad u'(0) = v_0, \tag{8}$$

where D^α denotes the Caputo fractional derivative

$$\mathrm{D}^\alpha f = \chi_{[\alpha]-\alpha+1} * \mathrm{D}^{[\alpha]} f \tag{9}$$

($[\alpha] := \sup\{n \in \mathbb{Z}_+ \mid n \leq \alpha\}$). We shall later need the assumption that $B_k, C, E_l, G \geq 0$. A $d \times d$ matrix inequality $M \geq 0$ is defined as $\langle v, Mv \rangle \geq 0$ for every $v \in \mathbb{R}^d$. An apparently more general problem

$$Au'' + \sum_{k=1}^{n} B_k \mathrm{D}^{\alpha_k} u + Cu' + \sum_{l=1}^{m} E_l \mathrm{D}^{\beta_l} u + Gu = f, \tag{10}$$

$$u(0) = u_0; \qquad u'(0) = v_0 \tag{11}$$

with $A > 0$ can be reduced to Eq. (7) by replacing the matrices B_k, C, E_l, G by $A^{-1/2} B_k A^{-1/2}, \ldots$ and u by $U = A^{1/2} u$. The kernels K_1, K_2 are in general not integrable over $\mathbb{R}_+$. The kernels (6) are however completely monotonic [17]

$$(-1)^n \mathrm{D}^n K_j(t) \geq 0 \qquad \forall t > 0, \quad \forall n \geq 0, \quad j = 1, 2$$

and locally integrable.

Theorem 1: Let $K_1, K_2 \in \mathscr{L}^1_{\mathrm{loc}}$.

Equation (3) has a unique solution $u \in W^1_{\mathrm{loc}}$.

Proof: Let R_1 be the resolvent of K_1 [18]. The resolvent equation

$$R_1 + R_1 * K_1 = K_1 \tag{12}$$

implies that

$$u'' = R_1 * F - F, \tag{13}$$

where

$$F := Cu' + K_2 * u' + Gu - f.$$

Equation (13) can be rewritten in terms of an auxiliary variable $v := u'$,

$$w_{,t} + Xw + Y * w = g, \tag{14}$$

$$w(0) = w_0, \tag{15}$$

where

$$w = \begin{bmatrix} u \\ v \end{bmatrix}, \quad w_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \tag{16}$$

$$X := \begin{bmatrix} 0, & -1 \\ G, & C \end{bmatrix}, \tag{17}$$

$$Y := \begin{bmatrix} 0, & 0 \\ -R_1 G, & K_2 - R_1 * K_2 - R_1 C \end{bmatrix}, \tag{18}$$

$$g = \begin{bmatrix} 0 \\ f \end{bmatrix}. \tag{19}$$

By the variation of constants formula the solution of Eq. (14) satisfies the Volterra equation

$$w + \mathrm{e}^{-tX} * Y * w = \mathrm{e}^{-tX} w_0 + \int_0^\infty \mathrm{e}^{-X(t-s)} g(s) ds. \tag{20}$$

Since $\mathrm{e}^{-tX} * Y \in \mathscr{L}^1_{\mathrm{loc}}$, Eq. (20) has a unique solution $w \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$.

The kernel K_2 given by Eq. (6) is nonnegative, nonincreasing and convex on $\mathbb{R}_+$. □

Stronger results can be obtained for the cases of either $K_1 = 0$ or $K_2 = 0$, with the remaining kernel K_2 or K_1 nonnegative, nonincreasing and convex. We recall that a matrix-valued function $M(t)$ is said to be nonnegative, nonincreasing and convex if for every vector y the real-valued function $y^\mathsf{T} M(t) y$ is nonnegative, nonincreasing and convex. Note that the kernels (6) have the required properties if

$$B_k, C, E_l, G \geq 0. \tag{21}$$

Theorem 2: If one of the following conditions is satisfied:

(i) $K_2 = 0$, $K_1 = K_{11} + K_{12}$, $K_{12} \in \mathscr{L}^1(\mathbb{R}_+)$, K_{11} is nonnegative, nonincreasing and convex and

$$\det\left\{s^2\left[1 + \tilde{K}_1(s)\right] + sC + G\right\} \neq 0;$$

(ii) $K_1 = 0$, $K_2 = K_{21} + K_{22}$, $K_{22} \in \mathscr{L}^1(\mathbb{R}_+)$, K_{21} is nonnegative, nonincreasing and convex and

$$\det\left\{s^2 I + sC + s\tilde{K}_2(p) + G\right\} \neq 0$$

and $f \in \mathscr{L}^q(\mathbb{R}_+)$, then $u \in W^2_q(\mathbb{R}_+)$.

Proof:

Case (i):

The equation

$$u'' + K_1 * u'' + Cu' + Gu = f$$

can be solved for u'' by applying the resolvent R_1 of K_1:

$$u'' + Cu' + Gu - R_1 * Cu' - R_1 * Gu = f - R_1 * f.$$

By the Shea-Wainger theorem ([19, Theorem 1]) $R_1 \in \mathscr{L}^1(\mathbb{R}_+) \cap \mathscr{C}(\overline{\mathbb{R}_+})$. An equivalent first-order system

$$u' - v = 0,$$

$$v' + Cv + gu - R_1 * Cv - R_1 * Gu = f - R_1 * f$$

has the form of (14). The solution (20) involves convolutions of R_1 with exponential functions. Such functions are integrable. The assertion of the theorem follows from the fact that convolution with an integrable kernel maps a Lebesgue space $\mathscr{L}^q(\mathbb{R}_+)$ into itself, for $1 \leq q \leq \infty$.

Case (ii):

Let R_2 denote the differential resolvent of K_2, i.e., the solution of the problem

$$R_2' + K_2 * R_2 = 0; \qquad R_2(0) = I,$$

where the prime denotes the derivative. Since K_2 is nonnegative, nonincreasing, convex on $\mathbb{R}_+$, integrable on $[0, 1]$ and $\det\left[pI + \tilde{K}\right] \neq 0$, Theorem 1 [19] implies that $R_2 \in \mathscr{L}^1(\mathbb{R}_+)$. Setting $v := u'$ and applying the differential resolvent to the equation

$$\mathrm{D}v + Cv + K_2 * v + Gu = f$$

with the initial condition $v(0) = v_0$, we get an equivalent formulation of the same problem

$$v + R_2C * v = R_2v_0 + R_2 * f - R_2G * u,$$

where $R_2G * u := R_2 * (Gu)$. Since

$$\det\left[1 + \tilde{R}_2(p)C\right] = \det\left[pI + \tilde{K}_2(p)\right]^{-1}\det\left[pI + C + \tilde{K}_2(p)\right] \neq 0$$

the Paley-Wiener theorem [20] implies the existence of a resolvent $R_3 \in \mathscr{L}^1(\mathbb{R}_+)$ of the kernel R_2C. Consequently

$$v = R_4v_0 + R_4 * f - R_4G * u,$$

where $R_4 := R_2 - R_3 * R_2 \in \mathscr{L}^1(\mathbb{R}_+) \cap \mathscr{C}(\overline{\mathbb{R}_+})$. □

In the following sections, we shall consider the nonlinear generalization of Eq. (3) with $f(t) = g(t, u(t), \mathrm{D}u(t))$. It is sufficient to assume that the substitution operator $N : (u(\cdot), v(\cdot)) \to g(\cdot, u(\cdot), v(\cdot))$ is a Nemytskii operator [21], i.e., g is a Carathéodory function on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ (measurable with respect to the first argument, jointly continuous in the last two arguments for almost all the values of the first argument), satisfying a growth condition

$$|g(t, u, v)| \leq a(t) + b|u|^{p/q} + c|v|^{p/q}$$

$p, q \geq 1,\ a \in \mathscr{L}^q,\ b, c \geq 0$. The substitution operator maps $\mathscr{L}^p$ into $\mathscr{L}^q$. The inclusion $\mathscr{L}^q([0, T]) \subset \mathscr{L}^p([0, T])$ holds for an arbitrary positive number T and $q \leq p$ and the convolution with an integrable resolvent kernel R maps $\mathscr{L}^p(\mathbb{R}_+)$ into itself. In the applications below $p/q = 2$. Since the kernel is also bounded on $[0, T]$ for a sufficiently small T, the Banach contraction theorem can be applied to show the existence and uniqueness of a local solution $u \in W_p^2([0, T])$.

3 Dissipativity

We now turn to dissipativity of Eq. (1), assuming that f is a potential operator, $f(t, u) = -\nabla_u F(u)$ and that the orders of the fractional derivatives satisfy the inequality $0 < \alpha_k < 1$. Multiplying Eq. (1) by u'^{T} we have

$$\frac{d}{dt}\left[\frac{1}{2}(u')^2 + F(u)\right] + u'^{\mathsf{T}}Cu' + \sum_{k=1}^{n} \frac{1}{\Gamma(1 - \alpha_k)} u'^{\mathsf{T}} B_k t^{-\alpha_k} * u' = 0. \tag{22}$$

The kernels $t^{-\alpha_k}$ are of positive type [18], $C, B_k \geq 0$, hence the elastic energy cannot increase above its original value

$$\left[\frac{1}{2}(u')^2 + F(u)\right]_0^T = -D([0,T]) \le 0, \tag{23}$$

where $D([0,T])$ denotes the accumulated dissipated energy

$$D([0,T]) := \int_0^T \left[u'^{\mathsf{T}} C u' + \sum_{k=1}^{n} \frac{1}{\Gamma(1-\alpha_k)} u'^{\mathsf{T}} B_k t^{-\alpha_k} * u' \right] \mathrm{d}t. \tag{24}$$

In order to prove dissipativity for an operator containing fractional derivatives of order $1 < \alpha < 2$, we consider the quadratic form defined for arbitrary smooth compactly supported functions $\phi : \mathbb{R} \to \mathbb{R}$,

$$Q[\phi] := \int_{-\infty}^{T} \phi(t) \chi_{2-\alpha} * \phi'(t) \mathrm{d}t,$$

where

$$\chi_\gamma(t) := \begin{cases} t^{\gamma-1}/\Gamma(\gamma) & t > 0 \\ 0 & t \le 0 \end{cases}$$

for arbitrary $\gamma \in \mathbb{C}$ [16]. Integration of the convolution integral by parts in the distributions sense yields another expression for $Q[\phi]$:

$$Q[\phi] = \int_{-\infty}^{T} \phi(t) \chi_{1-\alpha} * \phi(t) \mathrm{d}t.$$

Since $\chi_{1-\alpha}$ is a homogeneous distribution, it belongs to the Schwartz space $\mathscr{S}'(\mathbb{R})$ ([22], vol. I, Sec. 7.1). The Laplace transform of $\chi_{1-\alpha}$ is $s^{\alpha-1}$ and its real part $|s|^{\alpha-1}\cos((\alpha-1)\psi)$, $\psi = \arg s$, is nonnegative in the right half-plane $-\pi/2 \le \psi \le \pi/2$ if $1 < \alpha < 2$. By Theorem 8.3.11 in [23] $Q[\phi] \ge 0$ for every $\phi \in \mathscr{L}^2(\mathbb{R};\mathbb{R})$. In particular for $\phi(t) = u'(t)$, $t > 0$, $\phi(t) = 0$ for $t < 0$ the accumulated dissipated energy counted from $t = 0$ is nonnegative.

This shows that the accumulated dissipation is nonnegative if the dampers are represented by fractional time derivatives of order $0 < \alpha < 2$ with nonnegative coefficients. The elastic energy decreases with respect to its original value. In contrast with the classical dissipation model $C > 0$, $B_k, E_k = 0$ the decay of the elastic energy is not monotone. It is however possible to redefine the energy by adding a history-dependent part in such a way that the new energy decreases monotonely [10] or is conserved. Non-monotone energy decay can however be explained as due to a dissipative interaction of the pendulum with the damper. In this case the damper (the viscous fluid) returns part of the absorbed energy to the pendulum.

4 Reduction to a system of equations of fractional order

Lemma 3:

(1) If $0 < \alpha < 1$ and m is a positive integer then $D^\alpha D^m f = D^{\alpha+m} f$.
(2) If $m-1 < \beta < m$, $0 < \alpha < 1$, $\alpha + \beta \le m$ and $D^\beta f(0) = 0$, then $D^\alpha D^\beta f = D^{\alpha+\beta} f$.

We now consider an IVP

$$\sum_{k=1}^{m} A_k \mathrm{D}^{\alpha_k} u = f(u),$$

$$\mathrm{D}^k u(0) = u_k, \qquad k = 0, \ldots, n-1 \tag{25}$$

with $0 < \alpha_1 < \ldots \alpha_{k-1} < \alpha_k < \ldots \alpha_m$, $n - 1 < \alpha_m \leq n$,

$\{\alpha_k \mid k = 1, 2, \ldots m\} \subset \{p/q \mid p \in \mathbb{N}\}$

for some $q \in \mathbb{N}$. Let P be the largest value of p, $P/q = \alpha_m$.

We now define $y_p = \mathrm{D}^{p/q} u$ for all such p that the corresponding derivative appears explicitly either in Eq. (25) or in the initial data. In the latter case $p/q \in \mathbb{N}$. For every y_p which is a fractional derivative of u we impose the initial condition $y_p(0) = 0$. Lemma 3 implies that the IVP

$$\mathrm{D}^{1/q} y_0 - y_1 = 0$$

$$\ldots$$

$$\mathrm{D}^{1/q} y_{p-2} - y_{p-1} = 0 \tag{26}$$

$$A_m \mathrm{D}^{1/q} y_{P-1} + \sum_{p=1}^{P-1} y_p A_p - f(y_0) = 0$$

with the initial data for $y_p(0) = u_k$ such that $p/q = k \in \mathbb{N}$ and $y_p(0) = 0$ otherwise is equivalent to (25).

The homogeneous initial conditions for y_p such that $p/q \notin \mathbb{N}$ are justified a posteriori by the fact that the IVP (25) is equivalent to the original one. One can however prove that every solution $u \in \mathscr{C}^2$ satisfies these conditions [24], [25]. Existence, uniqueness and stability for equations $\mathrm{D}^\alpha u = f(t, u(t))$ is studied in [26]. In particular

Theorem 4: Let $\gamma > \beta > 0$ and suppose that F is a continuous function of $(t, Y, Z) \in [0, T] \times \Omega$, uniformly Lipschitz continuous with respect to $(Y, Z) \in \Omega$.

The initial-value problem

$$\mathrm{D}^\gamma Y = F(t, Y, \mathrm{D}^\beta Y), \tag{27}$$

$$Y(0) = Y_0 \tag{28}$$

for $Y : [0, T^*] \to \mathbb{R}^d$ has a unique solution for $T^* \leq T$, $T^* < \infty$ [27].

Equation (26) was integrated by the FraPECE algorithm described in [5] and [28]. The algorithm is a predictor-corrector algorithm of second-order accuracy. The error was controlled by checking the deviation from the energy balance

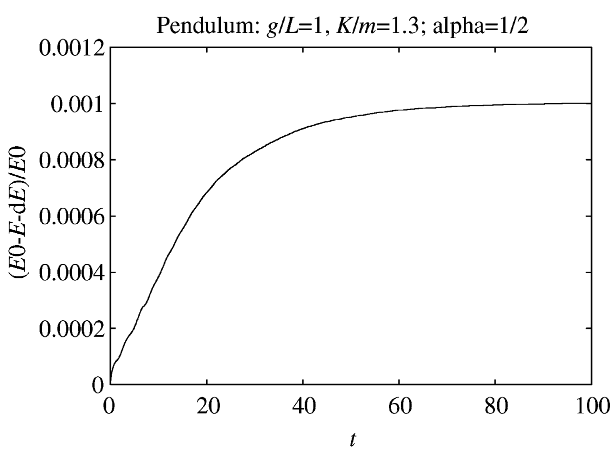


Fig. 1. The error measured by the ratio of $(E(0+) - E(t) - D([0, t]))/E(0+)$

$$E(0+) - E(t) - D([0,t]) \sim 0,$$

where $E(t)$ denotes the elastic energy at time t and the accumulated dissipated energy $D([0,t])$ is calculated by integrating an additional set of equations. In all the cases studied the error grows approximately linearly at the beginning and saturates at a value 10^{-4} for a step of 10^{-3} s (Fig. 1).

The order of the method is $1 + 1/q$ provided $\mathrm{D}^{1/q}u \in \mathscr{C}^2([0,T])$ [27].

5 Examples: pendulum with fractional damping

5.1 General remarks

The effects of fractional damping will be examined for the extensible and inextensible pendulum.

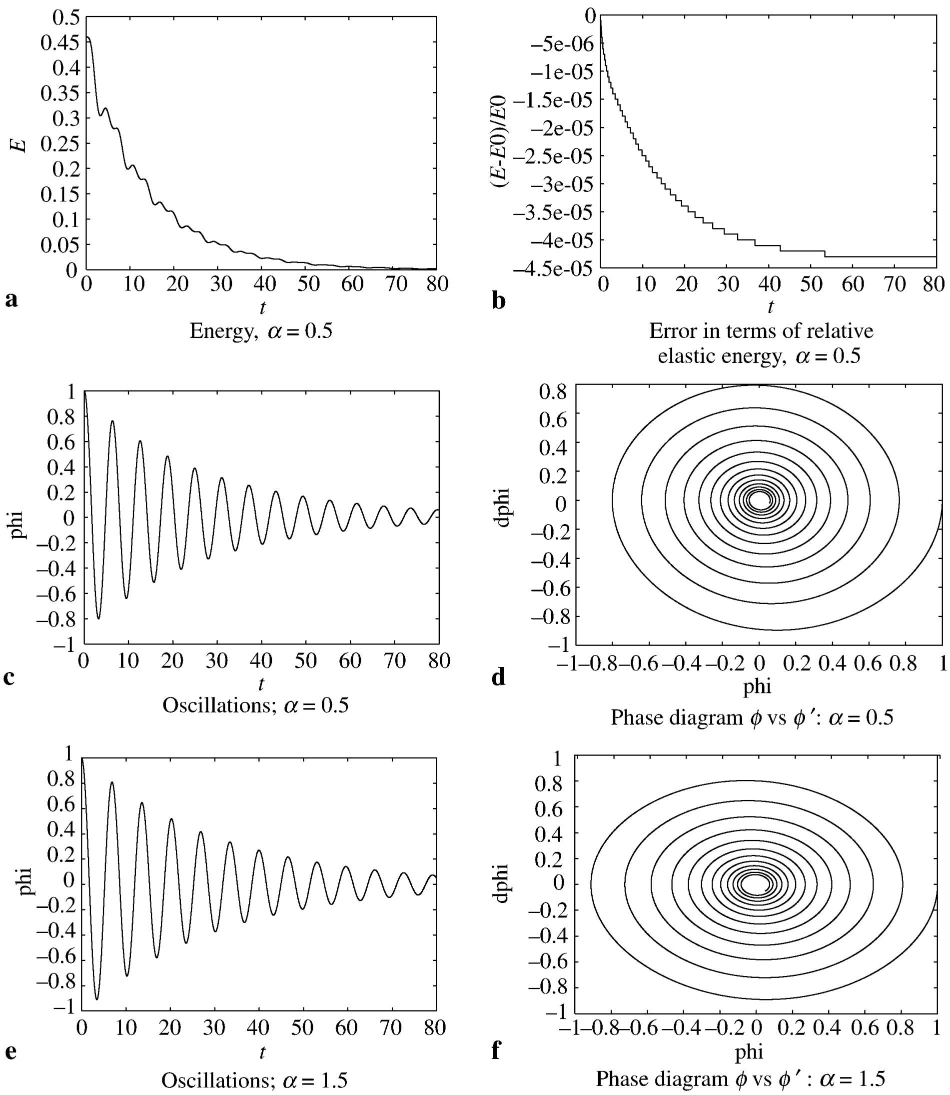


Fig. 2. Inextensible pendulum subject to ordinary and fractional damping of order 0.5 and 1.5

The main difference between fractional damping and first-order damping is due to nonlocality of the time derivatives. In agreement with Sect. 3 the elastic energy decays with respect to its initial value after the pendulum was deflected from the equilibrium. The elastic energy decays monotonely if and only if the order of the damping is 1. In this case it exhibits stationary points corresponding to the extremal positions of the pendulum (Fig. 4b). Fractional damping results in local energy minima at the extrema of the pendulum (Fig. 4a, c). The increases of the elastic energy are only temporary. Their origin can be traced back to the fact that the dissipation rate is proportional to the product $\phi' \mathrm{D}^{\alpha}\phi$, which changes sign, as can be seen from Fig. 3, due to the phase shift between the fractional derivatives.

5.2 Inextensible pendulum

The inextensible pendulum is defined by the equation of motion

$$\phi'' + (g/L)\sin\phi + \mu\tau^{\alpha}\mathrm{D}^{\alpha}\phi + \lambda\tau\phi' + v\tau^{\beta}\mathrm{D}^{\beta}\phi = 0, \tag{29}$$

$$\phi(0) = \phi_0; \qquad \phi'(0) = \dot{\phi}_0 \tag{30}$$

with the elastic energy

$$E(t) = H(\phi(t), \phi'(t)) = \frac{1}{2}L^2\phi'^2 + gL[1 - \cos(\phi)]. \tag{31}$$

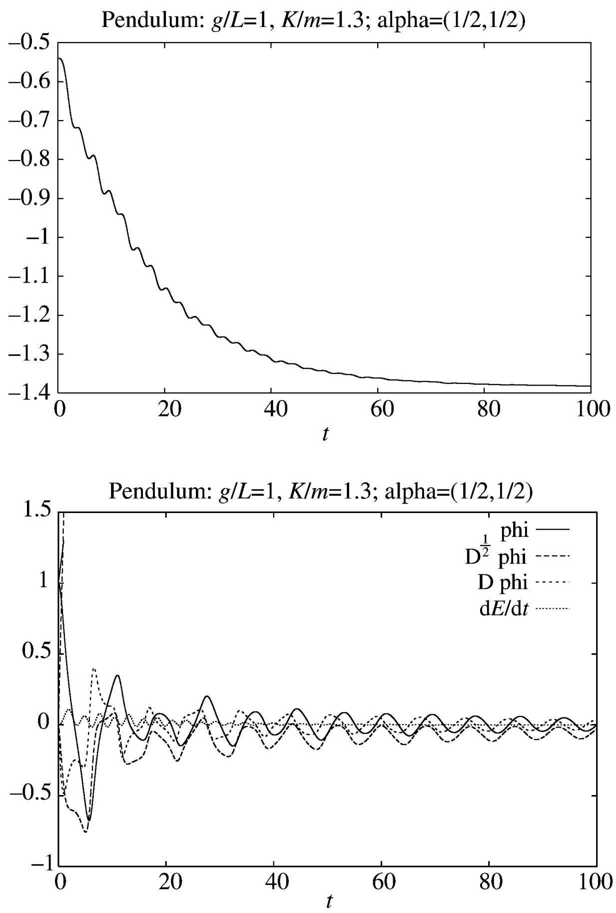


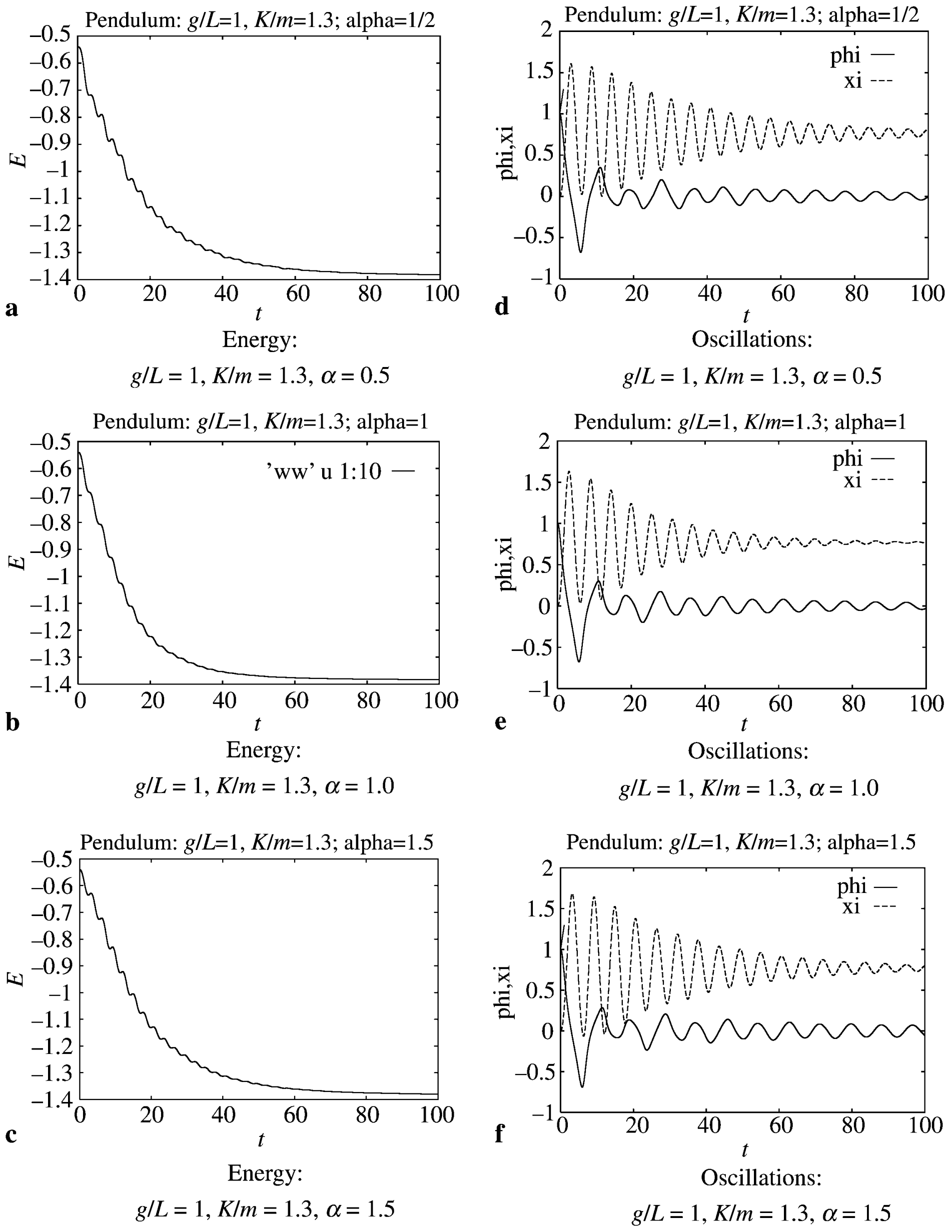
Fig. 3. Elastic energy decay and energy dissipation rate in a pendulum with fractional damping of order 1/2 of libration and spring extension. The phase shift of the fractional derivative is shown in the second figure

The energy dissipation rate is

$$\frac{dE}{dt} = [\mu\tau^{\alpha}\mathrm{D}^{\alpha}\phi + \lambda\tau\phi' + \nu\tau^{\beta}\mathrm{D}^{\beta}\phi]\phi'. \tag{32}$$

An accuracy test is obtained by including the last equation with the initial value 0 and checking the value of $[E(0) - E(t) - \Delta E(t)]/E(0)$, where $E(t) := H(\phi(t), \phi'(t))$ and ΔE is obtained by integrating Eq. (32).

The elastic energy does not fully account for the energy of the pendulum [10]. There is another ambiguity in the physical definition of the damping: the dampers can account for



a Energy: $g/L = 1$, $K/m = 1.3$, $\alpha = 0.5$

d Oscillations: $g/L = 1$, $K/m = 1.3$, $\alpha = 0.5$

b Energy: $g/L = 1$, $K/m = 1.3$, $\alpha = 1.0$

e Oscillations: $g/L = 1$, $K/m = 1.3$, $\alpha = 1.0$

c Energy: $g/L = 1$, $K/m = 1.3$, $\alpha = 1.5$

f Oscillations: $g/L = 1$, $K/m = 1.3$, $\alpha = 1.5$

Fig. 4. Energy decay and oscillations for half-integer and first-order extensible non-resonant pendulum

dissipative energy exchange with external agents. In particular, the Basset force [4] acting on a solid body accelerated in a viscous fluid is represented by a Caputo derivative of order 1/2 of its velocity. The pendulum exchanges energy with the fluid, which provides an additional explanation of non-monotone energy decay in this case.

5.3 Extensible pendulum

The extensible pendulum is a mass m suspended on a string in a gravitational field and allowed to move in a vertical plane. The spring is attached at the other end and is assumed to satisfy the linear Hooke's law with the spring constant K. The rest length of the string is L and its actual length, expressed in terms of elongation ξ, equals the distance $r = L(1+\xi)$ of the bob from the suspension point. The Hamiltonian is given by the formula

$$\begin{aligned} H &= \frac{1}{2}\left[(1+\xi)^2 \phi'^2 + \xi'^2\right] - (g/L)(1+\xi)\cos(\phi) + (K/mL^2)\xi^2/2 \\ &= \frac{1}{2}\left[p_\phi^2/(1+\xi)^2 + p_\xi^2\right] - (g/L)(1+\xi)\cos(\phi) + (K/mL^2)\xi^2/2, \end{aligned} \tag{33}$$

where $p_\xi := \xi'$, $p_\phi := (1+\xi)^2\phi'$. The equations of motion of a damped extensible elastic pendulum are

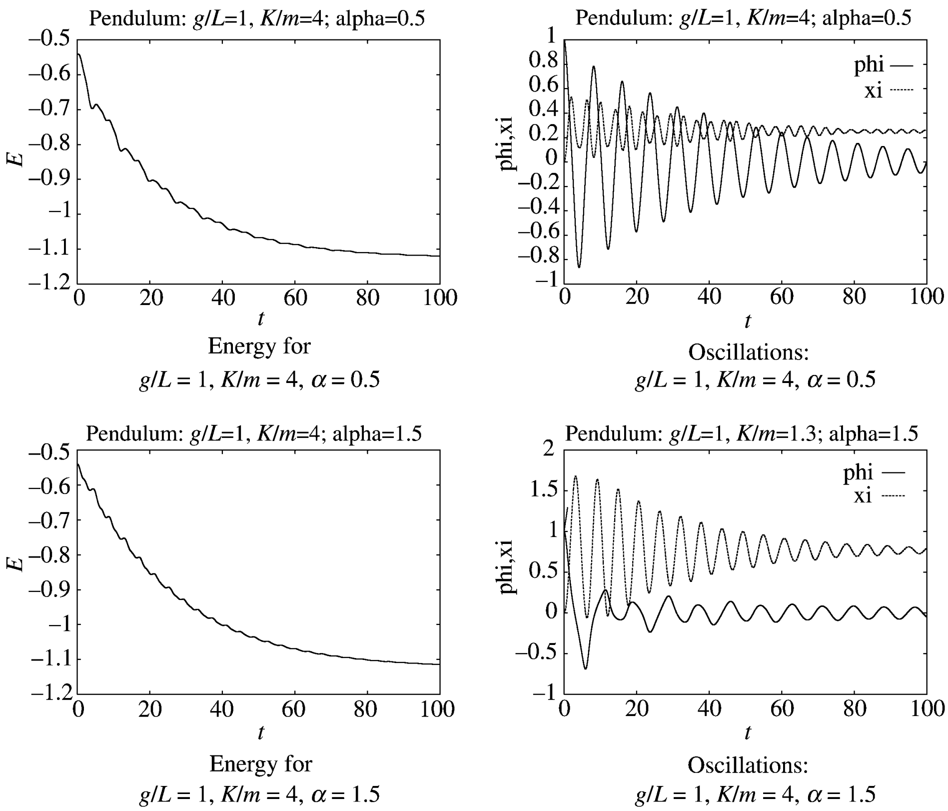


Fig. 5. Energy decay and oscillations for half-integer-order extensible resonant pendulum

$$
\begin{aligned}
\phi'' &= -\left[\frac{g}{L}\sin(\phi) + 2\phi'\xi'\right]/(1+\xi) - (\mu\tau^{\alpha}\mathrm{D}^{\alpha}\phi + \lambda\tau\phi' + \nu\tau^{\beta}\mathrm{D}^{\beta}\phi)/(1+\xi)^2 \\
\xi'' &= \frac{g}{L}\cos(\phi) + (1+\xi)\phi'^2 - \frac{K}{m}\xi - \mu_1\tau^{\alpha}\mathrm{D}^{\alpha}\xi - \lambda_1\tau\xi' - \nu_1\tau^{\beta}\mathrm{D}^{\beta}\xi,
\end{aligned}
\tag{34}
$$

where τ is a positive constant with the dimension $[T]$, while $\mu, \mu_1, \lambda, \lambda_1, \nu, \nu_1 \geq 0$ are dimensionless. The initial conditions are assumed as

$$\phi(0) = 1; \quad \dot{\phi}(0) = \xi(0) = \dot{\xi}(0) = 0.$$

An extensible pendulum without damping is an interacting system consisting of two oscillators: the pendulum and the spring. The interaction is highly non-linear. The extension of the string affects the period of the pendulum, which is a parametric excitation. The angular deviation of the pendulum modulates the force acting on the string. During one pendulum swing the gravitational force reaches its maximum four times. Resonant behavior is therefore expected if the ratio of the string characteristic frequency to the pendulum characteristic frequency is 2:1. A resonant behavior is however observed in the extensible pendulum for arbitrary frequencies of the linearized systems.

In the figures below the oscillations and elastic energy plots are shown for half-integer and integer-order oscillations. The same damping is applied to both degrees of freedom. Resonant and non-resonant cases are distinguished. The elastic energy decays monotonely in the case of an integer-order damping.

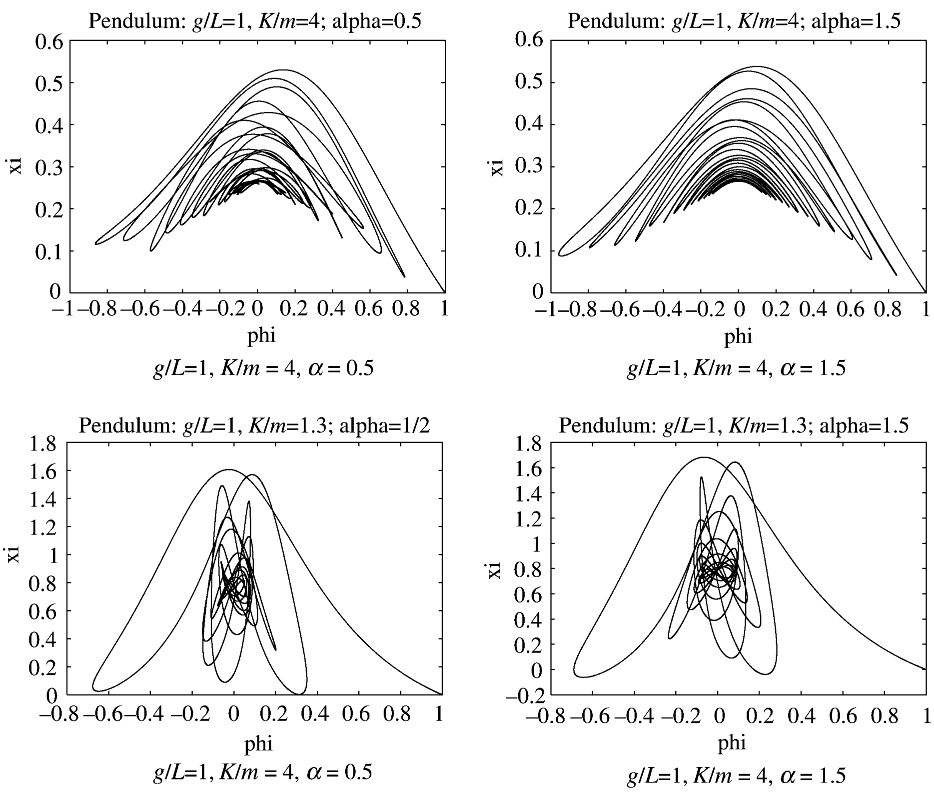


Fig. 6. Phase portraits for the half-integer-order extensible resonant and non-resonant pendulum

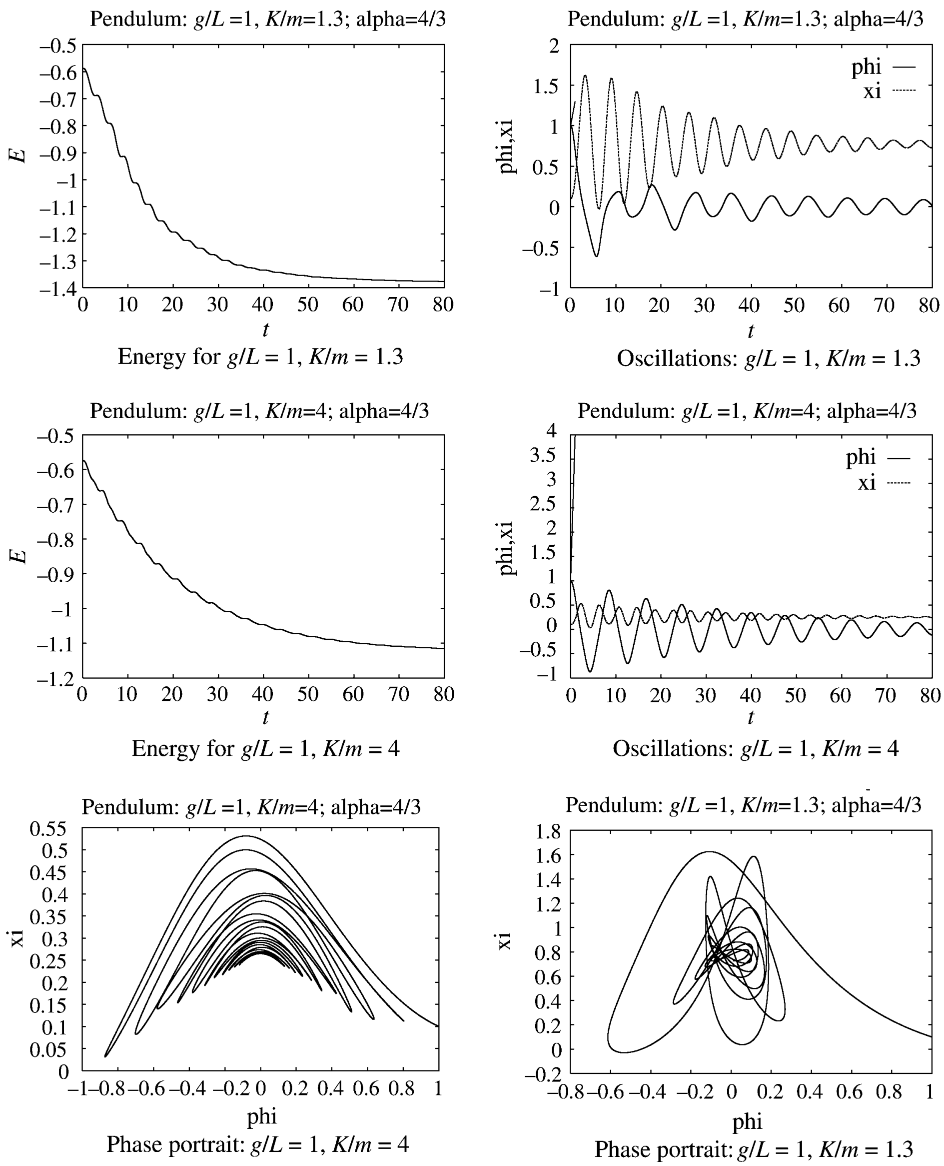


Fig. 7. Energy decay, oscillations and phase portraits for 4/3-order damping of an extensible resonant and non-resonant pendulum

Figure 6 shows a $\phi - \xi$ phase plot. A boomerang-shaped phase plot indicates a 1:2 resonance between the string and the pendulum.

Figure 7 shows some examples of oscillations and elastic energy decay for $\alpha = 4/3$.

6 Conclusions

Thermodynamic criteria for the damping effect have been extended to fractional derivatives of order $\alpha > 1$. The concept of a function of positive type allows imposing the dissipativity condition on the dynamics of a mechanical system. This applies in particular to fractional damping.

The energy of a thermodynamic system with memory is not uniquely defined. The dissipativity property is independent of the choice of the energy. In general nonlocal time operators result in non-monotone energy decay. A monotonely decaying energy can be constructed for fractional damping with $\alpha \leq 1$ at the expense of dealing with energy defined in terms of a history functional (or a function of internal state variables). More generally, Eq. (3) admits a monotonely decaying energy if $K_1 = 0$ [10]. The time evolution of such an energy can be calculated numerically if a different integration scheme is used [12], [11].

For numerical purposes of this paper we have defined the energy by the Hamiltonian. Even though it has been proved merely that the energy never exceeds its value at the time of the first deviation from equilibrium, numerical experiments show a general decrease of energy over time, with only temporary increases. In the case of a pendulum oscillating in a viscous fluid the temporary increases of energy can be associated with the energy exchange between the pendulum and the fluid.

Time evolution of a fractionally damped system with a rational order derivative can be calculated to high accuracy in terms of energy.

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