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Continued fractions, two-point Padé approximants and errors in the Stieltjes case

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Abstract

A Stieltjes function is expanded in mixed T- and S-continued fraction. The relations between approximants of this continued fraction and two-point Padé approximants are established. The method used by Gilewicz and Magnus (J. Comput. Appl. Math. 49 (1993) 79; Integral Transforms Special Functions 1 (1993) 9) has been adapted to obtain the exact relations between the errors of the contiguous two-point Padé approximants in the whole cut complex plane.

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1. Introduction

The special type of the two-point nonbalanced Padé approximants studied in this paper was earlier considered by Telega and Tokarzewski in a series of papers [8–12]. They constructed this type of approximants to approximate the Stieltjes functions appearing in the problems of the mechanics of composite materials. Although they considered only the real domain, most of their results remain

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valid in the complex domain. Certain mathematical properties of the method were verified only numerically. So there was a need for a more firm mathematical background of the method and for answers to some open problems that remained.

Then, our goal was to present a complete theoretical background to the nonbalanced two-point continued fractions and corresponding Padé approximation method. This was made in three papers: the paper [6] containing the analytic theory concerning the measures of the Stieltjes functions, the present paper containing all algebraic developements, and the paper [7] containing extentions and approximations of exact formulas given in Sections 6 and 7 allowing to estimate errors in some concrete situations.

The paper [6] mentioned briefly in Section 2 contains the solution of the old open problem: what is the relation between the measures defining two Stieltjes functions related by the inversion formula (2)? The analytic results given in Theorems 1 and 2, and resumed in the *Comment* are essential to justify the algebraic expansion in continued fraction (21) constructed in Section 3. In Section 4, we recall the definitions of two-point Padé approximants used before by the last two authors. In the following sections, we establish the three-term recurrence relations for approximants of mixed T- and S-continued fraction, called in the following *two-point TS-continued fraction*. If the continued fraction begins by S-fraction quotients and follows by T-fraction quotients, then we call it *ST-continued fraction*. Theorem 4 (Section 5) and Corollary 2 (Section 7) give the expression of the convergents of the TS- and ST-continued fractions, respectively, in terms of two-point Padé approximants. Finally, in a way analogous to the one used for ordinary Padé approximants to the Stieltjes functions [4,5], we obtain the order equilibrated relations between the errors of the contiguous two-point Padé approximants valid in the whole cut complex plane (Theorem 6 and Corollary 3). These results were used in [7] to obtain sharp estimates between the errors.

2. On the inverted Stieltjes functions

Let g be a Stieltjes function defined by

$$z \in \mathbb{C} \setminus (-\infty, -R]: \quad g(z) = \int_0^{1/R} \frac{\mathrm{d}\mu(t)}{1+tz} \tag{1}$$

and h the corresponding inverted Stieltjes function defined by

$$g(z) = \frac{g(0)}{1+zh(z)}, \qquad h(z) = \int_0^{1/R} \frac{\mathrm{d}\nu(t)}{1+tz}. \tag{2}$$

The term *inverted* is here improper and must be understood in the sense of Relation (2). We will discuss later some relations between the behaviour of g and h at infinity and obtain some characterization of respective measures $\mathrm{d}\mu$ and $\mathrm{d}\nu$.

Theorem 1. *Let μ be a nondecreasing function on $[0, 1/R]$ of the form*

$$\mu(t) = GH(t) + \sigma(t), \tag{3}$$

where $G \geqslant 0$, H is a Heaviside function and σ contains no jump at the origin. Then the Stieltjes function g has a finite limit G at infinity:

$$\lim_{z\to\infty} g(z) = G. \tag{4}$$

Remark. The above limit is positive if and only if the function μ has a jump at the origin, that is if $G \neq 0$. Otherwise, this limit is zero.

Proof. Suppose that we have (3), then for some positive ε we obtain

$$g(z) = G + \frac{1}{z}\int_0^{\varepsilon} \frac{\mathrm{d}\sigma(t)}{1/z+t} + \frac{1}{z}\int_{\varepsilon}^{1/R} \frac{\mathrm{d}\sigma(t)}{1/z+t} = G + I_1 + I_2.$$

If $z \gg 0$, then we can bound these integrals as follows:

$$I_1 + I_2 \leqslant \frac{1}{z}\int_0^{\varepsilon} \frac{\mathrm{d}\sigma(t)}{1/z} + \frac{1}{z}\int_{\varepsilon}^{1/R} \frac{\mathrm{d}\sigma(t)}{t} \underset{z\to\infty}{\to} \int_0^{\varepsilon} \mathrm{d}\sigma(t) \underset{\varepsilon\to 0}{\to} 0.$$

Let us observe that as $\int_0^{1/R} \mathrm{d}\sigma(t)$ contributes to the first moment of g, which exists, then even the integral $\int_0^{\varepsilon} \mathrm{d}\sigma(t)$ exists.

Conversely, the limit $G = 0$ in (4) implies that $\mu(t) = \sigma(t)$ in (3). This completes the proof. □

The following theorem indicates that the inversion of Stieltjes functions alternates the character of the corresponding measure at the origin.

Theorem 2. *If*

$$\lim_{z\to\infty} g(z) = 0 \tag{5}$$

and

(i) *if the integral*

$$\int_0^{1/R} \frac{\mathrm{d}\sigma(t)}{t} = C_1 < \infty \tag{6}$$

exists then

$$\lim_{z\to\infty} h(z) = \frac{g(0)}{C_1}; \tag{7}$$

(ii) *if integral* (6) *does not exist*, *then*

$$\lim_{z\to\infty} h(z) = 0. \tag{8}$$

Proof. Inverting (2) we get

$$h(z) = \frac{g(0)}{zg(z)} - \frac{1}{z}. \tag{9}$$

Consequently, we must analyse the behaviour of $zg(z)$ at infinity. Because integral (6) exists, we can write $\lim \int = \int \lim$ and thus:

$$\lim_{z\to\infty} zg(z) = \lim_{z\to\infty} \int_0^{1/R} \frac{\mathrm{d}\sigma(t)}{1/z+t} = \int_0^{1/R} \frac{\mathrm{d}\sigma(t)}{t} = C_1, \tag{10}$$

which proves (i) when approaching infinity with z in (9).

To prove (ii) we first show that if

$$\lim_{\varepsilon\to 0+} \int_\varepsilon^{1/R} \frac{\mathrm{d}\sigma(t)}{t} = \infty, \tag{11}$$

then $\lim_{z\to\infty} zg(z) = \infty$. Indeed,

$$zg(z) = \int_0^{1/R} \frac{\mathrm{d}\sigma(t)}{1/z+t} = \int_0^{\varepsilon} \frac{\mathrm{d}\sigma(t)}{1/z+t} + \int_\varepsilon^{1/R} \frac{\mathrm{d}\sigma(t)}{1/z+t} > \int_\varepsilon^{1/R} \frac{\mathrm{d}\sigma(t)}{1/z+t}$$

because the first integral is positive. Now,

$$\lim_{z\to\infty} zg(z) \geqslant \lim_{z\to\infty} \int_\varepsilon^{1/R} \frac{\mathrm{d}\sigma(t)}{1/z+t} = \int_\varepsilon^{1/R} \frac{\mathrm{d}\sigma(t)}{t} \quad \forall\varepsilon > 0.$$

Indeed, the above inequality shows that $\lim_{z\to\infty} zg(z) = \infty$ if (11) holds. So, again (9) implies (8). This completes the proof. $\square$

Comment. We have proved that if $\lim_{z\to\infty} g(z) = 0$ or, equivalently, if the measure $\mathrm{d}\mu$ defining g does not contain a δ measure at the origin and $\lim_{z\to\infty} zg(z)$ is finite, then $\lim_{z\to\infty} h(z) \neq 0$ and consequently, the measure $\mathrm{d}v$ defining h contains a δ measure at the origin. Conversely, if $\mathrm{d}\mu$ contains a δ measure at the origin, then the measure $\mathrm{d}v$ does not contain it.

For information we give without proof the general formula giving the measure $\mathrm{d}v$ in terms of $\mathrm{d}\mu$ [6]:

$$v'(t) = g(0)\left[\delta(t) \lim_{r\to 0} \frac{1}{2\pi i} \int_{C_r} \frac{\mathrm{d}\xi}{G + \xi \int_0^{1/R} \mathrm{d}\sigma(u)/(\xi - u)} + \frac{t\sigma'(t)}{|g(-1/t)|^2}\right], \tag{12}$$

where C_r is a little circle surrounding the origin of the cut and $\delta(t) = H'(t)$ is the Dirac distribution.

Numerical examples. Fig. 1 illustrates the measures corresponding to the inversion of two Stieltjes functions. The first is defined by the stepwise nondecreasing function $\tilde{\mu}$, and the second by its continuous analogue μ corresponding to the infinite number of steps. Both functions have no jump at the origin. The functions $\tilde{v}$ and v defining the respective inverted Stieltjes functions contain a characteristic jump at the origin. Let us observe the positions of the jumps of $\tilde{v}$ that interlace those of v.

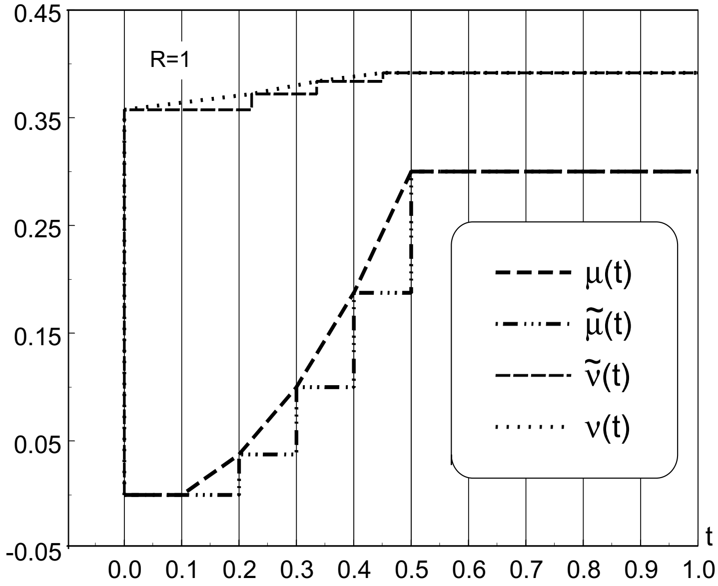

Fig. 1. Numerical examples of functions μ and ν (see (1) and (2)) and of the stepwise functions $\tilde{\mu}$ and $\tilde{\nu}$.

3. Two-point *TS*-continued fraction expansion of the Stieltjes functions

Let f_j be a Stieltjes function defined by a positive measure $\mathrm{d}\gamma_j$ not containing a δ measure at the origin

$$f_j(z) = \int_0^{1/R} \frac{\mathrm{d}\gamma_j(t)}{1+tz}. \tag{13}$$

Suppose that there exist the following expansions of the functions f_j at zero and at infinity:

$$S_0^{(j)}(z) = \sum_{n=0}^{\infty} c_n^{(j)} z^n, \quad c_n^{(j)} = (-1)^n \int_0^{1/R} t^n \,\mathrm{d}\gamma_j(t), \;\; n = 0, 1, \ldots, \tag{14}$$

$$S_\infty^{(j)}(z) = \sum_{n=1}^{\infty} C_n^{(j)} z^{-n}, \quad C_n^{(j)} = (-1)^{n+1} \int_0^{1/R} \frac{\mathrm{d}\gamma_j(t)}{t^n}, \;\; n = 1, 2, \ldots . \tag{15}$$

Let us remark that expansion (15) is, in principle, an asymptotic one. However, if $\mathrm{d}\gamma_j(t) = 0$ for $t \in [0, t_0]$, then it is convergent for $|1/z| < t_0$.

Following Theorems 1 and 2 and formula (10), our assumption on the behaviour of $\mathrm{d}\gamma_j$ at the origin and (15) leads to

$$\lim_{z\to\infty} f_j(z) = 0 \quad \text{and} \quad \lim_{z\to\infty} z f_j(z) = C_1^{(j)}. \tag{16}$$

However, the assumption contained in (15) is stronger. In fact, the existence of all coefficients $C_n^{(j)}$ implies that the measure $\mathrm{d}\gamma_j(t)$ tends to zero at the origin faster than any power of t.

To indicate clearly the change of the behaviour of measures, hereafter, we will denote by f_i^* the Stieltjes functions defined by the measures containing the δ measure at the origin such that

$$\mathrm{d}\mu_i(t) = \mathrm{d}(G_{i-1}H(t) + \gamma_i(t)), \quad G_{i-1} > 0. \tag{17}$$

Note that if $f_j(z) = f_j(0)/(1 + zf_{j+1}^*(z))$, then, assuming the existence of expansion (15), we obtain

$$\lim_{z\to\infty} f_{j+1}^*(z) = \frac{f_j(0)}{C_1^{(j)}} = G_j.$$

In other words assumption (15) implies that the measure generating f_{j+1}^* has form (17) with $G_j \neq 0$. Inverting the Stieltjes functions f_j or f_j^* we obtain Stieltjes functions f_{j+1}^* or f_{j+1}, respectively. This remark is essential for the following expansion in mixed *TS*-continued fractions. Indeed, we have two possibilities to operate with the functions of type f^*. We can, as in (18), extract the function of type f from the function of type f^* and then, invert the function of type f. Similarly, we can directly invert the function of type f^* as in (20). We can stop one or the other kind of expansion in our convenience. Hereafter, to simplify the notation, we introduce constants F_i and g_i:

$$f_{j+1}^*(z) = \frac{f_j(0)}{C_1^{(j)}} + f_{j+1}(z) = \frac{f_j(0)}{C_1^{(j)}} + \frac{f_{j+1}(0)}{1 + zf_{j+2}^*(z)} \stackrel{\text{def}}{=} G_j + \frac{F_{j+1}}{1 + zf_{j+2}^*(z)}, \quad 1 \leqslant j \leqslant k, \tag{18}$$

$$f_{k+2j-1}(z) = \frac{f_{k+2j-1}(0)}{1 + zf_{k+2j}^*(z)} \stackrel{\text{def}}{=} \frac{g_{k+2j-1}}{1 + zf_{k+2j}^*(z)}, \quad j \geqslant 1, \tag{19}$$

$$f_{k+2j}^*(z) = \frac{f_{k+2j}^*(0)}{1 + zf_{k+2j+1}(z)} \stackrel{\text{def}}{=} \frac{g_{k+2j}}{1 + zf_{k+2j+1}(z)}, \quad j \geqslant 1. \tag{20}$$

Now we can expand the function $f(z) = zf_1(z)$ in the nonbalanced two-point *TS*-continued fraction. The term *nonbalanced* means that the continued fraction is not matched equivalently at the origin and at infinity to the expansion of f. *TS* means that we begin by inversions (18) leading to the T-continued fraction part and we continue with inversions (19) and (20) leading to the S-continued fraction part of the resulting *TS*-continued fraction:

$$\begin{aligned} f(z) = zf_1(z) &= \frac{zF_1}{1+zG_1+}\,\frac{zF_2}{1+zG_2+}\cdots\frac{zF_k}{1+zG_k+}\,\frac{zg_{k+1}}{1+}\,\frac{zg_{k+2}}{1+}\cdots\frac{zg_{2M-k-1}}{1+}\,\frac{zf_{2M-k}^*(z)}{1.} \\ &= \frac{zF_1}{1+zG_1+}\,\frac{zF_2}{1+zG_2+}\cdots\frac{zF_k}{1+zG_k+}\,\frac{zg_{k+1}}{1+}\,\frac{zg_{k+2}}{1+}\cdots\frac{zg_{2M-k-1}}{1+}\,\frac{zg_{2M-k}}{1+}\,\frac{zf_{2M-k+1}(z)}{1.}. \end{aligned} \tag{21}$$

Please observe that by following step by step the expansion in S-fraction, we obtain in the last partial quotient alternatively functions of types f_j and of type f_j^*.

4. Two-point Padé approximants

The N-point Padé approximant to a function f is a rational function matching beginnings of expansions of f at N different points. The corresponding numbers of matched coefficients at each point are indicated by upper indices. Our two-point Padé approximant to f matches expansions of f at 0 and ∞ and is then written as follows:

$$[M/M]_f^{k,l}(z) = \frac{P_M(z)}{Q_M(z)} = \frac{p_1 z + \cdots + p_M z^M}{1 + q_1 z + \cdots + q_M z^M}. \tag{22}$$

It must agree at 0 with k coefficients of f (including trivially the first coefficient $f(0) = [M/M]_f^{k,l}(0) = 0$, because in our case $f(z) = z f_1(z)$) and with l coefficients of f at ∞, where $k + l = 2M + 1$. Using notations of (14) and (15) we then have

$$z S_0^{(1)}(z) Q_M(z) - P_M(z) = \mathrm{O}(z^k), \quad 0 \leqslant k \leqslant 2M + 1, \tag{23}$$

$$z S_\infty^{(1)}(z) Q_M(z) - P_M(z) = \mathrm{O}(z^{-(l+1-M)}), \quad 0 \leqslant l \leqslant 2M + 1. \tag{24}$$

We have already assumed that $Q_M(0) \neq 0$ and performed the normalization $Q_M(0) = 1$. The latter assumption means that Q_M has exactly degree M, that is $q_M \neq 0$. These two assumptions lead to the standard definition of "normal" two-point Padé approximant formulated in the following equations:

$$z S_0^{(1)}(z) - [M/M]_f^{k,l}(z) = \mathrm{O}(z^k), \tag{25}$$

$$z S_\infty^{(1)}(z) - [M/M]_f^{k,l}(z) = \mathrm{O}\left(\left(\frac{1}{z}\right)^{l+1}\right). \tag{26}$$

5. Two-point continued fraction approximants and two-point Padé approximants

Let A_m/B_m denote the mth approximant of a continued fraction, where m is not necessarily the degree of numerator or denominator. Let us recall [1,3] the following relations (remembering that in our case $f(0) = 0$):

Property 1. *Successive approximants*

$$\frac{A_m}{B_m} = \frac{zH_1}{1+}\frac{zH_2}{1+}\cdots\frac{zH_m}{1} \tag{27}$$

of S-fraction

$$f(z) = \frac{zH_1}{1+}\frac{zH_2}{1+}\cdots \tag{28}$$

are the ordinary one-point Padé approximants to f:

$$\frac{A_1}{B_1} = [1/0]_f^{2,0}(z), \quad \frac{A_2}{B_2} = [1/1]_f^{3,0}(z), \quad \frac{A_3}{B_3} = [2/1]_f^{4,0}(z), \ldots .$$

Now we can prove the following.

Theorem 3. *Successive approximants A_m/B_m of T-fraction*

$$f(z)=\frac{zF_1}{1+zG_1+}\frac{zF_2}{1+zG_2+}\cdots \tag{29}$$

are the two-point Padé approximants to f:

$$\frac{A_m}{B_m}=\frac{zF_1}{1+zG_1+}\cdots\frac{zF_m}{1+zG_m}=[m/m]_f^{m+1,m}(z),\quad m=1,2,\ldots . \tag{30}$$

Proof. It is now easy to note that $\deg A_m=\deg B_m=m$. Our task now is to prove that A_m/B_m matches $m+1$ coefficients (14) at 0 ($f(0)=0, c_0^{(1)},\ldots,c_{m-1}^{(1)}$) and m coefficients (15) at ∞ ($C_1^{(1)},\ldots,C_m^{(1)}$). Let us recall that

$$f(z)=zf_1(z)=\frac{zF_1}{1+zf_2^*(z)}\equiv\frac{zF_1}{1+zG_1+}\frac{zf_2(z)}{1.}. \tag{31}$$

Let us observe that the last T-continued fraction can be rewritten as the S-fraction:

$$f(z)=\frac{zF_1}{1+}\frac{zf_2^*(z)}{1.}\equiv\frac{zH_1}{1+}\frac{zh_2}{1.}. \tag{32}$$

We have introduced the notation $H_1\equiv F_1$ and $f_2^*\equiv h_2$. The first approximant of this fraction noted $A_1^*/B_1^*=zF_1/1.$ is, according to Property 1, the Padé approximant $[1/0]_f^{2,0}(z)$, that is, it matches two coefficients of f at the origin, which is exactly what we require. We will prove now that each finite T-continued fraction can be transformed in a unique way in the corresponding finite S-fraction:

$$f(z)=\frac{zF_1}{1+zG_1+}\frac{zF_2}{1+zG_2+}\cdots\frac{zF_m}{1+zG_m+}\frac{zf_{m+1}(z)}{1.}=\frac{zH_1}{1+}\frac{zH_2}{1+}\cdots\frac{zH_m}{1+}\frac{zh_{m+1}(z)}{1.}. \tag{33}$$

For $m=2$, the T-continued fraction is transformed in the following S-fraction:

$$f(z)=\frac{zF_1}{1+}\frac{z(G_1+F_2)}{1+}\frac{zh_3(z)}{1.},\quad \text{where } h_3(z)=\frac{f_3^*(z)F_2/(G_1+F_2)}{1+zf_3^*(z)G_1/(G_1+F_2)}.$$

The second approximant A_2^*/B_2^* of this S-fraction is $[1/1]_f^{3,0}(z)$ Padé approximant to f. It matches three coefficients of f at the origin according to Property 1. Of course $\frac{A_1^*}{B_1^*}\neq\frac{A_1}{B_1}$, $\frac{A_2^*}{B_2^*}\neq\frac{A_2}{B_2}$, and so on; however, according to Property 1, each approximant A_m^*/B_m^* matches the same number of coefficients at the origin as A_m/B_m, exactly $m+1$. Instead of performing directly the complicated transformations of the T-fractions (33) we can proceed simply. Let us remark that each function h_m is a Stieltjes function and then

$$h_m(z)=\frac{H_m}{1+}\frac{zh_{m+1}}{1.}. \tag{34}$$

This fact only proves the existence of transformation (33).

If we require, we can compute explicitly the constants $H_m\equiv h_m(0)$ in terms of constants F_i and G_i as follows. For $m=2$ we have $h_2(z)\equiv f_2^*(z)$ and then $H_2=h_2(0)=G_1+F_2$. For $m=3$ to compute

$H_3 \equiv h_3(0)$ we get $h_3(z)$ from the identities $h_2(z) = H_2/(1 + zh_3(z)) = G_1 + F_2/(1 + zf_3^*(z))$ and obtain finally $H_3 = F_2(G_2 + F_3)/(G_1 + F_2)$.

Now we perform the transformations of the T-fractions in S-fractions in the variable $1/z$. Let us introduce formally the notation $f(z) \equiv s(1/z)$. We begin by

$$f(z) = zf_1(z) = s\left(\frac{1}{z}\right) = \frac{zF_1}{1 + zG_1 + zf_2(z)} = \frac{\frac{F_1}{G_1}}{1+} \frac{\frac{1}{z}\frac{1}{G_1}(1 + zf_2(z))}{1.} \equiv \frac{H_1}{1+} \frac{\frac{1}{z}h_2'(\frac{1}{z})}{1.},$$

where $h_2'(1/z) = (1/G_1)(1+zf_2(z))$ is a Stieltjes function. We have introduced the appropriate notation of H_1' and h_2'. As previously, we obtain the general transformation

$$s\left(\frac{1}{z}\right) = \frac{zF_1}{1 + zG_1+} \frac{zF_2}{1 + zG_2+} \cdots \frac{zF_m}{1 + zG_m+} \frac{zf_{m+1}(z)}{1.} = \frac{H_1'}{1+} \frac{\frac{1}{z}H_2'}{1+} \cdots \frac{\frac{1}{z}H_m'}{1+} \frac{\frac{1}{z}h_{m+1}'(\frac{1}{z})}{1.}. \tag{35}$$

As previously, each function h_m' is a Stieltjes function. The successive approximants A_k'/B_k' of the last S-fraction in variable $1/z$ are clearly the following Padé approximants:

$$\frac{A_1'}{B_1'} = [0/0]_s^{1,0}\left(\frac{1}{z}\right), \quad \frac{A_2'}{B_2'} = [0/1]_s^{2,0}\left(\frac{1}{z}\right), \quad \frac{A_3'}{B_3'} = [1/1]_s^{3,0}\left(\frac{1}{z}\right), \ldots .$$

Therefore, the mth approximant matches m coefficients of expansion of s at the origin in the variable $1/z$, i.e. m coefficients of f at infinity in variable z. This completes the proof. $\square$

We are now able to prove the following.

Theorem 4. *Successive approximants of the two-point infinite TS-continued fraction*

$$f(z) = zf(z) = \frac{zF_1}{1 + zG_1+} \frac{zF_2}{1 + zG_2+} \cdots \frac{zF_k}{1 + zG_k+} \frac{zg_{k+1}}{1+} \frac{zg_{k+2}}{1+} \cdots \frac{zg_{2M-k}}{1+} \cdots \tag{36}$$

are the following two-point Padé approximants:

$$\frac{A_1}{B_1} = \frac{zF_1}{1 + zG_1.} = [1/1]_f^{2,1}(z),$$

$$\vdots$$

$$\frac{A_k}{B_k} = \frac{zF_1}{1 + zG_1+} \cdots \frac{zF_k}{1 + zG_k.} = [k/k]_f^{k+1,k}(z), \tag{37}$$

$$\frac{A_{k+1}}{B_{k+1}} = \frac{zF_1}{1 + zG_1+} \cdots \frac{zF_k}{1 + zG_k+} \frac{zg_{k+1}}{1.} = [k/k]_f^{k+2,k-1}(z), \tag{38}$$

$$\frac{A_{k+2}}{B_{k+2}} = \frac{zF_1}{1 + zG_1+} \cdots \frac{zF_k}{1 + zG_k+} \frac{zg_{k+1}}{1+} \frac{zg_{k+2}}{1.} = [k + 1/k + 1]_f^{k+3,k}(z), \tag{39}$$

$$\frac{A_{k+3}}{B_{k+3}} = \frac{zF_1}{1+zG_1+}\cdots\frac{zg_{k+3}}{1.} = [k+1/k+1]_f^{k+4,k-1}(z), \tag{40}$$

$$\vdots$$

$$\frac{A_{2M-k}}{B_{2M-k}} = \frac{zF_1}{1+zG_1+}\cdots\frac{zg_{2M-k}}{1.} = [M/M]_f^{2M-k+1,k}(z), \tag{41}$$

$$\vdots$$

Proof. The k first relations, up to (37), have been proved in Theorem 3. It is observed that the last denominator in (38) is $1+z(G_k+g_{k+1})$, i.e., the term g_{k+1} "contaminates" G_k. Therefore, the resulting two-point Padé approximant cannot match the kth coefficient of the expansion of f at infinity. On the other hand, the last partial quotient $zg_{k+1}/1.$ in fraction (38) is the first partial quotient of the beginning part of the expansion in S-fraction. In each new step of this fraction, following Property 1, the corresponding approximant gains one more matched coefficient of the expansion at the origin of f. Consequently, moving from (37) to (38), the number of coefficients matched at infinity decreases by one and the number of coefficients matched at the origin increases by one, while the degrees of numerators and denominators in (37) and (38) remain the same. In the next step, from (38) to (39), the degrees of numerator and denominator increase, so the approximant gains two newly matched coefficients. The coefficient G_k is "decontaminated". Therefore, the agreement at the origin and at infinity progresses by one, and so on. Please note that only the approximants A_{k+2l}/B_{k+2l} ($l \geqslant 0$) preserve the agreement of k coefficients at infinity. □

To compute the approximants of TS-continued fraction, one must split the classical recurrence formulae (see [1,2]) into two kinds of formulae. We formulate this in the following obvious theorem:

Theorem 5. *Successive approximants of the two-point infinite TS-continued fraction* (36)

$$f(z) = zf(z) = \frac{zF_1}{1+zG_1+}\frac{zF_2}{1+zG_2+}\cdots\frac{zF_k}{1+zG_k+}\frac{zg_{k+1}}{1+}\frac{zg_{k+2}}{1+}\cdots\frac{zg_{2M-k}}{1+}\cdots$$

can be calculated by the following three-term recurrence relations:

$$A_{-1} = 1, \quad A_0 = 0,$$

$$B_{-1} = 0, \quad B_0 = 1,$$

$$\begin{aligned} A_m &= (1+zG_m)A_{m-1} + zF_mA_{m-2}, \\ B_m &= (1+zG_m)B_{m-1} + zF_mB_{m-2}, \end{aligned} \qquad m = 1,2,\ldots,k, \tag{42}$$

$$\begin{aligned} A_m &= A_{m-1} + zg_mA_{m-2}, \\ B_m &= B_{m-1} + zg_mB_{m-2}, \end{aligned} \qquad m = k+1, k+2, \ldots . \tag{43}$$

Comment. All B_j can be directly identified as denominators of Padé approximants (22) because they are normalized as the denominators in question: $B_j(0)=1$.

6. Relations between the errors of two-point Padé approximants

To obtain such relations, we can apply methods similar to those used in [4,5].

Theorem 6. *Let $[n/n]=P_n/Q_n$ be a two-point Padé approximant to a Stieltjes function f defined by an approximant of two-point TS-continued fraction. Then the following relations between the two-point Padé approximant errors hold in the whole cut complex plane*:

$$z\in\mathbb{C}\setminus(-\infty,-R],\quad m=1,2,\ldots,k-1:$$

$$f(z)-[m+1/m+1]_f^{m+2,m+1}(z)=-z\frac{f_{m+2}(z)Q_m(z)}{Q_{m+1}(z)}\{f(z)-[m/m]_f^{m+1,m}(z)\},\tag{44}$$

$$M\geqslant k:\quad f(z)-[M+1/M+1]_f^{2M-k+3,k}(z)$$
$$=z^2\frac{f^*_{2M-k+2}(z)f_{2M-k+3}(z)Q_M(z)}{Q_{M+1}(z)}\{f(z)-[M/M]_f^{2M-k+1,k}(z)\}.\tag{45}$$

Proof. Let us consider the following T-fraction terminated by the S-fraction partial quotient

$$f(z)=zf_1(z)=\frac{zF_1}{1+zG_1+}\frac{zF_2}{1+zG_2+}\cdots\frac{zF_{m+1}}{1+zG_{m+1}+}\frac{zf_{m+2}(z)}{1.}$$

expressed as A_{m+2}/B_{m+2} approximant of itself:

$$f(z)=\frac{A_{m+1}+zf_{m+2}(z)A_m}{B_{m+1}+zf_{m+2}(z)B_m}.$$

Recalling (cf. Theorem 4) that $A_j/B_j=[j/j]$ for $j\leqslant k$, we can easily rearrange the previous relation to achieve form (44). To prove (45), let us consider two terminated *TS*-continued fractions of type (21), stopped at the indices $2M-k+2$ and $2M-k+3$, respectively. As previously, we can write $f(z)$ in the form of approximants of continued fractions:

$$f(z)=zf_1(z)=\frac{A_{2M-k+1}+zf^*_{2M-k+2}(z)A_{2M-k}}{B_{2M-k+1}+zf^*_{2M-k+2}(z)B_{2M-k}},\tag{46}$$

$$f(z)=zf_1(z)=\frac{A_{2M-k+2}+zf_{2M-k+3}(z)A_{2M-k+1}}{B_{2M-k+2}+zf_{2M-k+3}(z)B_{2M-k+1}}.\tag{47}$$

Recalling now that $A_{2M-k}/B_{2M-k}=[M/M]_f^{2M-k+1,k}=P_M/Q_M$ and $A_{2M-k+2}/B_{2M-k+2}=[M+1/M+1]_f^{2M-k+3,k}=P_{M+1}/Q_{M+1}$, we can rewrite (46) and (47) in the following forms:

$$f(z)-\frac{A_{2M-k+1}}{B_{2M-k+1}}=-z\frac{B_{2M-k}f^*_{2M-k+2}(z)}{B_{2M-k+1}}\{f(z)-[M/M]_f^{2M-k+1,k}(z)\},$$

$$f(z)-[M+1/M+1]_f^{2M-k+3,k}(z)=-z\frac{B_{2M-k+1}f_{2M-k+3}(z)}{B_{2M-k+2}}\left\{f(z)-\frac{A_{2M-k+1}}{B_{2M-k+1}}\right\},$$

respectively. Substituting the first expression into the second we obtain (45). □

7. Corollaries for the two-point ST-continued fractions

Let f be expanded now in ST-continued fraction:

$$\begin{aligned} f(z)=zf_1(z)&=\frac{zf_1(0)}{1+zf_2^*(z)}\\ &=\frac{zf_1(0)}{1+}\frac{zf_2^*(0)}{1+}\frac{zf_3(0)}{1+}\cdots\frac{zf_{2k}^*(0)}{1+}\frac{zf_{2k+1}(0)}{1+z\frac{f_{2k+1}(0)}{C_1^{(2k+2)}}+}\frac{zf_{2k+2}(0)}{1+z\frac{f_{2k+2}(0)}{C_1^{(2k+3)}}+}\cdots\\ &\equiv\frac{zg_1}{1+}\frac{zg_2}{1+}\cdots\frac{zg_{2k}}{1+}\frac{zF_{2k+1}}{1+zG_{2k+1}}\frac{zF_{2k+2}}{1+zG_{2k+2}}\cdots, \end{aligned}\tag{48}$$

where

$$m\leqslant k:\quad g_{2m-1}=f_{2m-1}(0);\quad g_{2m}=f_{2m}^*(0),$$

$$n\geqslant 1:\quad F_{2k+n}=f_{2k+n}(0);\quad G_{2k+n}=\frac{f_{2k+n}(0)}{C_1^{(2k+n+1)}}.$$

Corollary 1. *Successive approximants of the two-point infinite ST-continued fraction* (48) *can be calculated by the following three-term recurrence relations*:

$$A_{-1}=1,\quad A_0=0,$$

$$B_{-1}=0,\quad B_0=1,$$

$$\left.\begin{aligned} A_m&=A_{m-1}+zg_mA_{m-2},\\ B_m&=B_{m-1}+zg_mB_{m-2}, \end{aligned}\right.\quad m=1,2,\ldots,2k \tag{49}$$

$$\left.\begin{aligned} A_m&=(1+zG_m)A_{m-1}+zF_mA_{m-2},\\ B_m&=(1+zG_m)B_{m-1}+zF_mB_{m-2}, \end{aligned}\right.\quad m=2k+1,\ldots\ . \tag{50}$$

Only the order is inverted with respect to Theorem 5.

Corollary 2. *Successive approximants of the two-point infinite ST-continued fraction* (48) *are the following two-point Padé approximants*:

$$m \leqslant k: \quad \frac{A_{2m-1}}{B_{2m-1}} = [m/m-1]_f^{2m,0}(z),$$
$$\frac{A_{2m}}{B_{2m}} = [m/m]_f^{2m+1,0}(z), \tag{51}$$

$$n \geqslant 1: \quad \frac{A_{2k+n}}{B_{2k+n}} = [k+n/k+n]_f^{2k+n+1,n}(z). \tag{52}$$

Corollary 3. *Let* $[n/n] = P_n/Q_n$ *be a two-point Padé approximant to a Stieltjes function* f *defined by an approximant of the two-point ST-continued fraction* (48). *Then the following relation between the two-point Padé approximant errors holds in the whole cut complex plane*:

$$z \in \mathbb{C} \setminus (-\infty, -R], \quad m \geqslant 1:$$
$$f(z) - [k+m+1/k+m+1]_f^{2k+m+2,m+1}(z)$$
$$= -z\frac{f_{2k+m+2}(z)Q_{k+m}(z)}{Q_{k+m+1}(z)}\{f(z) - [k+m/k+m]_f^{2k+m+1,m}(z)\}. \tag{53}$$

Proof. The proof is similar to that of Theorem 6 and is based on the transformation of the approximant $f(z) = A_{2k+m+2}/B_{2k+m+2}$ of the terminated ST-continued fraction (48) stopped at $\ldots[zF_{2k+m+1}/(1+zG_{2k+m+1})]zf_{2k+m+2}(z)/1$. □

8. Conclusion

Our next goal is to generalize this theory to the nonbalanced N-point Padé approximants and, eventually, to find their relation with corresponding continued fractions. However, in the present time, we are not able to define properly those fractions.

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