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OPTIMAL DESIGN OF PARABOLIC DISK WITH RESPECT TO DUCTILE CREEP RUPTURE TIME

OPTYMALIZACJA PARABOLICZNEJ TARCZY PIERŚCIENIOWEJ Z UWAGI NA CZAS ZNISZCZENIA CIĄGLIWEGO

A b s t r a c t

Axisymmetric rotating disks optimal with respect to ductile creep rupture time are considered. Finite strain theory is applied. The material is described by the Norton-Bailey law generalized for true stresses and logarithmic strains. The set of four partial differential equations describes the creep conditions of parabolic disk. The optimal shape of the disk is found using parametric optimisation with two free parameters. The results are compared with disks of conical shape.

Keywords: finite strain theory, annular disk, ductile creep rupture time, biparametric optimization

S t r e s z c z e n i e

W artykule przedstawiono problem optymalizacji wirującej tarczy osiowosymetrycznej ze względu na czas zniszczenia ciągliwego. Do opisu materiału stosowano teorię nieliniowego pełzania Nortona-Baileya, uogólnioną dla naprężeń rzeczywistych i odkształceń logarytmicznych. Dla złożonych stanów naprężeń stosowano prawo podobieństwa dewiatorów w połączeniu z hipotezą Hubera-Misesa-Hencky'ego. Proces pełzania tarczy wirującej opisuje układ czterech nieliniowych równań różniczkowych. Wyniki otrzymano przez zastosowanie optymalizacji dwuparametrycznej.

Słowa kluczowe: teoria skończonych odkształceń, tarcza pierścieniowa, czas zniszczenia ciągliwego, pełzanie, optymalizacja dwuparametryczna

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1. Introduction

Application of rheology to problems of structural optimization offers a new wide scope of possible optimization criteria [3]. Detailed review of them and classification were given by Życzkowski [7]. From them, the most important, from practical point of view, seems to be time to creep rupture. Vast majority of papers devoted to problems of optimization with respect to creep rupture time took advantage of Kachanov's brittle rupture theory. It was caused by its relative simplicity – rigidification theorem may be applied.

Problem becomes much more complicated, when Hoff's ductile creep rupture is used. Then finite strain theory must be applied – the creep process must be analyzed from its beginning up to rupture. For the first time such an approach was made by Szuwalski [5]. The problem for annular disk was formulated and solved by Szuwalski, Ustrzycka [6], who were looking for the best conical disks.

In present paper we will look for better solutions among annular axisymmetric disks with initial profile described by quadratic function [2, 4]. The disk is clamped on the rigid shift of radius A, and rotates with constant angular velocity ω (fig. 1). The own mass of disk is taken into account, as well, as mass M uniformly distributed at the outer radius B.

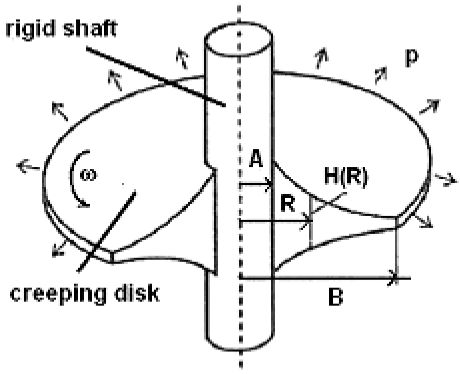

Fig. 1. Annular rotating disks

Rys. 1. Wirująca tarcza pierścieniowa

The problem is solved in spatial (Eulerian) coordinates and all parameters for initial disk are denoted by capital letters, while for current configuration by the same small ones.

2. Search domain

Because of difficulty of the problem: nonlinearities both – physical and geometrical and additional time factor, we decide on parametric optimization. The best disk will be sought among those, which initial shape is defined by quadratic function

$$\overline{H}(\overline{R}; b_0, b_1, b_2) = b_0 + b_1\overline{R} + b_2\overline{R}^2 \tag{1}$$

From three parameters in this function, only two may be treated as free ones, due to condition of constant volume

$$\bar{V} = \int_{\beta}^{1} 2\pi\left(b_0 + b_1\bar{R} + b_2\bar{R}^2\right)\bar{R}d\bar{R} = \pi \tag{2}$$

from which results

$$b_1 = \frac{2 - b_0\left(1-\beta^2\right) - b_2\left(1-\beta^4\right)}{\frac{4}{3}(1-\beta^3)} \tag{3}$$

where: β is the radio of given outer B and inner A radii.

In the process of optimization we will look for such values of parameter b_0 (the thickness of disk at the inner radius) and b_1, for which time to ductile creep rupture will be the longer. On the values of these parameters some limitation must be imposed. One may expect, that for discussed case of centrifugal forces – rotating disk, the thickness should diminish with growth of radius (though sometimes this limitation may be violated). It leads to

$$\frac{d\bar{H}(\bar{R})}{d\bar{R}} \le 0 \quad \rightarrow \quad b_1(b_0,\beta) < \frac{4\left(1-b_0\left(1-\beta^2\right)\right)}{\left(\frac{5}{3}+\beta^4-\frac{8}{3}\beta^3\right)} \tag{4}$$

Obviously the thickness at the outer radius, (and on the whole width of the disk), must be positive, what means

$$\bar{H}(1) > 0 \quad \rightarrow \quad b_1 < \frac{-2 + b_0\left(1-2\beta^2+\beta^4\right)}{-\frac{1}{3}+\frac{4}{3}\beta^3-\beta^4} \tag{5}$$

$$\bar{H}(\beta) > 0 \quad \rightarrow \quad b_1 < \frac{-2\beta^2 + 2b_0\beta^2\left(1-\beta^2\right) + b_0\left(1-\beta^4\right)}{-\frac{4}{3}\beta^2+\beta+\frac{1}{4}\beta^5} \tag{6}$$

Moreover, it may be anticipated that optimal shape should be convex

$$b_2 > 0 \quad \rightarrow \quad b_1 < \frac{\frac{3}{2}\left(1-b_0\left(1-\beta^2\right)\right)}{1-\beta^3} \tag{7}$$

Finally, the search domain in the plane of free parameters b_0 and b_2 will be restricted to the area shown in fig. 1 for β = 0.125.

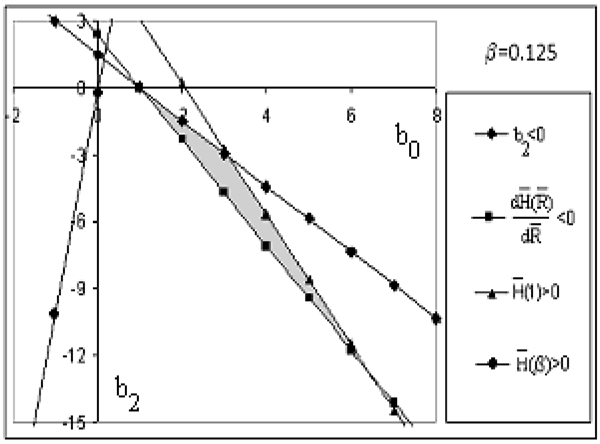

Fig. 2. Search domain in plane of free parameters

Rys. 2. Obszar poszukiwań parametrów swobodnych

3. Governing equations

Material of disk is described by the Norton low, generalized for complex stress state and finite strains

$$\dot{\varepsilon}_e = k\sigma_e^n \tag{8}$$

The process of creep for disk must at any moment satisfy the condition of internal equilibrium

$$\frac{1}{hr'}\frac{\partial}{\partial R}(h\sigma_r) + \frac{\sigma_r - \sigma_\vartheta}{r} + \frac{\gamma}{g}\omega^2 r = 0 \tag{9}$$

where: σ_r stands for current value of radial stress and σ_ϑ of circumferential one, h – for current thickness, γ – specific weight of material, g – acceleration of gravity, R – material (Lagrangian) radial coordinate and r – spatial (Eulerian) radial coordinate. Partial derivatives with respect to material coordinates are denoted by "primes", and with respect to time by dots.

Incompressibility of material is assumed

$$HRdR = hrdr \tag{10}$$

The shape change law, assumed in form of similarity of true stresses and velocities of logarithmic strains deviators [1] leads to

$$\frac{\dot{r}}{r} = \frac{1}{2}k\sigma_e^{n-1}(2\sigma_\vartheta - \sigma_r) \tag{11}$$

Compatibility condition, after some rearrangements, takes form

$$\sigma_\theta' = \frac{6\sigma_e^2\left(\sigma_r - \sigma_\theta\right)\frac{r'}{r} - \sigma_r'\left[\left(n-1\right)\left(2\sigma_r + \sigma_\theta\right)\left(2\sigma_\theta + \sigma_r\right) - 2\sigma_e^2\right]}{\left[\left(n-1\right)\left(2\sigma_\theta + \sigma_r\right)^2 + 4\sigma_e^2\right]} \tag{12}$$

In presented above equations we have four unknowns: true stresses σ_r and σ_θ, current thickness – h and spatial radial coordinate – r.

4. Numerical calculations

For sake of numerical calculations, dimensionless quantities, denoted by overbars, are introduced. Material and spatial coordinates are referred to the outer radius of the disk

$$\overline{R} = \frac{R}{B}, \qquad \overline{r} = \frac{r}{B} \tag{13}$$

Thickness of the disk is related to mean thickness h_m of annular disk of volume V

$$\overline{H} = \frac{\pi \cdot B^2}{V} H, \qquad \overline{h} = \frac{\pi \cdot B^2}{V} h \tag{14}$$

Radial loading at radius b of rotating disk is resulting from uniformly distributed on the outer edge mass M.

$$\sigma_r(b) = p = \frac{M\omega^2}{2\pi h(B)} \tag{15}$$

Dimensionless stresses are referred to calculated using rigidification theorem stress in motionless full plane disk subject to tension with uniform pressure p (15)

$$\overline{\sigma}_i = \frac{2V}{M\omega^2 B^2}\sigma_i, \qquad i = r, \vartheta \tag{16}$$

Dimensionless time is defined

$$\overline{t} = \frac{t}{\tau} \tag{17}$$

where: τ stands for the time of ductile rupture for full plane disk, rotating with the angle velocity ω, loaded by uniformly distributed at the external edge mass M, with neglected own mass. For such a disk relation between true stresses and logarithmic strains takes form

$$\dot{\varepsilon}_z = \frac{\dot{h}}{h} = \frac{3}{2} k \sigma_e^{n-1}(\sigma_z - \sigma_m) \tag{18}$$

and effective stress is equal

$$\sigma_e = \sigma_r = \sigma_\vartheta = p \tag{19}$$

Under assumption of plane stress ($\sigma_z \equiv 0$) and with help of initial condition

$$t = 0 \qquad h = h_m \tag{20}$$

time to rupture τ was established

$$\tau = \frac{1}{nk\left(\frac{M\omega^2}{2\pi h_m}\right)^n} = \frac{1}{nks^n} \tag{21}$$

Consequently dimensionless time is calculated

$$\bar{t} = nks^n t \tag{22}$$

Taking advantage of those dimensionless quantities the set of four equations describing the creep process takes finally form

$$\bar{\sigma}_r' = \frac{\bar{r}'}{\bar{r}}(\bar{\sigma}_r - \bar{\sigma}_\vartheta) - 2\cdot\bar{r}\bar{r}'\mu - \frac{\bar{h}'}{\bar{h}}\bar{\sigma}_r \tag{23}$$

$$\bar{\sigma}_\vartheta' = \frac{6\bar{\sigma}_e^2(\bar{\sigma}_r - \bar{\sigma}_\vartheta)\frac{\bar{r}'}{\bar{r}} - \bar{\sigma}_r'\left[(n-1)\left(5\bar{\sigma}_r\bar{\sigma}_\vartheta - 2\bar{\sigma}_r^2 + \bar{\sigma}_\vartheta^2\right) - 2\bar{\sigma}_e^2\right]}{(n-1)(2\bar{\sigma}_\vartheta + \bar{\sigma}_r)^2 + 4\bar{\sigma}_e^2} \tag{24}$$

$$\frac{d\bar{r}}{d\bar{t}} = \frac{\bar{r}}{2\cdot n}\left(\bar{\sigma}_r^2 + \bar{\sigma}_\vartheta^2 - \bar{\sigma}_r\bar{\sigma}_\vartheta\right)^{\frac{n-1}{2}}\left(2\bar{\sigma}_\vartheta - \bar{\sigma}_r\right) \tag{25}$$

$$\bar{h} = \frac{\bar{H}\bar{R}}{\bar{r}'\,\bar{r}} \tag{26}$$

At the beginning of the creep process (for $t = 0$) disk remains undeformed, therefore the initial conditions take form

$$\bar{r}(\bar{R}, 0) = \bar{R}\,, \qquad \bar{h}(\bar{R}, 0) = \bar{H}(\bar{R}) \tag{27}$$

For disk clamped on the rigid shaft, boundary conditions at internal radius may be written

$$\bar{r}(\beta, \bar{t}) = \beta\,, \qquad \dot{\bar{r}}\left(\beta, \bar{t}\right) = 0 \tag{28}$$

The condition at external radius (15), where the mass M is distributed, in dimensionless form

$$\overline{\sigma}_r(1,t) = \frac{1}{\overline{h}(1,t)} \tag{29}$$

In those equations an auxiliary quantity is used

$$\mu = \frac{\gamma \cdot V}{g \cdot M} \tag{30}$$

μ as ratio of own disk mass to mass distributed at the outer radius. Set written in given above form is very convenient for numerical calculations.

5. Numerical results

For given geometry of disk, described by pair of free parameters b_0 and b_2, assuming the value of radial strain at the radius a, from eqs. (23), (24) the distribution of true stresses may be found. They must satisfy the boundary condition at the outer radius (29), if not, the value of σ_r(A) must be changed and this procedure is repeated, until the condition (29) is satisfied.

Knowing distribution of true stresses, from (25) velocity of spatial radial coordinate is established, and for given time step, new coordinates are calculated. Finally incompressibility condition (26) makes it possible to find changed shape (thickness) of the disk. Thus, geometry of disk after the time step is defined. For this new geometry, from (25), (26) distribution of stresses in already deformed disk may be found, in a way described above. Repeating those calculation up to the moment the criterion of rupture is fulfilled, we may establish time to ductile rupture. For example, the process of creep for disk, for given initial geometry: $\overline{H}(\overline{R}) = 5.3 - 10\overline{R} + 6.1\overline{R}^2$, is shown in fig. 3.

The process of creep for disk accelerates in its final phase. Strains concentration effect occurs in places of locally weakened disk. With time, the deformations occur almost exclusively in these places.

Consequently, for the same b_0, we slightly change b_2, and the full cycle of mentioned above calculations is repeated, and time to rupture for a new disk is found. In such a way, the best value of b_2 for given b_0 may be found. Next, in the same way calculations are carried for changed value of b_0, and the best matching parameter b_2 is found. Finding maximum maximorum of obtained curves, we finally establish optimal solution for given parameters n, β, μ. The results are presented in fig. 4, 5 and 6.

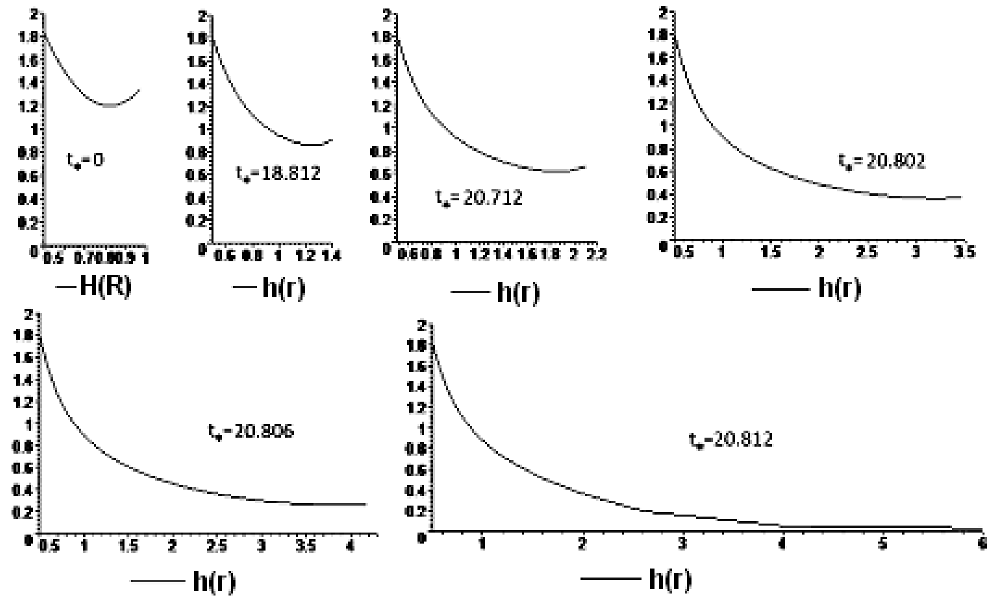

Fig. 3. The process of creep for disk, $\overline{H}(\overline{R}) = 5.3 - 10\overline{R} + 6.1\overline{R}^2$, n = 6, β = 0.5, μ = 0.1

Rys. 3. Proces pełzania tarczy, $\overline{H}(\overline{R}) = 5.3 - 10\overline{R} + 6.1\overline{R}^2$, n = 6, β = 0.5, μ = 0.1

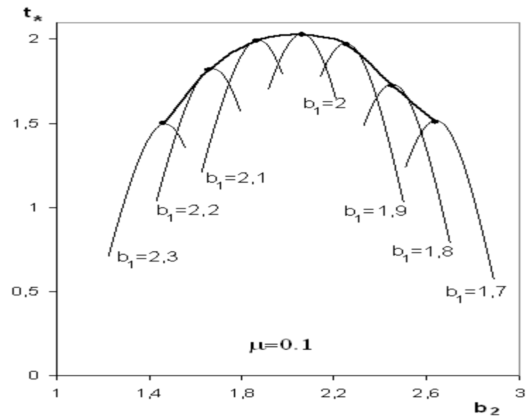

Fig. 4. The dependences of the ductile creep rupture time on the parameter b_2, n = 6, β = 0.125, μ = 0.1

Rys. 4. Zależność czasu zniszczenia ciągliwego od parametru b_2, n = 6, β = 0.125, μ = 0.1

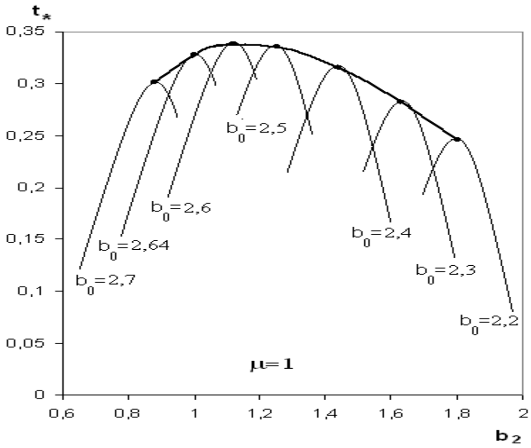

Fig. 5. The dependences of the ductile creep rupture time on the parameter b_2, n = 6, β = 0.25, μ = 1

Rys. 5. Zależność czasu zniszczenia ciągliwego od parametru b_2, n = 6, β = 0.25, μ = 1

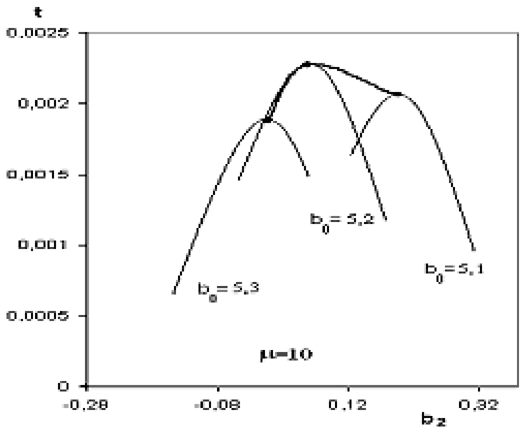

Fig. 6. The dependences of the ductile creep rupture time on the parameter b_2, n = 6, β = 0.5, μ = 10

Rys. 6. Zależność czasu zniszczenia ciągliwego od parametru b_2, n = 6, β = 0.5, μ = 10

As an instance, the dependence of time to rupture on free parameter b_2, for three different values of β and μ is shown. Maximum of curves, appointing the optimal value of parameter, becomes more "sharp" for larger values of μ. For larger values of μ maximum moves to the left, while for smaller (less significant influence of own mass) to the right. Observing the curves of optimal solutions for different values of μ (as ratio of own disk mass to mass distributed at the outer radius) can be seen, that for small values of μ (small influence of disk own mass) restriction (4) is broken. The optimal shape of the disk strives for reinforcement of the outer edge. Greater thickness of the optimal disk at outer edge results on growth of stiffness there. The mass of the disk cannot spread out freely at the disk may work to collapse longer. With the increase of the parameter μ, optimal solution tends to reduce the thickness of the disk at the outer edge.

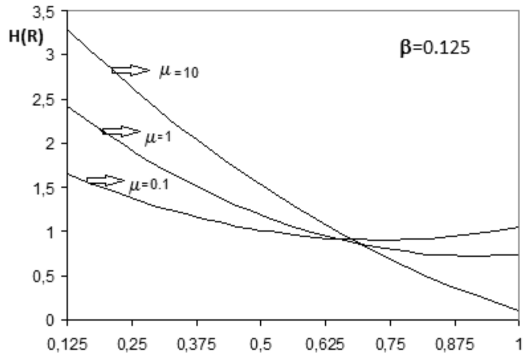

Fig. 7. Optimal design of parabolic disk, β = 0.125

Rys. 7. Optymalny profil traczy, β = 0.125

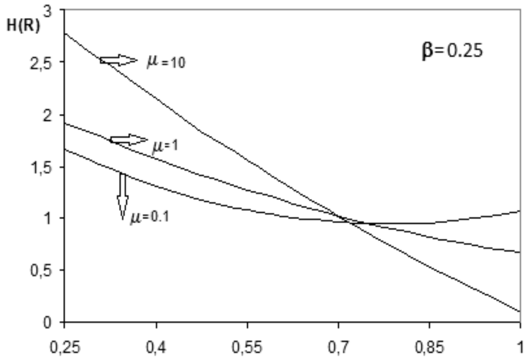

Fig. 8. Optimal design of parabolic disk, β = 0.25

Rys. 8. Optymalny profil traczy, β = 0.25

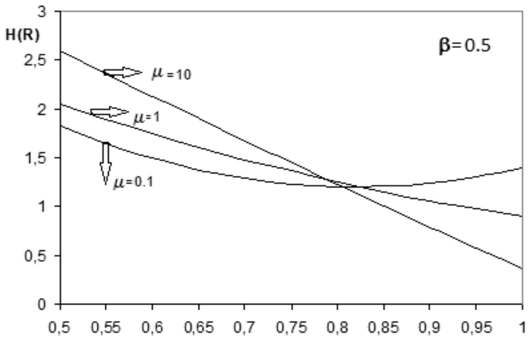

Fig. 9. Optimal design of parabolic disk, β = 0.5

Rys. 9. Optymalny profil traczy, β = 0.5

6. Conclusions

Applications of Hoff's ductile creep damage theory in optimization problems are rather scarce, as it requires finite strain theory, resulting in significant complication of problem. The complexity of such problems is connected with nonlinearities both: physical and geometric and moreover existence of additional time factor. In present paper the problem of optimal shape with respect to ductile rupture time for the parabolic rotating disk clamped on the rigid shaft is investigated. Obtained solutions strongly depend on ratio μ of own mass of the disk to mass uniformly distributed at the outer edge, causing there tensile pressure. When this mass is very large in comparison with own mass of the disk optimal disks are close to flat ones, on contrary, when influence of own mass is dominating, mass of the disk should be concentrated as close to the shaft as possible.

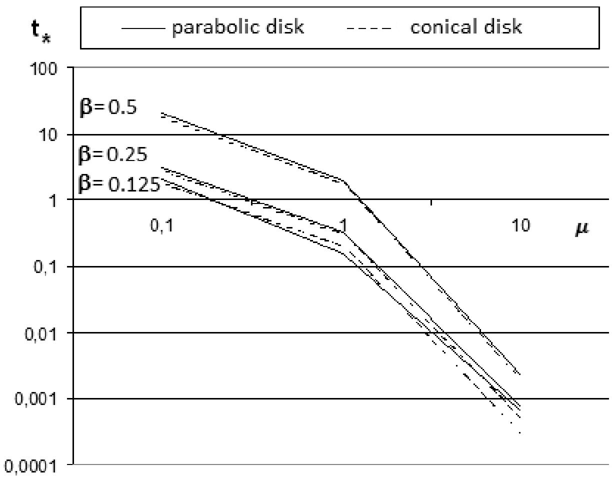

Fig. 10. The dependences of the ductile creep rupture time on the parameter μ

Rys. 10. Zależność czasu zniszczenia ciągliwego od parametru μ

For biparametric optimization some optimal solution have minimum inside the disk width – the thickness outer edge works as some kind of reinforcement and time to ductile rupture is longer. Optimization with one free parameter does not offer such opportunities.

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