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Propagation of weak discontinuities for quasilinear hyperbolic systems with coefficients functionally dependent on solutions

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Abstract. The propagation of weak discontinuities for quasilinear systems with coefficients functionally dependent on the solution is studied. We demonstrate that, similarly to the case of usual quasilinear systems, the transport equation for the intensity of weak discontinuity is quadratic in this intensity. However, the contribution from the (nonlocal) functional dependence appears to be in principle linear in the jump intensity (with some exceptions). For illustration, several examples, including two hyperbolic systems (with functional dependence), the dispersive Maxwell equations and fluid equations of the Hall plasma thruster, are considered.

1. Introduction. Hyperbolic systems of linear as well as nonlinear equations are characterized by several important properties: 1) the Cauchy problem is well posed (at least in a local sense), 2) causality property: the variation of the initial data over a compact domain can influence the solution after a finite time in a compact domain only. The latter property is related to the fact that weak discontinuities propagate along bicharacteristics with finite speed.

The question arises which of the above properties is preserved when we assume that the coefficients of the system are functionally dependent on the solution. In this case it is clear that, in general, causality cannot be preserved, at least not in the usual sense. Varying the initial data in a small region can influence the solution everywhere. Still, as will be demonstrated, weak discontinuities may appear only on characteristic surfaces. In addition, similarly to the case of usual PDEs, the transport equation for the jump intensity can be obtained. In contrast to the case of the usual nonlinear

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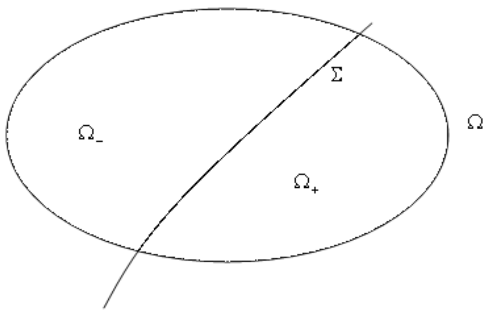
dependence when the transport equation is quadratic in the amplitude of the discontinuity, the contribution from nonlocal functional dependence is in principle linear, except the cases when a certain "resonant coupling" between discontinuities on different characteristics (or possibly the same one) occurs. We confine our attention to the case of quasilinear systems, allowing however for the functional dependence of their coefficients on solutions. In Section 2 we show that under fairly weak assumptions, weak discontinuity of the solution can occur only on characteristic surfaces. We restrict our considerations to the case of jump in the first derivatives. When higher derivatives are suffering a jump, the procedure is practically the same, except that one has to deal with prolonged systems.

2. Weak discontinuities for systems with functional dependence. Let us consider the following system of l quasilinear equations:

$$\sum_{j=1}^{m}\sum_{\nu=1}^{n} A_j^{s\nu}(x,u(x),u(\cdot))u^j_{,x^\nu} = f^s(x,u(x),u(\cdot)), \qquad s=1,\dots,l, \tag{2.1}$$

with coefficients $A_j^{s\nu}$, f^s defined for $x\in\tilde{\Omega}\subset\mathbb{R}^n$ and $u(\cdot)$ from some open subset B of $C(\tilde{\Omega})$. We are concerned with the case when the coefficients of the system can depend on the unknown functions $u:\tilde{\Omega}\to\mathbb{R}^m$ also in a nonlocal, functional way, which is expressed by the symbol $u(\cdot)$ in $A_j^{s\nu}$ and f^s. Let $\Omega\subset\tilde{\Omega}$ be an open set and Σ' a smooth hypersurface in $\tilde{\Omega}$, which divides Ω into two disjoint open parts, Ω_+ and Ω_-. Hence $\Omega=\Omega_+\cup\Omega_-\cup\Sigma$ and $\Sigma=\overline{\Omega}_+\cap\overline{\Omega}_-\cap\Omega$.



THEOREM 2.1. *Assume that the continuous function $u:\tilde{\Omega}\to\mathbb{R}^m$ and the coefficients of the system satisfy the following conditions:*

(i) $A(x,u(x),u(\cdot))$ *and* $f(x,u(x),u(\cdot))$ *are continuous in* $x\in\Omega$.
(ii) $u\in C^1(\Omega_+)\cap C^1(\Omega_-)$.
(iii) *For any* $x_0\in\Sigma$ *the derivatives* $u_{,x^\nu}$ *have finite limits* $u^+_{,x^\nu}(x_0)$ *for* $x\to x_0$ *with* $x\in\Omega_+$ *and similarly* $u^-_{,x^\nu}(x_0)$ *for* $x\to x_0$ *with* $x\in\Omega_-$. *Their difference will be denoted by* $[u_{,x^\nu}(x_0)]=u^+_{,x^\nu}(x_0)-u^-_{,x^\nu}(x_0)$.

(iv) *u satisfies* (2.1) *in Ω_+ and Ω_-.*

(v) *The jump $[u_{,x^\nu}]$ of the first derivatives of u is nonzero on Σ except possibly at isolated points.*

Then the surface Σ of weak discontinuity is a characteristic surface.

Notice that although all coefficients of (2.1) as well as the function u are defined on a bigger domain $\tilde{\Omega}$, we do not assume that u satisfies (2.1) outside of Ω_+ and Ω_-. Similarly the functional dependence can refer to a bigger domain than Ω.

Proof. First of all, recall that under the above assumptions, the derivatives of u in directions tangent to Σ are continuous on Σ. Indeed, taking any curve $\gamma : x = x(s)$ of class C^1 on Σ which passes through a point $x_1 \in \Sigma$, and integrating the one-sided gradient of u along this curve from x_1 to any other point x_2 of this curve, we obtain

$$u^j(x_2)^+ = u^j(x_1)^+ + \sum_{\nu=1}^{n} \int_\gamma u^{+j}_{,x^\nu}\, dx^\nu,$$

$$u^j(x_2)^- = u^j(x_1)^- + \sum_{\nu=1}^{n} \int_\gamma u^{-j}_{,x^\nu}\, dx^\nu.$$

Since u is continuous on Ω, we have

$$0 = u^j(x_2)^+ - u^j(x_2)^-, \qquad 0 = u^j(x_1)^+ - u^j(x_1)^-.$$

Thus, if $\sigma = dx(s)/ds$ denotes the vector tangent to γ at x_1 then

$$0 = \sum_{\nu=1}^{n} \int_\gamma [u^j_{,x^\nu}\sigma^\nu]\, ds.$$

Here $[g]$ denotes the jump of g across the surface of discontinuity.

In the limit when x_2 is approaching x_1 we obtain

$$[u^j_{,x^\nu}]\sigma^\nu = 0, \qquad \nu = 1, \ldots, n.$$

Clearly, since γ is an arbitrary curve on Σ it follows that σ can be any vector tangent to Σ. Consequently, the derivatives of u in directions tangent to Σ are continuous. Since at each point of Σ we have $n-1$ linearly independent tangent vectors, we conclude that $\operatorname{rank}[u^j_{,x^\nu}] \le 1$ and $[u^j_{,x^\nu}] = (X^j\lambda_\nu)$, where $X = (X^j) \in \mathbb{R}^m$, $\lambda = (\lambda_\nu) \in \mathbb{R}^n$, $\sum_{\nu=1}^n \lambda_\nu\sigma^\nu = 0$.

Let $x \in \Sigma$. Then on the two sides of Σ we have

$$\sum_{j=1}^{m}\sum_{\nu=1}^{n} A^{s,\nu}_j(x, u^+(x), u^+(\cdot))u^{+j}_{,x^\nu} = f^s(x, u^+(x), u^+(\cdot)), \qquad s = 1, \ldots, l,$$

$$\sum_{j=1}^{m}\sum_{\nu=1}^{n} A^{s,\nu}_j(x, u^-(x), u^-(\cdot))u^{-j}_{,x^\nu} = f^s(x, u^-(x), u^-(\cdot)), \qquad s = 1, \ldots, l.$$

Subtracting the second equation from the first, by the continuity of u, A, f in Ω we arrive at

$$\sum_{j=1}^{m}\sum_{\nu=1}^{n} A_j^{s,\nu}(x,u(x),u(\cdot))[u^j_{,x^\nu}]=0,\qquad s=1,\dots,l,$$

for $x\in\Sigma$. Since $[u^j_{,x^\nu}]=(X^j\lambda_\nu)$, we have

$$\sum_{j=1}^{m}\sum_{\nu=1}^{n}(A_j^{s,\nu}(x,u(x),u(\cdot))\lambda_\nu)X^j=0\quad\text{ for } x\in\Sigma.$$

This means that X is an eigenvector of the matrix $A\lambda$ $(=\sum_{\nu=1}^{n}A_j^{s\nu}\lambda_\nu)$, corresponding to the null eigenvalue of $A\lambda$, and λ is the "perpendicular" characteristic vector. Since, by definition, the surface perpendicular to λ is a characteristic surface, we conclude that the surface of weak discontinuity must be a characteristic surface. ■

As a simple example we take the Maxwell equations in two independent variables (t,x). We assume that the medium is dispersive (the dielectric constant depends on the frequency). For simplicity we take the magnetic permeability to be constant and equal to μ_0. Let $E,\mathcal{D},P,B,H$ denote the electric field, electric displacement field, electric polarization field, magnetic induction field and magnetic field respectively. Then we have the following constitutive relations:

$$\begin{aligned}
\mathcal{D}(t,x)&=\epsilon_0E(t,x)+P(t,x),\\
B(t,x)&=\frac{1}{\mu_0}H(t,x),\\
P(t,x)&=-\int_{-\infty}^{t}\Big(\int_{-\infty}^{\infty}\chi(t,t',x,x')E_2(t',x')\,dx'\Big)\,dt',
\end{aligned}$$

where ϵ_0 is the dielectric constant of the vacuum and $\chi(t,t',x,x')$ is the electric susceptibility of the medium. In the one-dimensional case, in the absence of electric charge and electric current, when additionally $B_2=\text{const}$ and $E_3=\text{const}$, the Maxwell equations reduce to

$$B_{3,t}+E_{2,x}=0,\qquad \mathcal{D}_{2,t}+H_{3,x}=0.$$

Using the constitutive relations we can write these equations in terms of E_2 and B_3, obtaining the following linear hyperbolic system:

$$\begin{aligned}
\text{(2.2)}\qquad \frac{\partial}{\partial t}\begin{pmatrix}E_2\\ B_3\end{pmatrix}+\begin{pmatrix}0 & \frac{1}{\epsilon_0\mu_0}\\ 1 & 0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}E_2\\ B_3\end{pmatrix}&\\
=\begin{pmatrix}-\int_{-\infty}^{t}(\int_{-\infty}^{\infty}\chi_{,t}(t,t',x,x')E_2(t',x')\,dx')\,dt'\\ 0\end{pmatrix}&,
\end{aligned}$$

where only the RHS includes functional dependence on the solution. Consequently, the discontinuities propagate with the vacuum light velocity.

However, if we assume that the function χ also depends on the current electric field $E(t,x)$, i.e.

$$P(t,x) = -\int_{-\infty}^{t}\Big(\int_{-\infty}^{\infty}\chi(E_2(t,x),t,t',x,x')E_2(t',x')\,dx'\Big)\,dt',$$

then our Maxwell equations become nonlinear with characteristic velocities functionally dependent on the solution:

$$\frac{\partial}{\partial t}\begin{pmatrix}E_2\\ B_3\end{pmatrix} + \begin{pmatrix}0 & c^2(I_1)\\ 1 & 0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}E_2\\ B_3\end{pmatrix} = \begin{pmatrix}I_2\\ 0\end{pmatrix}, \tag{2.3}$$

where

$$c^2(I_1) = \frac{1}{\epsilon_0\mu_0(1+I_1)}, \tag{2.4}$$

$$I_1 = \int_{-\infty}^{t}\Big(\int_{-\infty}^{\infty}\chi_{,E_2}(E_2(t,x),t,t',x,x')E_2(t',x')\,dx'\Big)\,dt', \tag{2.5}$$

$$I_2 = -\int_{-\infty}^{t}\Big(\int_{-\infty}^{\infty}\chi_{,t}(E_2(t,x),t,t',x,x')E_2(t',x')\,dx'\Big)\,dt'. \tag{2.6}$$

In this case the characteristic velocities depend on I_1 and are equal to $\pm c(I_1)$.

3. Some simple examples. In this section several nonlocal versions of nondissipative Burgers equations will be studied in order to elucidate the possible differences in propagation of discontinuities caused by nonlocal dependence. Let us consider the following four examples:

$$u_t(t,x) + u(t,x)u_x(t,x) = 0, \tag{3.1}$$

$$u_t(t,x) + u(t,x+1)u_x(t,x) = 0, \tag{3.2}$$

$$u_t(t,x) + u(t-1,x)u_x(t,x) = 0, \tag{3.3}$$

$$u_t(t,x) + u(t-1,x+1)u_x(t,x) = 0. \tag{3.4}$$

To make a clear distinction we will speak of the functional dependence of coefficients on the solution only in the case where we have nonlocal dependence. Hence (2.2), (2.3), (3.2), (3.3), (3.4) have coefficients functionally dependent on the unknown function u, whereas (3.1) does not. The first (local) of the four equations (3.1)–(3.4) is taken to compare its properties with the other three (nonlocal).

For all the above equations we will write the transport equation for the amplitude of weak discontinuity. To do this we differentiate (3.1)–(3.4) with respect to x and introduce the new function $p(t,x) = u_x(t,x)$. In this way we obtain the prolonged equations. Let $f(\chi^-(t),t)$ and $f(\chi^+(t),t)$ denote the one-sided limits of $f(t,x)$ when x tends to the characteristic curve $x = \chi(t)$. For example

$$f(\chi^+(t),t) = \lim_{\substack{x\to\chi(t)\\ x>\chi(t)}} f(t,x). \tag{3.5}$$

We will denote by $\sigma(t) = p(t,\chi^+(t)) - p(t,\chi^-(t))$ the jump of the first derivative u_x across the characteristic curve $x = \chi(t)$. The last step is to write the transport equation.

1. Differentiating (3.1) with respect to x we get

$$p_t(t,x) + u(t,x)p_x(t,x) = -p^2(t,x). \tag{3.6}$$

The left hand side is the derivative of p along the characteristic curve $x = \chi(t)$ $(d\chi/dt = u(t,\chi))$. Therefore we can write this equation as an ordinary differential equation along the characteristic curve,

$$\frac{dp(t,\chi(t))}{dt} = -p^2(t,\chi(t)). \tag{3.7}$$

If there is a discontinuity of $p(t,x)$ along the characteristic line then the jump

$$\sigma(t) = p(t,\chi^+(t)) - p(t,\chi^-(t)) \tag{3.8}$$

satisfies

$$\frac{d\sigma(t)}{dt} = -2p(t,\chi^+(t))\sigma(t) + \sigma^2(t). \tag{3.9}$$

The transport equation in this case is an ordinary differential equation with a quadratic term with respect to σ on the right hand side.

2. Differentiating (3.2) with respect to x we obtain

$$p_t(t,x) + u(t,x+1)p_x(t,x) = -p(t,x+1)p(t,x). \tag{3.10}$$

The equation of the characteristic is

$$\frac{d\chi(t)}{dt} = u(t,\chi(t)+1). \tag{3.11}$$

Usually for a given characteristic curve $x = \chi(t)$ its shift $x = \chi(t)+1$ is not a characteristic curve and therefore $p(t,x)$ is continuous on $x = \chi(t)+1$, $p(t,\chi^-(t)+1) = p(t,\chi^+(t)+1)$. In this case we arrive at the transport equation

$$\frac{d\sigma(t)}{dt} = -p(t,\chi(t)+1)\sigma(t), \tag{3.12}$$

which is again an ordinary differential equation, but this time linear in σ.

It may happen however that also the curve

$$x = \tilde{\chi}(t) := \chi(t) + 1$$

is a characteristic curve on which the solution has a weak discontinuity. In that case we have a sort of resonant interaction between discontinuities located at different (parallel) characteristic curves. As a result we obtain a system of two coupled equations for the respective jumps σ and $\tilde{\sigma}$ across both curves:

$$\frac{d\sigma(t)}{dt} = -p(t, \tilde{\chi}^+(t))\sigma(t) - p(t, \chi^+(t))\tilde{\sigma}(t) + \sigma(t)\tilde{\sigma}(t), \tag{3.13}$$

$$\frac{d\tilde{\sigma}(t)}{dt} = -p(t, \tilde{\chi}(t) + 1)\tilde{\sigma}(t). \tag{3.14}$$

Clearly, we have only two equations in the system if there are no other discontinuities coupled (e.g. $x = \chi(t) + 2$ is not a characteristic with weak discontinuity of the solution), otherwise we can end up with a still more complex system of transport equations.

3. For (3.3) we have

$$p_t(t, x) + u(t - 1, x)p_x(t, x) = -p(t - 1, x)p(t, x). \tag{3.15}$$

This time the characteristic curves are given by

$$\frac{d\chi(t)}{dt} = u(t - 1, \chi(t)). \tag{3.16}$$

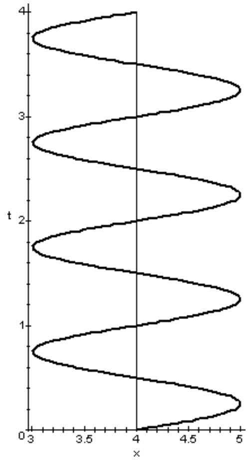
In the most typical case the translation of the characteristic curve along the t-axis by 1 is not a characteristic curve. In that case we obtain the following transport equation:

$$\frac{d\sigma(t)}{dt} = -p(t - 1, \chi(t))\sigma(t), \tag{3.17}$$

which is a linear ordinary differential equation equation for σ. If, however, the point $(t - 1, \chi(t))$ belongs to the same characteristic $x = \chi(t)$, we arrive at the following single functional differential transport equation:

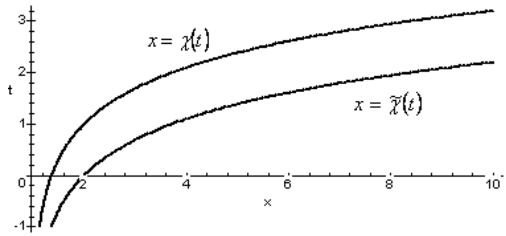
$$\begin{aligned} \frac{d\sigma(t)}{dt} = & -p(t - 1, \chi^+(t))\sigma(t) - p(t, \chi^+(t))\sigma(t - 1) \\ & + \sigma(t)\sigma(t - 1). \end{aligned} \tag{3.18}$$

This can happen when the characteristic is of the form $\chi(t) = \text{const}$ or if $\chi(t)$ is a 1-periodic function.



The characteristic $x = \chi(t)$ is a 1-periodic function.

The point $(t-1, \chi(t))$ can belong to another characteristic curve, $x = \tilde{\chi}(t)$ (i.e.$(t-1,\chi(t)) = (t-1, \tilde{\chi}(t-1))$), on which there is also weak discontinuity of u.



Translating the characteristic $x = \chi(t)$ along the t-axis may produce a new characteristic $x = \tilde{\chi}(t)$.

Then

$$\frac{d\sigma(t)}{dt} = -p(t-1, \chi^+(t))\sigma(t) - p(t, \chi^+(t))\sigma(t-1) + \sigma(t)\sigma(t-1), \tag{3.19}$$

$$\frac{d\tilde{\sigma}(t)}{dt} = -p(t-1, \tilde{\chi}^+(t))\tilde{\sigma}(t), \tag{3.20}$$

where $\tilde{\sigma}$ denotes the jump across the characteristic $x = \tilde{\chi}(t)$.

4. In the case of (3.4) we obtain the following prolonged equation:

$$p_t(t,x) + u(t-1, x+1)p_x(t,x) = -p(t-1, x+1)p(t,x). \tag{3.21}$$

Consequently, the characteristic curves satisfy

$$\frac{dx}{dt} = u(t-1, x+1). \tag{3.22}$$

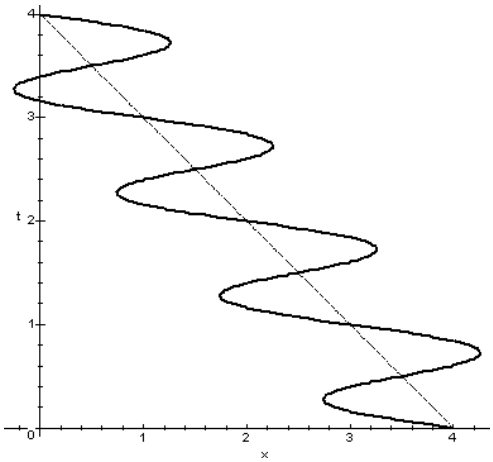
If $x = \chi(t)$ is a characteristic with discontinuity of u, then as before, the characteristic curve shifted along t is not usually a characteristic curve. In that case the transport equation will be a linear ordinary differential equation in σ:

$$\frac{d\sigma(t)}{dt} = -p(t-1, \chi(t)+1)\sigma(t). \tag{3.23}$$

If, however, after translation we fall on the same characteristic curve (i.e. the point $(t-1, \chi(t)+1)$ belongs to the curve $\{(t,x) : x = \chi(t)\}$), then the transport equation is an ordinary differential equation with retarded argument:

$$\begin{aligned} \frac{d\sigma(t)}{dt} &= -p(t-1, \chi^+(t)+1)\sigma(t) \\ &\quad - p(t, \chi^+(t))\sigma(t-1) + \sigma(t)\sigma(t-1). \end{aligned} \tag{3.24}$$

This can occur when the characteristic curve is defined by a periodic function g in the form $\chi(t) = x_0 - t - g(t)$, where $g \in C^1(-1,\infty)$ and $g(t) = g(t-1)$ for $t \geq 0$. Hence $\chi(t)+1 = \chi(t-1)$.



The plot of $\chi(t) = 4 - t - \sin(2\pi t)$.

It can happen that the point $(t-1, \chi(t)+1)$ belongs to another characteristic $x = \tilde{\chi}(t)$ and the function u also has a weak discontinuity there. In this case the transport equation (see (3.25) below) on the characteristic $x = \chi(t)$ is coupled with the transport equation (see (3.26)) on the characteristic $\tilde{\chi}(t) = \chi(t)+1$ (additionally, we assume here that u does not have a weak discontinuity on the curve $x = \tilde{\chi}(t)+1$):

$$\begin{aligned} \frac{d\sigma(t)}{dt} &= -p(t-1, \tilde{\chi}^+(t))\sigma(t) \\ &\quad - p(t, \chi^+(t))\tilde{\sigma}(t-1) + \sigma(t)\tilde{\sigma}(t-1), \end{aligned} \tag{3.25}$$

$$\frac{d\tilde{\sigma}(t)}{dt} = -p(t-1, \tilde{\chi}(t)+1)\tilde{\sigma}(t). \tag{3.26}$$

Equation (3.26) remains linear if $x = \tilde{\chi}(t)+1 = \chi(t)+2$ is not a characteristic curve, or if it is, but u does not suffer weak discontinuity there.

Although the above examples of transport equations do not exhaust all possible cases, they shed some light on the problem of transport of weak discontinuities in the case of equations with functional dependence. It can also be shown that by a suitable choice of initial data all the considered cases of transport equations for (3.2)–(3.4) can really occur. One may think that if instead of translation in space or time, the functional dependence is expressed by integral operators, then the "resonances" could not occur and the contribution from such functional dependence to the transport equation would be linear. However, for the equation

$$u_t(t,x) + u_x(t,x)\int_0^x K(t,s)u_s(t,s)\,ds = 0 \tag{3.27}$$

with a Volterra type operator, the transport equation is nonlinear. Notice that taking $K(t,x) \equiv 1$ we practically obtain (3.1). Similarly taking $x+1$ instead of x as an upper limit of integration we end up with an equation containing a shift along the x-axis.

4. The prolonged system. Now we will be concerned with a system consisting of m equations with m unknown functions and two independent variables $(t,x) \in [0,T] \times \mathbb{R}$:

$$u_t + A(t,x,u,u(\cdot))u_x = f(t,x,u,u(\cdot)), \tag{4.1}$$

where

$$u : [0,T] \times \mathbb{R} \to \mathbb{R}^m, \qquad u(t,x) = \begin{pmatrix} u^1(t,x) \\ \vdots \\ u^m(t,x) \end{pmatrix}, \qquad f = \begin{pmatrix} f^1 \\ \vdots \\ f^m \end{pmatrix}.$$

Assume that the matrix A has at least one real eigenvalue ξ of a constant multiplicity s. Without loss of generality we can assume that this is the first eigenvalue. Assume moreover that there exist s linearly independent left eigenvectors $L^1,\dots,L^s$ corresponding to this eigenvalue. Clearly ξ as well as the corresponding eigenvectors are (in general) functionally dependent on $u(\cdot)$. We also assume enough differentiability of A and f, to guarantee that all differentiations below can be performed.

We define the matrix L in such a way that the first s rows are the eigenvectors $L^1,\dots,L^s$. The next $m-s$ rows are linearly independent vectors $L^{s+1},\dots,L^m$ such that

$$L^kA \in \operatorname{span}\{L^{s+1},\dots,L^m\}, \qquad k = s+1,\dots,m.$$

In other words $L^{s+1}, \dots, L^m$ span the left invariant subspace for the matrix A complementary to $L^1, \dots, L^s$. After this choice of L the first s columns of the inverse $R = L^{-1}$ are right eigenvectors of A corresponding to ξ. The remaining columns of R span the complementary, $m - s$-dimensional right invariant subspace for A. In this case, A can be decomposed as follows:

$$A(t, x, u, u(\cdot)) = R(t, x, u, u(\cdot))\tilde{D}(t, x, u, u(\cdot))L(t, x, u, u(\cdot)),$$

where

$$L = \begin{pmatrix} L^1 \\ \vdots \\ L^m \end{pmatrix}, \qquad R = \begin{pmatrix} R_1 & \dots & R_m \end{pmatrix}, \qquad \tilde{D} = \begin{pmatrix} \xi\mathbb{I}_s & \mathbf{0} \\ \mathbf{0} & Q \end{pmatrix}.$$

Here Q is an $(m-s) \times (m-s)$ matrix, $\mathbb{I}_s$ stands for the $s \times s$ unit matrix, L^i, $i = 1, \dots, m$, are row vectors and R_i, $i = 1, \dots, m$, are column vectors. Since $L^1, \dots, L^s$ are linearly independent left eigenvectors corresponding to the eigenvalue ξ, the column vectors $R_1, \dots, R_s$ are right eigenvectors corresponding to the same eigenvalue.

For brevity we assume that the functional dependence of the coefficients of the system is realized by their dependence on a single, nonlinear integral operator $I[u]$. For example $I[u]$ can be a nonlinear operator of the form

$$I[u] = \int\limits_{\alpha_1(t)}^{\alpha_2(t)} \Big(\int\limits_{t-\theta(x)}^{t} \Phi(t, \tau, x, s, u(\tau, s))\, d\tau \Big) ds, \tag{4.2}$$

or

$$I[u] = \int\limits_{\alpha_1(t)}^{\alpha_2(t)} \tilde{\Phi}(t, x, s, u(t, s)) \cdot u_s(t, s)\, ds, \tag{4.3}$$

where

1. $\alpha_1, \alpha_2 : [0, T] \to \mathbb{R}$, $\alpha_1, \alpha_2 \in C^1([0, T])$, $-\infty \le \alpha_1 \le \alpha_2 \le \infty$; α_1 and α_2 are not characteristic curves;
2. $0 \le \theta(x) \le \infty$ for $x \in \mathbb{R}$;
3. $\Phi : [0, T] \times [-\theta(x), T] \times \mathbb{R}^{m+2} \to \mathbb{R}$ is differentiable with respect to all arguments and its partial derivatives are bounded;
4. $\tilde{\Phi} : [0, T] \times \mathbb{R}^{m+2} \to \mathbb{R}^m$ is bounded together with all its partial derivatives;
5. $\tilde{\Phi} \cdot u_s$ denotes the scalar product of $\tilde{\Phi}$ and u_s in $\mathbb{R}^m$.

In the case of the operator (4.2) the system has memory. The initial condition for u must also be given for negative time:

$$u(t, x) = u_0(t, x), \qquad (t, x) \in [-\theta(x), 0] \times \mathbb{R}. \tag{4.4}$$

We use the following notation for any function $g = g(t, x, u)$:

- $g_x = \frac{\partial}{\partial x} g(t,x,u)$ is the partial derivative with respect to x,
- $g_{,x}$ or $\frac{dg}{dx}$ denotes the total derivative,

$$g_{,x} = g_x + \sum_{i=1}^{m} g_{u^i} u^i_x + g_I I_x,$$

- $X \cdot \nabla_u g$ denotes the directional derivative of $g(t,x,u^1,\ldots,u^m,I)$ with respect to the variables u in the direction of the vector $X \in \mathbb{R}^m$,

$$X \cdot \nabla_u g = \sum_{i=1}^{m} X_i g_{u^i}.$$

Let us start from the case when the coefficients of (4.1) depend on the values of u and the operator (4.2). Multiplying (4.1) on the left by the matrix L and differentiating the result with respect to x we have

$$L_{,x} u_t + L u_{xt} + \tilde{D}_{,x} L u_x + \tilde{D}(L u_x)_{,x} = L_{,x} f + L f_{,x}. \tag{4.5}$$

After easy manipulations, noticing that $u_t = f - A u_x$ we arrive at

$$(L u_x)_{,t} + \tilde{D}(L u_x)_{,x} = L_{,t} u_x + L f_{,x} - \tilde{D}_{,x} L u_x + L_{,x} R \tilde{D} L u_x. \tag{4.6}$$

Since $LR = \mathbb{I}$, we have $L_{,x} R + L R_{,x} = 0$ and consequently

$$(L u_x)_{,t} + \tilde{D}(L u_x)_{,x} = L_{,t} u_x + L f_{,x} - \tilde{D}_{,x} L u_x - L R_{,x} \tilde{D} L u_x,$$

or

$$(L u_x)_{,t} + \tilde{D}(L u_x)_{,x} = L_{,t} u_x + L f_{,x} - L(R\tilde{D})_{,x} L u_x. \tag{4.7}$$

Let $p := L u_x$ and so $u_x = Rp$. In terms of p we can rewrite (4.7) as

$$p_t + \tilde{D}\, p_x = L_{,t} R p + L f_{,x} - L(R\tilde{D})_{,x} p. \tag{4.8}$$

Now we will show that the right hand side of (4.8) is in fact a second order polynomial in the variable $p = (p_1,\ldots,p_m)^T$.

Using directional derivatives we obtain

$$\begin{aligned} L_{,t} R p + L f_{,x} - L(R\tilde{D})_{,x} p &= L_t R p + (u_t \cdot \nabla_u L) R p + L_I I_{,t} R p \\ &\quad + L f_x + L(u_x \cdot \nabla_u f) + L f_I I_{,x} \\ &\quad - L(R\tilde{D})_x p - L(u_x \cdot \nabla_u (R\tilde{D})) p \\ &\quad - L(R\tilde{D})_I I_{,x} p \\ &= L_t R p + ((f - R\tilde{D} p) \cdot \nabla_u L) R p + L_I I_{,t} R p \\ &\quad + L f_x + L((Rp) \cdot \nabla_u f) + L f_I I_{,x} \\ &\quad - L(R\tilde{D})_x p - L((Rp) \cdot \nabla_u (R\tilde{D})) p \\ &\quad - L(R\tilde{D})_I I_{,x} p, \end{aligned}$$

where

$$I_{,x} = \int\limits_{\alpha_1(t)}^{\alpha_2(t)} \Bigg(\int\limits_{t-\theta(x)}^{t} \frac{\partial \Phi}{\partial x}\, d\tau - \frac{\partial \theta}{\partial x} \Phi\big(t, \theta(x), x, s, u(\theta(x), s)\big) \Bigg)\, ds, \tag{4.9}$$

$$\begin{aligned} I_{,t} = {} & \int\limits_{\alpha_1(t)}^{\alpha_2(t)} \Bigg(\int\limits_{t-\theta(x)}^{t} \frac{\partial \Phi}{\partial t}\, d\tau \Bigg)\, ds \\ & + \int\limits_{\alpha_1(t)}^{\alpha_2(t)} \big(\Phi\big(t, t, x, s, u(t,s)\big) - \Phi\big(t, t-\theta(x), x, s, u(t-\theta(x), s)\big)\big)\, ds \\ & + \frac{\partial \alpha_2}{\partial t} \int\limits_{t-\theta(x)}^{t} \Phi\big(t, \tau, x, \alpha_2(t), u(\tau, \alpha_2(t))\big)\, d\tau \\ & - \frac{\partial \alpha_1}{\partial t} \int\limits_{t-\theta(x)}^{t} \Phi\big(t, \tau, x, \alpha_1(t), u(\tau, \alpha_1(t))\big)\, d\tau. \end{aligned} \tag{4.10}$$

Applying the directional derivative to $RL = LR = \mathbb{I}$ along a vector X we have

$$(X \cdot \nabla_u L)R = -L(X \cdot \nabla_u R).$$

So we can transform our equality to obtain

$$\begin{aligned} L_{,t}Rp + Lf_{,x} - L(R\tilde{D})_{,x}p = {} & L_t Rp - L((f - R\tilde{D}p) \cdot \nabla_u R)p + L_I I_{,t} Rp \\ & + Lf_x + L((Rp) \cdot \nabla_u f) + Lf_I I_{,x} \\ & - L(R\tilde{D})_x p - L((Rp) \cdot \nabla_u (R\tilde{D}))p \\ & - L(R\tilde{D})_I I_x p. \end{aligned} \tag{4.11}$$

Using the last equality we can write the prolonged equation (4.8) in the form

$$p_t + \tilde{D}p_x = \varphi_0 + \psi_0 + \varphi_1 p + \psi_1 p + \varphi_2(p,p), \tag{4.12}$$

where

$$\varphi_0 = Lf_x, \tag{4.13}$$

$$\psi_0 = Lf_I I_{,x}, \tag{4.14}$$

$$\varphi_1 p = L((Rp) \cdot \nabla_u f) + \{L_t R - L(R\tilde{D})_x - Lf\}p, \tag{4.15}$$

$$\psi_1 p = \{-L(R\tilde{D})_I I_{,x} + L_I I_{,t} R\}p, \tag{4.16}$$

$$\varphi_2(p,p) = L\{(R\tilde{D}p) \cdot \nabla_u R - (Rp) \cdot \nabla_u (R\tilde{D})\}p. \tag{4.17}$$

The contribution from functional dependence comes through the terms ψ_0 and $\psi_1 p$, so it is linear in p.

Notice that all coefficients of the above polynomial in p as well as $I_{,x}$, $I_{,t}$ are continuous.

In principle we allow $\alpha_1 = -\infty$ or $\alpha_2 = +\infty$ but then it is necessary to assume some additional conditions which yield continuity of the integrals in $I_{,x}$, $I_{,t}$ with respect to (t,x).

Basically, in the case of the operator (4.3), the derivation of the prolonged system is almost the same. However, there will appear a new term with the Fréchet derivative of $I[u]$ at the "point" $u(\cdot)$ acting (linearly) on $u_t(\cdot)$. This is denoted by $I'(u;u_t)$. Since $u_t = f - R\tilde{D}p$, we finally obtain

$$\begin{aligned} p_t + \tilde{D}p_x = {} & Lf_x + L((Rp)\cdot\nabla_u f) + \{L_t R - L(R\tilde{D})_x\}p \\ & - L\{(f - R\tilde{D}p)\cdot\nabla_u R + (Rp)\cdot\nabla_u(R\tilde{D})\}p \\ & + Lf_I I_{,x} + \{-L(R\tilde{D})_I I_{,x} + L_I I_{,t} R\}p \\ & + L_I I'(u; f - R\tilde{D}p)Rp. \end{aligned} \tag{4.18}$$

In this case the total derivatives of I with respect to x and t are given by

$$I_{,x} = \int\limits_{\alpha_1(t)}^{\alpha_2(t)} \tilde{\Phi}_x(t,x,s,u(t,s))\cdot u_s(t,s)\,ds, \tag{4.19}$$

$$\begin{aligned} I_{,t} = {} & \int\limits_{\alpha_1(t)}^{\alpha_2(t)} \tilde{\Phi}_t(t,x,s,u(t,s))\cdot u_s(t,s)\,ds \\ & + \alpha_{2,t}\tilde{\Phi}(t,x,s,u(t,s))\cdot u_s(t,s)\big|_{s=\alpha_2(t)} \\ & - \alpha_{1,t}\tilde{\Phi}(t,x,s,u(t,s))\cdot u_s(t,s)\big|_{s=\alpha_1(t)}, \end{aligned} \tag{4.20}$$

$$\begin{aligned} I'(u;u_t) = {} & \int\limits_{\alpha_1(t)}^{\alpha_2(t)} \Big(\sum_{i=1}^{n} \tilde{\Phi}_{u^i} u_t^i\Big)\cdot u_s(t,s)\,ds + [\tilde{\Phi}\cdot u_t]_{s=\alpha_1(t)}^{s=\alpha_2(t)} \\ & - \int\limits_{\alpha_1(t)}^{\alpha_2(t)} \Big(\tilde{\Phi}_s + \sum_{i=1}^{n} \tilde{\Phi}_{u^i} u_s^i\Big)\cdot u_t\,ds. \end{aligned} \tag{4.21}$$

In spite of the similarity of (4.18) and (4.12) there is an important qualitative difference between them. The coefficients of the polynomial in p on the RHS of (4.18) are expressed by integrals containing u_x, so they depend functionally on p, whereas in the case of (4.12) they do not.

We are interested in the propagation of weak discontinuity in the open domain located between the curves $x = \alpha_1(t)$ and $x = \alpha_2(t)$. The discontinuity occurs on a characteristic which crosses the line $t = \text{const}$ at most at one point ([1]), therefore the functions defined by (4.19)–(4.21) are continuous in t, x.

([1]) This follows from the fact that the characteristic speed ξ is finite.

5. The transport equation. Suppose that $u(t,x)$ is a solution of the system under consideration. Let $x = \chi(t)$ be the characteristic curve corresponding to an eigenvalue ξ of the matrix A evaluated on u. We assume that ξ has constant multiplicity s. Let $R_1, \dots, R_s$ denote the corresponding right eigenvectors. The jump of p_k, $k = 1, \dots, m$, across the characteristic $x = \chi(t)$ will be denoted by

$$\sigma_k = p_k^+ - p_k^-, \qquad k = 1, \dots, m.$$

Along this characteristic line only σ_k for $k = 1, \dots, s$ can be different from zero, while for $k > s$ we have $\sigma_k = 0$. Indeed, by the equation

$$Lu_t + \tilde{D}Lu_x = Lf,$$

we obtain

$$L[u_t] + \tilde{D}L[u_x] = 0. \tag{5.1}$$

By the continuity of $u(t,x)$ on the curve of weak discontinuity $x = \chi(t)$ we have

$$\frac{du}{dt}(t, \chi^+(t)) = \frac{du}{dt}(t, \chi^-(t)), \tag{5.2}$$

or

$$\chi'(t)u_x^+ + u_t^+ = \chi'(t)u_x^- + u_t^-. \tag{5.3}$$

This could also be written as

$$[u_t] = -\chi'(t)[u_x]. \tag{5.4}$$

Substituting (5.4) into (5.1) we get

$$\begin{aligned}
&L(-\chi'(t)[u_x]) + \tilde{D}L[u_x] = 0,\\
&(\tilde{D} - \chi'(t)\mathbb{I}_m)[Lu_x] = 0,\\
&(\tilde{D} - \chi'(t)\mathbb{I}_m)[p] = 0.
\end{aligned}$$

Since ξ is an eigenvalue of the matrix A of multiplicity s, $\chi'(t) = \xi(t, \chi(t))$ and $\tilde{D}$ is composed of two square blocs: a diagonal one, $\mathbb{I}_s$ of dimension s, and an $(m-s)$-dimensional block Q. Obviously, according to our assumptions, ξ is not an eigenvalue of Q. Therefore all σ_k for $k > s$ must be equal to zero. Hence we conclude that only the functions p_k for $k = 1, \dots, s$ can have jump discontinuities on $x = \chi(t)$.

Now we will write a system of ordinary differential equations which governs the evolution of the jump intensity σ_k of p_k, $k = 1, \dots, s$, along the characteristic curve $x = \chi(t)$. Equations of this system are called transport equations. For convenience we denote by σ the m-dimensional column vector whose $m - s$ components $\sigma_{s+1}, \dots, \sigma_m$ are identically zero:

$$\sigma = (\sigma_1, \dots, \sigma_s, 0, \dots, 0). \tag{5.5}$$

To derive an equation for σ we start from (4.12) or from (4.18). In both cases we obtain the equations

$$\frac{d\sigma_k}{dt} = H_{1k}\sigma + H_{2k}(\sigma,\sigma), \qquad k = 1,\dots,s, \tag{5.6}$$

along the characteristic curve $x = \chi(t)$. In the case of (4.12) when the coefficients of the system depend on the operator (4.2) we have

$$\begin{aligned} H_{1k}\sigma = {} & L^k((R\sigma)\cdot\nabla_u f) + \{L^k_t R - L^k(R\tilde{D})_{,x}\}\sigma \\ & - L^k\{((f - R\tilde{D}p^+)\cdot\nabla_u R)\sigma - ((R\tilde{D}\sigma)\cdot\nabla_u R)p^+\} \\ & + L^k\{((Rp^+)\cdot\nabla(R\tilde{D}))\sigma + ((R\sigma)\cdot\nabla_u(R\tilde{D}))p^+\} \\ & + \{-L^k(R\tilde{D})_I I_{,x} + L^k_I I_{,t}R\}\sigma, \end{aligned} \tag{5.7}$$

$$H_{2k}(\sigma,\sigma) = -L^k((R\tilde{D}\sigma)\cdot\nabla_u R)\sigma - L^k((R\sigma)\cdot\nabla_u(R\tilde{D}))\sigma. \tag{5.8}$$

In the case of (4.18), when the coefficients of the system depend on the operator (4.3), we have

$$\begin{aligned} H_{1k}\sigma = {} & L^k((R\sigma)\cdot\nabla_u f) + \{L^k_t R - L^k(R\tilde{D})_x\}\sigma \\ & - L^k\{((f - R\tilde{D}p^+)\cdot(\nabla_u R))\sigma - ((R\tilde{D}\sigma)\cdot(\nabla_u R))p^+\} \\ & + L^k\{((Rp^+)\cdot\nabla_u(R\tilde{D}))\sigma + ((R\sigma)\cdot\nabla_u(R\tilde{D}))p^+\} \\ & + \{L^k_I I'(u; f - R\tilde{D}p)R - L^k(R\tilde{\tilde{D}})_I I_{,x} + L^k_I I_{,t}R\}\sigma, \end{aligned} \tag{5.9}$$

$$H_{2k}(\sigma,\sigma) = -L^k((R\tilde{D}\sigma)\cdot\nabla_u R)\sigma - L^k((R\sigma)\cdot\nabla_u(R\tilde{D}))\sigma. \tag{5.10}$$

Note that in both cases the terms resulting from dependence on $I[u(\cdot)]$ are linear in σ.

6. Examples from physics. After these general considerations, we now come back to the systems (2.2) and (2.3). In those examples the matrix A has real eigenvalues and can be diagonalized, i.e. $A = RDL$, where (in the case of (2.2)) we have

$$D = \begin{pmatrix} 1/\sqrt{\epsilon_0\mu_0} & 0 \\ 0 & -1/\sqrt{\epsilon_0\mu_0} \end{pmatrix}, \qquad L = \begin{pmatrix} \sqrt{\epsilon_0\mu_0} & 1 \\ -\sqrt{\epsilon_0\mu_0} & 1 \end{pmatrix},$$

$$R := L^{-1} = \begin{pmatrix} 1/(2\sqrt{\epsilon_0\mu_0}) & -1/(2\sqrt{\epsilon_0\mu_0}) \\ 1/2 & 1/2 \end{pmatrix}.$$

We denote by $u = (E_2, B_3)$ the unknown function and by f the right hand side of the relevant system ((2.2) or (2.3)):

$$u_t + Au_x = f. \tag{6.1}$$

Equations (2.2) are linear, and D, L, R are constant matrices. We multiply (6.1) on the left by the matrix L of left eigenvectors and differentiate with respect to x to obtain

$$Lu_{tx} + DLu_{xx} = Lf_{,x}.$$

Let $p = Lu_x$. Then we can rewrite the system (2.2) in the form

$$p_t + Dp_x = Lf_{,x}, \tag{6.2}$$

where

$$f_{,x} = \begin{pmatrix} -\int_{-\infty}^{t}(\int_{-\infty}^{\infty} \chi_{,tx}(t,t',x,x')E_2(t',x')\,dx')\,dt' \\ 0 \end{pmatrix}. \tag{6.3}$$

This system has two characteristics: $x = \chi_1(t,x_0) = t/\sqrt{\epsilon_0\mu_0} + x_0$ corresponding to the eigenvalue $1/\sqrt{\epsilon_0\mu_0}$ and $x = \chi_2(t,x_0) = -t/\sqrt{\epsilon_0\mu_0} + x_0$ corresponding to the eigenvalue $-1/\sqrt{\epsilon_0\mu_0}$.

Let σ_k, $k = 1,2$, denote the jump of the function p_k, $k = 1,2$, across the characteristic $x = \chi_k(t,x_0)$:

$$\sigma_k = p_k^+ - p_k^-, \qquad k = 1,2.$$

From (6.2) we obtain transport equations for the jumps:

- along the characteristic $x = \chi_1(t,x_0)$:

$$\sigma_{1,t} + \frac{1}{\sqrt{\epsilon_0\mu_0}}\sigma_{1,x} = \frac{d\sigma_1}{dt} = 0,$$

- along the characteristic $x = \chi_2(t,x_0)$:

$$\sigma_{2,t} - \frac{1}{\sqrt{\epsilon_0\mu_0}}\sigma_{2,x} = \frac{d\sigma_2}{dt} = 0.$$

For system (2.3) we have

$$D = \begin{pmatrix} \frac{1}{\sqrt{\epsilon_0\mu_0(1+I_1)}} & 0 \\ 0 & -\frac{1}{\sqrt{\epsilon_0\mu_0(1+I_1)}} \end{pmatrix}, \qquad L = \begin{pmatrix} \sqrt{\epsilon_0\mu_0(1+I_1)} & 1 \\ -\sqrt{\epsilon_0\mu_0(1+I_1)} & 1 \end{pmatrix},$$

$$R = \begin{pmatrix} \frac{1}{2\sqrt{\epsilon_0\mu_0(1+I_1)}} & -\frac{1}{2\sqrt{\epsilon_0\mu_0(1+I_1)}} \\ 1/2 & 1/2 \end{pmatrix}.$$

In a similar manner, but after much more involved calculations (cf. (4.8)) we arrive at

$$p_t + Dp_x = \begin{pmatrix} \sqrt{\epsilon_0\mu_0(1+I_1)}\,I_{2,x} \\ -\sqrt{\epsilon_0\mu_0(1+I_1)}\,I_{2,x} \end{pmatrix} \tag{6.4}$$

$$+ \begin{pmatrix} \frac{3I_{1,x}}{4\sqrt{\epsilon_0\mu_0}(1+I_1)^{3/2}} + \frac{I_{1,t}}{4(1+I_1)} & \frac{I_{1,x}}{4\sqrt{\epsilon_0\mu_0}(1+I_1)^{3/2}} - \frac{I_{1,t}}{4(1+I_1)} \\ -\frac{I_{1,x}}{4\sqrt{\epsilon_0\mu_0}(1+I_1)^{3/2}} - \frac{I_{1,t}}{4(1+I_1)} & -\frac{3I_{1,x}}{4\sqrt{\epsilon_0\mu_0}(1+I_1)^{3/2}} + \frac{I_{1,t}}{4(1+I_1)} \end{pmatrix} p,$$

where

$$I_{1,x} = \int_{-\infty}^{t}\Big(\int_{-\infty}^{\infty} \chi_{xE_2}(E_2(t,x),t,t',x,x')E_2(t',x')\,dx'\Big)\,dt' + E_{2,x}(t,x)\int_{-\infty}^{t}\Big(\int_{-\infty}^{\infty} \chi_{E_2^2}(E_2(t,x),t,t',x,x')E_2(t',x')\,dx'\Big)\,dt', \tag{6.5}$$

$$I_{1,t} = \int_{-\infty}^{t}\Big(\int_{-\infty}^{\infty} \chi_{tE_2}(E_2(t,x),t,t',x,x')E_2(t',x')\,dx'\Big)\,dt' + E_{2,t}(t,x)\int_{-\infty}^{t}\Big(\int_{-\infty}^{\infty} \chi_{E_2^2}(E_2(t,x),t,t',x,x')E_2(t',x')\,dx'\Big)\,dt' + \int_{-\infty}^{\infty} \chi_{E_2}(E_2(t,x),t,t,x,x')E_2(t,x')\,dx', \tag{6.6}$$

$$I_{2,x} = -\int_{-\infty}^{t}\Big(\int_{-\infty}^{\infty} \chi_{tx}(E_2(t,x),t,t',x,x')E_2(t',x')\,dx'\Big)\,dt' - E_{2,x}(t,x)\int_{-\infty}^{t}\Big(\int_{-\infty}^{\infty} \chi_{tE_2}(E_2(t,x),t,t',x,x')E_2(t',x')\,dx'\Big)\,dt'. \tag{6.7}$$

Finally we obtain the following transport equations:

- on the characteristic defined by $\frac{dx}{dt} = \frac{1}{\sqrt{\epsilon_0\mu_0(1+I_1)}}$ the intensity σ_1 evolves according to

$$\frac{d\sigma_1}{dt} = \sigma_1\bigg\{\frac{p_1^+ + p_2}{4\epsilon_0\mu_0(1+I_1)^2}\int_{-\infty}^{t}\Big(\int_{-\infty}^{\infty} \chi_{E_2^2}E_2(t',x')\,dx'\Big)\,dt' - \frac{1}{2}\int_{-\infty}^{t}\Big(\int_{-\infty}^{\infty} \chi_{tE_2}E_2(t',x')\,dx'\Big)\,dt' + \frac{3I_{1,x}^+}{4\sqrt{\epsilon_0\mu_0}(1+I_1)^{3/2}} + \frac{I_{1,t}^+}{4(1+I_1)}\bigg\} - \sigma_1^2\frac{1}{4\epsilon_0\mu_0(1+I_1)^2}\int_{-\infty}^{t}\Big(\int_{-\infty}^{\infty} \chi_{E_2^2}E_2(t',x')\,dx'\Big)\,dt'; \tag{6.8}$$

- on the characteristic defined by $\frac{dx}{dt} = -\frac{1}{\sqrt{\epsilon_0\mu_0(1+I_1)}}$ we have

$$\frac{d\sigma_2}{dt} = \sigma_2\bigg\{\frac{p_1 + p_2^+}{4\epsilon_0\mu_0(1+I_1)^2}\int_{-\infty}^{t}\Big(\int_{-\infty}^{\infty} \chi_{E_2^2}E_2(t',x')\,dx'\Big)\,dt' \tag{6.9}$$

$$
\begin{aligned}
&-\frac{1}{2}\int\limits_{-\infty}^{t}\Big(\int\limits_{-\infty}^{\infty}\chi_{tE_2}E_2(t',x')\,dx'\Big)\,dt' \\
&-\frac{3I^{+}_{1,x}}{4\sqrt{\epsilon_0\mu_0}(1+I_1)^{3/2}}+\frac{I^{+}_{1,t}}{4(1+I_1)}\Big\} \\
-\,\sigma_2^2&\frac{1}{4\epsilon_0\mu_0(1+I_1)^2}\int\limits_{-\infty}^{t}\Big(\int\limits_{-\infty}^{\infty}\chi_{E_2^2}E_2(t',x')\,dx'\Big)\,dt'.
\end{aligned}
$$

Comparing these two examples one may note that the nonlinear dependence of the polarization field on the current electric field leads to much more complex transport equations. In this case we are dealing with local and nonlocal dependence as well.

The system describing the dynamics of 3-component plasma in the Hall thruster [BMPGD], [ZP] can serve as an example of a system with an operator of the type (4.3):

$$
(6.10)\qquad
\begin{bmatrix} N_{a,t}\\ n_{,t}\\ V_{,t}\\ T_{,t}\end{bmatrix}
+
\begin{bmatrix}
V_a & 0 & 0 & 0\\
0 & V & n & 0\\
0 & \frac{kT}{nm} & V & \frac{k}{m}\\
0 & \frac{2T}{3n}\frac{I}{ne} & \frac{2}{3}T & V-\frac{I}{ne}
\end{bmatrix}
\begin{bmatrix} N_{a,x}\\ n_{,x}\\ V_{,x}\\ T_{,x}\end{bmatrix}
=
\begin{bmatrix}
-\beta N_a n\\
\beta N_a n\\
\nu_{\mathrm{eff}}\big(\frac{I}{en}-V\big)+\beta N_a(V_a-V)\\
-\frac{2}{3k}\beta N_a(\gamma e E_{\mathrm{ion}}-E_{\mathrm{ke}})+\frac{4\nu_m}{3k}E_{\mathrm{ke}}-\frac{2\nu_{\mathrm{ew}}}{3k}E_{\mathrm{ke}}-\frac{4}{3}\nu_{\mathrm{ew}}T
\end{bmatrix}.
$$

The unknown functions $N_a(t,x)$, $n(t,x)$, $V(t,x)$, $T(t,x)$ represent the density of neutral atoms, the density of ions (which is equal to the density of electrons) and the electron temperature. The functional I appearing in the matrix of the system is given by

$$
I=\Big(\int\limits_0^L\frac{\nu_{\mathrm{eff}}}{en}\,dx\Big)^{-1}\cdot\Big[\frac{e}{m}U_0+\int\limits_0^L\Big(\nu_{\mathrm{eff}}V+\frac{1}{n}\frac{\partial}{\partial x}\Big(\frac{kT}{m}n\Big)\Big)\,dx\Big].
$$

Although the last example was the inspiration for this paper, we do not give the transport equations for system (6.10), because of their technical complexity.

7. The Cauchy problem for the initial discontinuity. Now we assume that the system under consideration, i.e. (4.1), is hyperbolic with coefficients functionally dependent on the solution. So we assume that the matrix A has real eigenvalues (not necessarily distinct) which are functionally dependent on the solution and consequently (by hyperbolicity) that A has a complete set of m independent eigenvectors. To avoid complications we assume that the eigenvalues have constant multiplicity. Under these assumptions it follows that, for a given $u(t,x)$, the matrix $A(t,x,u(\cdot))$ can be diagonalised:

$$A = RDL, \quad R := L^{-1}, \quad D = \operatorname{diag}[\xi_1, \ldots, \xi_m].$$

The matrix L is nonsingular and its rows are linearly independent left eigenvectors corresponding to the eigenvalues $\xi_1, \ldots, \xi_m$. The columns of R are linearly independent right eigenvectors of A.

In particular our quasilinear hyperbolic system of m equations with m unknowns functions may have the form

$$u_{,t} + A(t,x,u,I[u])u_{,x} = f(t,x,u,I[u]),$$

where I is an operator from $C^1(\mathbb{R})$ into itself and the matrix A is differentiable with respect to all arguments. Depending on the form of the nonlocal operator $I[u]$, we assume that the function u satisfies appropriate initial conditions, e.g.

$$u(t,x) = u_0(t,x), \quad (t,x) \in [-\theta(x), 0] \times \mathbb{R},$$

for (4.2) or $u(0,x) = u_0(t,x)$ for (4.3). Let us concentrate on the simple case when the contribution from the functional dependence to the transport equations is linear (no resonant cases which are somewhat pathological). The dependence of the coefficients of the system on the local value of $u(t,x)$ can give rise also to quadratic terms in the transport equations. In that case, according to our considerations, the transport equations are ordinary differential equations. If $u(0,x)$ has a weak discontinuity at some point x_0 then the initial jump of $p = Lu_{0,x}$ defines an m-dimensional vector $[p(0,x_0)]$. This vector can be decomposed in the basis formed by the right eigenvectors of the matrix A,

$$[p(0,x_0)] = \sigma_1 R_1 + \cdots + \sigma_m R_m.$$

The coefficients σ_k of this decomposition constitute the initial conditions for the transport equations defined along characteristic curves starting from x_0. Thus the jump discontinuity in the initial values will generate weak discontinuity, in general, on all characteristic curves starting from x_0. Thus in the case of nonlocal functional dependence given by (4.2) or (4.3) the weak discontinuity behaves exactly in the same manner as in the case of usual quasilinear hyperbolic systems.

8. Conclusions. It appears that from the point of view of propagation of weak singularities, systems of quasilinear first order partial differential equations with coefficients functionally dependent on their solution behave in a manner similar to the usual first order quasilinear systems. If there is a jump in derivatives of the solution then it must occur on a characteristic surface. The transport equations for the amplitude of the jump can also be derived. In principle they are ordinary differential equations along appropriate bicharacteristic curves. In the case of systems with two independent variables (t, x), the bicharacteristics are simply the characteristic curves. In this particular case, we derived the transport equations governing the evolution of the jump intensity along the characteristic curves. To avoid various difficulties we assumed that the multiplicity of the eigenvalue does not change in the region under consideration. Although typically transport equations are ordinary differential equations which are quadratic in the amplitude of weak discontinuity, some complications can appear in resonant cases, when discontinuities occurring on various characteristic curves (or on the same curve) interact. This is possible because of the nonlocal dependence of solutions. Typically, the contribution coming from nonlocal dependence is linear in the jump intensity (as in the case of operators (4.2), (4.3)) and the transport equation is an ordinary differential equation. However, as demonstrated in Sec. 3, "resonant" interaction between discontinuities from different characteristics can lead to systems of ordinary equations or even to functional differential equations. In typical cases, however, when functional dependence is sufficiently regular (e.g. (4.2), (4.3)) and when the coefficients are also local functions of u, there is full similarity between the propagation of singularities for the quasilinear system with functional dependence and for the usual quasilinear hyperbolic system. In the case of a hyperbolic system with initial data exhibiting weak discontinuity, the singularity propagates along all possible characteristic curves which start from singularity points of the initial data.

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