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PLASMA INSTABILITY

Nonlinear Regimes of Farley–Buneman Instability

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Abstract—The dynamic suppression of the instability of a quasi-monochromatic wave by nonlinear wave–wave interaction is considered. It is shown that, near the threshold of linear instability, the process of decay into two strongly damped waves leads to the onset of a quasi-periodic or a stochastic nonlinear stabilization regime involving a small number of modes. A case study is made of the Farley–Buneman instability in an isothermal magnetized current-carrying plasma in which particle collisions play an important role. Typical characteristic features of different stabilization regimes are analyzed as functions of current and other plasma parameters.

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1. INTRODUCTION

In the E region of the Earth's lower ionosphere, complicated processes occurring in the interaction of the solar wind with the Earth's magnetosphere generate an electrojet and the accompanying quasi-constant electric field. When the strength of this field exceeds a certain threshold value, the Farley–Buneman (FB) instability develops, specifically, instability of collisional plasma with an electric current transverse to the magnetic field. Since its discovery more than 30 years ago [1], the linear theory of this instability has been well developed [1–4], which made it possible to explain some properties of small-scale nonuniformities occurring in the lower ionosphere. Short waves propagating nearly perpendicular to the external magnetic field are excited at the fastest rate. The problem of how the growth rates and thresholds of the FB instability depend on kinetic effects, plasma thermal conductivity, and plasma nonquasineutrality is still being studied. Thus, in [5, 6] the instability growth rates were thoroughly calculated with allowance for electron kinetics, in particular, the dependence of the collision frequency on velocity was obtained. However, in order to explain the modulation depth of the inhomogeneities and their other properties, it is necessary to take into account the instability suppression mechanisms. In [7, 8], it was supposed that the FB instability saturates because of the nonlinear interaction of unstable with damped modes and the specific nonlinear mechanism was proposed. It is obvious that, since the characteristic time scales on which the density oscillates are longer than the cyclotron period, it is the vector nonlinearity due to the nonlinear nature of the electron drift motion that plays a governing role in the relaxation of the spectrum of plasma inhomogeneities as well as electrostatic waves in a weakly collisional plasma [9, 11].

It should be noted that, in a weakly collisional, magnetized plasma, drift-wave-like plasma nonuniformities associated with the large-scale gradients of the plasma density and temperature can coexist with the classical FB instability, associated with the electric current. Although the instabilities leading to the growth of electrostatic waves are different, the mechanisms for the nonlinear interaction between these inhomogeneities are the same as those for the FB instability and are typical of the nonlinear interaction between drift waves in plasma. For a plasma in which dissipative processes play a substantial role, it seems justified to suppose that the developing spectrum (turbulence) of the density waves can be described by a finite number of modes, in which case the developing few-mode turbulence can be efficiently described by the sets of hydrodynamic-like equations for the wave amplitudes with allowance for the growth and damping effects. A familiar Lorentz attractor can be considered, in particular, as belonging to this class of systems.

Analytic approaches to calculating the spectra are all interesting, but often contradict each other and have to be tested by numerical simulations and also by justifying the assumptions made, especially those about the random phases of the waves. Full-scale kinetic simulations run into obvious difficulties, such as the lack of computer resources, complexity of an adequate description of collisions, presence of numerical noise, and great difference in the time scales of particle oscillations and in the rates of nonlinear processes. In some other papers [7, 11], it was proposed to simulate the FB instability by using a hydrodynamic approach and taking into account plasma quasineutrality. The authors of those papers had to make special efforts in order to solve the nonlinear equation relating the density and potential fluctuations. In addition, the authors ignored thermal conductivity and Landau damping by ions—effects that come into play in the short-wavelength range and become especially important when a cascade develops that brings energy to the damping region. There also are papers that made use of a hybrid simulation method in which the electron motion was described hydrodynamically and the ion motion, kinetically. On the whole, it can be concluded that, although some interesting results have been obtained (in particular, it was shown that the FB instability can be stabilized by nonlinear wave–wave interaction processes), the overall picture is far from being complete and it is not clear which regimes and which nonuniformity spectra the plasma relaxes to.

It is therefore of interest to consider an alternative approach to simulating the spectra of nonuniform waves by solving ordinary differential equations for the amplitudes of different spatial modes, i.e., by deriving and analyzing hydrodynamic-like equations with quadratic nonlinear terms in which the linear components describe wave dispersion and also dissipative effects. An advantage of this approach is that the simulation results are illustrative and easy to analyze and that it is possible to readily modify the equations by incorporating additional effects via changing the corresponding coefficients and/or using a larger number of modes. In particular, in the linear approximation, Landau damping by ions is taken into account by supplementing the equation of ion motion with the corresponding linear term.

2. LINEAR THEORY

The linear theory of the FB instability was considered by using various models for describing magnetized plasma hydrodynamically and kinetically. In this section, we do not pretend to give a novel analysis, but only present the main results of the linear theory that will be used to construct a nonlinear model. For the conditions under consideration, the plasma can be assumed to be uniform. In the linear approximation, the properties of electrostatic waves in such plasma are described by the dispersion relation

$$1 + \varepsilon_e + \varepsilon_i = 0, \qquad (1)$$

where $\varepsilon_e(\omega, \mathbf{k})$ and $\varepsilon_i(\omega, \mathbf{k})$ are the electron and ion dielectric functions, respectively. The analysis is carried out in a frame of reference where the ions are at rest as a whole. Since the friction due to collisions with neutrals at the frequency ν_i is strong, this frame essentially coincides with the rest frame of the neutral component of the weakly ionized ionospheric plasma. Using the general expression for the dielectric function of a collisional plasma [12] with unmagnetized ions $\left(\omega \gg \omega_i = \dfrac{eB_0}{m_i c}\right)$ and assuming that the ion thermal velocity v_{T_i} is low in comparison with the wave phase velocity $kv_{T_i} \ll \omega$, or, more precisely, $|\omega + i\nu_i| \gg kv_{T_i}$, we obtain

$$\varepsilon_i = \frac{1}{k^2\lambda_{Di}^2} \frac{1 + \dfrac{\omega + i\nu_i}{\sqrt{2}kv_{T_i}} Z\left(\dfrac{\omega + i\nu_i}{\sqrt{2}kv_{T_i}}\right)}{1 + \dfrac{i\nu_i}{\sqrt{2}kv_{T_i}} Z\left(\dfrac{\omega + i\nu_i}{\sqrt{2}kv_{T_i}}\right)} \simeq -\frac{\omega_{pi}^2}{(\omega + i\nu_i)\omega}\left[1 - i\sqrt{\frac{\pi}{2}}\left(\frac{\omega + i\nu_i}{kv_{T_i}}\right)^3 \exp\left(-\left(\frac{\omega + i\nu_i}{\sqrt{2}kv_{T_i}}\right)^2\right)\right], \qquad (2)$$

where we have used the familiar expansion of the plasma dispersion function $Z(x)$, which is related to the Kramp function.

Under the assumption that the wave frequency is much lower than the electron gyrofrequency $\omega \ll \omega_e = \dfrac{eB}{m_e c}$, the electrons can be considered to be magnetized and to drift as a whole with the velocity $\mathbf{v}_d = c[\mathbf{E}_0 \times \mathbf{B}]/\mathbf{B}^2$ with respect to the ions under the action of the constant electric field $\mathbf{E}_0$. The electron component of the dielectric function of collisional plasma is then expressed as [12]

$$\varepsilon_e = \frac{1}{k^2\lambda_{De}^2} \frac{1 + \dfrac{\omega - \mathbf{k}\cdot\mathbf{v}_d + i\nu_e}{\sqrt{2}k_\parallel v_{Te}} \sum\limits_{n=-\infty}^{\infty} e^{-\mu_e}\mathbf{I}_n(\mu_e) Z\left(\dfrac{\omega - \mathbf{k}\cdot\mathbf{v}_d + i\nu_e - n\omega_e}{\sqrt{2}k_\parallel v_{Te}}\right)}{1 + \dfrac{i\nu_e}{\sqrt{2}k_\parallel v_{Te}} \sum\limits_{n=-\infty}^{\infty} e^{-\mu_e}\mathbf{I}_n(\mu_e) Z\left(\dfrac{\omega - \mathbf{k}\cdot\mathbf{v}_d + i\nu_e - n\omega_e}{\sqrt{2}k_\parallel v_{Te}}\right)}.$$

Here, $\mu_e = \dfrac{k_\perp^2 v_{Te}^2}{\omega_e^2}$, $\lambda_D^2 = T_e 4\pi n_0 e^2$, and $v_{T_{i,e}} = \sqrt{\dfrac{T_{i,e}}{m_{i,e}}}$, with T_i and T_e being the ion and electron temperatures and n_0 being the plasma density, and $\mathbf{I}_n$ is the modified Bessel function. Setting $k_\parallel v_{Te} \ll \omega_e, \nu_e$, $|\omega - \mathbf{k}\cdot\mathbf{v}_d| \ll \nu_e$, and $\mu_e \ll 1$, so that

$$e^{-\mu_e}\mathbf{I}_0(\mu_e) = 1 - \left(\frac{k_\perp v_{Te}}{\omega_e}\right)^2,$$

$$Z\left(\frac{\omega - \mathbf{k}\cdot\mathbf{v}_d + i\nu_e}{k_\parallel v_{Te}}\right) \approx -\frac{\sqrt{2}k_\parallel v_{Te}}{\omega - \mathbf{k}\cdot\mathbf{v}_d + i\nu_e}\left(1 + \frac{k_\parallel^2 v_{Te}^2}{(\omega - \mathbf{k}\cdot\mathbf{v}_d + i\nu_e)^2}\right),$$

we then arrive at the following approximate expression for ε_e:

$$\varepsilon_e \simeq \frac{1 - \left(1 - \left(\frac{k_\perp v_{Te}}{\omega_e}\right)^2\right)\left(1 + \frac{k_\parallel^2 v_{Te}^2}{(\omega - \mathbf{k}\cdot\mathbf{v}_d + i\nu_e)^2}\right)}{k^2\lambda_D^2\left[1 - \frac{i\nu_e}{\omega - \mathbf{k}\cdot\mathbf{v}_d + i\nu_e}\left(1 - \left(\frac{k_\perp v_{Te}}{\omega_e}\right)^2\right)\left(1 + \frac{k_\parallel^2 v_{Te}^2}{(\omega - \mathbf{k}\cdot\mathbf{v}_d + i\nu_e)^2}\right)\right]}$$
$$\simeq \frac{\omega_p^2}{\omega_e^2}\frac{(\omega - \mathbf{k}\cdot\mathbf{v}_d + i\nu_e)\left(\frac{k_\perp^2}{k^2} + \frac{k_\parallel^2}{k^2}\frac{\omega_e^2}{\nu_e^2}\right)}{\omega - \mathbf{k}\cdot\mathbf{v}_d + i\nu_e\left(\frac{k_\perp^2 v_{Te}^2}{\omega_e^2} + \frac{k_\parallel^2 v_{Te}^2}{\nu_e^2}\right)}. \quad (3)$$

With allowance for the condition $\nu_e \gg |\omega - \mathbf{k}\cdot\mathbf{v}_d|$, this expression can be represented as

$$\varepsilon_e \simeq \frac{\omega_p^2}{\omega_e^2}\frac{i\nu_e\zeta}{\omega - \mathbf{k}\cdot\mathbf{v}_d + i\nu_e k_\perp^2\rho_e^2\zeta}, \quad (4)$$

where $\rho_e = \sqrt{T_e/m_e\omega_e^2}$ is the electron gyroradius and the notation $\zeta = \frac{k_\perp^2}{k^2} + \frac{k_\parallel^2}{k^2}\frac{\omega_e^2}{\nu_e^2}$ has been introduced to make the formulas more compact.

In the long-wavelength range, Landau damping by ions can be ignored, so dispersion relation (1) and expressions (2) and (4) yield the dispersion relation

$$\frac{\omega_p^2}{\omega_e^2}\frac{i\nu_e\zeta}{\omega - \mathbf{k}\cdot\mathbf{v}_d + i\nu_e k_\perp^2\rho_e^2\zeta} = \frac{\omega_{pi}^2}{\omega(\omega + i\nu_i)} \quad (5)$$

or

$$\omega(\omega + i\nu_i) - c_s^2k^2 = \frac{\omega_{\mathrm{lh}}^2(\omega - \mathbf{k}\cdot\mathbf{v}_d)}{i\nu_e\zeta}, \quad (6)$$

where $\omega_{\mathrm{lh}}^2 = \omega_{pi}^2\omega_e^2/\omega_p^2$ and $c_s^2 = (T_e + T_i)/m_i$. An approximate solution to this relation that is valid for long wavelengths describes quasineutral plasma waves with the frequency

$$\mathrm{Re}\,\omega = \Omega_k \simeq \frac{\mathbf{k}\cdot\mathbf{v}_d}{1 + \Psi_0}. \quad (7)$$

For $\omega^2 \geq c_s^2k^2$, these waves grow at the rate

$$\gamma_{\mathrm{FB}} = \mathrm{Im}\,\omega \simeq \nu_e\zeta\frac{\omega^2 - c_s^2k^2}{\omega_{\mathrm{lh}}^2(1 + \Psi_0)}, \quad (8)$$

where

$$\Psi_0 = 1 + \frac{\nu_i\nu_e}{\omega_{\mathrm{lh}}^2}\zeta \simeq \frac{\nu_e\nu_i}{\omega_{\mathrm{lh}}^2}\left(\frac{\omega_e^2}{\nu_e^2}\frac{k_z^2}{k_\perp^2} + 1\right). \quad (9)$$

The waves are seen to be unstable when

$$\mathbf{k}\cdot\mathbf{v}_d > kc_s(1 + \Psi_0).$$

Note that, with increasing wavenumber k, the growth rate γ_{FB} also increases as $\gamma_{\mathrm{FB}} \sim k^2$. The dependence of the frequency and growth rate of the FB instability on the wave vector $k_x \parallel \mathbf{v}_d$ is illustrated in Fig. 1, which shows the results obtained by solving dispersion relation (5) numerically and those calculated from formulas (7) and (8). We can see that, for $k_x\rho_e \geq 0.1$, approximate solution (7), (8) is insufficiently accurate.

In studying the FB instability, Landau damping by ions is usually ignored. However, since the wave phase velocity $V_{\mathrm{ph}} = \Omega_k/k = \frac{v_d}{1 + \Psi_0}\frac{k_x}{k} = \frac{v_d}{1 + \Psi_0}\cos\phi$ is on the order of the ion thermal velocity, the damping of waves in their resonant interaction with ions can be important, thereby restricting the range of propagation angles of the unstable waves. In addition, Landau damping by ions stabilizes the instability in the short-wavelength range [1, 13]. The effect of Landau damping by ions should be described kinetically. In a linear description, this can be done by retaining the imaginary part of ε_i in dispersion relation (1). In this case, the dispersion relation has the form

$$1 + \frac{\omega_p^2}{\omega_e^2}\frac{i\nu_e\zeta}{\omega - \mathbf{k}\cdot\mathbf{v}_d + i\nu_e k_\perp^2\rho_e^2\zeta} - \frac{\omega_{pi}^2(1 - i\chi)}{(\omega + i\nu_i)\omega} \simeq 0, \quad (10)$$

where we have introduced the notation

$$\chi = \sqrt{\frac{\pi}{2}}\left(\frac{\omega + i\nu_i}{kv_{T_i}}\right)^3\exp\left(-\left(\frac{\omega + i\nu_i}{\sqrt{2}kv_{T_i}}\right)^2\right). \quad (11)$$

For $\omega_e^2 \ll \omega_p^2$, the plasma nonquasineutrality, accounted for by unity on the left-hand side of dispersion relation (10), is unimportant, so we obtain the following expression for describing the wave dispersion properties:

$$\omega = \frac{\mathbf{k}\cdot\mathbf{v}_d(1-i\chi) + i\frac{\nu_e}{\omega_{\rm lh}^2}\zeta\left[\omega^2 - (1-\chi)c_s^2k^2\right]}{1 - i\chi + \frac{\nu_e\nu_i}{\omega_{\rm lh}^2}\zeta}. \quad (12)$$

For $|\chi| \ll 1$, the effect of ion kinetics on the frequency of FB waves is small. In this case, under the above assumption $\mathrm{Re}\,\omega \gg \mathrm{Im}\,\omega$, we find that the wave frequency differs insignificantly from that given by expression (7),

$$\mathrm{Re}\omega \simeq \frac{\mathbf{k}\cdot\mathbf{v}_d(1+\mathrm{Im}\chi)}{1+\Psi_0+\mathrm{Im}\chi},$$

and that the resulting growth rate is approximately equal to the sum of classical growth rate (8) and the rate of Landau damping by ions,

$$\gamma = \mathrm{Im}\omega \simeq \frac{-\mathbf{k}\cdot\mathbf{v}_d\mathrm{Re}\chi + \frac{\mathrm{Re}\chi}{1+\Psi_0+\mathrm{Im}\chi} + \frac{\nu_e\zeta}{\omega_{\rm lh}^2}\left(\omega^2 - \omega_{\rm lh}^2k^2v_{Te}^2\right)}{1+\Psi_0+\mathrm{Im}\chi} \simeq \gamma_{\rm FB} - \mathrm{Re}\omega\mathrm{Re}\chi. \quad (13)$$

Another approximate solution (5), specifically,

$$\mathrm{Re}\omega \simeq \sqrt{\frac{\omega_{pi}^2}{\omega_p^2}k^2v_{Te}^2} = kc_s,$$

$$\gamma = \mathrm{Im}\omega \simeq \frac{\omega_{\rm lh}^2}{\nu_e\zeta}(\mathbf{k}\cdot\mathbf{v}_d - kc_s) - \nu_i kc_s$$

refers to ion sound, damped when $\mathbf{k}\cdot\mathbf{v}_d < kc_s$, and will not be considered in further analysis.

The condition for the effect of Landau damping by ions to be small, $|\chi| \ll 1$, is satisfied when $\left|\frac{\omega+i\nu_i}{kv_{T_i}}\right| \ll 1$; however, this case, $\frac{\mathbf{k}\cdot\mathbf{v}_d}{kv_{T_i}} \ll 1+\Psi_0$, is of no interest because there is no instability. In the opposite limiting case, $\left|\frac{\omega+i\nu_i}{kv_{T_i}}\right| \gg 1$, or when $\frac{\mathbf{k}\cdot\mathbf{v}_d}{kv_{T_i}} \gg 1+\Psi_0$, Landau damping by ions is also insignificant. Thus, formula (13) implies that, in the short-wavelength range, the waves are damped when

$$\frac{\gamma_{\rm FB}}{\mathrm{Re}\omega} = \nu_e\zeta\frac{\omega^2 - c_s^2k^2}{\omega\omega_{\rm lh}^2(1+\Psi_0)} \simeq \frac{\Psi_0}{1+\Psi_0}\frac{\omega}{\nu_i} \le \mathrm{Re}\chi$$

so an additional condition is imposed on the drift velocity and, consequently, on the constant electric field strength required for instability.

Setting $|\chi| \ll 1$ and $\mathrm{Re}\,\omega \gg \mathrm{Im}\,\omega$, we can determine the wave energy density (or equivalently, the energy density of a low-frequency plasma density wave) from the general formula for electrostatic waves:

$$W_k = \frac{\partial\mathrm{Re}\omega\varepsilon(\omega,k)}{\partial\omega}\frac{|\mathbf{k}\varphi_k|^2}{8\pi} = \omega\frac{\partial\mathrm{Re}\varepsilon(\omega,k)}{\partial\omega}\frac{|\mathbf{k}\varphi_k|^2}{8\pi}.$$

For $\nu_e \gg \omega \gg \nu_i$, we have

$$\omega\mathrm{Re}\frac{\partial\varepsilon(\omega,k)}{\partial\omega} \simeq \frac{\omega_{pi}^2}{\omega^2}\left[2 - \frac{(\omega-\mathbf{k}\cdot\mathbf{v}_d)\omega}{(\omega-\mathbf{k}\cdot\mathbf{v}_d)^2 + \left(\nu_e k_\perp^2\rho_e^2\zeta\right)^2}\right].$$

Substituting $\omega \simeq \frac{\mathbf{k}\cdot\mathbf{v}_d}{1+\Psi_0}$ into this formula yields

$$\mathrm{Re}\frac{\partial\varepsilon(\omega,k)}{\partial\omega} \simeq \frac{\omega_{pi}^2(1+\Psi_0)^3}{(\mathbf{k}\cdot\mathbf{v}_d)^3} \times\left[2 + \frac{\Psi_0(\mathbf{k}\cdot\mathbf{v}_d)^2}{\Psi_0^2(\mathbf{k}\cdot\mathbf{v}_d)^2 + (1+\Psi_0)^2\left(\nu_e k_\perp^2\rho_e^2\zeta\right)^2}\right] > 0.$$

Instability growth rate (13) can also be found by using the formulas

$$\frac{\partial}{\partial t}W_k = \mathrm{Re}\mathbf{j}_k\cdot\mathbf{E}_k^*, \quad \gamma = \frac{\partial}{\partial t}\ln W_k = \frac{\mathrm{Im}\varepsilon(\omega,k)}{\frac{\partial\mathrm{Re}\varepsilon(\omega,k)}{\partial\omega}},$$

this can be verified by direct calculations.

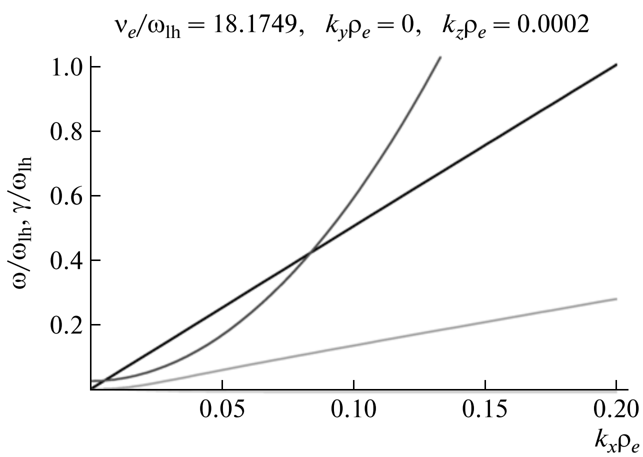

Fig. 1. Frequency and growth rate of the density waves vs. wave vector.

The results of the present section serve as a basis for constructing a mathematical model of nonlinear interaction between waves, which will be considered in the next section.

3. TWO-FLUID MODEL

In what follows, we will consider low-frequency electrostatic waves in a collisional plasma under the conditions $\nu_i \le \frac{\partial}{\partial t} \ll \nu_e$. Such waves are described by the two-fluid hydrodynamic equations [14]

$$m_e n_e \nu_e \mathbf{v}_e + \nabla P_e = -en_e\mathbf{E} - \frac{en_e}{c}[\mathbf{v}_e \times \mathbf{B}], \quad (14)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_e \cdot \nabla\right) n_e + n_e \nabla \cdot \mathbf{v}_e = 0, \quad (15)$$

$$\frac{3}{2} n_e \left(\frac{\partial}{\partial t} + \mathbf{v}_e \cdot \nabla\right) T_e + n_e T_e \nabla \cdot \mathbf{v}_e = -\nabla \cdot \mathbf{q}_e, \quad (16)$$

$$m_i n_i \left(\frac{\partial}{\partial t} + \mathbf{v}_i \cdot \nabla + \nu_i\right) \mathbf{v}_i + \nabla P_i = en_i \mathbf{E}, \quad (17)$$

$$\frac{3}{2} n_i \left(\frac{\partial}{\partial t} + \mathbf{v}_i \cdot \nabla\right) T_i + n_i T_i \nabla \cdot \mathbf{v}_i = -\nabla \cdot \mathbf{q}_i = \nabla \cdot \kappa_i \nabla T_i, \quad (18)$$

with the heat fluxes

$$\mathbf{q}_i = -\kappa_i \nabla T_i, \; \mathbf{q}_e = -\kappa_\parallel^e \nabla_\parallel T_e - \kappa_\perp^e \nabla_\perp T_e - \kappa_\wedge^e \mathbf{b} \times \nabla_\perp T_e, \quad (19)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_e \cdot \nabla\right) \ln \frac{T_e^{\frac{3}{2}}}{n_e} = \frac{1}{n_e T_e} \nabla_\parallel \kappa_\parallel^e \nabla_\parallel T_e + \frac{1}{n_e T_e} \nabla_\perp \kappa_\perp^e \nabla_\perp T_e. \quad (20)$$

Here, φ is the electrostatic potential, $n_{e,i}$ are the electron and ion densities, and $\mathbf{v}_{e,i}$ are electron and ion fluid velocities. For a plasma with infrequent collisions, the electron and ion pressure tensors, P_e and P_i, are scalars, $P_{e,i} = n_{e,i} T_{e,i}$, and the temperatures of the ion and electron components of the plasma differ insignificantly from the temperature of its neutral component. For the lower ionosphere with a neutral density of $n_a \sim 10^{13}\ \mathrm{cm}^{-3}$, a situation is typical in which the behavior of ions and electrons is governed by their collisions with neutrals at the frequencies $\nu_i = \sigma_{ia} n_a v_{T_i} \sim 300$ and $\nu_e = \sigma_{ea} n_a v_{T_e} \sim 30000$, respectively. Under the condition

$$\frac{n_e}{n_a} \ge \frac{3\sigma_{ea} v_{T_e} T_e^{3/2} \sqrt{m_e}}{4\sqrt{2\pi} e^4 Z_i \Lambda} = \frac{0.3}{Z_i \Lambda} \frac{\sigma_{ea}}{\left(\frac{e^2}{T_e}\right)^2} \sim \frac{0.15}{Z_i \Lambda} 10^{14} \sigma_{ea} \left(\frac{T_e}{1\ \mathrm{eV}}\right)^2 \sim 10^{-3}$$

the electrons collide with ions more frequently than with neutrals; i.e., the electron collision frequency is $\nu_e = \tau_e^{-1} \simeq \frac{Z_i \sqrt{2}\Lambda}{9\sqrt{\pi}} \frac{\omega_p}{N_D}$, where $v_{T_e} = \sqrt{\frac{T_e}{m_e}}$, Λ is the Coulomb logarithm, and $N_D = \frac{4\pi}{3} n_e \frac{v_{T_e}^3}{\omega_p^3} = \frac{4\pi}{3} r_D^3 n_e$ is the number of particles in the Debye sphere. For definiteness, we will restrict ourselves to considering only this situation, in which the electron thermal conductivities are $\kappa_\parallel^e = 3.2 \frac{n T_e \tau_e}{m_e}$, $\kappa_\perp^e = 4.7 \frac{n T_e \nu_e}{m_e \omega_{ce}^2}$, and $\kappa_\wedge^e = \frac{5 n T_e}{2 m_e \omega_{ce}}$.

It is also necessary to take into account the large-scale electric field $\mathbf{E}_0$, which maintains the mean electric current in a direction transverse to the magnetic field. To do this, we set $\mathbf{E} = \mathbf{E}_0 - \nabla\varphi$, where the potential φ satisfies Poisson's equation

$$\Delta\varphi = 4\pi e(n_e - n_i - n_{\mathrm{res}}), \quad (21)$$

which closes the above set of equations. We distinguish between two ion groups: main ions with the density n_i and resonant ions with the density $n_{\mathrm{res}} \ll |n_i - n_0|$, which are responsible for Landau damping and the motion of which should be described kinetically. The resonant ions satisfy the continuity equation $\frac{\partial}{\partial t} n_{\mathrm{res}} + \frac{1}{Ze} \nabla \cdot \mathbf{j}_{\mathrm{res}} = 0$, where $\mathbf{j}_{\mathrm{res}}$ is their electric current and Ze is their charge.

Under the conditions adopted here $\left(\frac{\partial}{\partial t} \ll \nu_e \ll \omega_e\right)$, the electron motion can be described in the drift approximation. It is convenient to introduce a new potential, $\Psi = \varphi - \frac{\gamma_e T_e}{e} \ln P_e \simeq \varphi - \frac{T_e}{e}\left(\frac{\delta T_e}{T_e} + \frac{\delta n_e}{n_e}\right)$, so we have $\mathbf{E} + \frac{\nabla P_e}{en} = \mathbf{E}_0 - \nabla\Psi$. The electron velocity then has the components

$$v_{ze} = \frac{1}{m_e \nu_e}\left(\frac{e\partial\varphi}{\partial z} - \frac{1}{n_e}\frac{\partial P_e}{\partial z}\right) = \frac{e}{m_e \nu_e}\frac{\partial \Psi}{\partial z}, \quad (22)$$

$$\begin{aligned} \mathbf{v}_{e\perp} &= \mathbf{v}_d + \frac{c}{\mathbf{B}^2}[\mathbf{B} \times \nabla\Psi] - \frac{m_e \nu_e}{e}\frac{c}{\mathbf{B}^2}[\mathbf{B} \times \mathbf{v}_e] \\ &\simeq \mathbf{v}_d + \frac{c}{B}[\mathbf{z} \times \nabla\Psi] - \frac{\nu_e}{\omega_e}[\mathbf{z} \times \mathbf{v}_d] + \frac{c}{B}\frac{\nu_e}{\omega_e}\nabla_\perp\Psi, \end{aligned} \quad (23)$$

where $\mathbf{v}_d$ is the transverse drift velocity in the large-scale electric field and $\mathbf{z} = \mathbf{B}/B$. The total derivative is given by the relationship

$$\frac{\partial}{\partial t} + \mathbf{v}_e \cdot \nabla \simeq \frac{\partial}{\partial t} + \frac{e}{m_e \nu_e}\frac{\partial \Psi}{\partial z}\frac{\partial}{\partial z} + \mathbf{v}_d \cdot \nabla_\perp - \frac{c}{B}[\nabla\Psi \times \nabla]_z + \frac{c}{B}\frac{\nu_e}{\omega_e}\nabla_\perp \cdot \Psi \nabla_\perp,$$

so Eq. (15) for the electron density can be represents as

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_d\cdot\nabla_\perp\right)\frac{\delta n_e}{n}+\frac{c}{B}[\nabla\ln n_0\times\nabla\Psi]_z$$
$$+\frac{c}{B}\frac{\nu_e}{\omega_e}\left(\nabla_\perp^2+\frac{\omega_e^2}{\nu_e^2}\frac{\partial^2}{\partial z^2}\right)\Psi \quad (24)$$
$$=\frac{c}{B}\left[\nabla\Psi\times\nabla\frac{\delta n_e}{n}\right]_z-\frac{\nu_e}{\omega_e}\frac{c}{B}\left(\frac{\omega_e^2}{\nu_e^2}\frac{\partial\Psi}{\partial z}\frac{\partial}{\partial z}+\nabla_\perp\times\Psi\nabla_\perp\right)\frac{\delta n_e}{n}.$$

It is easy to see that the main nonlinear term in these equations is estimated by $\sim\frac{c}{B}\left[\nabla\frac{\delta n_e}{n}\times\nabla\Psi\right]_z$.

Estimates of the nonlinear terms in the equation of ion motion show that, for $\omega\ge\omega_i$, the nonlinearity of the motion can be ignored, so we can set $\left(\frac{\partial}{\partial t}+\mathbf{v}_i\cdot\nabla\right)\simeq\frac{\partial}{\partial t}$. Indeed, comparing between the characteristic time scales of the nonlinear motion of the ions and electrons,

$$\frac{1}{\tau_i}\sim\frac{|(\mathbf{v}_i\cdot\nabla)\mathbf{v}_i|}{|\mathbf{v}_i|}\sim k|\mathbf{v}_i|\sim k\frac{ek'\varphi}{m_i\omega},$$

$$\frac{1}{\tau_e}\sim\frac{\left|\frac{c}{B}[\nabla_\perp n\times\nabla\varphi]\right|}{n}\sim\frac{c}{B}[\mathbf{k}'\times\mathbf{k}]_z\varphi\sim\frac{cm_i}{eB}[\mathbf{k}'\times\mathbf{k}]_z\frac{e\varphi}{m_i},$$

$$\frac{1}{\tau_e}\ge\frac{1}{\tau_i}\quad\text{for}\quad\frac{\omega}{\omega_i}[\mathbf{k}'\times\mathbf{k}]_z\ge kk'$$

we can see that the first time scale is much longer than the second. Consequently, in the fluid approximation, the dynamics of the main ion component can be described by the equations

$$\frac{\partial}{\partial t}\frac{\delta n_i}{n_0}=\frac{\partial}{\partial t}\frac{\delta n}{n_0}=-\nabla\cdot\mathbf{v}_i=-\nabla^2\Phi \quad (25)$$

and

$$\left(\frac{\partial}{\partial t}+\nu_i\right)\nabla^2\Phi=\left(\frac{\partial}{\partial t}+\nu_i\right)\nabla\cdot\mathbf{v}_i=\nabla\left(-\frac{e}{m_i}\nabla\varphi-\frac{1}{m_in_i}\nabla P_i\right)$$

or by the equation

$$\nabla^2\left[\left(\frac{\partial}{\partial t}+\nu_i\right)\Phi+\frac{e\varphi}{m_i}+\frac{T_i}{m_i}\left(\frac{\delta T_i}{T_i}+\frac{\delta n_i}{n_0}\right)\right]=0, \quad (26)$$

where Φ is the ion velocity potential, defined by $\mathbf{v}_i=\nabla\Phi$.

As regards waves with wavelengths much greater than the Debye radius, there are grounds to assume that the waves are quasineutral, $n_e\simeq n_i=n$ (see, e.g., [7]). In the linear approximation, this assumption is almost obvious, and conditions for the applicability of the quasineutral description are easy to check. In what follows, we will assume that the plasma is quasineutral in the nonlinear regime as well, keeping in mind, however, that, when the density perturbation grows to large amplitudes, the plasma can become nonquasineutral and the description of waves should be refined accordingly.

When the deviation from quasineutrality can be ignored, $\delta n_e\simeq\delta n_i+n_{\rm res}\simeq\delta n+n_{\rm res}$, the condition for the electron and ion motions to be consistent, $\frac{\partial}{\partial t}(\delta n_e-\delta n_i-n_{\rm res})=\nabla\cdot n(\mathbf{v}_i-\mathbf{v}_e)+\frac{1}{Zen_0}\nabla\cdot\mathbf{j}_{\rm res}\simeq 0$, imposes the restrictions with which to determine the potential ψ. In fact, comparing Eqs. (25) and (24), we arrive at the consistency condition

$$\frac{c}{B}\frac{\nu_e}{\omega_e}\left(\nabla_\perp^2\Psi+\frac{\omega_e^2}{\nu_e^2}\frac{\partial^2\Psi}{\partial z^2}\right)+\frac{c}{B}[\nabla\Psi\times\nabla_\perp\ln n]_z$$
$$+(\mathbf{v}_d\cdot\nabla_\perp)\frac{\delta n}{n}-\frac{\nu_e}{\omega_e}[\mathbf{v}_d\times\nabla_\perp]\frac{\delta n}{n}+\frac{c}{B}\left[\nabla\Psi\times\nabla_\perp\frac{\delta n}{n}\right] \quad (27)$$
$$+\frac{c}{B}\frac{\nu_e}{\omega_e}\left(\nabla_\perp\cdot\frac{\delta n}{n}\nabla_\perp\Psi+\frac{\omega_e^2}{\nu_e^2}\frac{\partial}{\partial z}\frac{\delta n}{n}\frac{\partial\Psi}{\partial z}\right)=\nabla^2\Phi+\frac{1}{Zen_0}\nabla\cdot\mathbf{j}_{\rm res},$$

which does not contain time derivatives and is thereby nonevolutionary.

If we ignore thermal conductivity $\left(\frac{\delta T_{e,i}}{T_i}=\frac{2}{3}\frac{\delta n}{n}\right)$ and the interaction of waves with resonant particles, then we obtain the set of FB equations (the classical FB model)

$$\frac{\partial}{\partial t}\frac{\delta n}{n}+\frac{T_e}{m_i}\nabla^2\phi=0, \quad (28)$$

$$\left(\frac{\partial}{\partial t}+\nu_i\right)\phi+\frac{e\varphi}{T_e}+\frac{5}{3}\frac{T_i}{T_e}\frac{\delta n}{n}=0, \quad (29)$$

$$\mathbf{v}_d\cdot\nabla_\perp\frac{\delta n}{n}+\omega_e\rho_e^2\left[\nabla\ln\left(n_0+\frac{\delta n}{n}\right)\times\nabla\frac{e\Psi}{T_e}\right]_z$$
$$+\nu_e\rho_e^2\left(\nabla_\perp^2+\frac{\omega_e^2}{\nu_e^2}\frac{\partial^2}{\partial z^2}\right)\frac{e\Psi}{T_e}=\frac{T_e}{m_i}\nabla^2\phi, \quad (30)$$

$$\frac{e\varphi}{T_e}=\frac{e\Psi}{T_e}+\frac{5}{3}\frac{\delta n}{n}. \quad (31)$$

In this set, only the third (nonevolutionary) equation contains the electron nonlinearity. Above, we have introduced the normalized potential $\phi=\frac{m_i\Phi}{T_e}$. In k space, we can use the expansion $\delta n/n=\sum n_k\exp i\mathbf{k}\cdot\mathbf{r}$ to represent these equations as

$$\frac{\partial}{\partial t}n_k=c_e^2k^2\phi_k, \quad (32)$$

$$\left(\frac{\partial}{\partial t}+\nu_i\right)\phi_k+\psi_k+\frac{5}{3}\frac{T_e+T_i}{T_e}n_k=0, \quad (33)$$

$$i\mathbf{v}_d\cdot\mathbf{k}_\perp n_k + i\omega_e\rho_e^2[\nabla\ln n_0\times\mathbf{k}]_z\psi_k - \nu_e\rho_e^2k_\perp^2\zeta\psi_k + c_e^2k^2\phi_k = -\omega_e\rho_e^2\left[\nabla\frac{\delta n}{n}\times\nabla\psi\right]_{zk}, \quad (34)$$

where $\psi = \dfrac{e\Psi}{T_e}$ and $c_e^2 = \dfrac{T_e}{m_i}$.

The classical FB model was used by Otani and Oppenheim [8]. In the linear approximation, Eqs. (32)–(34) can be reduced to the equation

$$\left(\frac{\partial}{\partial t} + i\mathbf{v}_d\cdot\mathbf{k}_\perp\right)n_k = \frac{i\omega_e\rho_e^2[\nabla\ln n_0\times\mathbf{k}]_z - \nu_e\mathbf{k}_\perp^2\rho_e^2\zeta}{c_e^2k^2}\left[\left(\frac{\partial}{\partial t}+\nu_i\right)\frac{\partial}{\partial t} + c_s^2k^2\right]n_k, \quad (35)$$

which yields the dispersion relation

$$\omega - \mathbf{v}_d\cdot\mathbf{k}_\perp = \frac{\omega_e\rho_e^2[\nabla\ln n_0\times\mathbf{k}]_z + i\nu_ek_\perp^2\rho_e^2\zeta}{c_e^2k^2}\times\left[\omega(\omega+i\nu_i) - c_s^2k^2\right]. \quad (36)$$

The solution to this dispersion relation was obtained in the very first papers (see [15]). For $\nabla\ln n_0 = 0$, this solution was considered in the previous section.

In the short-wavelength range, the equations in question should be refined in order to correct their main drawback, specifically, the presence of linear instability at arbitrarily large values of the wave vectors (see above). This should be done by taking into account the resonant interaction with ions (Landau damping) and also the electron thermal conductivity, which changes the properties of waves at large k values. In this case, in the quasineutral approximation, Eqs. (32)–(34) should be supplemented with the thermal conductivity equation, which can be written in terms of variation in the electron entropy,

$$S_e = \ln\frac{T_e^{\frac{3}{2}}}{n_e}, \quad \delta S_e = s_e, \quad \frac{\delta T_e}{T_e} = \frac{2}{3}\left(\frac{\delta n_e}{n} + s_e\right)$$

in the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_e\cdot\nabla\right)s_e = \nu_e\left(\alpha_{ez}\lambda_e^2\nabla_\parallel^2 + \alpha_{e\perp}\rho_e^2\nabla_\perp^2\right)\frac{\delta T_e}{T_e}, \quad (37)$$

or in k space,

$$\left(\frac{\partial}{\partial t} + i\mathbf{v}_d\cdot\mathbf{k}_\perp\right)s_k = \frac{2}{3}\nu_e\mathbf{k}_\perp^2\rho_e^2\alpha(n_k + s_k), \quad (38)$$

where $\alpha = \alpha_{e\perp} + \alpha_{ez}\dfrac{\omega_e^2k_z^2}{\nu_e^2k_\perp^2}$ and

$$\psi = \frac{e\Psi}{T_e} = \frac{e\varphi}{T_e} - \frac{5}{3}\frac{\delta n_e}{n} - \frac{2}{3}s_e, \quad \frac{e\varphi}{T_e} = \psi + \frac{5}{3}\frac{\delta n}{n} + \frac{2}{3}s_e.$$

In the linear approximation, we introduce the notation $c_s^2 = \dfrac{5}{3}\dfrac{T_e+T_i}{T_e}$ and $c_e^2 = \dfrac{5}{3}\dfrac{T_e+T_i}{m_i}$ to arrive at the set of equations

$$\left[\left(\frac{\partial}{\partial t}+\nu_i\right)\frac{\partial}{\partial t} + c_s^2k^2\right]n_k + c_e^2k^2\left(\psi_k + \frac{2}{3}s_k\right) = 0,$$

$$\left(\frac{\partial}{\partial t} + i\mathbf{v}_d\cdot\mathbf{k}_\perp\right)n_k = \nu_e\mathbf{k}_\perp^2\rho_e^2\zeta\psi_k, \quad (39)$$

$$\left(\frac{\partial}{\partial t} + i\mathbf{v}_d\cdot\mathbf{k}_\perp - \frac{2}{3}\nu_ek_\perp^2\rho_e^2\alpha\right)s_k = \frac{2}{3}\nu_e\mathbf{k}_\perp^2\rho_e^2\alpha n_k,$$

which yields the dispersion relation

$$\Omega(\Omega+i\nu_i) - c_s^2k^2 = c_e^2k^2\left(\frac{-i(\Omega - \mathbf{v}_d\cdot\mathbf{k}_\perp)}{\nu_ek_\perp^2\rho_e^2\zeta} + \frac{i\frac{4}{9}\nu_ek_\perp^2\rho_e^2\alpha}{\Omega - \mathbf{v}_d\cdot\mathbf{k}_\perp - i\frac{2}{3}\nu_ek_\perp^2\rho_e^2\alpha}\right) \quad (40)$$

or

$$\Omega(\Omega+i\nu_i) - \left(c_s^2 - \frac{2}{3}c_e^2\right)k^2 = (\Omega - \mathbf{v}_d\cdot\mathbf{k}_\perp)\left(\frac{\omega_{\mathrm{lh}}^2}{i\nu_e\zeta} + \frac{\frac{2}{3}c_e^2k^2}{\Omega - \mathbf{v}_d\cdot\mathbf{k}_\perp - i\frac{2}{3}\nu_ek_\perp^2\rho_e^2\alpha}\right). \quad (41)$$

We can see that the thermal conductivity does not come into play when

$$\Omega - \mathbf{v}_d\cdot\mathbf{k}_\perp \gg \frac{2}{3}i\nu_ek_\perp^2\rho_e^2\alpha,$$

or, according to approximate solution (7), when

$$\mathbf{k}\cdot\mathbf{v}_d\left(\frac{\Psi_0}{1+\Psi_0}\right) \gg \frac{2}{3}\alpha\nu_ek_\perp^2\rho_e^2.$$

For $k_\perp^2\rho_e^2 \sim \mathbf{k}\cdot\mathbf{v}_d/\nu_e \ll 1$, the instability is suppressed.

4. WEAK INTERACTION APPROXIMATION

In [7, 11], the FB instability was simulated based on a quasineutral hydrodynamic model. The authors of those papers had to make special efforts to solve the nonlinear nonevolutionary equation relating the fluctuations of the density and potential. It was mentioned above that, in considering small-scale waves, it is also necessary to account for Landau damping by ions as well as the electron thermal conductivity. The mathematical model can be further simplified in such a way that it will be capable of determining the wave spectra, while capturing all important features of the wave interaction. Indeed, linear instability theory implies that the rate at which the waves grow is much less than their period, so it is natural to assume that, when the instability is suppressed, the reciprocal of the characteristic time scale on which the waves interact is also much less than the wave period. Consequently, we can seek the solution in the weak interaction approximation such that $n_k \simeq n_k(t)\exp(-i\Omega_k t)$ and $\frac{\partial}{\partial t}n_k(t) \ll \Omega_k n_k(t)$. This approach, which is similar to that used by Volosevich and Galperin [16], makes it possible to substantially reduce computational difficulties because it reduces the problem to that of analyzing the set of equations for the slowly varying amplitudes of the density waves.

Hence, we substitute $n_k \simeq n_k(t)\exp(-i\Omega_k t)$ into the equations following from the complete set of Eqs. (32)–(38) in the Fourier representation,

$$\frac{\partial}{\partial t}n_k = c_e^2 k^2 \phi_k, \tag{42}$$

$$\left(\frac{\partial}{\partial t} + \nu_i\right)\phi_k + \varphi_k + \frac{5T_i}{3T_e}n_k = 0, \tag{43}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} + i\mathbf{v}_d\cdot\mathbf{k}_\perp\right)n_k &= \nu_e\rho_e^2 k_\perp^2 \zeta\psi_k \\ &+ \frac{\omega_e}{2}\rho_e^2 \sum_{\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2} [\mathbf{k}_1\times\mathbf{k}_2]_z(\psi_{k_2}n_{k_1} - \psi_{k_1}n_{k_2}) \\ &+ \frac{\nu_e}{2}\rho_e^2 \sum_{\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2}\left(\frac{\omega_e^2}{\nu_e^2}k_{2z}k_{1z} + \mathbf{k}_{2\perp}\cdot\mathbf{k}_{1\perp}\right)n_{k_1}\psi_{k_2},\end{aligned} \tag{44}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} + i\mathbf{v}_d\cdot\mathbf{k}_\perp - \frac{2}{3}\alpha\nu_e k_\perp^2\rho_e^2\right)s_k &= \frac{2}{3}\alpha\nu_e k_\perp^2\rho_e^2 n_k, \\ \varphi_k &= \psi_k + \frac{5}{3}n_k + \frac{2}{3}s_k\end{aligned} \tag{45}$$

and, in the weak interaction approximation, systematically eliminate the quantities ϕ_k, φ_k, and s_k in accordance with the formulas

$$\varphi_k = \frac{1}{c_e^2 k^2}\left(-i\Omega_k + \frac{\partial}{\partial t}\right)n_k, \tag{46}$$

$$\begin{aligned}\varphi_k &= -\frac{5T_i}{3T_e}n_k - \left(-i\Omega_k + \frac{\partial}{\partial t} + \nu_i\right)\phi_k \\ &= -\frac{5T_i}{3T_e}n_k + \frac{1}{c_e^2k^2}\left(\Omega_k^2 + i\Omega_k\left(2\frac{\partial}{\partial t} + \nu_i\right)\right)n_k,\end{aligned} \tag{47}$$

$$\begin{aligned}s_k &= \frac{i\frac{2}{3}\alpha\nu_e k_\perp^2\rho_e^2}{\Omega_k - \mathbf{v}_d\cdot\mathbf{k}_\perp - i\frac{2}{3}\alpha\nu_e k_\perp^2\rho_e^2} \\ &\times\left(n_k - \frac{i\frac{\partial}{\partial t}n_k}{\Omega_k - \mathbf{v}_d\cdot\mathbf{k}_\perp - i\frac{2}{3}\alpha\nu_e k_\perp^2\rho_e^2}\right).\end{aligned} \tag{48}$$

Inserting

$$\begin{aligned}\psi_k &= \varphi_k - \frac{5}{3}n_k - \frac{2}{3}s_k \\ &= \left\{\frac{\Omega_k(\Omega_k + i\nu_i)}{c_e^2k^2} - \frac{5}{3}\left(1 + \frac{T_i}{T_e}\right) - \frac{i\frac{4}{9}\alpha\nu_e k_\perp^2\rho_e^2}{\Omega_k - \mathbf{v}_d\cdot\mathbf{k}_\perp - i\frac{2}{3}\alpha\nu_e k_\perp^2\rho_e^2}\right\}n_k \\ &+ \left\{\frac{2\Omega_k}{c_e^2k^2} + \frac{i\frac{4}{9}\alpha\nu_e k_\perp^2\rho_e^2}{\left[\Omega_k - \mathbf{v}_d\cdot\mathbf{k}_\perp - i\frac{2}{3}\alpha\nu_e k_\perp^2\rho_e^2\right]^2}\right\}i\frac{\partial}{\partial t}n_k\end{aligned}$$

into Eq. (44), we then arrive at the equation

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - i(\Omega_k - \mathbf{v}_d\cdot\mathbf{k}_\perp)\right)n_k \\ &= \frac{\nu_e}{\omega_{\text{lh}}^2}\zeta\left\{\Omega_k(\Omega_k + i\nu_i) - c_s^2k^2 - \frac{i\frac{4}{9}\alpha\nu_e k_\perp^2\rho_e^2}{\Omega_k - \mathbf{v}_d\cdot\mathbf{k}_\perp - i\frac{2}{3}\alpha\nu_e k_\perp^2\rho_e^2}\right\}n_k \\ &+ \frac{\nu_e}{\omega_{\text{lh}}^2}\zeta\left\{2\Omega_k + \frac{i\frac{4}{9}\alpha\nu_e k_\perp^2\rho_e^2 c_e^2k^2}{\left[\Omega_k - \mathbf{v}_d\cdot\mathbf{k}_\perp - i\frac{2}{3}\alpha\nu_e k_\perp^2\rho_e^2\right]^2}\right\}i\frac{\partial}{\partial t}n_k \\ &+ \frac{\omega_e}{2}\rho_e^2\sum_{\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2}[\mathbf{k}_1\times\mathbf{k}_2]_z(\psi_{k_2}n_{k_1} - \psi_{k_1}n_{k_2}) \\ &+ \frac{\nu_e}{2}\rho_e^2\sum_{\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2}\left(\frac{\omega_e^2}{\nu_e^2}k_{2z}k_{1z} + \mathbf{k}_{2\perp}\cdot\mathbf{k}_{1\perp}\right)n_{k_1}\psi_{k_2}.\end{aligned} \tag{49}$$

The assumption that the interaction is weak implies that (note that Ω_k are real!)

$$\begin{aligned}&\Omega_k - \mathbf{v}_d\cdot\mathbf{k}_\perp \\ &= \frac{\nu_e k_\perp^2\rho_e^2}{c_e^2k^2}\zeta\,\mathrm{Re}\left\{i\Omega_k(\Omega_k + i\nu_i) - ic_s^2k^2 + \frac{\frac{4}{9}\alpha\nu_e k_\perp^2\rho_e^2 c_e^2k^2}{\Omega_k - \mathbf{v}_d\cdot\mathbf{k}_\perp - i\frac{2}{3}\alpha\nu_e k_\perp^2\rho_e^2}\right\} \\ &= \frac{\nu_e}{\omega_{\text{lh}}^2}\zeta\,\mathrm{Re}\left\{-\Omega_k\nu_i + \frac{\frac{4}{9}\alpha\nu_e\omega_{\text{lh}}^2k_\perp^2k^2\rho_e^4}{\Omega_k - \mathbf{v}_d\cdot\mathbf{k}_\perp - i\frac{2}{3}\alpha\nu_e k_\perp^2\rho_e^2}\right\},\end{aligned}$$

which coincides with the expression for the real part of dispersion relation (40). Introducing the notation

$$Z_k = \zeta \operatorname{Im}\left\{ i\frac{\nu_e\left(\Omega_k(\Omega_k + i\nu_i) - c_s^2k^2\right)}{\omega_{\rm lh}^2} + \frac{\frac{4}{9}\alpha\nu_e^2 k_\perp^2 k^2 \rho_e^4}{\Omega_k - \mathbf{v}_d\cdot\mathbf{k}_\perp - i\frac{2}{3}\alpha\nu_e k_\perp^2\rho_e^2}\right\} \ll \Omega_k$$

and ignoring the small components proportional to $\sim\frac{\partial}{\partial l}n_k$ in the nonlinear terms, we obtain the sought-for equation for the amplitudes of the density waves:

$$\frac{\partial n_k}{\partial t} = (\gamma_k + i\delta\Omega_k)n_k + \frac{\omega_e}{2}\sum_{\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2} G(k_1,k_2)n_{k_1}n_{k_2}\exp i\left(\Omega_k - \Omega_{k_1} - \Omega_{k_2}\right)t\,.$$

Here, γ_k are real. Taking into account the dependence on the fast phase yields the following nonlinear equation for these amplitudes:

$$\left(\frac{\partial}{\partial t} + i\Omega_k - \gamma_k\right)n_k = \frac{\omega_e}{2}\sum_{\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2} G(k_1,k_2)n_{k_1}n_{k_2}.$$

The quantities in this equation are given by fairly complicated formulas:

$$\gamma_k + i\delta\Omega_k = \frac{Z_k}{A_k},$$

$$A_k = 1 + i\frac{\nu_e}{\omega_{\rm lh}^2}\zeta\left[2\Omega_k + \frac{i\frac{4}{9}\alpha\nu_e\omega_{\rm lh}^2k^2k_\perp^2\rho_e^4}{\left[\Omega_k - \mathbf{v}_d\cdot\mathbf{k}_\perp - i\frac{2}{3}\alpha\nu_e k_\perp^2\rho_e^2\right]^2}\right],$$

$$\begin{aligned} G(k_1,k_2) = \frac{\rho_e^2[\mathbf{k}_1\times\mathbf{k}_2]_z}{A_k}&\left(\frac{1}{c_e^2k_1^2}\left\{\Omega_{k_1}(\Omega_{k_1}+i\nu_i) - c_s^2k_1^2 - \frac{i\frac{4}{9}\alpha_1\nu_e\omega_{\rm lh}^2k_1^2k_{1\perp}^2\rho_e^4}{\Omega_{k_1} - \mathbf{v}_d\cdot\mathbf{k}_1 - i\frac{2}{3}\alpha_1\nu_e k_{1\perp}^2\rho_e^2}\right\}\right.\\ &\left.-\frac{1}{c_e^2k_2}\left\{\Omega_{k_2}(\Omega_{k_2}+i\nu_i) - c_s^2k_2 - \frac{i\frac{4}{9}\alpha_2\nu_e\omega_{\rm lh}^2k_2^2k_2^2\rho_e^4}{\Omega_{k_2} - \mathbf{v}_d\cdot\mathbf{k}_2 - i\frac{2}{3}\alpha_2\nu_e k_2^2\rho_e^2}\right\}\right)\\ &+\frac{\nu_e}{A_k\omega_e}\rho_e^2\left(\frac{\omega_e^2}{\nu_e^2}k_{2z}k_{1z} + \mathbf{k}_{2\perp}\cdot\mathbf{k}_{1\perp}\right)\\ &\times\left(\frac{1}{c_e^2k_1^2}\left\{\Omega_{k_1}(\Omega_{k_1}+i\nu_i) - c_s^2k_1^2 - \frac{i\frac{4}{9}\alpha_1\nu_e\omega_{\rm lh}^2k_1^2k_{1\perp}^2\rho_e^4}{\Omega_{k_1} - \mathbf{v}_d\cdot\mathbf{k}_1 - i\frac{2}{3}\alpha_1\nu_e k_{1\perp}^2\rho_e^2}\right\}\right.\\ &\left.+\frac{1}{c_e^2k_2^2}\left\{\Omega_{k_2}(\Omega_{k_2}+i\nu_i) - c_s^2k_2 - \frac{i\frac{4}{9}\alpha_2\nu_e\omega_{\rm lh}^2k_2^2k_2^2\rho_e^4}{\Omega_{k_2} - \mathbf{v}_d\cdot\mathbf{k}_2 - i\frac{2}{3}\alpha_2\nu_e k_2^2\rho_e^2}\right\}\right). \end{aligned} \tag{50}$$

Let us write out approximate expressions that are valid in the region where the thermal-conductivity-induced corrections are unimportant. From formulas (50) we correspondingly obtain

$$\Omega_k \simeq \frac{\mathbf{v}_d\cdot\mathbf{k}_\perp}{1 + \frac{\nu_i\nu_e\rho_e^2}{c_e^2}\zeta},$$

$$\gamma_k \simeq Z_k = \zeta\frac{\nu_e\left(\Omega_k^2 - c_s^2k^2\right)}{\omega_{\rm lh}^2},$$

$$A_k \simeq 1 + i2\frac{\Omega_k\nu_e}{\omega_{\rm lh}^2}\zeta \simeq 1,$$

$$G(k_1,k_2) \simeq \frac{\rho_e^2[\mathbf{k}_1\times\mathbf{k}_2]_z}{A_k}\left(\frac{\Omega_{k_1}(\Omega_{k_1}+i\nu_i)}{c_e^2k_1^2} - \frac{\Omega_{k_2}(\Omega_{k_2}+i\nu_i)}{c_e^2k_2^2}\right) \simeq \frac{\rho_e^2[\mathbf{k}_1\times\mathbf{k}_2]_z}{A_k}\left(\frac{\Omega_{k_1}^2}{c_e^2k_1^2} - \frac{\Omega_{k_2}^2}{c_e^2k_2^2}\right),$$

$$G(k_1,k_2) \equiv G(k_2,k_1) \simeq \frac{\rho_e^2[\mathbf{k}_1\times\mathbf{k}_2]_z}{A_k}\left(\frac{\Omega_{k_1}^2}{c_e^2k_1^2} - \frac{\Omega_{k_2}^2}{c_e^2k_2^2}\right).$$

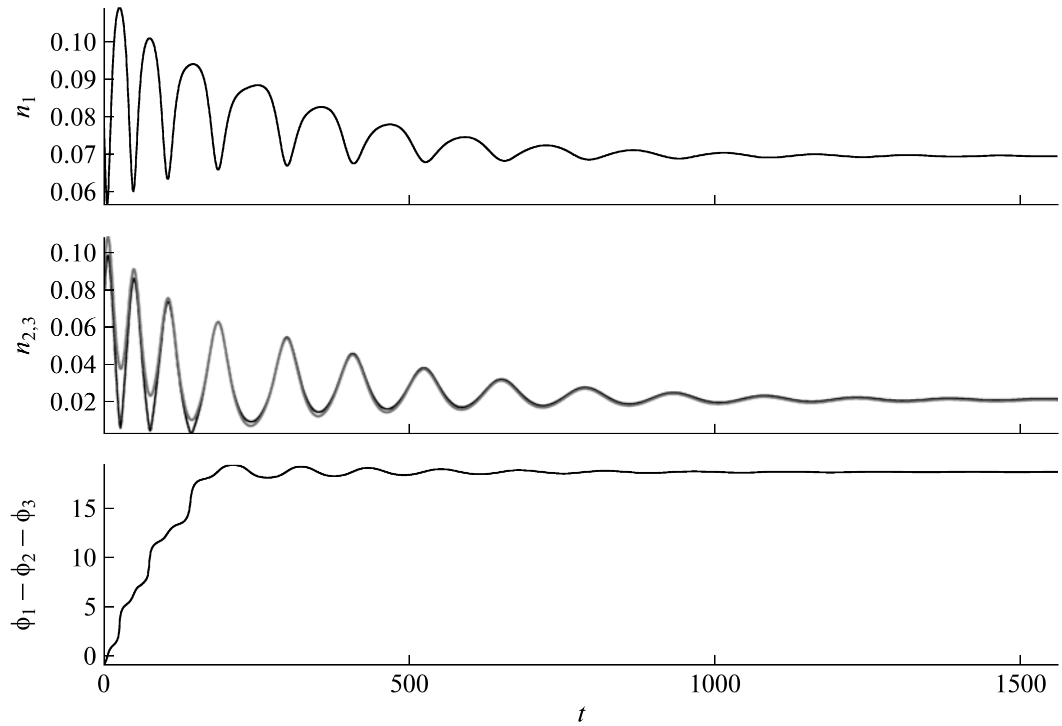

Fig. 2. Calculated time evolution of the normalized amplitudes of the density waves for $\gamma_1 = 1$, $\gamma_2 = -2.3$, and $\gamma_3 = -5.1$.

The latter equation describes the evolution of the spectrum of the density waves during the development of the FB instability with allowance for the nonlinear wave–wave interaction. The FB instability can be stabilized by the efficient mechanism for energy transfer from unstable waves propagating inside the cone to the strongly damped waves outside the cone. This energy transfer can be provided by a cascade of three-wave processes that is accompanied by a decrease in the wavelengths and brings energy to the region of strong Landau damping by ions. However, the energy can also be transferred to the damping region in a shorter way: in the three-wave processes, the unstable wave can transfer its energy directly to the damped waves [17]. Let us consider three waves the interaction between which is described by the following set of equations ($\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$):

$$\left(\frac{\partial}{\partial t} + i\Omega_1 - \gamma_1\right) n_1 = \frac{\omega_e}{2} G(k_2, k_3) n_2 n_3, \qquad (51)$$

$$\left(\frac{\partial}{\partial t} + i\Omega_2 - \gamma_2\right) n_2 = \frac{\omega_e}{2} G(k_1, -k_3) n_1 n_3^*, \qquad (52)$$

$$\left(\frac{\partial}{\partial t} + i\Omega_3 - \gamma_3\right) n_3 = \frac{\omega_e}{2} G(k_1, -k_2) n_1 n_2^*. \qquad (53)$$

The three-wave interaction process can be highly efficient under the synchronization condition

$$\Omega_1 \simeq \Omega_2 + \Omega_3. \qquad (54)$$

This condition alone is insufficient, however. The reason is that the efficiency of the nonlinearity can be low, $G \simeq 0$, because of its vector nature: if the waves are collinear, then it is necessary to additionally require that the coefficients G be large. It is easy to see that the necessary conditions are achieved, e.g., when $k_{2y} \simeq -k_{3y} \gg k_{1y}$, in which case we have $[\mathbf{k}_3 \times \mathbf{k}_2]_z = (k_{2x}k_{3y} - k_{2y}k_{3x}) \simeq k_{2y}(k_{2x} + k_{3x}) = k_{2y}k_{1x}$ and $[\mathbf{k}_1 \times \mathbf{k}_2]_z \simeq (k_{2x}k_{1y} - k_{2y}k_{1x}) \simeq k_{2y}k_{1x}$; that is, all the coefficients are of the same order of magnitude and the equality $k_{2x} + k_{3x} = k_{1x}$ ensures that synchronization condition (54) is satisfied with good accuracy.

Assuming that the amplitude of one of the waves (that with k_0) is much greater than the amplitudes of the two others, from Eqs. (51)–(53) we can conveniently obtain the following relationships for the decay instability:

$$(\Gamma - \gamma_{k_2}) n_{k_2}^* = \frac{\omega_e}{2} k_2^2 \rho_e^2 G_{01}^* n_{k_0}^* n_{k_1},$$

$$(\Gamma - \gamma_{k_1}) n_{k_1} = \frac{\omega_e}{2} k_1^2 \rho_e^2 G_{02} n_{k_0} n_{k_2}^*,$$

$$(\Gamma - \gamma_{k_2})(\Gamma - \gamma_{k_1}) = \frac{\omega_e}{2} k_2^2 \rho_e^2 \frac{\omega_e}{2} k_1^2 \rho_e^2 G_{01}^* G_{02} |n_{k_0}|^2.$$

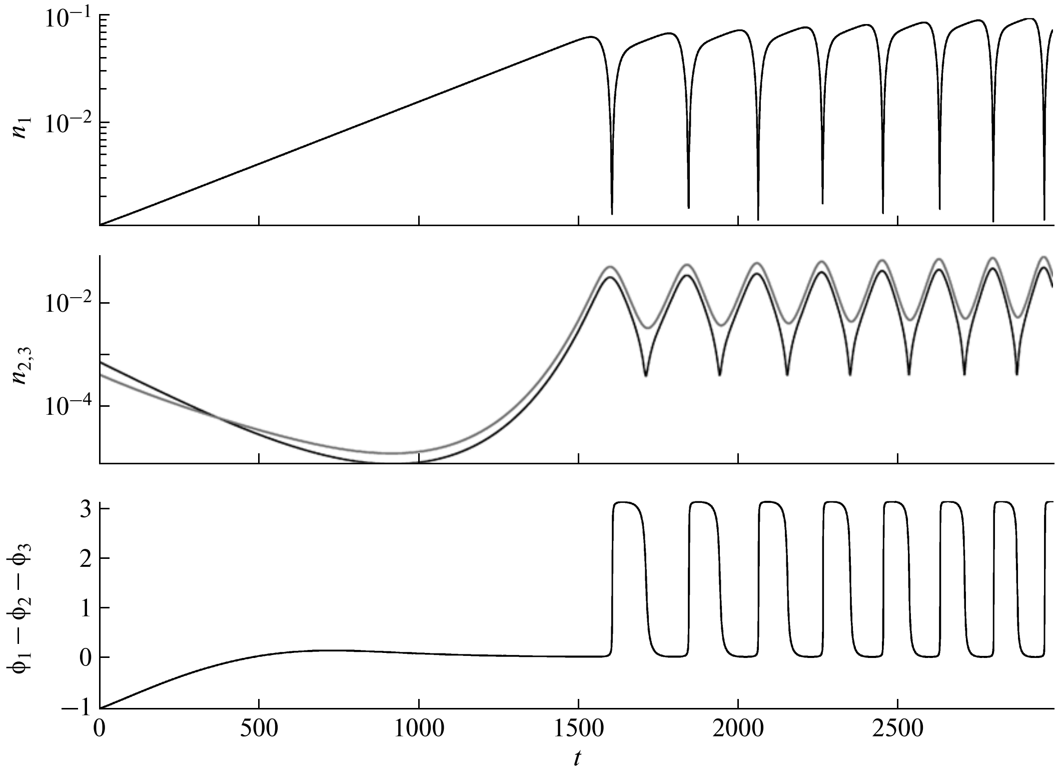

Fig. 3. Numerical solution for $\gamma_1 = 1$, $\gamma_2 = -3.3$, and $\gamma_3 = -6.1$: periodic regime of nonlinear suppression of the instability.

In particular, these relationships yield the threshold condition for the development of the decay instability:

$$|n_{k_0}|^2 = \frac{\omega_e^2}{4\gamma_{k_2}\gamma_{k_1}} k_2^2\rho_e^2 k_1^2\rho_e^2 G_{01}^* G_{02}$$

$$\sim \frac{\omega_e^2}{4\gamma_{k_2}\gamma_{k_1}} \frac{[\mathbf{k}_1 \times \mathbf{k}_2]_z^2 k_1^2 k_2^2 \rho_e^8}{(1 + \Psi_0)^2} \frac{\mathbf{v}_d^2}{c_e^2}.$$

Numerical investigations of the solutions to the equations show that, under certain conditions (see below), the FB instability can be suppressed by a nonlinear (three-wave) mechanism. It can be seen that there exist different nonlinear regimes of the instability. Figure 2 illustrates the results of solving Eqs. (51)–(53) numerically for $\gamma_1 = 1$, $\gamma_2 = -2.3$, and $\gamma_3 = -5.1$. Note that simulations were carried out with the quantities normalized so that $\frac{\omega_e}{2} G(k_1, -k_2) = \frac{\omega_e}{2} G(k_1, -k_3) = \frac{\omega_e}{2} G(k_2, k_3)$. Under these conditions, after several wave periods, the instability relaxes to a steady nonlinear regime, in which the amplitudes of the density waves remain essentially unchanged.

Another regime of nonlinear suppression of the instability is exemplified in Fig. 3, which presents the results of numerical solution of Eqs. (51)–(53) for $\gamma_1 = 1$, $\gamma_2 = -3.3$, and $\gamma_3 = -6.1$. In this case, the instability relaxes to a periodic regime of nonlinear waves in which the energy is regularly exchanged between the modes.

Finally, Fig. 4 shows the results of numerical solution of Eqs. (51)–(53) for $\gamma_1 = 1$, $\gamma_2 = -3.3$, and $\gamma_3 = 7.1$. In such circumstances, the instability relaxes nonlinearly to a stochastic regime in which the waves are irregular and exchange their energy in three-wave interactions.

Mathematically, the above patterns of the instability saturation correspond to an attractor of the solutions to Eqs. (51)–(53); moreover, under certain conditions, we deal with a strange attractor, as in the case of stochastic regime. It should be noted that, according to numerical solutions, the attractor does not always exist; that is, for certain values of the parameters of the system, the solution increases without bound with time and it may be that, for a particular choice of the wave vectors of the satellites, there will be no nonlinear stabilization. That this can be the case is especially obvious when the vectors for all the three waves are chosen to be collinear, i.e., when $\frac{\omega_e}{2} G(k_1, -k_3) = 0$ and the system becomes nonlinear. It is of interest to note that, in some cases when the instability does not saturate nonlinearity with time, there nevertheless exists a fairly long time interval during which the waves intensively exchange energy with each

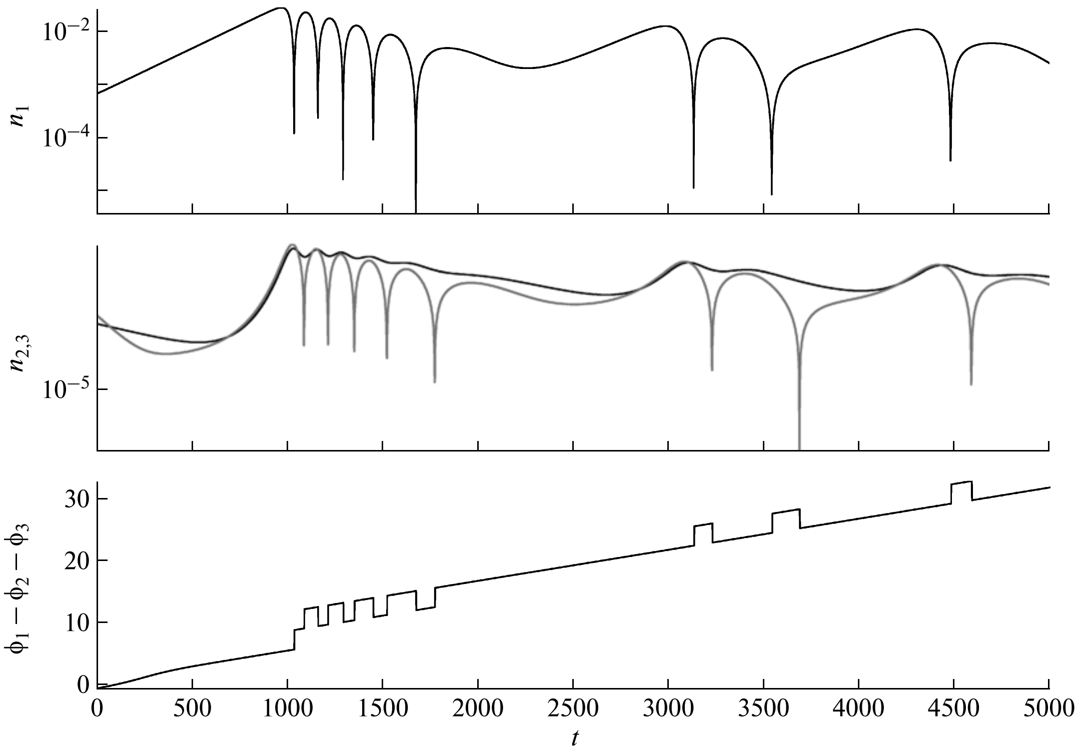

Fig. 4. Numerical solution for $\gamma_1 = 1$, $\gamma_2 = -3.3$, and $\gamma_3 = -7.1$: nonlinear saturation of the instability of the density waves in the stochastic regime.

other and the unstable wave grows very slowly. In view of the approximate character of a few-mode description, this situation may serve as a piece of evidence for stabilization by the three-wave interaction.

5. CONCLUSIONS

In considering the problem of whether the FB instability can be suppressed by nonlinear three-wave interaction processes, it is possible to develop an approximate description of the dynamics of the system on the basis of a set of hydrodynamic-like equations for the wave amplitudes, even though the dissipative processes play a substantial role. These equations, which are similar to those describing the nonlinear interaction of waves in collisionless plasma, are naturally derived from the equations of motion of a fluid (plasma) by expanding the physical fields in spatial modes. The model that makes a quasineutrality assumption but does not assume that the wave–wave interaction is weak could also be regarded as belonging to the same type if it were not for the presence of nonevolutionary coupling. This coupling, which cannot be resolved in the general case, leads to a more involved, nonquadratic nonlinearity, so the analysis becomes far more complicated and the description loses its universal character. On the other hand, if the quasineutrality assumption is ruled out, then eliminating the electric field potential with the help of Poisson's equation again leads to hydrodynamic-like equations. These equations, however, have large coefficients and as such are more difficult to solve under conditions close to quasineutrality, a case in which large terms in physically meaningful solutions should be canceled out. Hence, the proposed description of the nonlinear interaction between FB waves seems to be optimal for numerical analysis.

Our investigation of the nonlinear instability saturation by the three-wave mechanism has revealed different types (regimes) of the dynamics of the system. Each of the regimes corresponds to an attractor of the solutions to the corresponding set of differential equations. Numerical simulations for the case of three-wave interaction in a system with given parameters show that, if the attractor exists, then it is unique. Of course, this conclusion, which was obtained in simulating only few cases, has to be justified in more detail. Yet, given this hypothesis, we can conclude that the suppression of instability in a three-wave system ends with a certain state that depends exclusively on the physical plasma parameters and on the wave parameters (vectors). It seems reasonable to suppose that, slightly above the instability threshold, the number of waves required for an actual description of the system may be larger, always remaining finite, however. As in the case of three waves, we can anticipate that there exist attractors representing stabilized states with a finite number of modes. The problem to be addressed in a future study is that of clarifying whether there can exist several attractors or, equivalently, several different stabilized states of a system with the same plasma parameters. Another important problem is that of determining the wave parameters and the optimum number of waves with which to provide the best representation of the perturbed plasma states as functions of the drift velocity of the electrons with respect to the ions and also of such parameters as the composition, density, and temperature of the plasma.

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