Flow of Stokesian fluid through a cellular medium and thermal effects

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Abstract. The thermal effects of a stationary Stokesian flow through an elastic micro-porous medium are compared with the entropy produced by Darcy's flow. A micro-cellular elastic medium is considered as an approximation of the elastic porous medium. It is shown that after asymptotic two-scale analysis these two approaches, one analytical, starting from Stoke's equation and the second phenomenological, starting from Darcy's law give the same result. The incompressible and linearly compressible fluids are considered, and it is shown that in micro-porous systems the seepage of both types of fluids is described by the same equations.

Key words: porous media, Onsager's principle, entropy, heat production.

1. Introduction

In more or less degree, natural solid aggregates such as rocks and soils (in particular, travertine, pumice stones, aquifers, petroleum reservoirs, shale gas resources), zeolites, biological tissues (woods, cork, sponges, cartilage, bones), and artificial products such as cements and ceramics (molecular sieves, bricks, concrete) can be considered as porous media. The concept of porous media is used in many areas of applied science and engineering: filtration, mechanics (acoustics, geomechanics, soil mechanics, rock mechanics), engineering (petroleum engineering, bio-remediation, construction engineering), geophysics (hydrogeology, petroleum geology), biology and biophysics, material science. Thus, the fluid flow through porous media is a subject of common interest, and emerged as a separate field of science, and numerous are works related to the field [1–8].

A special case of flow through porous medium is realised in devices that provide liquid flow through parallel channels, to achieve high heat dissipation rate for the needs of micro-electronic and optical technologies, cf. [9]

It is known from the experimental evidence and from theoretical motivation that the flow through porous medium may be well approximated by Darcy's law, which belongs to large family of transport laws such as Fourier's, Ohm's or Fick's. For example, by analogy to Fourier's law in which the temperature gradient is the thermodynamical force which drives the stream of heat, in Darcy's flow it is the pressure gradient which enforces the movement of fluid through porous medium, cf. [10–12]. It would be interesting to verify whether Darcy's law can be embedded into a framework of Onsager's theory, in a similar manner as other transport equations can [13, 14].

The viscous flow is the entropy generation process, and in its description only mechanical quantities, the velocity and the pressure gradient are applied. To place Darcy's flow into On-

sager's scheme we use two ways of its description. Firstly, the entropy production is evaluated only by the aforementioned analogy with other processes satisfying Onsager's theory.

On the other hand, because the flow of fluid through porous medium is still described by Stokes' equation of hydrodynamics, we calculate the entropy production in the flow through a porous medium calculated directly from Stokesian flow [13], and next compare the result with that obtained from Darcy's approach in Onsager's theory.

To this end, we consider a flow of Stokesian fluid through a cellular elastic medium using two scale asymptotic approach, what can be regarded as an approximation to the process of flow through porous material, cf. [15]. This means that we consider a system composed of porous solid and viscous liquid in flow. In a micro-scale, when the flow through separate micro-canals is considered, it can be regarded as a micro-flow, as it is realized in small and narrow canals of the porous medium. In a macro-scale, when the individual canals are smeared out, the flow is described by Darcy's law

$$\mathbf{v} = -\mathbf{K}\,\nabla p, \qquad (1)$$

here $\mathbf{v}$ is the velocity of flow, $\mathbf{K}$ is the permeability tensor and ∇p is the gradient of the pressure.

Darcy's law can be derived from Stokes' equation of viscous fluid flow endowed in the appropriate boundary conditions on the interface between the viscous fluid and the rigid skeleton of the cellular material, namely, it is assumed that the fluid velocity vanishes on the interface.

1.1. Cellular medium. The porous medium, if identified with a cellular material, it is a bounded set $\overline{\Omega}$ of three-dimensional space (the bar denotes the closure of the set), where Ω is a sufficiently regular domain, with the boundary $\partial\Omega$.

We assume that the porous medium reveals a micro-periodic structure, cf. [15, 16]. The basic cell Y has a form

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of cube and consists of two parts (two disjoint open sets): Y_S and Y_L, where the subscript S and L denote the solid and fluid parts, respectively, cf. Fig. 1. Obviously,

$$Y = Y_S \cup Y_L \cup \Gamma,$$

where Γ stands for the interface between Y_S and Y_L.
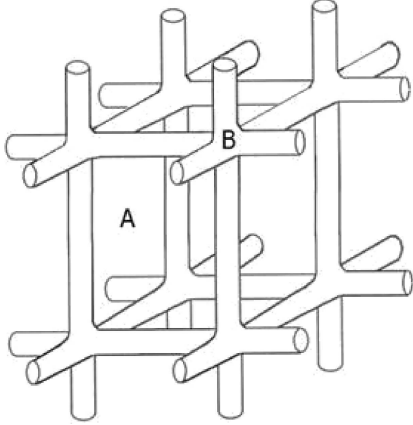

Fig. 1. Basic cell Y is composed of two simply-connected parts. Each part, either A or B, can be solid or liquid

A small parameter ε such that $(0 < \varepsilon < 1)$ characterizes the microstructure of the porous medium considered. Namely,

$$\varepsilon = \frac{\ell}{\mathcal{L}}, \tag{2}$$

where ℓ and $\mathcal{L}$ are typical length scales associated with the dimensions of micro-pores and the domain Ω, respectively.

The domain Ω is assumed to have an εY – periodic structure. The set Ω is covered with a regular mesh of size ε, each cell being a cube Y.

In analogy to the structure of elementary cell Y, the total volume of the porous body is composed of two parts $\Omega = \Omega_S \cup \Omega_L$, where Ω_S denotes the volume of solid (skeleton) and Ω_L – the volume of pores (filled with the liquid). The average over the basic cell is introduced

$$\langle (\cdots) \rangle = \frac{1}{|Y|} \int\limits_Y (\cdots) d\mathbf{y}. \tag{3}$$

The porosity is defined as

$$f = \frac{|Y_L|}{|Y|} \tag{4}$$

it is as the fraction of the liquid in the considered solid–liquid composite medium. Of course, also $f = \Omega_L/\Omega$, and $1 - f = |Y_S|/|Y| = \Omega_S/\Omega$.

1.2. Elastic skeleton. The porous solid (skeleton) is described by the linear elasticity

$$\sigma_{ij} = C_{ijmn} \frac{\partial u_m}{\partial x_n}, \tag{5}$$

here σ_{ij} and C_{ijmn} denote the stress tensor and the elasticity tensor of the skeleton, respectively, while $\mathbf{u} = \mathbf{u}(\mathbf{x})$ is the displacement vector field in the skeleton. The field depends only on the position $\mathbf{x}$ and not on the time t, as we are considering stationary processes only. The micro-periodic dependence of C_{ijmn} on position x is admitted also. The tensor C_{ijmn} subdues the usual conditions of symmetry and coercivity, see below Subsec. 1.4.

1.3. Viscous fluids. In a viscous fluid due to internal friction irreversible processes of energy dissipation occur, which lead to the energy dissipation, cf. [13],

$$E_{\text{diss}} = \sigma'_{ij} \frac{\partial v_i}{\partial x_j}. \tag{6}$$

The viscous stress tensor σ' depends on the space derivatives of velocity $\mathbf{v}$ only,

$$\sigma'_{ij} = \eta_{ijmn} \frac{\partial v_m}{\partial x_n}, \tag{7}$$

where η_{ijmn} is the viscosity tensor, independent of velocity. Also, the tensor η_{ijmn} satisfies the symmetry and coercivity conditions, see Subsec. 1.4.

Despite the fact that Stokesian fluids belong to the class of simple fluids and are isotropic, cf. [17], we apply here the tensorial description of the viscosity. This is done for brevity and compactness of the deduction, cf. lengths of the left and right hand sides of Eq. (9).

Substituting (7) into (6) we obtain

$$E_{\text{diss}} = \eta_{ijmn} \frac{\partial v_m}{\partial x_n} \frac{\partial v_i}{\partial x_j} \tag{8}$$

For isotropic fluid

$$\eta_{ijmn} = \eta \left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} \right) + \left(\zeta - \frac{2}{3}\eta \right) \delta_{mn}\delta_{ij} \tag{9}$$

and

$$\sigma'_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \left(\zeta - \frac{2}{3}\eta \right) \frac{\partial v_k}{\partial x_k} \delta_{ij} \tag{10}$$

with the scalar coefficients of viscosity, η and ζ. If the fluid may be regarded as incompressible that is if $\text{div}\mathbf{v} = 0$, only one coefficient of viscosity, namely η is of importance, as the second term in (10) vanishes.

1.4. The composite body with microperiodic structure. In Subsec. 1.1 the small parameter $\varepsilon > 0$ was introduced, and it was assumed that the body is made of elementary cubic cells $\varepsilon Y = \varepsilon [0, Y_1] \times [0, Y_2] \times [0, Y_3]$. This means that
(i) The functions C_{ijkl} and η_{ijkl} are Y – periodic, it is for fixed $\varepsilon > 0$ the material functions are

$$C^\varepsilon_{ijkl}(\mathbf{x}) = C_{ijkl}\left(\frac{\mathbf{x}}{\varepsilon}\right) \qquad \text{and} \qquad \eta^\varepsilon_{ijkl}(\mathbf{x}) = \eta_{ijkl}\left(\frac{\mathbf{x}}{\varepsilon}\right)$$

(ii) The matrices C_{ijkl} and η_{ijkl} are assumed to be positive definite, what means that there exists a constant $\alpha > 0$ such that (on Y): $C_{ijkl}\xi_{ij}\xi_{lk} > \alpha\xi_{ij}\xi_{lk}$ and $\eta_{ijkl}\xi_{ij}\xi_{lk} > \alpha\xi_{ij}\xi_{lk}$ for all symmetric matrices ξ_{ij}, $\xi_{ij} = \xi_{ji}$.

From the mathematical point of view the homogenization means the passage with ε to zero in an appropriate meaning, cf. [18].

1.5. Stokes' equation. If the fluid velocities are very slow, the viscosities are large, or the length-scales of the flow are very small, the substantial time derivative of velocity can be approximated simply by the time partial derivative, $d\mathbf{v}/dt \equiv \partial\mathbf{v}/\partial t + \mathbf{v}\nabla\mathbf{v} \approx \partial\mathbf{v}/\partial t$ and the viscous flow is described by the equation

$$\rho\frac{\partial v_i}{\partial t} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma'_{ij}}{\partial x_j}$$

where σ'_{ij} is given by Eq. (7), ρ denotes the density and p – the pressure. In stationary flow, when $\partial v_i/\partial t = 0$,

$$-\frac{\partial p}{\partial x_i} + \frac{\partial \sigma'_{ij}}{\partial x_j} = 0. \qquad (11)$$

This, stationary version of Stokes' equation will be considered in sequel.

1.6. Continuity equation. This equation expresses the conservation of the fluid material. It is of the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0$$

and in the stationary process

$$\frac{\partial(\rho v_i)}{\partial x_i} = 0. \qquad (12)$$

For incompressible fluids, when $\rho = $ constant, the continuity equation becomes

$$\frac{\partial v_i}{\partial x_i} = 0. \qquad (13)$$

In sequel, for compressible fluids the linear dependence between density and pressure is assumed

$$\rho = \rho_0 + \beta p, \qquad (14)$$

where ρ_0 is a constant, and β is the compressibility of the fluid.

1.7. Entropy production in the viscous flow. The equation of entropy density production in such a flow has the form [13],

$$\rho T\frac{ds}{dt} = \sigma'_{ij}\frac{\partial v_i}{\partial x_j}, \qquad (15)$$

where s is the entropy per unit mass of the fluid. The last equation is valuable in non-stationary as well as in stationary flows.

After Eq. (8) we can also write

$$\rho T\frac{ds}{dt} = \eta_{ijmn}\frac{\partial v_m}{\partial x_n}\frac{\partial v_i}{\partial x_j}. \qquad (16)$$

In stationary processes, when the right hand side of (16) is time-independent, the growth of entropy is uniform.

1.8. Compressibility of water and solids. In thermodynamics and mechanics of continuous media, compressibility is defined as the relative volume change of a fluid or solid as a response to a pressure (or mean stress) change.

$$\beta = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T.$$

The specification above is incomplete, because for any object or system the magnitude of the compressibility depends strongly on whether the process is adiabatic or isothermal. For low velocities of flow through porous medium (seepage) the isothermal compressibility should be applied, and above it is defined by the subscript T which indicates that the partial differential is to be taken at constant temperature.

Compressibility is used in the Earth sciences to quantify the ability of a soil or rock to reduce in volume with applied pressure. This concept is important for specific storage, when estimating groundwater reserves in confined aquifers. The domain occupied by the porous materials can be considered as composed of two volumes: one occupied by solids and second created by voids. The void space can be filled by liquid or gas.

The analysis of the Stokesian flow through porous medium is commonly realised under the assumption of the incompressibility of the fluid, e.g. [15–24]. However in such an analysis it is neglected an experimental evidence that the compressibility of fluid (water) is sometimes comparable with that of the skeleton material. For example, the compressibility of the water at 25^0C equal 4.6×10^{-10} Pa^{-1}, while the compressibility of the rock is of order of 3×10^{-10} Pa^{-1}, cf. the Table of compressibility, also [25, 26].

Compressibility

material	β [m^2/N] or [Pa^{-1}]
clay	$\approx 1 \times 10^{-7}$
dense sand, sandy gravel	$\approx 1 \times 10^{-8}$
rock	$\leq 3 \times 10^{-10}$
water at 25°C	$\approx 4.6 \times 10^{-10}$

Under very specific conditions the compressibility can be negative [27, 28].

The compressibility of water can be neglected, if one considers seepage through sands, but not when the flow through a fissured rock is analysed.

1.9. Scope of the paper. A stationary process of seepage through the elastic porous medium is studied. Firstly, Onsager's theory is applied to obtain the entropy production by flow through porous medium in description by Darcy's law, considered as a phenomenological law, Sec. 2. Next, we discuss the properties micro-porous body, their mathematical formulation (Sec. 3) and we derive on Darcy's law from Stokes' equation for slow stationary flows. Next, Darcy's law, recuperated in this manner is used to verify the Onsager approach, for incompressible flow (Sec. 4), and compressible flows (Sec. 5). In Sec. 5 we prove that the results obtained for incompressible fluid flows through micro-periodic porous media still hold true for the compressible flows. The total entropy production obtained from the Stokes equation *via* asymptotic expansions is compared with the Onsager's theory presented in Sec. 2.

2. Entropy production in Darcy's flow after Onsager

Now, we apply Onsager's theory to determine the entropy production in Darcy's flow.

Let $x_a = x_1, x_2, \cdots$ be some quantities characterising the state of some closed system. Their equilibrium values are given by the condition that in statistical equilibrium, the entropy S of the whole system must reach a maximum. If $X_a \equiv -\partial S/\partial x_a$, in equilibrium $X_a = 0$ for all a.

In a state near of equilibrium all the x_a are very little different from their equilibrium values and the X_a are small. Processes will occur which tend to bring the system again into equilibrium. The quantities x_a are functions of time, and their rate of change is given by the time derivatives $\dot{x}_a$. In the linear approximation

$$\dot{x}_i = -\sum_b \gamma_{ab} X_b,$$

where γ_{ab} are called the kinetic coefficients. Onsager's principle for the kinetic coefficients states that the γ_{ab} are symmetrical with respect to suffices a and b, it is $\gamma_{ab} = \gamma_{ba}$.

The rate of change of the entropy S is

$$\dot{S} = -\sum_a X_a \dot{x}_a.$$

In Darcy's flow, cf. Eq. (1),

$$v_i = -K_{ij}\frac{\partial p}{\partial x_j} \tag{17}$$

we identify the velocity $\mathbf{v}$ with $\dot{x}_a$, the pressure gradient ∇p with X_a, and the permeability coefficient K_{ij} with the kinetic coefficient γ_{ab}. Then, the total entropy production should be given by

$$\dot{S} = C \int_\Omega K_{ij}\frac{\partial p}{\partial x_i}\frac{\partial p}{\partial x_j}d\mathbf{x} \tag{18}$$

with Ω being the volume of the porous body, and C - a constant, dependent on the definition of streams and thermodynamic forces.

3. Stationary viscous flow through a porous medium

3.1. Rescaling of the viscosity. In order to obtain the macroscopic description of the medium with micro-structure we should pass with the smallnes parameter ε to zero. This, however leads to a difficulty, as it is explained below in an example of Hagen-Poiseuille's flow.

For a pipe whose cross-section is an equilateral triangle with side a, cf. Fig. 2, the volumetric discharge, it is the volume Q of incompressible fluid passing per unit time through any cross-section of the pipe is

$$Q = \frac{\sqrt{3}}{320\eta}a^4\frac{\Delta p}{l},$$

where Δp is the pressure difference between the ends of the pipe and l is its length [13], and the quotient $(-\Delta p/l)$ denotes the pressure gradient. The discharge Q of the fluid is thus proportional to the fourth power of the side of the triangle being a cross-section of the pipe.
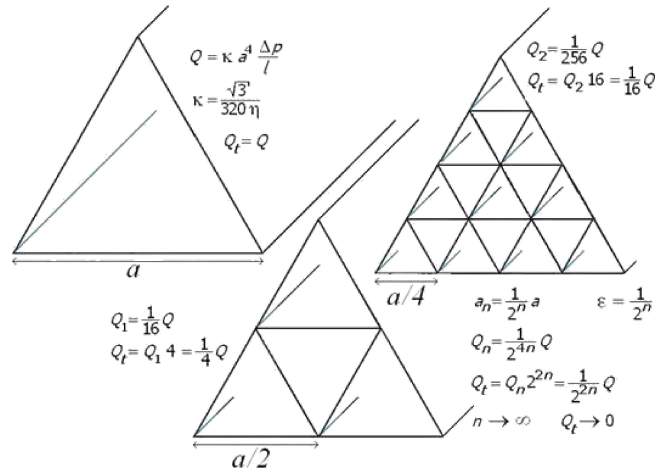

Fig. 2. Drop of the total discharge Q_{tot} with gradual division of the initial cross-section. If a pipe has still triangular cross-section, but the triangle has side $a/2$, then the output is 16 times smaller. If the pipe initially with side a is divided in 4 pipes with still triangular cross-sections, everyone with side $a/2$ and with infinitely thin but rigid walls, then the total discharge is $4 \times (1/16) = 1/4$ of the initial discharge Q, and so on

Let a pipe have still triangular cross-section, and the corresponding equilateral triangle has the side

$$a_n = \frac{1}{2^n}a,$$

where n is a integer, $n > 1$. Then the discharge is

$$Q_n = \frac{1}{2^{4n}}Q.$$

If the pipe initially with the side a is divided in 2^{2n} sub-pipes with still triangular cross-sections but each with side $a/(2n)$ and every sub-pipe has infinitely thin but rigid walls, then the total output of all sub-pipes is

$$Q_{\text{t}} = Q_n 2^{2n} = \frac{1}{2^{4n}}2^{2n}Q = \frac{1}{2^{2n}}Q$$

and for $n \to \infty$ the total discharge vanishes, $Q_{\text{t}} \to 0$, cf. Fig. 2. To overcome such situation, as we still are to describe the real porous medium which has a finite discharge, and the passing with ε to zero is only a mathematical method to smear out the inhomogenity of porous medium, we have to rescale the viscosity, and put

$$\eta^\varepsilon = \varepsilon^2\eta. \tag{19}$$

In our example $\varepsilon = 1/2^n$. This rescaling assures the finite discharge for $n \to \infty$, cf. also [29].

3.2. Two-scale asymptotic approach. For a fixed $\varepsilon > 0$ all the relevant quantities are denoted with the superscript ε. Let us denote by $\mathbf{u}^\varepsilon$ the displacement field in the solid part Ω_S^ε of the porous medium – liquid system, and let $\mathbf{v}^\varepsilon$ be the velocity

field and by p^ε the pressure in the liquid occupying Ω_L^ε. These fields satisfy equations of Lamé and Stokes, respectively,

$$\frac{\partial}{\partial x_i}\left(C_{ijmn}\frac{\partial u_m^\varepsilon}{\partial x_n}\right) = 0,$$

$$-\frac{\partial p^\varepsilon}{\partial x_i} + \eta_{ijmn}^\varepsilon\frac{\partial^2 v_m^\varepsilon}{\partial x_j \partial x_n} = 0. \tag{20}$$

According to previous subsection, cf. Eq. (17), the following rescaling should be introduced

$$\eta_{ijmn}^\varepsilon = \varepsilon^2 \eta_{ijmn}. \tag{21}$$

The condition imposed on the solid-liquid interface Γ^ε are

$$\|\sigma_{ij}^\varepsilon\|n_j = 0 \quad \text{and} \quad \mathbf{v}^\varepsilon = \mathbf{0}. \tag{22}$$

Here $\|\sigma_{ij}^\varepsilon\|n_j$ denotes the jump of stresses, $\|\sigma_{ij}^\varepsilon\|n_j = \sigma_{ij}^{L\varepsilon}n_j - \sigma_{ij}^{S\varepsilon}n_j$, and $\sigma_{ij}^{L\varepsilon}$ and $\sigma_{ij}^{S\varepsilon}$ denote the stresses on liquid and solid sides of the interface, respectively.

The appropriate continuity equations, depending on whether the liquid is incompressible or compressible, should be added.

The stress tensor is specified by

$$\sigma_{ij}^\varepsilon = \begin{cases} C_{ijmn}\dfrac{\partial u_m^\varepsilon}{\partial x_n} & \text{in} \quad \Omega_S^\varepsilon \\[2mm] -p\delta_{ij} + \varepsilon^2\eta_{ijmn}\dfrac{\partial v_m^\varepsilon}{\partial x_n} & \text{in} \quad \Omega_L^\varepsilon \end{cases} \tag{23}$$

Accordingly to two-scale asymptotic approach, instead of one space variable $\mathbf{x}$, we introduce two variables, macroscopic $\mathbf{x}$ and microscopic $\mathbf{y}$, where $\mathbf{y} = \mathbf{x}/\varepsilon$ and instead of a function $f(\mathbf{x})$ consider the function $f(\mathbf{x}, \mathbf{y})$. Taking into account the formula for the total derivative (known as the chain rule) we have

$$\frac{\partial f(\mathbf{x}, \mathbf{y})}{\partial x_i} = \frac{\partial f(\mathbf{x}, \mathbf{y})}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial f(\mathbf{x}, \mathbf{y})}{\partial y_i} \quad \text{with} \quad \mathbf{y} = \frac{\mathbf{x}}{\varepsilon}.$$

Finally, the asymptotic expansions are introduced

$$\mathbf{u}^\varepsilon(\mathbf{x}) = \mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}) + \varepsilon\mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}) + \varepsilon^2\mathbf{u}^{(2)}(\mathbf{x}, \mathbf{y}) + \cdots$$

$$p^\varepsilon(\mathbf{x}) = p^{(0)}(\mathbf{x}, \mathbf{y}) + \varepsilon p^{(1)}(\mathbf{x}, \mathbf{y}) + \varepsilon^2 p^{(2)}(\mathbf{x}, \mathbf{y}) + \cdots \tag{24}$$

$$\mathbf{v}^\varepsilon(\mathbf{x}) = \mathbf{v}^{(0)}(\mathbf{x}, \mathbf{y}) + \varepsilon\mathbf{v}^{(1)}(\mathbf{x}, \mathbf{y}) + \varepsilon^2\mathbf{v}^{(2)}(\mathbf{x}, \mathbf{y}) + \cdots,$$

where functions $\mathbf{u}^{(i)}(\mathbf{x}, \mathbf{y})$, $p^{(i)}(\mathbf{x}, \mathbf{y})$ and $\mathbf{v}^{(i)}(\mathbf{x}, \mathbf{y})$, $i = 0, 1, 2, \cdots$ are Y-periodic.

4. Incompressible flow

If the liquid is incompressible, we have the following incompressibility condition

$$\frac{\partial v_k^\varepsilon}{\partial x_k} = 0. \tag{25}$$

Substituting expansion $(24)_3$ into condition (25) we obtain

$$\left(\frac{\partial}{\partial x_k} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_k}\right)\left(v_k^{(0)} + \varepsilon v_k^{(1)} + \varepsilon^2 v_k^{(2)} + \cdots\right) = 0 \tag{26}$$

which provides a chain of equations for coefficients at different degrees of ε, for $\varepsilon \to 0$.

In order to avoid the singularity ε^{-1}, the coefficients at ε^{-1} in Eq. (26) should vanish. Hence

$$\frac{\partial v_k^{(0)}}{\partial y_k} = 0 \tag{27}$$

and at ε^0

$$\frac{\partial v_k^{(0)}}{\partial x_k} + \frac{\partial v_k^{(1)}}{\partial y_k} = 0. \tag{28}$$

Averaging, cf. Eq. (3), the last equation over the basic cell Y and using periodic boundary conditions gives

$$\frac{\partial\langle v_k^{(0)}\rangle}{\partial x_k} = 0. \tag{29}$$

Next, substituting expansions (24) into Eqs.(20) together with the rescaling of viscosity (21) and comparing terms associated with the same power of ε, we arrive at the homogenized set of equations.

The coefficient at ε^{-2} appears in Eqs. $(20)_1$ only. We must put

$$\frac{\partial}{\partial y_i}\left(C_{ijmn}\frac{\partial u_m^{(0)}}{\partial y_n}\right) = 0. \tag{30}$$

Hence, we find that $\mathbf{u}^{(0)}$ can depend only on $\mathbf{x}$ but not on $\mathbf{y}$,

$$\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(\mathbf{x}^{(0)}). \tag{31}$$

The coefficients at ε^{-1} in Eqs. (20) should vanish also

$$\frac{\partial}{\partial y_i}\left\{C_{ijmn}\left(\frac{\partial u_m^{(0)}}{\partial x_n} + \frac{\partial u_m^{(1)}}{\partial y_n}\right)\right\} = 0,$$

$$-\frac{\partial p^{(0)}}{\partial y_i} = 0. \tag{32}$$

From Eq. $(32)_2$ we find that $p^{(0)}$ can depend only on $\mathbf{x}$ but not on $\mathbf{y}$,

$$p^{(0)} = p^{(0)}(\mathbf{x}). \tag{33}$$

To satisfy Eq. $(32)_1$ we substitute

$$u_m^{(1)}(\mathbf{x}, \mathbf{y}) = A_{mpq}(\mathbf{y})\frac{\partial u_p^{(0)}(\mathbf{x})}{\partial x_q} + P_m(\mathbf{y})p^{(0)}(\mathbf{x}). \tag{34}$$

Here the functions $A_{mpq}(\mathbf{y})$ and $P_m(\mathbf{y})$ are Y-periodic solutions to the following local equations

$$\frac{\partial}{\partial y_j}\left(C_{ijpq} + C_{ijmn}\frac{\partial A_{mpq}}{\partial y_n}\right) = 0,$$

$$\frac{\partial}{\partial y_j}\left(\delta_{ij} + C_{ijmn}\frac{\partial P_m}{\partial y_n}\right) = 0. \tag{35}$$

The coefficients at ε^0 in Eqs. (20) yield

$$\frac{\partial}{\partial x_i}\left\{C_{ijmn}\left(\frac{\partial u_m^{(0)}}{\partial x_n}+\frac{\partial u_m^{(1)}}{\partial y_n}\right)\right\}$$

$$+\frac{\partial}{\partial y_i}\left\{C_{ijmn}\left(\frac{\partial u_m^{(1)}}{\partial x_n}+\frac{\partial u_m^{(2)}}{\partial y_n}\right)\right\}=0, \qquad (36)$$

$$-\left(\frac{\partial p^{(0)}}{\partial x_i}+\frac{\partial p^{(1)}}{\partial y_i}\right)+\eta_{ijmn}\frac{\partial^2 v_m^{(0)}}{\partial y_j \partial y_n}=0.$$

The last equality $(36)_2$ is satisfied if

$$p^{(1)}=\xi_j(\mathbf{y})\frac{\partial p^{(0)}(\mathbf{x})}{\partial x_j} \qquad (37)$$

and

$$v_i^{(0)}=-\chi_{ik}(\mathbf{y})\frac{\partial p^{(0)}(\mathbf{x})}{\partial x_k}, \qquad (38)$$

where $\xi=\xi(\mathbf{y})$ and $\chi_{ik}(\mathbf{y}$ are Y – periodic functions solutions to local problems, defined on the elementary cell Y, cf. [15].

Substituting expressions (37) and (38) into Eq. $(36)_1$ gives

$$\delta_{ik}+\frac{\partial \xi_k}{\partial y_i}+\eta_{ijmn}\frac{\partial^2 \chi_{mk}}{\partial y_j \partial y_n}=0. \qquad (39)$$

As the velocity $\mathbf{v}^{(0)}$ satisfies also the incompressibility Eq. (27), we have the additional condition

$$\frac{\partial \chi_{ik}}{\partial y_k}=0. \qquad (40)$$

Averaging the expression (38) over the basic cell Y, cf. Eq. (3), gives Darcy's law

$$v_i^{(0)}=-K_{ij}\frac{\partial p^{(0)}(\mathbf{x})}{\partial x_j}, \qquad (41)$$

where

$$K_{ij}=\langle \chi_{ij}(\mathbf{y})\rangle. \qquad (42)$$

Applying the asymptotic expansions (24) to the material relation (23), comparing the terms at ε^0, and performing averaging over the basic cell, we get

$$\langle \sigma^{(0)}\rangle=C_{ijmn}^h\frac{\partial u_m^{(0)}}{\partial x_n}+\left(\left\langle C_{ijmn}\frac{\partial P_m}{\partial y_n}\right\rangle-\delta_{ij}\right)p^{(0)}, \qquad (43)$$

where

$$C_{ijpq}^h=\left\langle C_{ijpq}+C_{ijmn}\frac{\partial A_{mpq}}{\partial y_n}\right\rangle. \qquad (44)$$

Equation (43) expresses the material law for a porous body in which the skeleton is elastic and the fluid incompressible.

4.1. Entropy production in incompressible Darcy's flow.
After Eq. (16) we have

$$\rho T^\varepsilon \frac{ds^\varepsilon}{dt}=\eta_{ijmn}^\varepsilon \frac{\partial v_m^\varepsilon}{\partial x_n}\frac{\partial v_i^\varepsilon}{\partial x_j}. \qquad (45)$$

We expand the density of entropy s (*per* unit volume) and the temperature T into the series

$$s^\varepsilon(x,t)=s^{(0)}(x,y,t)+\varepsilon s^{(1)}(x,y,t)$$

$$+\varepsilon^2 s^{(2)}(x,y,t)+\cdots,$$

$$T^\varepsilon(x,t)=T^{(0)}(x,y,t)+\varepsilon T^{(1)}(x,y,t)$$

$$+\varepsilon^2 T^{(2)}(x,y,t)+\cdots \qquad (46)$$

and substitute these expansions together with (21) and (24) into (45) to obtain

$$\rho\left(T^{(0)}(x,y,t)+\varepsilon T^{(1)}(x,y,t)+\varepsilon^2 T^{(2)}(x,y,t)+\cdots\right)$$

$$\times\frac{d}{dt}\left(s^{(0)}(x,y,t)+\varepsilon s^{(1)}(x,y,t)+\varepsilon^2 s^{(2)}(x,y,t)+\cdots\right)$$

$$=\varepsilon^2 \eta_{ijmn}\left\{\left(\frac{\partial}{\partial x_n}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_n}\right)\left(v_m^{(0)}+\varepsilon v_m^{(1)}+\varepsilon^2 v_m^{(2)}+\cdots\right)\right\}$$

$$\times\left\{\left(\frac{\partial}{\partial x_j}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\right)\left(v_i^{(0)}+\varepsilon v_i^{(1)}+\varepsilon^2 v_i^{(2)}+\cdots\right)\right\}. \qquad (47)$$

At ε^0 we obtain

$$\rho T^{(0)}\frac{ds^{(0)}}{dt}=\eta_{ijmn}\frac{\partial v_m^{(0)}}{\partial y_n}\frac{\partial v_i^{(0)}}{\partial y_j} \qquad (48)$$

or, after using (38)

$$\rho T^{(0)}\frac{ds^{(0)}}{dt}=\eta_{ijmn}\frac{\partial \chi_{mq}}{\partial y_n}\frac{\partial \chi_{ik}}{\partial y_j}\frac{\partial p^{(0)}}{\partial x_q}\frac{\partial p^{(0)}}{\partial x_k}. \qquad (49)$$

We remember that $p^{(0)}$ depend on $\mathbf{x}$ only, cf. Eq. (33). Integrating by parts over the basic cell Y we have

$$\rho T^{(0)}\frac{ds^{(0)}}{dt}=\frac{1}{|Y|}\int\limits_Y d\mathbf{y}$$

$$\left\{\frac{\partial}{\partial y_j}\left(\eta_{ijmn}\frac{\partial \chi_{mq}}{\partial y_n}\chi_{ik}\right)-\eta_{ijmn}\frac{\partial^2 \chi_{mq}}{\partial y_j \partial y_n}\chi_{ik}\right\}$$

$$\frac{\partial p^{(0)}}{\partial x_q}\frac{\partial p^{(0)}}{\partial x_k}$$

and after taking into account periodic boundary conditions

$$\rho T^{(0)}\frac{ds^{(0)}}{dt}=-\frac{1}{|Y|}\int\limits_Y d\mathbf{y}\left\{\eta_{ijmn}\frac{\partial^2 \chi_{mq}}{\partial y_j \partial y_n}\chi_{ik}\right\}\frac{\partial p^{(0)}}{\partial x_q}\frac{\partial p^{(0)}}{\partial x_k}$$

or, after using Eq. (39)

$$\rho T^{(0)}\frac{ds^{(0)}}{dt}=\frac{1}{|Y|}\int\limits_Y\left(\delta_{iq}+\frac{\partial \xi_q}{\partial y_i}\right)\chi_{ik}d\mathbf{y}\frac{\partial p^{(0)}}{\partial x_q}\frac{\partial p^{(0)}}{\partial x_k}$$

The second term in parentheses written according to the relation

$$\frac{\partial \xi_q}{\partial y_i}\chi_{ik}=\frac{\partial}{\partial y_i}(\xi_q \chi_{ik})-\xi_q\frac{\partial \chi_{ik}}{\partial y_i}$$

vanishes after integration: the first terms because of the periodic boundary conditions, and the second after Eq. (40). Hence

$$\rho T^{(0)} \frac{ds^{(0)}}{dt} = K_{iq} \frac{\partial p^{(0)}}{\partial x_q} \frac{\partial p^{(0)}}{\partial x_k}, \qquad (50)$$

where K_{iq} is given by (42).

After integration over the whole volume Ω we get

$$\rho T \frac{dS^{(0)}}{dt} = \int_{\Omega} K_{jm} \frac{\partial p^{(0)}}{\partial x_j} \frac{\partial p^{(0)}}{\partial x_m} d\mathbf{x} \qquad (51)$$

and this is an analog of the result for the total entropy production (18).

5. Compressible flow

The general scheme of homogenization in this case is the same as in the case of incompressible flow. Only the equation of continuity is different, and this fact can lead to another definition of the local function χ_{ij}. However, we show below that what concerns the micro-periodic porous medium, the results obtained for incompressible fluid still hold true for the compressible one.

The continuity equation reads now

$$\frac{\partial}{\partial x_i} \left(\rho^{\varepsilon} v_i^{\varepsilon} \right) = 0 \qquad (52)$$

and it is assumed that the liquid is described by a linear state equation expressing the density ρ by the pressure p,

$$\rho^{\varepsilon} = \rho_0 + \beta p^{\varepsilon} \qquad (53)$$

where ρ_0 and β are constant.

Additionally to previous expansions, the density ρ is also expanded

$$\rho^{\varepsilon}(x) = \rho^{(0)}(x, y) + \varepsilon \rho^{(1)}(x, y) + \varepsilon^2 \rho^{(2)}(x, y) + \cdots . \quad (54)$$

The continuity equation takes the form

$$\left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial}{\partial y_i} \right) \left\{ \left(\rho^{(0)} + \varepsilon \rho^{(1)} + \varepsilon^2 \rho^{(2)} + \cdots \right) \right.$$
$$\left. \left(v_i^{(0)} + \varepsilon v_i^{(1)} + \varepsilon^2 v_i^{(2)} + \cdots \right) \right\} = 0. \qquad (55)$$

The state equation

$$\left(\rho^{(0)} + \varepsilon \rho^{(1)} + \varepsilon^2 \rho^{(2)} + \cdots \right)$$
$$= \rho_0 + \beta \left(p^{(0)} + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + \cdots \right) \qquad (56)$$

at $\varepsilon \to 0$ reduces to the form

$$\rho^{(0)} = \rho_0 + \beta p^{(0)}. \qquad (57)$$

The continuity Eq. (55) yields at ε^{-1}

$$\frac{\partial}{\partial y_i} \left(\rho^{(0)} v_i^{(0)} \right) = 0 \qquad (58)$$

and at ε^0

$$\frac{\partial}{\partial x_i} \left(\rho^{(0)} v_i^{(0)} \right) + \frac{\partial}{\partial y_i} \left(\rho^{(0)} v_i^{(1)} + \rho^{(1)} v_i^{(0)} \right) = 0. \quad (59)$$

But from Eq. (33) we know that $p^{(0)}$ does not depend on $\mathbf{y}$. Hence from (57) we deduce that

$$\rho^{(0)} = \rho^{(0)}(\mathbf{x}). \qquad (60)$$

We observe Eq. (58) for linearly compressible fluid is the same as Eq. (27) of incompressible fluid. Hence all discussion on the entropy production in compressible Darcy's flow can be repeated after the discussion of the incompressible flow, and leads to the same results.

6. Final remarks

6.1. Results. We have derived expressions characterizing the seepage through micro-periodic porous media for compressible fluid. In particular, we have found that the result for the entropy production can be included into Onsager's scheme.

We have shown also that what concerns the micro-periodic porous medium, the results obtained for incompressible fluid still hold true for the compressible one.

The entropy growth in flow through micro-porous medium was analysed using two formulations. The Stokes equation and Darcy's law, and it was shown that the entropy rate of growth in both approaches is the same. It means that the both descriptions satisfy a requirement of thermodynamic consistency, cf. [30].

Our result once more affirms opinion that Onsager's reciprocal relations can be regarded as a universal natural law, the scope and importance of which becomes clear only after being put in a proper relation to complicated questions in border areas between physics and chemistry. For his discovery of these reciprocal relations, Lars Onsager was awarded the 1968 Nobel Prize in Chemistry. The presentation speech referred to the three laws of thermodynamics and then added "It can be said that Onsager's reciprocal relations represent a further law making a thermodynamic study of irreversible processes possible", cf. [31]. Some authors have even described Onsager's relations as the "Fourth law of thermodynamics".

6.2. Viscosity tensor. The viscosity of the fluid is important characteristic, as the starting point of the above analysis is Stokes' equation.

There were two reasons to use the tensor description of viscosity. First, as it was explained, it gives more compact notation. Second reason, it is a hope that such, a more general description can find an application after a possible discovery of new kinds of fluids. For an analogy, in the theory of liquid crystals a second rank symmetric traceless tensor order parameter is used to describe the orientational order of nematics. On the other hand, A. Cemal Eringen considers linear constitutive equations of anisotropic micropolar fluids [32], cf. also [33].

The molecular theory of the elastic constants of simple liquids has been given a firm foundation by Zwanzig and Mountain [34]. The elastic constants of liquids are high-frequency limits of the corresponding generalized viscosities.

6.3. Positives and negatives of the used model. We considered a two-component cellular structure as a model of porous medium. The structure is made up from an elastic skeleton and of a viscous fluid (incompressible or compressible one). The walls of canals in the skeleton are supposed to be impermeable for the fluid, and the accepted boundary conditions at the solid-liquid interface were actually the same as would be accepted for the walls of any macroscopic canal. It is an obvious idealisation, however it represents the main features of the real porosity. In limit passage $\varepsilon \to 0$ it assures, at least qualitatively Darcy's flow, and as it was shown above the irreversible thermodynamics relation on the entropy growth.

Properly speaking, the fluid flows through canals are formally macro-canals, as its walls are impermeable (they are made of continuous solid and do not demonstrate any granular or molecular structure). Only after homogenisation, $\varepsilon \to 0$, the macro-canals become micro-canals and Darcy's flow is regained, and only then the cellular structure can be considered as a porous medium (for which, among others, Darcy's law is valid). The discussed flow is a creeping, also known as Stokes' flow, in which inertial forces are small compared with viscous forces, the fluid velocities are low, the viscosities are large, and hence Reynolds' number is low. The argumentation itself is patterned on Sanchez-Palencia derivation [15], to which an explanation of scaling of the viscosity by the ε^2 factor was added.

More accurate, but also more cumbersome description of the seepage can be realised by analysing the double scale porosity solid, cf. [35]. We did not use this model, as the one scale porosity has revealed to be sufficient to verify the entropy growth property of the flow.

As a result of the fluid pressure the elastic skeleton is subject to a displacement field, constant in the time. For unsteady flows the inertia forces appear and the homogenisation leads to an extended Darcy's law (with time convolutions), cf. [21, 22]. Thermodynamics of such flows needs further investigation.

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