STANDARD SINE FITTING ALGORITHMS APPLIED TO BLADE TIP TIMING DATA

ZASTOSOWANIE STANDARDOWYCH ALGORYTMÓW DOPASOWANIA FUNKCJI SINUS DO DANYCH POMIAROWYCH DRGAŃ ŁOPATEK

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Abstract: Blade Tip Timing (BTT) is non-intrusive method to measure blade vibration in turbomachinery. Time of Arrival (TOA) is recorded when a blade is passing a stationary sensor. The measurement data, in form of undersampled (aliased) tip-deflection signal, are difficult to analyze with standard signal processing methods like digital filters or Fourier Transform. Several indirect methods are applied to process TOA sequences, such as reconstruction of aliased spectrum and Least-Squares Fitting to harmonic oscillator model. We used standard sine fitting algorithms provided by IEEE-STD-1057 to estimate blade vibration parameters. Blade-tip displacement was simulated in time domain using SDOF model, sampled by stationary sensors and then processed by the sinefit.m toolkit. We evaluated several configurations of different sensor placement, noise level and the number of data. Results of the linear sine fitting, performed with the frequency known a priori, were compared with the non-linear one's. Some of non-linear iterations were not convergent. The algorithms and testing results are aimed to be used in analysis of asynchronous blade vibration.

Keywords: turbomachinery, blade vibration, tip-timing, sine fitting, least squares, harmonic oscillator

Streszczenie: Metoda dyskretno-fazowa (ang. tip-timing) jest nieinwazyjnym sposobem pomiaru drgań łopatek w przepływowych maszynach wirnikowych. Czas przyjścia zapisywany jest w momencie, kiedy łopatka mija stacjonarny czujnik. Do analizy danych pomiarowych w formie rzadko próbkowanego sygnału odkształcenia wierzchołka (aliasing) wykorzystano standardowe metody dopasowania funkcji sinus dostarczone przez normę IEEE-STD-1057 w celu estymacji parametrów drgań łopatek. Przemieszczenia wierzchołków łopatki w dziedzinie czasu zasymulowano wykorzystując model oscylatora o jednym stopniu swobody. Wyniki liniowego dopasowania, wykonanego ze znaną a priori częstotliwością, porównano z wynikami dopasowania nieliniowego. Uzyskane algorytmy i wyniki testowania mają być wykorzystane do analizy drgań asynchronicznych łopatek.

Słowa kluczowe: przepływowe maszyny wirnikowe, drgania łopatek, metoda dyskretno-fazowa, dopasowanie funkcji sinus, metoda najmniejszych kwadratów, oscylator harmoniczny

1. Introduction

The experimental results are subject to measurement uncertainty and noise. Data fitting to the model with the method of least squares is one of the most common numerical techniques. Fitting of lines and polynomial curves is a well-known procedure, but trigonometric functions are more demanding. However sine fitting is widely used in engineering, including testing of AD converters used in audio industry. This work uses algorithms and tools developed as a result of the international cooperation [[1], [2], [3]], which are available in the form of IEEE-Std-1057 standard [[4]] and the corresponding MATLAB toolkit (sinefit.m).

Only a few points per the cycle of sine function can be used to estimate parameters of blade vibration. This is due to the limited number of sensors that can be installed into the casing. Unlike common signals, TOA data are non-uniformly sampled.

2. Sine fitting algorithms

As pointed out in 2008 in IEEE Std 1057 document, there are two main ways to fit the sine function. There are three-parameter or four-parameter least squares fitting algorithms using matrix operations. The first is applied, when searched frequency of a sinusoidal signal is known. Otherwise, the second algorithm is applied.

Three-parameter algorithm is used when applying the method of least squares to fit a sine wave to the registered points in the case of the knowledge of the frequency of the sinusoid. Matrix notation of used solution is based on the following sine wave equation:

$$x[n] = A_0 \cos(2\pi f_0 t_n) + B_0 \sin(2\pi f_0 t_n) + C_0 \tag{1}$$

Assuming that the registered data consist of a sequence of x[1], x[2], …, x[M] samples, respectively stored at time t_1, t_2, …, t_M, the algorithm seeks the values of A_0, B_0 and C_0, which are minimizing the square sum of the differences represented by the following equation:

$$\sum_{n=1}^{M}[x[n] - A_0 \cos(2\pi f_0 t_n) - B_0 sin(2\pi f_0 t_n) - C_0]^2 \tag{2}$$

where f_0 is a known sine wave frequency value matched to the recorded data.

To come to a solution, the values of A_0, B_0 and C_0 must be calculated, which is not a difficult process, notwithstanding the fact that the equation is a linear function. At the beginning, we have to create the following two matrices:

$$D_0 = \begin{bmatrix} \cos(2\pi f_0 t_1) & \sin(2\pi f_0 t_1) & 1 \\ \cos(2\pi f_0 t_2) & \sin(2\pi f_0 t_2) & 1 \\ \vdots & \vdots & \vdots \\ \cos(2\pi f_0 t_M) & \sin(2\pi f_0 t_M) & 1 \end{bmatrix} \tag{3}$$

$$x = \begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[M] \end{bmatrix} \tag{4}$$

and assume that:

$$s_0 = \begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} \tag{5}$$

In matrix notation, the sum of square differences from the equation nr. 2 can be represented by the equation:

$$(x - D_0 s_0)^T (x - D_0 s_0) \tag{6}$$

where T denotes the transpose of the matrix.

Least-squares solution which minimizes the above equation is represented by the equation:

$$\hat{s_0} = (D_0^T D_0)^{-1}(D_0^T x) \tag{7}$$

Finally, the obtained vector $\hat{s_0}$ components, can be used to solve the fitted function. In order to determine its amplitude and phase, which can be found in the formula $x[n] = A\cos(2\pi f_0 t_n + \varphi) + C$, the following equation should be solved:

$$A = \sqrt{A_0^2 + B_0^2} \tag{8}$$

$$\varphi = -\arctan(B_0, A_0) \tag{9}$$

The residuals of the fit and its error can be obtained from equations:

$$r[n] = x[n] - A_0 \cos(2\pi f_0 t_n) - B_0 \sin(2\pi f_0 t_n) - C_0 \tag{10}$$

$$\varepsilon_{rms} = \sqrt{\frac{1}{M} \sum_{n=1}^{M} r[n]^2} \tag{11}$$

Four-parameter fit algorithm is used in case of not knowing the frequency of matched sinusoidal signal. Sine wave equation is similar to that described in three-parameter algorithm, with the only difference that the parameter f_0 is unknown. This entails, unfortunately, the consequences of changing the nature of non-linear equations, which also involves the way to solve it. In this case, an iterative method must be applied.

Assuming that the recorded data contain the sequence of samples x[1], x[2], …, x[M] respectively stored at time t_1, t_2, …, t_M, the algorithm,

in iterative process, is looking for A_i, B_i, C_i and f_i values which are minimizing the square sum of the differences represented by the following equation:

$$\sum_{n=1}^{M}[x[n] - A_i \cos(2\pi f_i t_n) - B_i \sin(2\pi f_i t_n) - C_i]^2 \tag{12}$$

where "i" is the number of the next iteration.

Appropriate algorithm, which should be followed to get the solution, is presented below:

a) set the iteration index $i = 0$,
b) make an initial estimation of the recorded data frequency f_0. For this task, you can use the DFT (for all or a part of the record), taking the inverse of the average period between zero crossings or taking into account the best result when applied input frequency. A very effective method is to use a fast Fourier interpolation, which were presented in the work of Schoukens and Bilau. This method operates interpolated formula on DFT coefficients of the recorded data and is often used in applications for testing ADCs,
c) perform a preliminary matching using three-parameter fit algorithm to determine the A_0, B_0 i C_0,
d) set $i = i + 1$ for the next iteration,
e) create matrices presented below:

$$x = \begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[M] \end{bmatrix} \tag{13}$$

$$D_0 = \begin{bmatrix} \cos(2\pi f_i t_1) & \sin(2\pi f_i t_1) & 1 & -A_{i-1}t_1 \sin(2\pi f_i t_1) & B_{i-1}t_1 \sin(2\pi f_i t_1) \\ \cos(2\pi f_i t_2) & \sin(2\pi f_i t_2) & 1 & -A_{i-1}t_2 \sin(2\pi f_i t_2) & B_{i-1}t_2 \sin(2\pi f_i t_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cos(2\pi f_i t_M) & \sin(2\pi f_i t_M) & 1 & -A_{i-1}t_M \sin(2\pi f_i t_M) & B_{i-1}t_M \sin(2\pi f_i t_M) \end{bmatrix} \tag{14}$$

$$x = \begin{bmatrix} A_i \\ B_i \\ C_i \\ \Delta f_i \end{bmatrix} \tag{15}$$

f) determine the solution of least squares using the following equation:

$$\hat{s}_i = \left(D_i^T D_i\right)^{-1}\left(D_i^T x\right) \tag{16}$$

g) update frequency estimation using the relationship:

$$f_i = f_{i-1} + \Delta f_{i-1} \tag{17}$$

for $i = 1$ the value $\Delta f_{i-1} = 0$.

h) determine the amplitude and phase in the equation $x[n] = A\cos(2\pi f_i t_n + \varphi) + C$ using relationships shown below:

$$A = \sqrt{A_i^2 + B_i^2} \tag{18}$$

$$\varphi = -\arctan(B_i, A_i) \tag{19}$$

i) repeat sequence of steps from (d) to (h), converting the model using the new values of Ai, Bi and fi calculated from the previous iteration. On the basis of experiments, determined that the number of iterations should be accurately specified. The best results were achieved with six repetitions for the entire calculations cycle. This method doubles the number of significant digits in the parameter "f" with each iteration and converges very quickly.

The residuals of the fit and its error can be obtained from equations:

$$r[n] = x[n] - A_i\cos(2\pi f_i t_n) - B_i\sin(2\pi f_i t_n) - C_i \tag{20}$$

$$\varepsilon_{rms} = \sqrt{\frac{1}{M}\sum_{n=1}^{M} r[n]^2} \tag{21}$$

3. Blade Tip Timing simulation

All matching methods will be tested with use of the data generated by the tip-timing measurement simulator. Data obtained from simulation is representing a certain number of blades times to come before the face of the sensor. The resulting measurement points will provide a set of data used in the amplitude estimation in the matching method.

It is assumed that the blade rotates with frequency $\Omega = 2\pi f_{obr}$ and passes in front of the sensor's face once per revolution. Next assumption is, that at the beginning of the simulation, for time t = 0, the blade's in angular position ε. Hence, the time required, for not vibrating blade, to pass in front of the sensor face during the k number of revolutions, is:

$$t_{czujnik} = \frac{\varepsilon + \phi}{\Omega} + \frac{2\pi k}{\Omega} \tag{22}$$

where Ω is the rotational speed of the rotor for the specified measurement, ε is the angle used to position the blade corresponding to the aisle of the first blade under first sensor and ϕ is the angle of the sensor regarding to the first, which is the reference.

On the basis of the above formula, times for the transitions of the blade under a given sensor, will be computed. In this case it is assumed that four sensors are arranged in positions:

- sensor no. 0 - 0 degrees,
- sensor no. 1 - 18 degrees,
- sensor no. 2 - 33 degrees,
- sensor no. 3 - 40 degrees.

The appointed times of blade transitions are afterward stored in arrays t_czujnik_0, t_czujnik_1, t_czujnik_2, t_czujnik_3.

In the last step of creating a simulator, all received data should be put in a single array t_czujnik, so that, on the basis of transition times and blade vibration amplitudes, appoint a temporary vibration amplitudes for the blade transitions under the sensor - A_czujnik matrix.

As a result, a simulation graph showing the variation of blade vibrations and its vibration amplitude at individual measurement points, corresponding to transitions under forefront of each of the four sensors, will be obtained.
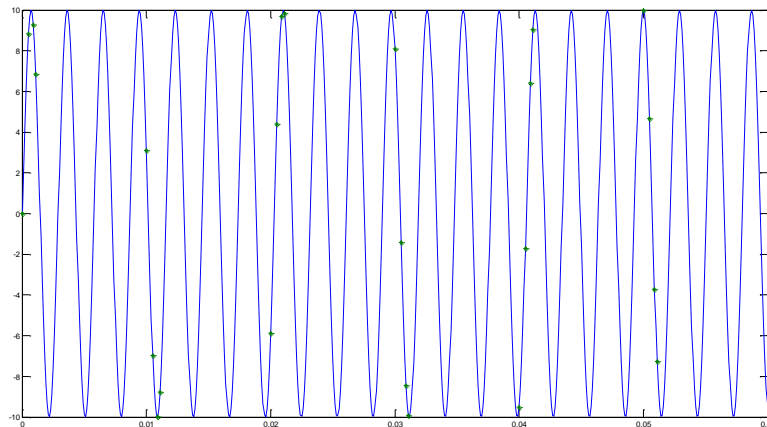

Fig. 1 Amplitude of blade vibrations

4. Analysis of amplitude estimation algorithm using data from tip-timing simulator

Points generated by the tip-timing simulator described above, were used to test the vibration amplitude estimation algorithm with use of sine function matching method. The testing process was divided into two cases, respectively for three and four - parameters algorithms.

Sine fitting using 3-parameter algorithm

In the case of use of the 3-parameter fit algorithm using least squares method, it was assumed that the frequency of wanted function is known. Amplitude estimation algorithm was tested for a number of situations:

a) analyzing the fitting process depending on the amount of generated points used in the matching algorithm,

All tests were carried out with the revealed frequency of 345 Hz. The amount of data used for the analysis is the parameter which was not changed. As a result, the calculated values were offset, phase shift, amplitude and the number of iterations.
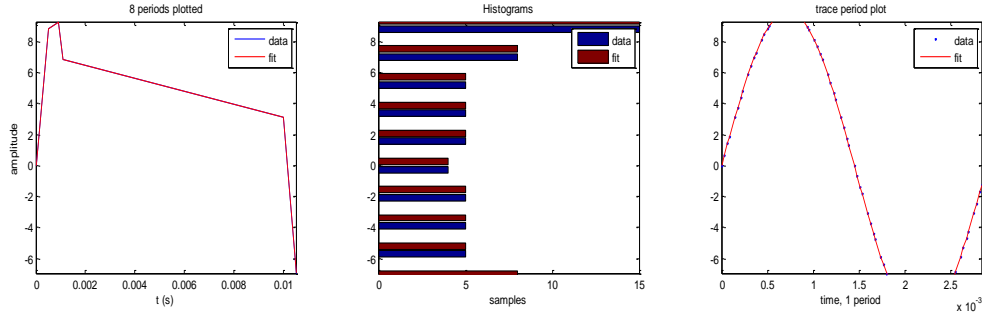

Fig. 2 An example of the result of sine function matching
a) waveform generated on the basis of samples, b) histogram containing a quantity
of samples and their utilization in implementing the matching function,
c) the result of the matching function for one period

The results from the experiments have shown that the number of points generated from the simulation, that are used by matching algorithm using a sine function during seeking blade vibration function parameters, has no effect on the number of iterations, the value of the offset and amplitude in the case of 3 or 6 rotations. Phase shift in each case was corrupted. When using only eight data points for the two periods of the vibration signal, the number of calculations dropped to 1 iteration. However, the use of measurement points from one rotation, increases the number of iterations dramatically to 22. At the same time, increasing the amount of point generated from simulation (20 revolutions), is causing a noticeable increase in the value of the phase shift, without affecting other parameters.

b) analyzing the fitting process depending on the position of the sensors in the tip-timing simulation,

During carrying out a series of adjustments for the different positions of the sensors, it was noted that their position does not affect the outcome of matches. The resulting parameters of the fitted sine functions are identical:

Tab. 1. Computed function parameters for different sensor locations

Offset	0
Amplitude	10
Frequency	345 Hz
Phase shift	-1.5708 rad

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c) analyzing the fitting process depending on the number of sensors in the tip-timing simulation.

Firstly, it was assumed that more sensors are deployed on the perimeter of the rotor body, including the availability of the places of possible assembly. Therefore, it was found that sensors can be arranged at +/- 60 degrees from the reference sensor (0 degrees).

Further analysis of the matching process was consisting in increasing number of sensors. To the base amount (4 pieces) were added 4 more, which had a random distribution in the above mentioned range. The results, however, strongly suggest that the increased number of sensors does not affect the correctness of the matching algorithm and the resulting parameters of the sine function are identical to those for the base settings.

In the case of reducing the number of sensors - up to 1, the same results were obtained as in the previous paragraphs. Conducted simulations show that the adjustment is impossible only in the case where the number of sampling points for one sensor is less than 2.

Sine fitting using 4-parameter algorithm

In the second solution, the rough (+/- 20%) value of frequency has to be given. The problem is non-linear, therefore a 4-parameter algorithm for least squares fitting must operate iteratively. For this method, the following experiments were performed:

a) increasing the number of data points:
   - The best result was obtained for one rotation, for each start frequency,
   - In case of 20 revolutions, the result was obtained when the given frequency was equal to expected one (345 Hz) and slightly higher in one of the cases (370 Hz)
   - In all cases the error of the phase estimation was in the range of 90°-270°,
   - Except one case, the amplitude was estimated with the accuracy of +/- 5%
   - A number close to 445 Hz was often received in the result of the frequency estimation instead of the expected value of 345 Hz,

b) modifying the placement of tip-timing sensors:
   - The estimation was performed in two cases: the first case using data for one rotation from four sensors, and the second - for six rotations and the same sensor count.
   - The best results were achieved for simulated sensors mounted only in the selected quarter of the circumference of the casing: 0°-40° or 300°-340°. Sensors placed in two quarters at the same time strongly reduced the chance of getting the correct estimation result.
   - More data, available in the case of 6-rotation fitting, did not increase the accuracy of estimation. Despite correctly placed sensors, the procedure was not able to identify any of the four parameters when the start frequency

differed significantly from the expected value. The best results were achieved from sensors placed in the range of 300°-340°.

c) changing the number of sensors:

- *The attempt to fit the data for one rotation*

  Increasing the number of sensors above the primary 4 caused no noticeable change in the estimated parameters. The algorithm is unable to determine the frequency and other vibration parameters when the number of sensors is less than 4.

- *The attempt to fit the data for 6 rotations*

  Higher number of points in 6-rotation dataset worsen operation of sine fit procedure. When number of sensors is increased, number of cases with the correct estimation results is reduced. However, reducing the number of sensors improved the convergence. If the sensor were placed in the position of 0, 40 and 300, estimates of amplitude and frequency were found correctly for all tested initial frequencies. If more than one sensor were used, there was an expected error in determining of the frequency but the amplitude and offset were estimated correctly.

5. Summary

The following observations can be made:

− The procedure is not optimized for non-uniform sampling and sensor placement.

− Phase shift is always estimated incorrectly, regardless of the initial conditions, the algorithm used, the number of sensors and their placement, and the number of data points.

− In case of 3-parameter estimation (with the frequency known a priori) the obtained amplitude and offset are independent of the rotations' number and the number of sensors and their spacing.

− In the case of the 4-parameter estimation (when frequency is unknown):

  - Probability of convergence is acceptable when the given frequency is very close to the expected value ($\pm 1\%$);

  - Best results were obtained when multi-sensor data from one rotation were used;

  - Sensors placed in one quarter of the circumference give better results than ones distributed around the circle;

  - The number of fitted data points should match (or be a multiply of) the sensor count.

6. References

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