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APPLICATIONS OF THE BURZYŃSKI HYPOTHESIS OF MATERIAL EFFORT FOR ISOTROPIC SOLIDS

SUMMARY

The paper contains a short presentation of main idea of energy-based hypothesis of material effort proposed by Burzyński with discussion of the resulting failure criteria. Some examples illustrating applications of these criteria are discussed and visualizations of limit surfaces in the space of principal stresses are presented.

***Keywords**: Burzyński yield condition, yield surface, strength differential effect, energy-based yield criteria*

ZASTOSOWANIA HIPOTEZY WYTĘŻENIA BURZYŃSKIEGO DLA CIAŁ IZOTROPOWYCH

Praca zawiera krótką prezentację głównej idei energetycznej hipotezy wytężenia zaproponowanej przez Burzyńskiego wraz z dyskusją wynikających z niej kryteriów wytrzymałości. Przedstawiono kilka przykładów zastosowań tych kryteriów wraz z wizualizacją granicznych powierzchni w przestrzeni naprężeń głównych.

***Słowa kluczowe**: kryterium plastyczności Burzyńskiego, powierzchnia plastyczności, efekt różnicy wytrzymałości, energetyczne kryteria plastyczności*

1. INTRODUCTION

In the beginning, the used in the title notion "material effort" and its subtle otherness to the often too hastily used term "material strength" should be clarified. It was made in the work of Włodzimierz Burzyński: "*The main subject of the theory of elasticity is to mathematically determine the state of strain or stress in a solid body being under the conditions determined by the action of a system of external forces, the specific shape of the body and its elastic properties. The solution of this question exhausts the role of the elasticity theory and next the theory of strength of materials comes into play. Its equally important task is to give the dimensions of the considered body with determined exactness with respect to the states unwanted regarding the body safety on the one hand and most advantageous economical conditions on the other hand. This problem, very simple in the case of a uniaxial state of stress, becomes so complicated in a general case that from the beginning of the mentioned theories, special attention has been paid to this question and an intermediate chapter, being at the same time the final part of the theory of elasticity and the introduction to the strength of materials theory, has been introduced. This new passage deals with material effort and different hypotheses related with this notion.*" – p. 4 (Burzyński 1928), and then a crystal clear definition follows: *"Generally, under the notion: material effort we understand the physical state of a body, comprehended in the sense of elasticity or plasticity or material strength, and generated by a system of stresses, and related with them strains, in the body"* – p. 40 (Burzyński 1928) *(translation of the both passages by Anna Stręk).*

The aim of this paper is to present an energetic approach to the measure of material effort for a wide range of materials which, in general, indicate asymmetry in failure characteristics. It means that in results of tension and compression tests there is observed a difference in the failure strength: values of elastic, yield (plastic) or strength limits, since the failure of a material is usually classified into brittle failure (fracture) or ductile failure (yield). The energy-based hypothesis of material effort proposed originally by W. Burzyński is presented (Burzyński 1928, 1929a, 1929b) and the resulting failure criteria phrased for stress tensor components in an arbitrary Cartesian coordinate system or, in particular, with the use of principal stresses are discussed. The paper contains a consistent presentation of facts concerning Burzyński's failure criteria, which still remain rather not widely known despite their having been presented persistently by (Życzkowski 1981, 1999) and (Skrzypek 1993); for the references of further citations, cf. (Pęcherski 2008). As for new results own applications of Burzyński's failure criteria for traditional and new materials are presented. There has been formulated own algorithm, prepared by one of the coauthors (TF) with use of MathCad, by means of which there are presented visualizations of failure surfaces in the system of principal axes. The algorithm could be also used to estimate the strength of the examined materials. The discussed results can be also applied as a teaching material during courses of mechanics of materials.

2. BURZYŃSKI'S FAILURE CRITERIA

Włodzimierz Burzyński, in his doctoral thesis (Burzyński 1928) not only systematized the contemporary knowledge about yield criteria but also presented a new approach to determine the measure of material effort for materials which reveal difference in the failure strength (in particular: the elastic limit) for tension and compression. According to the

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original Burzyński's hypothesis, *the measure of material effort defining the limit of elastic range is a sum of the density of elastic energy of distortion and a part of density of elastic energy of volume change being a function of the state of stress and particular material properties.* The mathematical formula for so stated the hypothesis reads as follows:

$$\Phi_f + \eta\Phi_v = K$$
$$\eta = \omega + \frac{\delta}{3p}, \quad p = \frac{\sigma_1+\sigma_2+\sigma_3}{3} = \frac{\sigma_x+\sigma_y+\sigma_z}{3} \tag{1}$$

where:

$$\Phi_f = \frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right]$$

means the density of elastic energy of distortion, and:

$$\Phi_v = \frac{1-2\nu}{6E}(\sigma_1+\sigma_2+\sigma_3)^2 = \frac{1-2\nu}{12G(1+\nu)}(\sigma_1+\sigma_2+\sigma_3)^2$$

is the density of elastic energy of volume change. The constant K corresponds to the value of the density of elastic energy in a limit state, while ω, δ are material parameters dependent on the contribution of the density of elastic energy of volume change influenced by the mean stress p. By the symbols σ_1, σ_2, σ_3 are meant principal stresses and by: σ_x, σ_y, σ_z – normal stresses in an arbitrary Cartesian coordinate system. By introducing the function η Burzyński took into account the experimentally based observation that the increase of the mean stress p results in the decrease of the density of elastic energy of volume change Φ_ν in the measure of material effort. The above formulation of the measure of material effort is precise for the limit states of linear elasticity, typical for brittle behaviour of materials. When the limit state is related with the lost of material strength preceded by certain plastic strain, then the measure of material effort (1) loses its foundations of linear elasticity, because in this case inelastic states of material may occur. This is the reason why W. Burzyński suggested to treat functions Φ_f i Φ_ν in equation (1) as general strain functions and he emphasized this fact by the word "quasi-energies" of strain.

In the discussed measure of material effort (1) there are introduced three material parameters: ω, δ, K. The final form of Burzyński's failure hypothesis reads:

$$\frac{1}{3}\sigma_f^2 + 3\frac{1-2\nu}{(1+\nu)}\omega p^2 + \frac{1-2\nu}{(1+\nu)}\delta p = 4GK \tag{2}$$

where $\sigma_f^2 = 12G\Phi_f$. The essence of Burzyński's derivation lies in the smart conversion of variables. Instead of demanding of direct identification of the triplet (ω, δ, K), another one is substituted, which results from commonly performed strength tests: elastic (plastic) limit in uniaxial tension – k_t, uniaxial compression – k_c, and torsion – k_s: (ω, δ, K) $\rightarrow$ (k_t, k_c, k_s).

After solving the system of three equations obtained with use of (2) for the above mentioned tests, the following substitutions are obtained:

$$\frac{1-2\nu}{(1+\nu)}\omega = \frac{2\left(3k_s^2 - k_c k_t\right)}{k_c k_t}, \quad \frac{1-2\nu}{(1+\nu)}\delta = \frac{6k_s^2\left(k_c-k_t\right)}{k_c k_t},$$
$$K = \frac{k_s^2}{2G}.$$

Due to this (2) transforms into the form discussed earlier in (Życzkowski 1999):

$$\frac{k_c k_t}{3k_s^2}\sigma_e^2 + \left(9 - \frac{3k_c k_t}{k_s^2}\right)p^2 + 3\left(k_c-k_t\right)p - k_c k_t = 0 \tag{3}$$

where $\sigma_e{}^2 = \frac{1}{2}\sigma_f{}^2$ is a symbol of equivalent stress used in the theory of plasticity. According to the discussion conducted in (Burzyński 1928) and (Życzkowski 1999) the equation (3) in the space of principal stresses, depending on the relations among material constants (k_t, k_c, k_s), describes the surfaces: an ellipsoid for $3k_s^2 > k_t k_c$ or a hyperboloid for $3k_s^2 < k_t k_c$, which, however, does not have any practical application. W. Burzyński also noticed that there occur interesting cases if these three material constants are particularly connected, for example if they are bound together as the geometrical average: $\sqrt{3}k_s = \sqrt{k_t k_c}$, then (3) takes the form (Burzyński 1928; Życzkowski 1999):

$$\sigma_e^2 + 3\left(k_c - k_t\right)p - k_c k_t = 0 \tag{4}$$

The above equation presents the formula of a paraboloid of revolution in the space of principal stresses. The original energy-based hypothesis of Burzyński (Burzyński 1928) and his comprehensive phenomenological theory of material effort was forgotten and repeatedly 'rediscovered', often in parts, by later authors. Discussion of other works containing the latter equation is presented in (Życzkowski1981, 1999). If the relation among the limit constants has the form of the harmonic average: $\sqrt{3}k_s = \frac{2k_t k_c}{k_c + k_t}$, then (3) reads (Burzyński 1928; Życzkowski 1999):

$$\sigma_e^2 + 3\frac{k_c - k_t}{k_c + k_t}p - 2\frac{k_c k_t}{k_c - k_t} = 0 \tag{5}$$

The above equation represents in the space of principal stresses a cone of revolution. The failure condition (5) is known in the literature as the criterion of Drucker-Prager (Drucker and Prager 1952) and it has found application mainly for the analysis of limit states in soils and other geological materials. It should be acknowledged that some foreign authors refer also in this context to W. Burzyński's criterion (cf. e.g. Jirašek and Bažant 2002; Nardin *et al.* 2003).

3. EXAMPLES OF APPLICATIONS OF THE BURZYŃSKI FAILURE CRITERIA

Defining the strength differential factor $\kappa = \frac{k_c}{k_t}$ allows to determine particular cases of the criterion, for example: for $\kappa = 1$ there is $k_c = k_t = k$ and then $k_s = \frac{k}{\sqrt{3}}$, which suits the condition assumed in the Huber-Mises-Hencky criterion. After suitable transformation (3) takes the form expressed by stress tensor components in an arbitrary system of Cartesian coordinates:

$$\sigma_x^2+\sigma_y^2+\sigma_z^2-2\left(\frac{k_c k_t}{2k_s^2}-1\right)\left(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x\right)+ \\ +\frac{k_t k_c}{k_s^2}\left(\tau_x^2+\tau_y^2+\tau_z^2\right)+\left(k_c-k_t\right)\left(\sigma_x+\sigma_y+\sigma_z\right)=k_c k_t \quad (6)$$

and in the system of principal axes:

$$\sigma_1^2+\sigma_2^2+\sigma_3^2-2\left(\frac{k_c k_t}{2k_s^2}-1\right)\left(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1\right)+ \\ +\left(k_c-k_t\right)\left(\sigma_1+\sigma_2+\sigma_3\right)=k_c k_t \quad (7)$$

If $\sigma_3 = 0$, there is obtained a plane state of stress, for which:

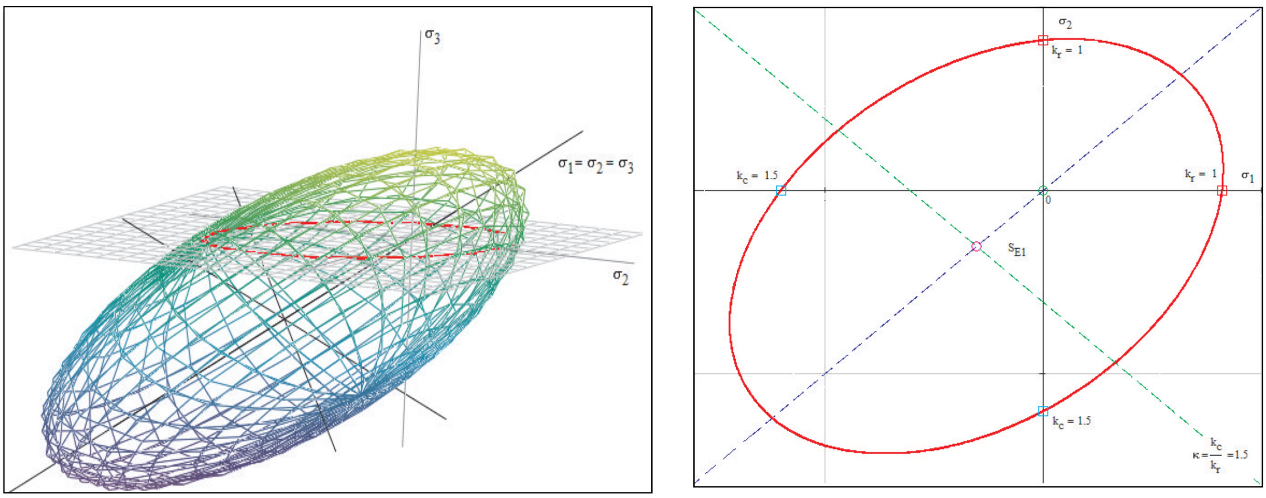
$$(\sigma_1^2+\sigma_2^2)-2\left(\frac{k_c k_t}{2k_s^2}-1\right)\sigma_1\sigma_2+ \\ +(k_c-k_t)(\sigma_1+\sigma_2)=k_c k_t \quad (8)$$

According to the criterion, for the plane stress state the limit curve is an ellipse for which the center of symmetry is defined by: $S_e=\left(\frac{k_s^2(k_c-k_t)}{k_c k_t-4k_s^2}, \frac{k_s^2(k_c-k_t)}{k_c k_t-4k_s^2}\right)$ and the axes of symmetry by: $\sigma_2=\sigma_1$, $\sigma_2=-\sigma_1+\frac{2k_s^2(k_c-k_t)}{k_c k_t-4k_s^2}$.

Example 1

The example of application of the Burzyński failure criterion for the ceramic preform Al_2O_3, which is used in the production of the metallic-ceramic composites, is presented. Such a structure represents cellular materials with open cells and a brittle matrix (Kowalewski 2009). In this case the Burzyński criterion (3) is used with the assumption: $3k_s^2 > k_r k_c$, for which the limit surface has the shape of an ellipsoid, Fig. 1.

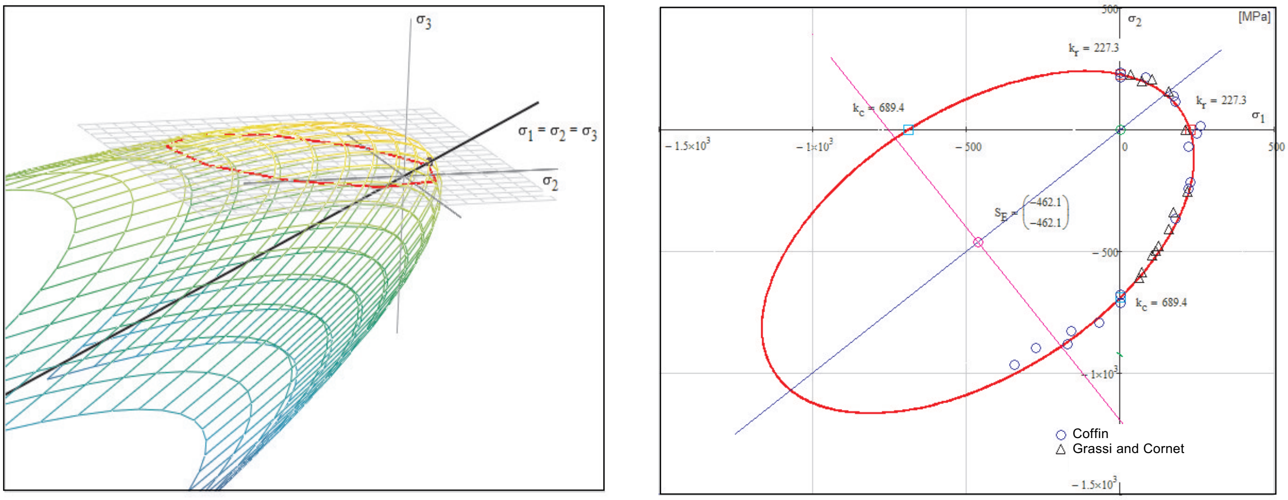
In the space of principal stresses for $\sqrt{3}k_s = \sqrt{k_t k_c}$ the graphical representation of the criterion (3) is a paraboloid of revolution with the axis of symmetry given by the axis of hydrostatic compression: $\sigma_1 = \sigma_2 = \sigma_3$. In the plane state of stress for $\sigma_3 = 0$ the graphical representation of the Burzyński hypothesis is an ellipse. The centre of symmetry of such an ellipse is defined by $S_e=\left(\frac{k_s^2(k_c-k_t)}{k_c k_t-4k_s^2}, \frac{k_s^2(k_c-k_t)}{k_c k_t-4k_s^2}\right)$ and the axes of symmetry are given by: $\sigma_2=\sigma_1, \sigma_2=-\sigma_1+\frac{2k_s^2(k_c-k_t)}{k_c k_t-4k_s^2}$. If $k_c = k_t$ then the centre of the ellipse is given by the beginning of the coordinate system and the Burzyński hypothesis is equal to the Huber hypothesis; in this case the graphical representation of the yield surface is a cylinder of revolution with the axis of symmetry: $\sigma_1 = \sigma_2 = \sigma_3$.



The ellipsoid in the space of principal stresses with the selected cross-section for σ_3=0

The ellipse in the plane state of stress

Fig. 1. Graphical representation of the Burzyński failure criterion for a ceramic preform Al_2O_3



The ellipsoid in the space of principal stresses with the selected cross-section for σ_3=0

The ellipse in the plane state of stress

Fig. 2. Graphical representation of the Burzyński failure criterion for gray cast iron

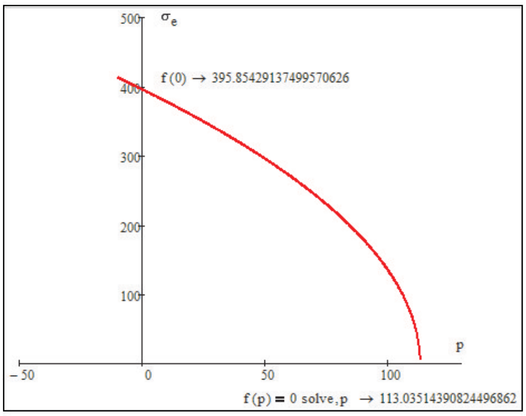


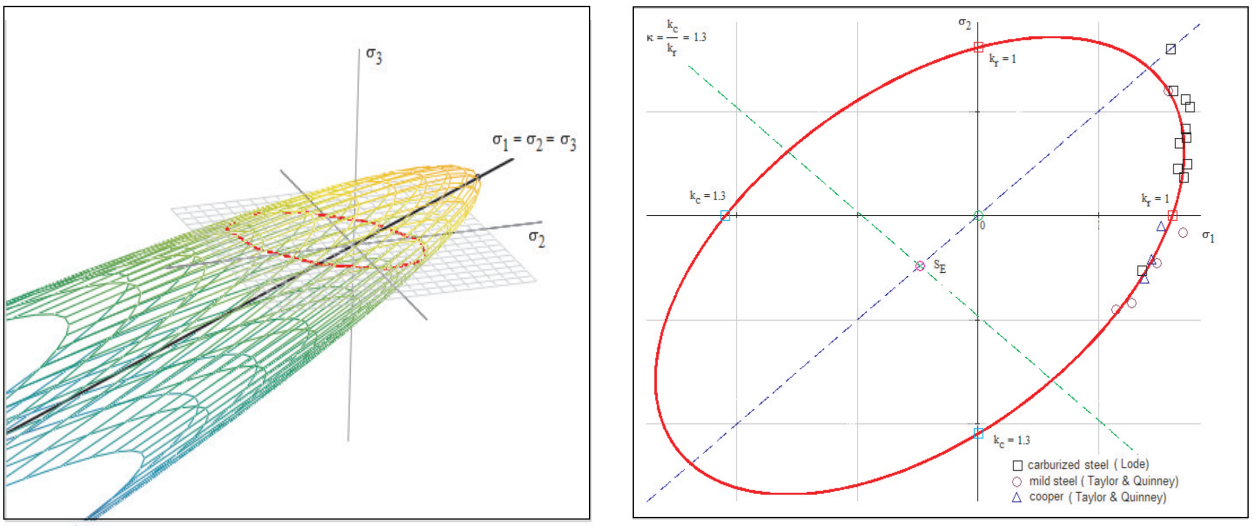
Fig. 3. Half of the parabola being the representation of the Burzyński failure criterion for gray cast iron in the coordinates (σ_e, p)

Example 2

The example is based on the results discussed by P.S. Theocaris (Theocaris 1995). It is a representation of strength tests for gray cast iron. The strength limits are set by the uniaxial tension and compression: k_c = 689.47 MPa, k_t = 227.5 MPa; κ = 3.0. The experimental data are taken from the works of Coffin, Grassi and Cornet (Theoacaris 1995) (Fig. 2 and 3).

Example 3

The idea of this example is based on the paper (Theocaris 1995). The results of experimental tests for two kinds of steel and copper are discussed. The experimental data are taken, however, from the original works of G.I. Taylor and H. Quinney (1931) and H. Lode (1926) (Fig. 4 and 5).



The paraboloid of revolution in the space of principal stresses with the selected cross-section for σ_3=0

The ellipse in the plane state of stress

Fig. 4. Graphical representation of the Burzyński yield criterion for two kinds of steel and copper

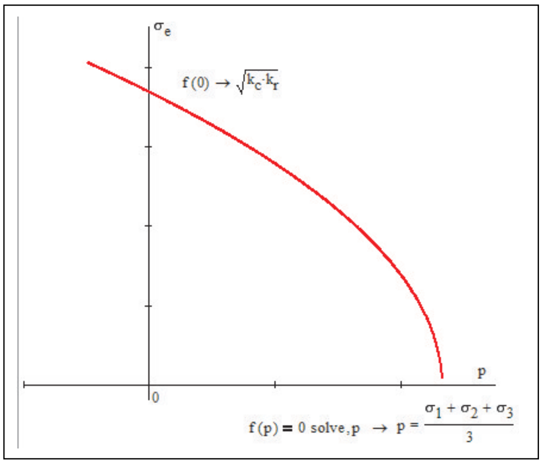


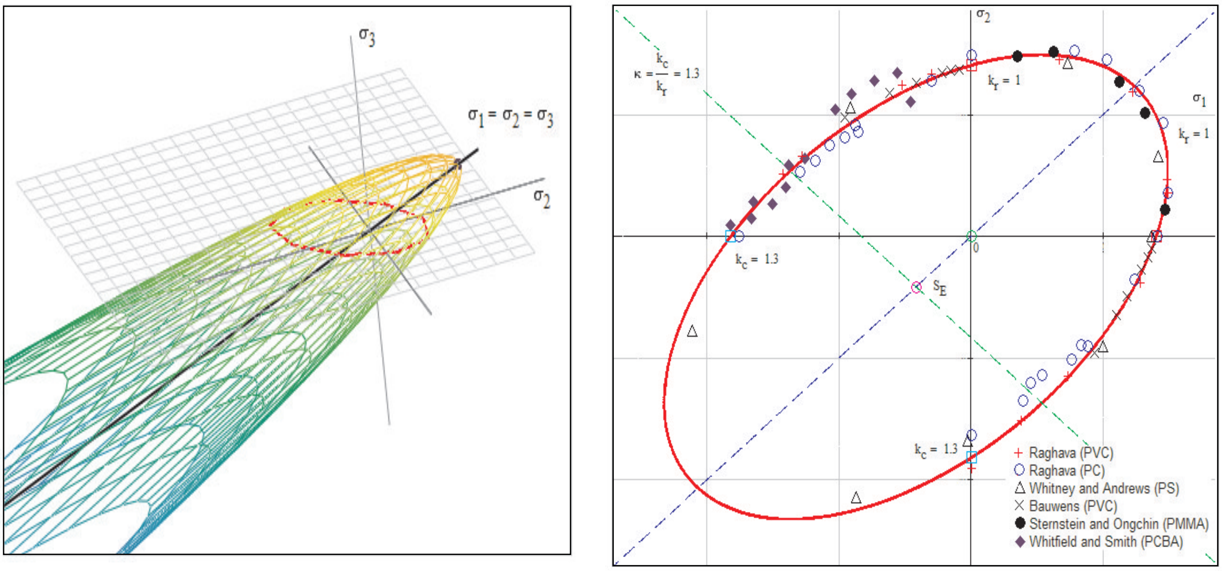
Fig. 5. Half of the parabola being the representation of the Burzyński failure criterion in the coordinates (σ_e, p)

Example 4

The example is based on the paper (Theocaris 1995) and presents the results of strength tests for different kinds of polymers (Fig. 6 and 7).

Example 5

Graphical representation of the identification of the limit function for the composite MMC with metallic matrix: alumina alloy 6061 and particle inclusions: zircon and corundum, 6061 + 2Zr + 20Al_2O_3 (Dutkiewicz 2009) (Fig. 8 and 9).



The paraboloid of revolution in the space of principal stresses with the selected cross-section for σ_3=0

The ellipse in the plane state of stress

Fig. 6. Graphical representation of the Burzyński yield criterion for polymers

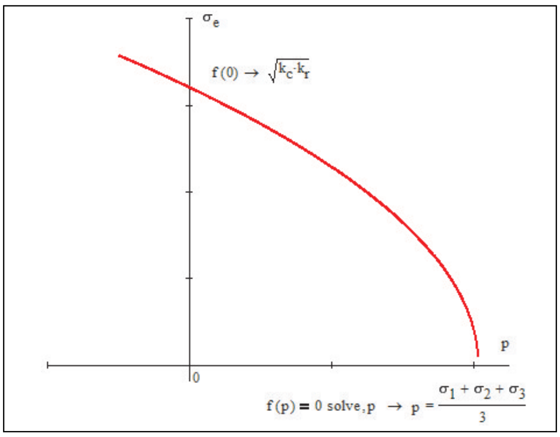
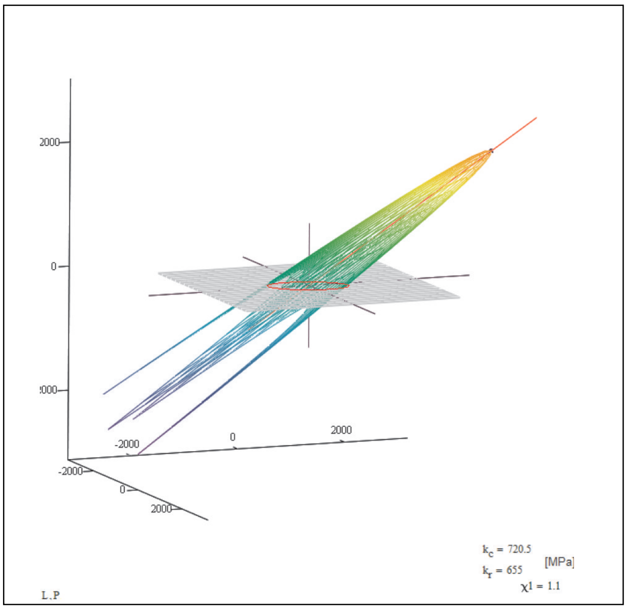


Fig. 7. Half of the parabola being the representation of the Burzyński failure criterion for polymers in the coordinates (σ_e, p)

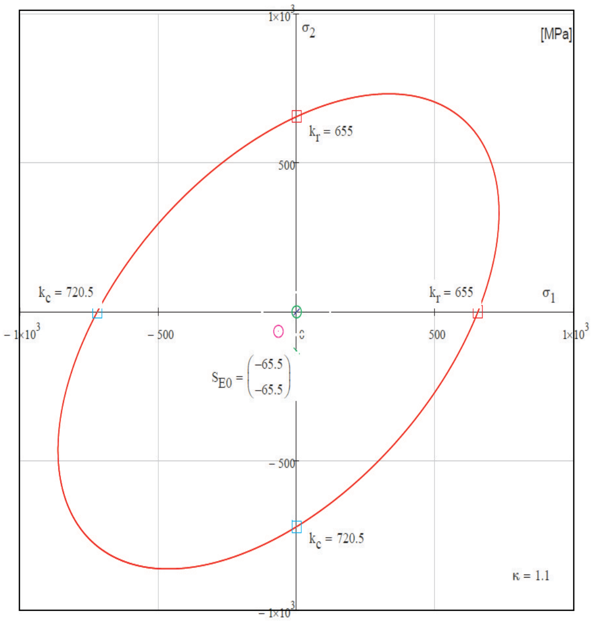
4. CONCLUSIONS

It is worthy of emphasizing that W. Burzyński proposed the hypothesis which was universal in the sense of energy. Therefore, it can be applied not only to isotropic materials. It is also applicable to different kinds of anisotropic solids revealing, in particular, characteristic asymmetry of elastic range.

In his thesis Burzyński (Burzyński 1928) presented for the first time the energetic approach to determine the measure of material effort for a certain class of orthotropic materials. The issue of yielding condition of orthotropic materials raised by Burzyński is worth further studies because of its promising possibilities of application for modern materials.



The paraboloid of revolution in the space of principal stress together with the selected cross-section for σ_3=0



The ellipse in the plane state of stress

Fig. 8. Graphical representation of the Burzyński yield criterion for the MMC composite

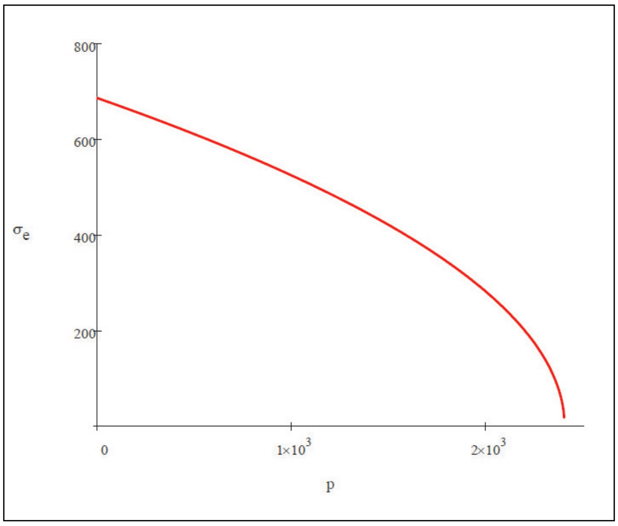


Fig. 9. Half of the parabola being the representation of the Burzyński failure criterion for the MMC composite in the coordinates (σ_e, *p*)

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References

Burzyński W. 1929, *Ueber die Anstrengungshypothesen*. Schweizerische Bauzeitung, 94, Nr. 21, 259–262.

Burzyński W. 1928, *Studjum nad hipotezami wytężenia*. Akademija Nauk Technicznych, Lwów; also: Burzyński W. 1982, Dzieła wybrane, part 1, PWN, Warszawa, 67–258 and English translation: Study on Material Effort Hypotheses (selected passage from doctoral dissertation), Enging. Trans., 57, 2009, 185–215.

Burzyński W. 2008, *Teoretyczne podstawy hipotez wytężenia*. Czasopismo Techniczne, 1–41, 47, 1929; also English translation „Theoretical foundations of the hypotheses of material effort, Enging. Trans., 56, 269–305.

Drucker D.C., Prager W. 1952, *Soil mechanics and plastic analysis or limit design*. Quart. Appl. Math. 10, 157.

Dutkiewicz J. 2009, *Raport D2-1.1*. KomCerMet, 1.07.2009–30.09.2009

Jirašek, M., Bažant, Z.P. 2002, *Inelastic Analysis of Structures*. J. Wiley, Chichester.

Kowalewski Z. 2009, *Raport D2-2.1*. KomCerMet, 1.07.2009–30.09.2009.

Pęcherski R.B. 2008, *Burzyński yield condition vis-ŕ-vis the related studies reported in the literature*. Engineering Transactions, 47–55, 56.

Theocaris P.S. 1995, *Failure criteria for isotropic bodies revisited*. Engineering Fracture Mechanics, 51, 239–264.

Lode W. 1925, *Versuche über den Einfluss der mittleren Hauptspannung auf die Fliessgrenze*. ZAMM, 5.

Skrzypek J.J. 1993, *Plasticity and Creep. Theory, Examples, and Problems*. R.B. Hetnarski (English Edition Editor), CRC Press, Boca Raton.

Taylor G.I., Quinney H. 1931, *The plastic distortion of metals*. Phil. Trans.Roy. Soc., A230, 323–362.

Nardin, A., Zawarise, G., Schrefler, B.A. 2003, *Modelling of cutting tool – soil interaction – Part I: contact behaviour*. Computational Mechanics, 31, 327–339.

Życzkowski M. 1981, *Combined Loadings in the Theory of Plasticity*. PWN-Polish Scientific Publishers, Warszawa.

Życzkowski M. 1999, *Discontinuous bifurcation in the case of the Burzynski – Torre yield condition*. Acta Mechanica, 132, 19–35.