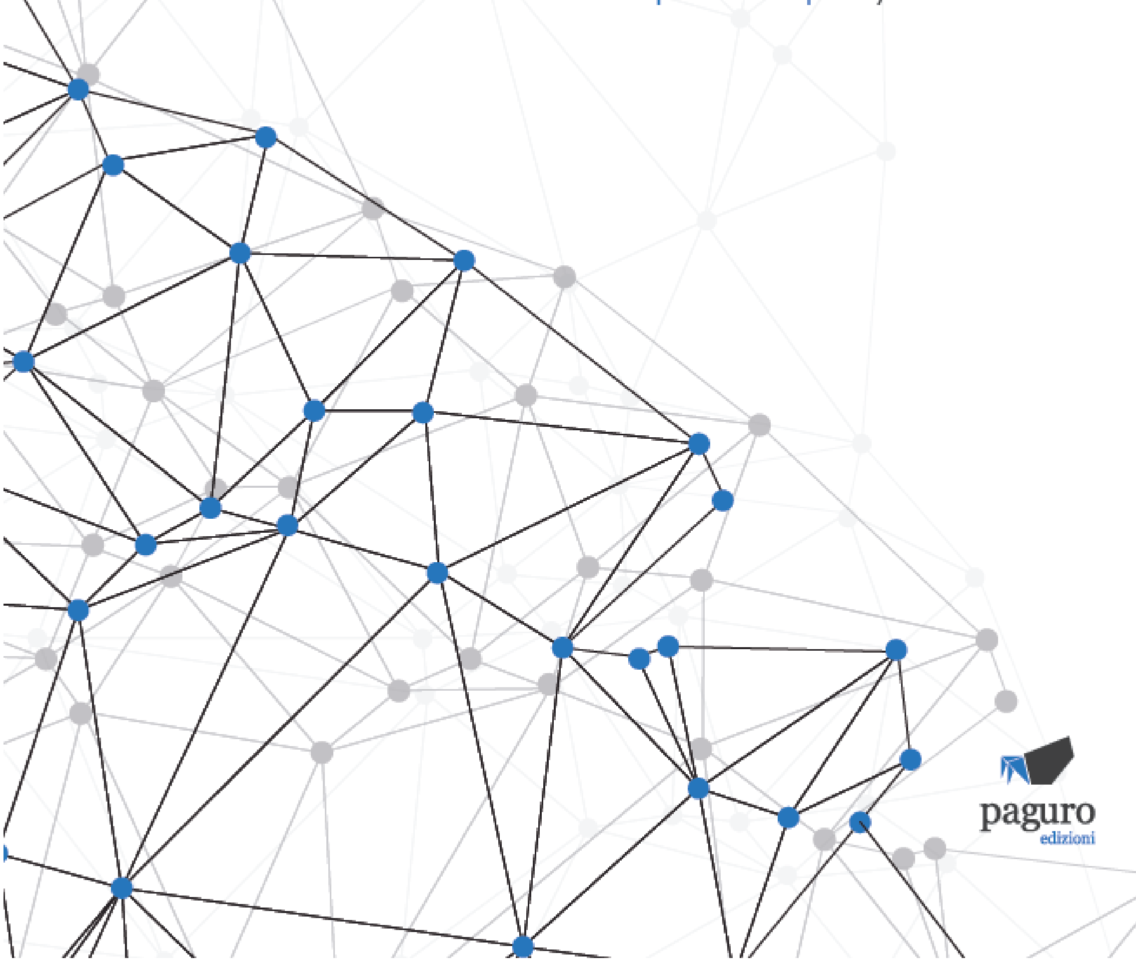
ISBN 978-88-99509-14-9

Proceedings of The 11th International Congress on THERMAL STRESSES 2016

5-9 June | Salerno | Italy



paguro
edizioni

One-dimensional model of nonlinear thermo-elasticity at low temperatures and small strains

J. Ignaczak[1] and W. Domański[2]

[1] *Institute of Fundamental Technological Research, Polish Academy of Sciences, Warsaw, Poland*

[2] *Institute of Mathematics and Cryptology, Faculty of Cybernetics, Military University of Technology, Warsaw, Poland*

A one-dimensional nonlinear homogeneous isotropic thermo-elastic model with an elastic heat flow at low temperatures and small strains is analyzed using the method of weakly nonlinear asymptotics. For such a model both the free energy and the heat flux vector depend not only on the absolute temperature and strain tensor but also on an elastic heat flow that satisfies an evolution equation. The governing equations are reduced to a matrix PDE, and the associated Cauchy problem with a weakly perturbed initial condition is solved. The solution is given in the form of a power series with respect to a small parameter the coefficients of which are functions of a slow variable that satisfy a system of nonlinear second-order ordinary differential transport equations. For a particular Cauchy problem in which the initial data are generated by a closed-form solution to the transport equations, the principal part of the asymptotic solution is a sum of four travelling thermo-elastic waves admitting blow-up amplitudes.

Key Words: *Nonlinear Thermo-elasticity; Low Temperatures; Small Strains; Weakly Nonlinear Asymptotics*

1 Introduction

We are interested in the asymptotic analysis of a model describing a thermally nonlinear homogeneous isotropic thermo-elastic solid with an elastic heat flow at low temperatures and small strains. The equations of the full 3-D model, after non-dimensionalization, are given by the Eqs. (29) in [1]. When reduced to one-space dimension they look as follows:

$$
\begin{aligned}
&\frac{\partial T}{\partial t} - \frac{B}{T}\frac{\partial T}{\partial x} + \frac{\partial B}{\partial x} + T\,\frac{\partial V}{\partial x} - \frac{\partial^2 T}{\partial x^2} = 0\\
&\frac{\partial B}{\partial t} + \frac{1}{T}\frac{\partial T}{\partial x} = 0\\
&\frac{\partial V}{\partial t} + \frac{\in}{\zeta^2}\frac{\partial T}{\partial x} - \frac{1}{\zeta^2}\frac{\partial W}{\partial x} = 0\\
&\frac{\partial W}{\partial t} - \frac{\partial V}{\partial x} = 0
\end{aligned}
\qquad (1)
$$

Here, T is the absolute temperature; and B, V, and W denote the x component of the elastic heat flow, velocity, and strain fields, respectively; in addition, $\in > 0$ and $\zeta > 0$ denote a thermoelastic coupling constant and an inertia coefficient, respectively. It is assumed that a heat supply and the body forces are absent.

Other low-temperature nonlinear thermoelastic models have been discussed in a survey article [5].

In the present paper the 1-D model described by (1) is explored using the method of weakly nonlinear asymptotics [2], [3].

In the following Section the governing equations in a matrix form are recalled.

In Section 3 an eigenvalue problem for the matrix equation is solved in a closed-form.

In Section 4 the Cauchy problem for the matrix equation with a weakly perturbed initial condition is formulated, and an asymptotic solution to the problem is postulated in the form of a power series with respect to a small parameter the coefficients of which are the functions of a slow variable. The slow variable functions are then determined in terms of the wave-amplitudes that satisfy the nonlinear ordinary second order differential transport equations.

In Section 5 an asymptotic solution to a particular Cauchy problem involving the initial data generated by a closed-form solution to the transport equations with blow-up amplitudes is obtained.

2 The governing equations in a matrix form

The model described by equations (1) can be represented as a quasi-linear system of partial differential equation of the second order for an unknown vector field

$$\mathbf{u} = [T, B, V, W]^{\mathrm{T}};\ \ \mathbf{u} = \mathbf{u}(t, x);\ t \geq 0;\ |x| < \infty;$$

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{A}(\mathbf{u})\frac{\partial}{\partial x}\mathbf{u} + \mathbf{B}\frac{\partial^2}{\partial x^2}\mathbf{u} = \mathbf{0} \qquad (2)$$

where $\mathbf{A}(\mathbf{u})$ is the 4×4 matrix depending on $\mathbf{u}$

$$\mathbf{A}(\mathbf{u}) = \begin{bmatrix} -B/T & 1 & T & 0 \\ 1/T & 0 & 0 & 0 \\ \in/\zeta^2 & 0 & 0 & -1/\zeta^2 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad (3)$$

and $\mathbf{B}(\mathbf{u})$ is the 4×4 matrix given by

$$\mathbf{B} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3 Eigenvalue problem for the matrix equation

The eigenvalues λ of the matrix $\mathbf{A}(\mathbf{u})$ satisfy the algebraic equation

$$\det\left(\mathbf{A}(\mathbf{u}) - \lambda\,\mathbf{I}\right) = 0 \qquad (4)$$

By expanding the determinant (4), we obtain

$$\lambda^4 + \lambda^3\frac{B}{T} - \lambda^2\left(\frac{1}{T} + \frac{1}{\zeta^2} + \frac{\in}{\zeta^2}T\right) - \lambda\frac{B}{\zeta^2 T} + \frac{1}{\zeta^2 T} = 0 \qquad (5)$$

We are interested in solving the eigenvalue problem for a particular constant state

$$\mathbf{u}_0 = [T_0, 0, 0, 0]^{\mathrm{T}} \text{ with } T_0 > 0\,. \qquad (6)$$

We obtain the following

Lemma 1. There are four explicit right $\mathbf{r}_{\mathrm{i}}(\mathbf{u}_0)$ and four left $\mathit{l}_{\mathrm{i}}(\mathbf{u}_0)$ eigenvectors corresponding to four *real* eigenvalues of the matrix $\mathbf{A}(\mathbf{u}_0)$. Moreover we may choose the eigenvectors so that the orthogonality conditions

$$\mathbf{r}_{\mathrm{i}}(\mathbf{u}_0)\cdot \mathit{l}_{\mathrm{j}}(\mathbf{u}_0) = \delta_{\mathrm{ij}} \qquad \mathrm{i, j} = 1,2,3,4 \qquad (7)$$

hold true.

The proof of Lemma 1 is given in [4].

4 Asymptotic solution to the Cauchy problem

Let us consider the initial-value problem for the following matrix equation with perturbed initial data (ε is a small parameter).

$$\frac{\partial}{\partial t}\mathbf{u}^{\varepsilon} + \mathbf{A}(\mathbf{u}^{\varepsilon})\frac{\partial}{\partial x}\mathbf{u}^{\varepsilon} + \mathbf{B}\frac{\partial^2}{\partial x^2}\mathbf{u}^{\varepsilon} = \mathbf{0} \qquad (8)$$

subject to the initial condition

$$\mathbf{u}^{\varepsilon}(0, x) = \mathbf{u}_0 + \varepsilon\,\mathbf{u}^{*}(\varepsilon\, x) + \mathbf{0}(\varepsilon^2) \qquad (9)$$

where ε is a small positive number and $\mathbf{u}^{*} = \mathbf{u}^{*}(y)$ is a prescribed function on $|y| < \infty$.

An asymptotic solution to the problem (8)-(9) is postulated in the form

$$\mathbf{u}^{\varepsilon}(t, x) = \mathbf{u}_0 + \varepsilon\,\tilde{\mathbf{u}}(\eta) \qquad (10)$$

where $\eta = \varepsilon(x - \lambda\, t)$ is a new slow variable with λ an eigenvalue of the matrix $\mathbf{A}(\mathbf{u}_0)$ and

$$\tilde{\mathbf{u}}(\eta) = \mathbf{u}_1(\eta) + \varepsilon\ \ \mathbf{u}_2(\eta) + \mathbf{0}(\varepsilon^2) \qquad (11)$$

First we use the Taylor's expansion

$$\mathbf{A}(\mathbf{u}_0 + \varepsilon\,\tilde{\mathbf{u}}) = \mathbf{A}(\mathbf{u}_0) + \varepsilon\,\nabla_{\mathbf{u}}[\mathbf{A}(\mathbf{u}_0)\tilde{\mathbf{u}}] + \mathbf{0}(\varepsilon^2) \qquad (12)$$

Next, by treating η as a new independent variable we express all derivatives from equation (8) in terms of derivatives with respect to η. Then equation (8) is reduced to the form

$$\varepsilon^2[\mathbf{A}(\mathbf{u}_0) - \lambda\,\mathbf{I}]\frac{d}{d\eta}\mathbf{u}_1 + \varepsilon^3\{[\mathbf{A}(\mathbf{u}_0) - \lambda\,\mathbf{I}]\frac{d}{d\eta}\mathbf{u}_2 +$$

$$\nabla_u[\mathbf{A}(\mathbf{u}_0)\mathbf{u}_1]\frac{d}{d\eta}\mathbf{u}_1 + \mathbf{B}\frac{d^2}{d\eta^2}\mathbf{u}_1\} + \mathbf{0}(\varepsilon^4) = 0 \qquad (13)$$

Equating to zero consecutively alike powers of ε we obtain first the terms with ε^2:

$$[\mathbf{A}(\mathbf{u}_0) - \lambda\,\mathbf{I}]\frac{d}{d\eta}\mathbf{u}_1 = \mathbf{0} \qquad (14)$$

Treating the above as the algebraic equation we conclude that its solution must be proportional to the right eigenvector $\mathbf{r}$ of the matrix $\mathbf{A}(\mathbf{u}_0)$ corresponding to the eigenvalue λ. Therefore we can write

$$\mathbf{u}_1 = a(\eta)\mathbf{r} \qquad (15)$$

where $a = a(\eta)$ is an unknown wave's amplitude.

Next we equate to zero the terms with ε^3 and get:

$$[\mathbf{A}(\mathbf{u}_0) - \lambda\,\mathbf{I}]\frac{d}{d\eta}\mathbf{u}_2 = -\mathbf{f} \qquad (16)$$

where

$$\mathbf{f} = \nabla_{\mathbf{u}}[\mathbf{A}(\mathbf{u}_0)\mathbf{u}_1]\frac{d}{d\eta}\mathbf{u}_1 + \mathbf{B}\frac{d^2}{d\eta^2}\mathbf{u}_1 \qquad (17)$$

Treating (16) as the nonhomogeneous algebraic equation and using linear algebra, we get from Fredholm's alternative that this equation has a solution provided that the right hand side of it is orthogonal to the left eigenvectors of the matrix $\mathbf{A}(\mathbf{u}_0)$. Hence

$$\boldsymbol{l}\cdot\mathbf{f} = 0 \qquad (18)$$

Writing the above equation explicitly we get

$$\boldsymbol{l}\cdot\{\nabla_{\mathbf{u}}[\mathbf{A}(\mathbf{u}_0)\mathbf{u}_1]\frac{d}{d\eta}\mathbf{u}_1 + \mathbf{B}\frac{d^2}{d\eta^2}\mathbf{u}_1\} = 0 \qquad (19)$$

Next taking into account (15) we obtain the transport evolution equation for the unknown amplitude $a(\eta)$

$$\Gamma a(\eta)\frac{d}{d\eta}a(\eta) + \Lambda\frac{d^2}{d\eta^2}a(\eta) = 0 \qquad (20)$$

where

$$\Gamma = \boldsymbol{l}\cdot\{\nabla_{\mathbf{u}}[\mathbf{A}(\mathbf{u}_0)\mathbf{r}]\}\mathbf{r}, \qquad \Lambda = \boldsymbol{l}\cdot\mathbf{Br} \qquad (21)$$

As a result, the following theorem holds true:

Theorem 1 An asymptotic solution to the Cauchy problem described by the equations (8) and (9) that represents a nonlinear low-temperature and small-strain thermo-elastic wave propagating along the *x*-axis takes the form

$$\mathbf{u}^{\varepsilon}(t,x) = \mathbf{u}_0 + \varepsilon\sum_{i=1}^{4} a_i(\eta_i)\mathbf{r}_i(\mathbf{u}_0) + \mathbf{0}(\varepsilon^2) \qquad (22)$$

where $a_i = a_i(\eta_i), |\eta_i| < \infty$, is a solution to the transport equations

$$\frac{d}{d\eta_i}\left[\mathrm{H}_i a_i^2(\eta_i) + \frac{d}{d\eta_i}a_i(\eta_i)\right] = 0 \qquad (23)$$

Here,

$$\mathrm{H}_i = \frac{1}{2}\frac{\Gamma_i}{\Lambda_i} \qquad i = 1,2,3,4 \qquad (24)$$

$$\Gamma_i = \mathrm{r}_{i1}\left[l_{i1}\left(\mathrm{r}_{i3} - \frac{\mathrm{r}_{i2}}{T_0}\right) - \frac{\mathrm{r}_{i1}l_{i2}}{T_0^2}\right] \qquad (25)$$

$$\Lambda_i = -l_{i1}\,\mathrm{r}_{i1} \qquad (26)$$

In equations (25)-(26) r_{ik} (l_{ik}) denotes the k-th component of the unit right (left) eigenvector $\mathbf{r}_i$ $(\boldsymbol{l}_i)$ of the matrix $\mathbf{A}(\mathbf{u}_0)$.

The proof of Theorem 1 is given in [4].

5 A particular Cauchy problem

The particular Cauchy problem is related to a particular set of the dimensionless parameters T_0, $\in$, and ζ as well as to a particular initial condition. It is assumed that

$$T_0 = 5, \quad \in = 0.0356, \quad \zeta = 1/\sqrt{2\in} = 2.6499 \qquad (27)$$

and using (24)-(27) we obtain the inequalities

$$\mathrm{H}_1 < 0, \quad \mathrm{H}_2 > 0, \quad \mathrm{H}_3 > 0, \quad \mathrm{H}_4 < 0 \qquad (28)$$

The initial condition is postulated in the form

$$\mathbf{u}(0,x) = \mathbf{u}_0 + \varepsilon \sum_{i=1}^{4} a_j(\varepsilon x)\mathbf{r}_i(\mathbf{u}_0) + \mathbf{0}(\varepsilon^2) \quad (29)$$

where $a_i = a_i(\eta_i)$, $|\eta_i| < \infty$, is a solution to the transport equations (23) subject to the conditions

$$\frac{d}{d\eta_i} a_i(\eta_i) \to 0 \ \text{ as } \eta_i \to -\infty \ \ (i = 1,2,3,4)$$
$$a_1(-\infty) = a_4(-\infty) = 1,\ a_2(-\infty) = a_3(-\infty) = -1 \quad (30)$$

It is easy to show that such a solution is represented by the <u>two-valued function</u> defined by

$$a_i(\eta_i) = \begin{cases} \coth(H_i\eta_i) \text{ for } -\infty < \eta_i < 0 \\ \tanh(H_i\eta_i) \text{ for } 0 < \eta_i < \infty \end{cases} \quad (31)$$

or

$$a_i(\eta_i) = \begin{cases} \tanh(H_i\eta_i) \text{ for } -\infty < \eta_i < 0 \\ \coth(H_i\eta_i) \text{ for } 0 < \eta_i < \infty \end{cases} \quad (32)$$

and the following theorem holds true

Theorem 2. An asymptotic solution to the particular Cauchy problem in which the function $a_i = a_i(\varepsilon x)$ is a restriction of the two-valued function (31)-(32) for $t = 0$ takes the form

$$\mathbf{u}(t,x) = \mathbf{u}_0 + \varepsilon \sum_{i=1}^{4} a_j(\eta_i)\mathbf{r}_i(\mathbf{u}_0) + \mathbf{0}(\varepsilon^2) \quad (33)$$

where $a_i = a_i(\eta_i)$, $|\eta_i| < \infty$, is the two-valued function defined by (31)-(32). If $a_i = a_i(\varepsilon x)$ in equation (29) is identified with one of the two branches of the two-valued function at $t = 0$, then (33) is a unique asymptotic solution to the particular Cauchy problem.

The proof of Theorem 2 is given in [4].

Conclusions

One-dimensional nonlinear low-temperature and small-strain thermoelastic model with an elastic heat flow is revisited using the method of weakly nonlinear asymptotics.

The governing equations are cast to a matrix PDE, and the associated Cauchy problem with a weakly perturbed initial condition is solved.

The solution is given in the form of a power series with respect to a small parameter the coefficients of which are functions of a slow variable that satisfy a system of nonlinear second order ordinary differential transport equations.

For a particular Cauchy problem in which the initial data are generated by a closed-form solution to the transport equations, the principal term of the asymptotic solution is a sum of four travelling thermo-elastic waves admitting blow-up amplitudes.

Since the model is to cover low temperatures and small strains, the asymptotic solution to the particular Cauchy problem should be restricted to a region inside of a support of the finite amplitudes.

References

[1] Ignaczak, J., Soliton-like solutions in a nonlinear dynamic coupled thermoelasticity, Journal of Thermal Stresses, Vol. 13, 73-98, 1990.

[2] Domanski , W., The complex Burgers equation as a model for collinear interactions of weakly nonlinear shear plane waves in anisotropic elastic materials, J. of Engineering Mathematics, Vol. 95, 267-278, 2015.

[3] Holmes, M. H., Introduction to Perturbation Methods, Springer-Verlag, New York, 1995.

[4] Ignaczak J. and Domanski W., An asymptotic approach to one-dimensional model of nonlinear thermo-elasticity at low temperatures and small strains, to appear.

[5] Chandrasekharaiah, D. S., Hyperbolic Thermoelasticity: A review of recent literature, Applied Mechanics Reviews, Vol. 51, 705-729, 1998.