TEST RIGID BODIES IN RIEMANNIAN SPACES AND THEIR QUANTIZATION

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Discussed are some classical and quantization problems of rigid bodies of infinitesimal size moving in Riemannian spaces. The rigorous meaning of “infinitesimal size” consists in replacing an extended body by the structured material point with internal degrees of freedom (co-moving orthonormal frame). The special case of constant curvature two-dimensional spaces is discussed. The Sommerfeld polynomial method is used to perform the quantization of such problems.

Keywords: test rigid body, Riemannian manifolds, Sommerfeld polynomial method.

1. Introduction

The general formulation of mechanics of systems with internal degrees of freedom in non-Euclidean spaces was presented in [26, 28]. Here we consider in some details two kinds of two-dimensional problems, namely motion of structured material points on the sphere and pseudosphere (Lobachevsky space). Many interesting dynamical models, including some quite realistic ones, may be effectively investigated in analytical terms. It is not very surprising in the spherical and pseudospherical geometries because of exceptional properties of constant-curvature spaces. Perhaps this may have something to do with that all these manifolds are algebraic ones (of the second degree in the spherical and pseudospherical cases). What concerns a dynamical model, the special stress is laid on something that may be called a “nonharmonic harmonic oscillator”. In a neighbourhood of the equilibrium situation the potential energy behaves in a harmonic way, but anharmonic corrections become relevant for large deflections. Moreover, they are unavoidable because of some topological reasons. Certain alternative models of this kind were suggested in [16, 22–24]. It is important that our oscillator models are rigorously solvable in terms of special functions.

Strictly speaking, we discuss the Schrödinger quantization procedure for a test rigid body. We follow the standard procedure of quantization in Riemannian manifolds, i.e. we use the L^2-Hilbert space of wave functions in the sense of the usual Riemannian measure (volume element). The classical kinetic energy is replaced by the corresponding quantum expression based on the Laplace–Beltrami operator. The separation of variables is performed and then the corresponding one-dimensional Schrödinger equations are solved using the Sommerfeld polynomial method [13, 14]. This quantization of two-dimensional problems may have something to do with the dynamics of graphens, fullerens and nanotubes [8–10, 29]. Our problem is also closely related to the so-called restricted problems of rigid body dynamics [4, 6, 7, 12].

2. Classical description

In generic Riemannian manifold (M, g) there is obviously no concept of isometry, except for the trivial isometry (the identity transformation). So, there is no concept of extended rigid body. Similarly, in general, there are no finite affine transformations (with an exception of the trivial one), and therefore, there exists no concept of extended affine bodies (homogeneously deformable gyroscopes). But we can consider some models of infinitesimal affinely-rigid body and metrically-rigid body.

The treatment consists in replacing extended bodies by structured material points, i.e. by material points with attached linear frames (affine body) or orthonormal frames (gyroscope). These bases describe internal degrees of freedom. This means that degrees of freedom are analytically described by the spatial coordinates x^i $(i = 1, \ldots, n)$ and the components $e^i{}_A$ of the attached co-moving bases e_A $(A = 1, \ldots, n)$. In the gyroscopic case, the quantities $e^i{}_A$ are constrained by the orthogonality condition

$$g_{ij} e^i{}_A e^j{}_B = \delta_{AB}. \tag{1}$$

Obviously, the metric tensor g_{ij} is always taken at the point $x \in M$, where the body is instantaneously placed, and the basis $(\ldots, e_A, \ldots)$ is attached, so $e_A \in T_x M$. Therefore, the quantities $e^i{}_A$ are then functionally constrained by (1), and they are not generalized coordinates. So, they are not very suitable for analytical calculations.

The configuration space Q of infinitesimal rigid body in (M, g) may be identified with $F(M, g)$, i.e. the manifold of all g-orthonormal frames in all tangent spaces of M. Obviously, $F(M, g)$ is an $n(n+1)/2$-dimensional manifold; there is n translational degrees of freedom and $n(n-1)/2$ rotational ones

$$\dim Q = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

To obtain an effective analytical description, one fixes some, usually nonholonomic field of frames E_A, $A = 1, \ldots, n$, usually somehow distinguished by the geometry of (M, g). Then we take the expansion

$$e_A(t) = E_B(x(t)) R^B{}_A(t), \tag{2}$$

where $R(t)$ is a time-dependent orthogonal matrix, i.e.

$$\delta_{CD} R^C{}_A R^D{}_B = \delta_{AB}. \tag{3}$$

The angular velocity ω in the co-moving representation is defined by

$$\frac{De_B}{Dt} := e_A \omega^A{}_B. \tag{4}$$

One can show that

$$\omega^A{}_B = \rho^A{}_B + d^A{}_B, \qquad \rho^A{}_B = (R^{-1})^A{}_C \frac{d}{dt} R^C{}_B,$$
$$d^A{}_B = (R^{-1})^A{}_F \Gamma^F{}_{DC} R^D{}_B R^C{}_G v^G,$$

we omit the simple proof. Here ρ is the "relative" angular velocity of the co-moving frame e with respect to the fixed reference frame E, the object d ("drive") describes angular velocity with which E itself rotates along the trajectory of motion. The symbols

$$v^G = e^G{}_i \frac{dx^i}{dt}$$

denote the co-moving components of the translational velocity,

$$\Gamma^A{}_{BC} = E^A{}_i \Gamma^i{}_{jk} E^j{}_B E^k{}_C - E^A{}_{i,j} E^i{}_B E^j{}_C$$

are anholonomic components of the Levi-Civita affine connection with respect to E_A.

In analogy to [28] we have the following expression for the total kinetic energy,

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{m}{2} g_{ij} v^i v^j + \frac{1}{2} \delta_{AB} \omega^A{}_C \omega^B{}_D J^{CD}. \tag{5}$$

In this formula the descriptors "tr" and "int" refer obviously to the translational and internal parts, m denotes the mass, and $J^{CD} = J^{DC}$ are co-moving components of the tensor of internal inertia.

Here we are interested mainly in the two-dimensional gyroscope, however this procedure is also convenient when dealing with infinitesimal affinely-rigid body [26]. The reason for this is that also in the case of affine motion there is a distinction between the compact $n(n-1)/2$-dimensional subgroup of rotations and the $n(n+1)/2$-dimensional quotient manifold. Therefore, even in this case it may be convenient to distinguish between analytical formulae for rotations and deformations.

The formulae above, first of all (5), are very convenient, almost indispensible in the technical procedures of solving equations of motion. However, their disadvantage is that some geometric aspects are rather hidden. Let us repeat some of them. In a more general case of affine motion, i.e. one without constraints (1), the expression for the kinetic energy has the form

$$T = T_{\mathrm{tr}} + T_{\mathrm{int}} = \frac{m}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{1}{2} g_{ij} \frac{De^i{}_A}{Dt} \frac{De^j{}_B}{Dt} J^{AB}. \tag{6}$$

Obviously, it remains also true when (1) is imposed. For Lagrangians of the potential form

$$L = T - V(x, e)$$

the resulting equations of motion read

$$m\frac{Dv^a}{Dt} = \frac{1}{2}S^k{}_l R^l{}_k{}^a{}_j v^j + F^a, \qquad e^a{}_K \frac{D^2 e^b{}_L}{Dt^2} J^{KL} = N^{ab}. \tag{7}$$

The meaning of symbols is as follows:

$$v^a = \frac{dx^a}{dt}, \qquad S^{kl} = S^k{}_m g^{ml} = K^{kl} - K^{lk}, \qquad K^{ab} = e^a{}_A \frac{De^b{}_B}{Dt} J^{AB},$$
$$F^a = g^{ab}F_b = -g^{ab}\left(\frac{\partial V}{\partial x^b} - \Gamma^i{}_{jb} e^j{}_B \frac{\partial V}{\partial e^i{}_B}\right), \tag{8}$$
$$N^{ab} = N^a{}_c g^{cb} = -g^{bc} e^a{}_K \frac{\partial V}{\partial e^c{}_K},$$

and $R^i{}_{jkl}$ is the curvature tensor of (M, g).

Therefore, v^a are components of the translational velocity, S^{kl} are components of spin (intrinsic angular momentum), F^a are coordinates of the translational force, and N^{ab} are components of the affine torque. It is important that the covariant components of F in general differ from $-\partial V/\partial x^b$; moreover, the latter ones are not covector components in M. It is only the total F, the covariant exterior differential of V that is a good M-covector. When the metrical constraints are imposed, i.e. when we deal with the metrically-rigid body, (7) becomes

$$\begin{gathered} m\frac{Dv^a}{Dt} = \frac{1}{2}S^k{}_l R^l{}_k{}^a{}_j v^j + F^a, \\ \frac{DS^{ab}}{Dt} = e^a{}_K \frac{D^2 e^b{}_L}{Dt^2} J^{KL} - e^b{}_K \frac{D^2 e^a{}_L}{Dt^2} J^{KL} = \mathcal{N}^{ab} = N^{ab} - N^{ba}, \end{gathered} \tag{9}$$

obviously, with algebraically substituted (1). This is a nice balance of linear momentum and spin, geometrically suggestive, but computationally not so effective as equations derived from (5). Nevertheless, (9) presents a nice description of the mutual interaction between the translational and the attitude motion [5].

3. Special two-dimensional cases

Now we shall consider some special two-dimensional cases. Therefore, for the infinitesimal rigid body (infinitesimal gyroscope) we are dealing with three degrees of freedom: two translational and one internal, rotational. The resulting models are interesting in themselves from the point of view of pure analytical mechanics, in particular, some integrability and hyperintegrability (degeneracy) problems may be effectively studied [17, 28]. Obviously, the explicit analytical results exist only in Riemann manifolds (M, g) with some peculiar structure, first of all (but not only) in constant-curvature spaces. Some practical applications of classical two-dimensional models also seem to be possible, e.g. in geophysical problems, in mechanics of structured micropolar and micromorphic shells, etc. What concerns geophysics, we mean, e.g. motions of continental plates. The motion of pollutions like oil spots on the oceanic surface is another suggestive example.

Let us now quote some instructive special examples, namely a two-dimensional rigid body moving in constant-curvature spaces like the spherical space $\mathrm{S}^2(0, R)$ and pseudo-spherical Lobachevsky space $\mathrm{H}^{2,2,+}(0, R)$ [11]. Certain aspects of these models were discussed in [28, 30], thus we present here only general ideas. The corresponding metric elements are given respectively by

$$ds^2 = dr^2 + R^2 \sin^2 \frac{r}{R} d\varphi^2, \qquad ds^2 = dr^2 + R^2 \sinh^2 \frac{r}{R} d\varphi^2, \tag{10}$$

with the proviso that in the spherical case all situations with $r = \pi R$ and arbitrary values of φ correspond to the same point (the "southern" pole, or if $r = 0$—the "northern" pole). The range of r is respectively $[0, \pi R]$, and $[0, \infty]$.

The most convenient choice of the reference frame is

$$E_r = \frac{\partial}{\partial r}; \qquad E_\varphi = \frac{1}{R \sin \frac{r}{R}} \frac{\partial}{\partial \varphi}, \qquad E_\varphi = \frac{1}{R \sinh \frac{r}{R}} \frac{\partial}{\partial \varphi},$$

respectively, in the spherical and pseudospherical case.

In two dimensions the angular velocity matrix has only one essential component, namely

$$\omega^1{}_2 = -\omega_2{}^1 = \omega, \qquad \rho^1{}_2 = -\rho_2{}^1 = \rho, \qquad d^1{}_2 = -d_2{}^1 = d,$$

the diagonal entries obviously vanish.

After some calculations the above formulae give:

(i) sphere:

$$\rho = \frac{d\psi}{dt}, \qquad d = \cos \frac{r}{R} \frac{d\varphi}{dt}, \qquad \omega = \frac{d\psi}{dt} + \cos \frac{r}{R} \frac{d\varphi}{dt}, \tag{11}$$

(ii) pseudosphere:

$$\rho = \frac{d\psi}{dt}, \qquad d = \cosh \frac{r}{R} \frac{d\varphi}{dt}, \qquad \omega = \frac{d\psi}{dt} + \cosh \frac{r}{R} \frac{d\varphi}{dt}. \tag{12}$$

Therefore, using formula (5), we obtain for the kinetic energy $T = T_{\mathrm{tr}} + T_{\mathrm{int}}$ the expression below. Depending on the considered manifold, it has the following form:

(i) sphere:

$$T = \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + R^2 \sin^2 \frac{r}{R}\left(\frac{d\varphi}{dt}\right)^2\right) + \frac{I}{2}\left(\frac{d\psi}{dt} + \cos \frac{r}{R} \frac{d\varphi}{dt}\right)^2, \tag{13}$$

where I is the moment of inertia. In the absence of deformations, the internal inertia is controlled only by this single scalar. This is the peculiarity of the "two-dimensional world".

Even for the purely translational motion some interesting questions arise, e.g. what are spherically symmetric potentials $V(r)$ for which all orbits are closed?

Obviously we mean the problems based on Lagrangians of the type

$$L_{\mathrm{tr}} = T_{\mathrm{tr}} - V(r).$$

This is a counterpart of the famous Bertrand problem in $\mathbb{R}^2$. And it may be shown that the answer is similar [23, 25], i.e. the possible potentials are as follows:

(a) oscillatory potentials

$$V(r) = \frac{\gamma}{2} R^2 \tan^2 \frac{r}{R}, \tag{14}$$

(b) Kepler–Coulomb potentials

$$V(r) = -\frac{\alpha}{R} \cot \frac{r}{R}. \tag{15}$$

Obviously, with the spherical topology also the geodetic problem belongs here:

(c) $V(r) = 0$, i.e. (in a sense) the special case of (a) or (b) when $\gamma = 0$, $\alpha = 0$. There is an obvious correspondence with the flat-space Bertrand problem; it is suggested by the very asymptotics for $r \approx 0$, i.e.

$$V(r) \approx \frac{\gamma}{2} r^2, \qquad V(r) \approx -\frac{\alpha}{r}.$$

Obviously, this is a rough argument, but it may be shown [23, 25] that there exists a rigorous isomorphism based on the projective geometry.

The mentioned Bertrand models lead to completely integrable and maximally degenerate (hyperintegrable) problems. But even for the simplest, i.e. geodetic, models with the internal degrees of freedom the situation changes drastically. There exist interesting and practically applicable integrable models, but as a rule interaction with internal degrees of freedom reduces or completely removes degeneracy.

For certain reasons it will be convenient to rewrite the formula (13) in terms of the new variable $\vartheta = r/R$, the modified "geographic latitude", i.e.

$$T = \frac{mR^2}{2} \left(\left(\frac{d\vartheta}{dt} \right)^2 + \sin^2 \vartheta \left(\frac{d\varphi}{dt} \right)^2 \right) + \frac{I}{2} \left(\frac{d\psi}{dt} + \cos \vartheta \frac{d\varphi}{dt} \right)^2. \tag{16}$$

It is seen that if formally $(\varphi, \vartheta, \psi)$ are interpreted as Euler angles (respectively the precession, nutation and rotation), the above expression is formally identical with the kinetic energy of the three-dimensional symmetric rigid body (without translations) with the main moments of inertia given by

$$I_1 = I_2 = mR^2, \qquad I_3 = I.$$

If $I = mR^2$ one obtains the expression for the spherical top in three dimensions.

There is nothing surprising in the mentioned isomorphism because the quotient manifold $\mathrm{SO}(3, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$ may be in a natural way identified with $\mathrm{S}^2(0, 1)$ (or with any $\mathrm{S}^2(0, R)$). Projecting the motion of the three-dimensional symmetric top onto the quotient sphere-manifold we obtain a two-dimensional translational motion; the one-dimensional subgroup of rotations about the z-axis refers to the internal motion of the two-dimensional rotator.

The projection procedure is exactly compatible with the mentioned correspondence between Euler angles in $SO(3, \mathbb{R})$ and our generalized coordinates $(\varphi, \vartheta = r/R, \psi)$ of the infinitesimal rotator in $S^2(0, R)$.

Let $U(\varphi, \vartheta, \psi) \in SO(3, \mathbb{R})$ be just the element labelled by the Euler angles $(\varphi, \vartheta, \psi)$, thus

$$U(\varphi, \vartheta, \psi) = U_z(\varphi) U_x(\vartheta) U_z(\psi), \tag{17}$$

where U_z, U_x are rotations respectively around the z- and x-axes; the angles of rotations are indicated as the arguments and

$$U(\varphi, \vartheta, \psi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{18}$$

Calculating the "co-moving angular velocity"

$$\widehat{\varkappa} = U^{-1} \frac{dU}{dt} \tag{19}$$

of this fictitious three-dimensional top one obtains that

$$\widehat{\varkappa} = \widehat{\varkappa}_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \widehat{\varkappa}_2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \widehat{\varkappa}_3 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{20}$$

where

$$\widehat{\varkappa}_1 = \sin\vartheta \sin\psi \frac{d\varphi}{dt} + \cos\psi \frac{d\vartheta}{dt}, \tag{21}$$

$$\widehat{\varkappa}_2 = \sin\vartheta \cos\psi \frac{d\varphi}{dt} - \sin\psi \frac{d\vartheta}{dt}, \tag{22}$$

$$\widehat{\varkappa}_3 = \cos\vartheta \frac{d\varphi}{dt} + \frac{d\psi}{dt}. \tag{23}$$

In (23) we easily recognize ω in (11), i.e. the expression for the one-component angular velocity of the two-dimensional rotator. Calculating formally the kinetic energy of the three-dimensional symmetric $SO(3, \mathbb{R})$-top, i.e.

$$T = \frac{K}{2} (\widehat{\varkappa}_1)^2 + \frac{K}{2} (\widehat{\varkappa}_2)^2 + \frac{I}{2} (\widehat{\varkappa}_3)^2, \tag{24}$$

and substituting $K = mR^2$, $\vartheta = r/R$, we obtain exactly (13), i.e. (16).

As usual in analytical mechanics, the kinetic energy (13), (16) may be identified with some Riemannian structure on the configuration space. Let us write down our kinetic energy in the following form with the explicitly separated mass factor,

$$T = \frac{m}{2} G_{ij}(q) \frac{dq^i}{dt} \frac{dq^j}{dt}. \tag{25}$$

Just as above, our generalized coordinates q^i, $i = 1, 2, 3$, are the variables (r, φ, ψ) written just in this direction. After some calculations we obtain that

$$[G_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 \sin^2 \frac{r}{R} + \frac{I}{m} \cos^2 \frac{r}{R} & \frac{I}{m} \cos \frac{r}{R} \\ 0 & \frac{I}{m} \cos \frac{r}{R} & \frac{I}{m} \end{bmatrix}. \tag{26}$$

In the special case $I = mR^2$ one obtains that G simplifies to $\breve{G}$, where

$$\left[\breve{G}_{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & R^2 \cos \frac{r}{R} \\ 0 & R^2 \cos \frac{r}{R} & R^2 \end{bmatrix}. \tag{27}$$

In analogy to (13), (16) we obtain that

(ii) pseudosphere:

$$T = \frac{m}{2} \left(\left(\frac{dr}{dt} \right)^2 + R^2 \sinh^2 \frac{r}{R} \left(\frac{d\varphi}{dt} \right)^2 \right) + \frac{I}{2} \left(\frac{d\psi}{dt} + \cosh \frac{r}{R} \frac{d\varphi}{dt} \right)^2, \tag{28}$$

i.e.

$$T = \frac{mR^2}{2} \left(\left(\frac{d\vartheta}{dt} \right)^2 + \sinh^2 \vartheta \left(\frac{d\varphi}{dt} \right)^2 \right) + \frac{I}{2} \left(\frac{d\psi}{dt} + \cosh \vartheta \frac{d\varphi}{dt} \right)^2. \tag{29}$$

The kinetic energy is based on the metric tensor G_{ij} with the components

$$[G_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 \sinh^2 \frac{r}{R} + \frac{I}{m} \cosh^2 \frac{r}{R} & \frac{I}{m} \cosh \frac{r}{R} \\ 0 & \frac{I}{m} \cosh \frac{r}{R} & \frac{I}{m} \end{bmatrix}. \tag{30}$$

It is seen that the spherically very special case $I = mR^2$ here, in the pseudo-spherical case also leads to some simplification of $\left[G_{ij}\right]$, but not in so striking way as previously. This fact has deep geometric reasons which will be explained in the sequel.

In the spherical space the essential point is the natural identification between the quotient manifold $\mathrm{SO}(3, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$ and the spheres $\mathrm{S}^2(0, R)$, $\mathrm{S}^2(0, 1)$. And this has to do with the formal identification between two-dimensional rigid body moving over the spherical surface and the three-dimensional symmetrical top without translational degrees of freedom. The special case $I = mR^2$ corresponds to the spherical top.

In general, the kinetic energy is then invariant under $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(2,\mathbb{R})$. In the three-dimensional top analogy, $\mathrm{SO}(3,\mathbb{R})$ is acting as left regular translations and $\mathrm{SO}(2,\mathbb{R})$ as right regular translations corresponding to the group of rotations around the body-fixed z-axis. If $I = mR^2$ we have the full invariance under $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})$.

In the hyperbolic pseudospherical geometry the problem is isomorphic with the three-dimensional Lorentzian (Minkowskian) top on $\mathbb{R}^3$. The rotation group $\mathrm{SO}(3,\mathbb{R})$ is replaced by the three-dimensional Lorentz group $\mathrm{SO}(1,2)$. And still an important role is played by $\mathrm{SO}(2,\mathbb{R})$ interpreted again as the group of usual rotations in Euclidean space of (x, y)-variables (thus, not affecting the z-direction). The above kinetic energy (28), (29) is invariant under $\mathrm{SO}(1,2) \times \mathrm{SO}(2,\mathbb{R})$. But it is never invariant under $\mathrm{SO}(1,2) \times \mathrm{SO}(1,2)$, i.e. under the left and right Lorentz regular translations in the $\mathrm{SO}(1,2)$-sense. The spherical special case $I = mR^2$ does not help here. Indeed, the underlying metric G (and the kinetic energy itself) is positively definite. But the doubly-invariant ($\mathrm{SO}(1,2) \times \mathrm{SO}(1,2)$-invariant) metric on $\mathrm{SO}(1,2)$, i.e. its Killing metric is not positively definite. Instead it has the normal-hyperbolic signature $(++-)$. The reason is that it is semi-simple (even simple) noncompact group. This brings about the question about nonpositive kinetic energies (metric tensors) on our configuration space. As the negative contribution to the Killing metric tensor on $\mathrm{SO}(1,2)$ comes from its compact subgroup $\mathrm{SO}(2,\mathbb{R})$ of (x, y)-rotations, i.e. from the gyroscopic degree of freedom in the language of $\mathrm{H}^{2,2,+}(0, R)$, there is a natural suggestion to invert the sign of the gyroscopic contribution to (13), (16), i.e. to make it negative. One is naturally reluctant to indefinite kinetic energies but there are examples when they are just convenient and very useful as tools for describing some kinds of physical interactions [27], just encoding them even without any use of potentials.

So, we can try to use, or at least mathematically analyze, the "Lorentz-type kinetic energies" T_{L} of the form

$$\begin{aligned} T_{\mathrm{L}} &= \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + R^2 \sinh^2 \frac{r}{R}\left(\frac{d\varphi}{dt}\right)^2\right) - \frac{I}{2}\left(\frac{d\psi}{dt} + \cosh \frac{r}{R}\frac{d\varphi}{dt}\right)^2 \\ &= \frac{mR^2}{2}\left(\left(\frac{d\vartheta}{dt}\right)^2 + \sinh^2 \vartheta \left(\frac{d\varphi}{dt}\right)^2\right) - \frac{I}{2}\left(\frac{d\psi}{dt} + \cosh \vartheta \frac{d\varphi}{dt}\right)^2. \end{aligned} \tag{31}$$

Thus, it is so as if the extra rotation diminished effectively the kinetic energy of translational motion. If we write as usual that

$$T_{\mathrm{L}} = \frac{m}{2}{}_{\mathrm{L}}G_{ij}(q)\frac{dq^i}{dt}\frac{dq^j}{dt},$$

then, with the same convention concerning the ordering of coordinates (r, φ, ψ), we have that

$$
\left[{}_{\mathrm{L}}G_{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2\sinh^2\dfrac{r}{R} - \dfrac{I}{m}\cosh^2\dfrac{r}{R} & -\dfrac{I}{m}\cosh\dfrac{r}{R} \\ 0 & -\dfrac{I}{m}\cosh\dfrac{r}{R} & -\dfrac{I}{m} \end{bmatrix} \tag{32}
$$

(compare this with (30)).

And now, obviously, the remarkable simplification occurs in the very special case $I = mR^2$ just as in the spherical symmetry. This has to do "as usual" with extending of the symmetry group from $\mathrm{SO}(1,2) \times \mathrm{SO}(2,\mathbb{R})$ to $\mathrm{SO}(1,2) \times \mathrm{SO}(1,2)$ (two additional parameters of symmetry). Namely, ${}_{\mathrm{L}}G$ becomes then ${}_{\mathrm{L}}\breve{G}$, i.e.

$$
\left[{}_{\mathrm{L}}\breve{G}_{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -R^2 & -R^2\cosh\dfrac{r}{R} \\ 0 & -R^2\cosh\dfrac{r}{R} & -R^2 \end{bmatrix} \tag{33}
$$

(compare this with (27) and notice the characteristic sign differences).

Obviously, if we use the above isomorphism between the two-dimensional top sliding over the Lobachevsky plane with the three-dimensional Lorentz top without translational motion in $\mathbb{R}^3$, then it is clear that ${}_{\mathrm{L}}G$ is, up to normalization, identical with the Killing metric tensor of $\mathrm{SO}(1,2)$. Let us quote some formulae and concepts analogous to three-dimensional angular velocities, i.e. to (19), (20). Then the kinetic energy will be expressed like in (24).

First of all we parameterize $\mathrm{SO}(1,2)$ with the help of what we call the "pseudo-Euler angles". So, let us write that

$$
\mathrm{SO}(1,2) \ni L(\varphi,\vartheta,\psi) = U_z(\varphi)L_x(\vartheta)U_z(\psi),
$$

where the meaning of U_z is like in (17) and L_x denotes some Lorentz transformation in $\mathbb{R}^3$, namely the "boost" along the x-axis, i.e.

$$
L_x(\vartheta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\vartheta & \sinh\vartheta \\ 0 & \sinh\vartheta & \cosh\vartheta \end{bmatrix}.
$$

During the motion all these quantities are functions of time and we can calculate the corresponding Lie-algebraic element

$$
\widehat{\lambda} = L^{-1}\frac{dL}{dt},
$$

i.e. the co-moving pseudo-angular velocity. After some calculations we obtain formulae analogous to (20)–(23), namely,

$$
\widehat{\lambda} = \widehat{\lambda}_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \widehat{\lambda}_2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \widehat{\lambda}_3 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

where

$$\widehat{\lambda}_1 = \sinh\vartheta \sin\psi \frac{d\varphi}{dt} + \cos\psi \frac{d\vartheta}{dt},$$
$$\widehat{\lambda}_2 = -\sinh\vartheta \cos\psi \frac{d\varphi}{dt} + \sin\psi \frac{d\vartheta}{dt},$$
$$\widehat{\lambda}_3 = \cosh\vartheta \frac{d\varphi}{dt} + \frac{d\psi}{dt}.$$

The similarities and differences in comparison with the corresponding spherical formulae are easily seen.

And now we can write two formulae analogous to (24), i.e.

$$T = \frac{K}{2}(\widehat{\lambda}_1)^2 + \frac{K}{2}(\widehat{\lambda}_2)^2 + \frac{I}{2}(\widehat{\lambda}_3)^2, \tag{34}$$
$$T = \frac{K}{2}(\widehat{\lambda}_1)^2 + \frac{K}{2}(\widehat{\lambda}_2)^2 - \frac{I}{2}(\widehat{\lambda}_3)^2, \tag{35}$$

where $K > 0$ and $I > 0$. This is the symmetric $\mathrm{SO}(1,2)$-top in $\mathbb{R}^3$. The indefinite expression (35) is structurally suited to the normal-hyperbolic signature of $\mathrm{SO}(1,2)$. When $K = I$, then it becomes the spherical Lorentz top in $\mathbb{R}^3$ in the indefinite version based on the Killing metric.

It is easy to see that both expressions (34) and (35) are invariant under $\mathrm{SO}(1,2)\times \mathrm{SO}(2,\mathbb{R})$, where $\mathrm{SO}(1,2)$ and $\mathrm{SO}(2,\mathbb{R})$ act on $\mathrm{SO}(1,2)$ through respectively the left and right regular translations. The form (35) with $K = I$ is invariant under all regular translations (both left and right), i.e. under $\mathrm{SO}(1,2)\times \mathrm{SO}(1,2)$. And specifying $K = mR^2$ in (34) and (35), we obtain respectively (28) and (31).

Again there are two Bertrand-type potentials, [23, 25], i.e.

(a) the "harmonic oscillator"-type potential:

$$V(r) = \frac{\gamma}{2}R^2 \tan^2\frac{r}{R}, \qquad \gamma > 0, \tag{36}$$

(b) the "attractive Kepler-Coulomb"-type one,

$$V(r) = -\frac{\alpha}{R}\cot\frac{r}{R}, \qquad \alpha > 0. \tag{37}$$

With these and only these potentials all bounded orbits are closed. And now the term "bounded" is essential because the "physical space" is now not compact. Indeed, there exist unbounded motions corresponding to energy values exceeding some thresholds. It is interesting that unlike in the spherical world, in the Lobachevsky space the isotropic degenerate oscillator has an open subset of unbounded trajectories because the potential (36) has a finite upper bound, i.e.

$$\operatorname{Sup} V = \frac{\gamma}{2}R^2.$$

For energy values above this threshold all trajectories are unbounded, the motion is infinite. Below this threshold all trajectories are not only bounded but also periodic.

The existence of threshold in (37) is not surprising, it is like in the usual Kepler problem in $\mathbb{R}^2$. But the threshold for the isotropic degenerate oscillator is a very interesting feature of the Lobachevsky "world".

4. The quantized problems

We formulate now the quantized version of our models. Before doing this, let us remind briefly the general ideas of quantization in Riemannian configuration spaces [15]. Considered is a classical geodetic system in a Riemannian manifold (Q, G), where Q denotes the configuration space, and G is the "metric" tensor field on Q underlying the kinetic energy form. In terms of generalized coordinates we have

$$T = \frac{1}{2} G_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt}.$$

As usual, the metric tensor G gives rise to the natural measure μ on Q,

$$d\mu(q) = \sqrt{\left|\det[G_{ij}]\right|} dq^1 \cdots dq^f,$$

where f denotes the number of degrees of freedom, i.e. $f = \dim Q$. For simplicity the square-root expression will be denoted by $\sqrt{|G|}$. The mathematical framework of Schrödinger quantization is based on $L^2(Q, \mu)$, i.e. the Hilbert space of complex-valued wave functions on Q, which are square-integrable in the μ-sense. Their scalar product is given by the usual formula

$$\langle \Psi_1 | \Psi_2 \rangle = \int \overline{\Psi_1}(q) \Psi_2(q) d\mu(q).$$

The classical kinetic energy expression is replaced by the operator

$$\hat{T} = -\frac{\hbar^2}{2} \Delta,$$

where $\hbar$ denotes the ("crossed") Planck constant, and Δ is the Laplace–Beltrami operator corresponding to G,

$$\Delta = \frac{1}{\sqrt{|G|}} \sum_{i,j} \partial_i \sqrt{|G|} G^{ij} \partial_j = G^{ij} \nabla_i \nabla_j,$$

where ∇ denotes the Levi-Civita covariant differentiation in the G-sense.

If the problem is not geodetic and some potential $V(q)$ is admitted, the corresponding Hamilton (energy) operator is given by

$$\hat{H} = \hat{T} + \hat{V},$$

where the operator $\hat{V}$ acts on wave functions simply multiplying them by V,

$$(\hat{V} \Psi)(q) = V(q) \Psi(q).$$

This is the reason why very often one does not distinguish graphically between $\hat{V}$ and V.

REMARK. There is a delicate problem concerning quantization which cannot be discussed here, and, fortunately, does not interfere directly with the main subjects of our analysis. Strictly speaking, wave functions are not scalars but complex densities of the weight $1/2$ so that the bilinear expression $\overline{\Psi}\Psi$ is a real scalar density of weight one, thus, a proper object for describing probability distributions [15]. But in all realistic models, and the our one is not an exception, the configuration space is endowed with some Riemannian structure. And this enables one to factorize scalar (and tensor) densities into products of scalars (tensors) and some standard densities built of the metric tensor. Therefore, the wave function may be finally identified with the complex scalar field (multicomponent one when there are internal degrees of freedom).

From now on we concentrate ourselves on the special case of a test rigid body moving on the two-dimensional sphere and pseudosphere. The previous notation is used to denote the variables. The Hamiltonian operator is given by the expression

$$\hat{H} = \hat{T} + V(r) = -\frac{\hbar^2}{2m}\Delta + V(r). \tag{38}$$

The variables φ, ψ have the cyclic character in T [25]. This focuses our attention on dynamical models where the potential energy is also independent of the angles φ, ψ.

After some calculations we obtain for the Laplace–Beltrami operator the expression below. Depending on the considered manifold, it has the following form:

(i) sphere:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{R}\cot\frac{r}{R}\frac{\partial}{\partial r} - \frac{2\cos(r/R)}{R^2\sin^2(r/R)}\frac{\partial^2}{\partial\varphi\partial\psi} + \frac{mR^2\sin^2(r/R) + I\cos^2(r/R)}{IR^2\sin^2(r/R)}\frac{\partial^2}{\partial\psi^2} + \frac{1}{R^2\sin^2(r/R)}\frac{\partial^2}{\partial\varphi^2}. \tag{39}$$

In the special case, when $I = mR^2$, we obtain

$$\breve{\Delta} = \frac{\partial^2}{\partial r^2} + \frac{1}{R}\cot\frac{r}{R}\frac{\partial}{\partial r} - \frac{2\cos(r/R)}{R^2\sin^2(r/R)}\frac{\partial^2}{\partial\varphi\partial\psi} + \frac{1}{R^2\sin^2(r/R)}\frac{\partial^2}{\partial\psi^2} + \frac{1}{R^2\sin^2(r/R)}\frac{\partial^2}{\partial\varphi^2}, \tag{40}$$

as was mentioned above, the problem is isomorphic with the three-dimensional spherical top (without translations). Similarly

(ii) pseudosphere:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{R}\coth\frac{r}{R}\frac{\partial}{\partial r} - \frac{2\cosh(r/R)}{R^2\sinh^2(r/R)}\frac{\partial^2}{\partial\varphi\partial\psi} + \frac{mR^2\sinh^2(r/R) + I\cosh^2(r/R)}{IR^2\sinh^2(r/R)}\frac{\partial^2}{\partial\psi^2} + \frac{1}{R^2\sinh^2(r/R)}\frac{\partial^2}{\partial\varphi^2}. \tag{41}$$

If we assume the rotational kinetic energy to contribute with the negative sign, then the expression (41) becomes

$$
\begin{aligned}
{}_{\mathrm{L}}\Delta = {} & \frac{\partial^2}{\partial r^2} + \frac{1}{R}\coth\frac{r}{R}\frac{\partial}{\partial r} - \frac{2\cosh(r/R)}{R^2\sinh^2(r/R)}\frac{\partial^2}{\partial\varphi\partial\psi} \\
& + \frac{-mR^2\sinh^2(r/R) + I\cosh^2(r/R)}{IR^2\sinh^2(r/R)}\frac{\partial^2}{\partial\psi^2} + \frac{1}{R^2\sinh^2(r/R)}\frac{\partial^2}{\partial\varphi^2}. \qquad (42)
\end{aligned}
$$

In particular, in the very special case $I = mR^2$, these operators have the following form

$$
\begin{aligned}
\breve{\Delta} = {} & \frac{\partial^2}{\partial r^2} + \frac{1}{R}\coth\frac{r}{R}\frac{\partial}{\partial r} - \frac{2\cosh(r/R)}{R^2\sinh^2(r/R)}\frac{\partial^2}{\partial\varphi\partial\psi} \\
& + \frac{1}{R^2\sinh^2(r/R)}\frac{\partial^2}{\partial\psi^2} + \frac{1}{R^2\sinh^2(r/R)}\frac{\partial^2}{\partial\varphi^2}. \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
{}_{\mathrm{L}}\breve{\Delta} = {} & \frac{\partial^2}{\partial r^2} + \frac{1}{R}\coth\frac{r}{R}\frac{\partial}{\partial r} - \frac{2\cosh(r/R)}{R^2\sinh^2(r/R)}\frac{\partial^2}{\partial\varphi\partial\psi} \\
& + \frac{1}{R^2\sinh^2(r/R)}\frac{\partial^2}{\partial\psi^2} + \frac{1}{R^2\sinh^2(r/R)}\frac{\partial^2}{\partial\varphi^2}. \qquad (44)
\end{aligned}
$$

Let us observe that Δ, and therefore also the kinetic energy operator $\hat{T}$ for (39) are left-invariant under the action of the group $\mathrm{SO}(3,\mathbb{R})$, while (42) and the corresponding $\hat{T}$ are left-invariant under $\mathrm{SO}(1,2)$. On the right they are invariant only under $\mathrm{SO}(2,\mathbb{R})$, the subgroup of rotations about the "material" z-axis. In the special case $I = mR^2$, they are invariant also under the total right actions of $\mathrm{SO}(3,\mathbb{R})$, $\mathrm{SO}(1,2)$. Obviously, even in the case $I = mR^2$, (41) fails to be right-Lorentz-invariant, but of course (44) is right-invariant. But in applications, when some potential V is admitted, the invariance is lost. In any case, it is seen that the quantum operators of the kinetic energy have invariance properties quite analogous to the corresponding classical ones. Obviously, the symmetry operations are meant in the sense of the argument-wise action of unitary operators representing transformations of wave functions. Infinitesimal generators of the corresponding one-parameter subgroups simply do commute with the kinetic energy operator.

A basis of solutions of the stationary Schrödinger equation $\hat{H}\Psi = E\Psi$ has the form

$$\Psi(r,\varphi,\psi) = f_r(r)f_\varphi(\varphi)f_\psi(\psi). \qquad (45)$$

It is convenient to use the variable $\vartheta = r/R$ for our calculations, then we put

$$\Psi(\vartheta,\varphi,\psi) = f_\vartheta(\vartheta)e^{in\varphi}e^{il\psi}, \qquad (46)$$

where n, l are integers.

The true quantum dynamics is contained in the factor f_ϑ. The separation of variables and the procedure of Sommerfeld polynomials guarantee that our wave functions are proper global solutions. However, some comments are necessary here. Namely, in a sense one can admit some additional "solutions". The point is that the configuration space, in the case e.g. of the motion on sphere, being isomorphic with the rotation group $\mathrm{SO}(3,\mathbb{R})$ is doubly connected. Its universal covering is $\mathrm{SU}(2)$, obviously the homomorphism ratio is $2:1$. Therefore, according to certain ideas of Pauli, in the form developed in [1–3], one can try to use the covering manifold as a modified configuration space. The same holds in principle in the hyperbolic case. This "covering space-philosophy" seems to suggest us to admit the numbers n, l in our product formula for Ψ to be simultaneously integers or simultaneously half-integers. Let us mention that there are systematically returning ideas of spin as the "internal angular momentum" of a quantized rigid (or deformable) body [1–3].

We followed this line in [19, 24]. Nevertheless, we were dealing there with the three-dimensional rigid body without translational motion. One can quite reasonably expect the corresponding "half-integer" solutions to be then realistic. It is not clear if this is the case also for the body moving on the sphere $\mathrm{S}^2(0,R)$ with one internal/rotational degree of freedom, although the configuration space is then the same, i.e. $\mathrm{SO}(3,\mathbb{R})$.

Hence, the stationary Schrödinger equation with an arbitrary potential $V(\vartheta)$ leads after the standard separation procedure to the following one-dimensional "radial" eigenequations:

(i) sphere:

$$\frac{d^2 f_\vartheta(\vartheta)}{d\vartheta^2} + \cot\vartheta \frac{d f_\vartheta(\vartheta)}{d\vartheta} - \left(\frac{((m/I)R^2\sin^2\vartheta + \cos^2\vartheta)n^2 + l^2 - 2nl\cos\vartheta}{\sin^2\vartheta} - \frac{2mR^2}{\hbar^2}(E - V(\vartheta))\right) f_\vartheta(\vartheta) = 0, \tag{47}$$

(ii) pseudosphere:

$$\frac{d^2 f_\vartheta(\vartheta)}{d\vartheta^2} + \coth\vartheta \frac{d f_\vartheta(\vartheta)}{d\vartheta} - \left(\frac{(\pm(m/I)R^2\sinh^2\vartheta + \cosh^2\vartheta)n^2 + l^2 - 2nl\cosh\vartheta}{\sinh^2\vartheta} - \frac{2mR^2}{\hbar^2}(E - V(\vartheta))\right) f_\vartheta(\vartheta) = 0 \tag{48}$$

with the mentioned above meaning of the $\pm$ signs.

It is natural to expect that for Bertrand potentials discussed in [25] the resulting Schrödinger equations should be rigorously solvable in terms of some standard special functions. The most convenient way of solving them is to use the Sommerfeld polynomial method [16, 18, 20, 21]. In this method the solutions are expressed by the usual or confluent Riemann P-functions. They are deeply related to the hypergeometric functions (usual F or confluent F_1 respectively). If the usual convergence demands are imposed, then the hypergeometric functions become polynomials and our solutions are expressed by elementary functions. At the same time the energy levels and separation constants are expressed by the eigenvalues of the corresponding operators. There exists some special class of potentials to which the Sommerfeld polynomial method is applicable. The restriction to solutions expressible in terms of Riemann P-functions is reasonable, because this class of functions is well known and many special functions used in physics may be expressed by them.

5. Examples

Eqs. (47) and (48) may be solved only when the explicit form of potential is specified. We consider a special case, when the translational part of the potential energy $V(\vartheta)$ $(V(r))$ has the Bertrand structure, i.e. with the "frozen" rotations all orbits would be closed [25].

(i) sphere:

$$V(r) = \frac{\gamma}{2} R^2 \tan^2 \frac{r}{R}.$$

Here we consider the model of the oscillatory potential (14). Let us mention, it is a kind of the anharmonic potential which in the neighbourhood of equilibrium resembles some properties of the harmonic oscillator like in [16, 22–24]. This is the reason why we are interested in it.

Applying the Sommerfeld polynomial method we obtain the energy levels E as

$$E = \frac{1}{2}\hbar\Omega\left(\left(2k + 1 + |n - l| + \sqrt{(n + l)^2 + \frac{\gamma m R^4}{\hbar^2}}\right)^2 + 4n^2\left(\frac{m}{I}R^2 - 1\right) - \frac{4\gamma m R^4}{\hbar^2} - 1\right), \tag{49}$$

where $\Omega = \hbar\omega/4mR^2$, $\omega = \sqrt{\gamma/m}$ and $k = 0, 1, \ldots$. After some calculations we obtain the function $f_r(r)$ in the form

$$f_r(r) = \left(\cos\frac{r}{R}\right)^{\kappa}\left(\sin\frac{r}{R}\right)^{\nu} F\left(-k, k + 1 + \kappa + \nu; 1 + \kappa; \cos^2\frac{r}{R}\right), \tag{50}$$

where

$$\kappa = \sqrt{(n + l)^2 + \frac{\gamma m R^4}{\hbar^2}}, \qquad \nu = |n - l|.$$

(ii) pseudosphere:

$$V(r) = \frac{\gamma}{2} R^2 \tanh^2 \frac{r}{R}, \qquad \gamma > 0.$$

We take the "harmonic oscillator"-type potential (36).

We find the energy levels E in the form

$$E = \frac{1}{2}\hbar\Omega\Bigg(\left(2k+1+|n-l|+\sqrt{(n+l)^2+\frac{\gamma m R^4}{\hbar^2}}\right)^2 - 4n^2\left(\pm\frac{m}{I}R^2-1\right)-\frac{4\gamma m R^4}{\hbar^2}-1\Bigg). \tag{51}$$

The function $f_r(r)$ has the form

$$f_r(r) = \left(\cosh\frac{r}{R}\right)^{\kappa}\left(\sinh\frac{r}{R}\right)^{\nu} F\left(-k, k+1+\kappa+\nu; 1+\kappa; \cosh^2\frac{r}{R}\right). \tag{52}$$

Let us notice that

$$\lim_{r\to\infty}\frac{\gamma}{2}R^2\tanh^2\frac{r}{R} = \frac{\gamma}{2}R^2.$$

This is the upper bound of the potential V (36). Therefore, the formula (51) is correct only for such quantum numbers that

$$E < \operatorname{Sup} V = \frac{\gamma}{2}R^2.$$

Above this threshold we are dealing with the continuous spectrum and the classically nonrestricted motion.

The considered systems are completely nondegenerate. On the quantum level this fact is reflected by the existence of three quantum numbers labelling the energy levels. They cannot be combined into a single quantum number, i.e. there is no total quantum degeneracy, i.e. hyperintegrability, with respect to them. The interaction between translational and rotational degrees of freedom completely removes degeneracy. As yet it is not clear for us if some weaker degeneracy does occur for some relationships between constants m, I, R, γ. This is to be discussed later on. We are also going to investigate the Kepler–Coulomb potential models (15), (37).

Nevertheless, in the spherical, resonance $I = mR^2$ model,

$$\frac{d^2 f_\vartheta(\vartheta)}{d\vartheta^2} + \cot\vartheta\frac{d f_\vartheta(\vartheta)}{d\vartheta} - \left(\frac{n^2+l^2-2nl\cos\vartheta}{\sin^2\vartheta} - \frac{2I}{\hbar^2}(E - V(\vartheta))\right) f_\vartheta(\vartheta) = 0, \tag{53}$$

some special case of the total degeneracy is seen, namely $\gamma = 0$. This is the geodetic motion $V = 0$. The point is that this problem is isomorphic, as we mentioned above, with the quantum mechanics of the spherical rigid body without translational motion in three dimensions, i.e. with evidently completely degenerate model.

In the pseudospherical case, when $I = mR^2$

$$\frac{d^2 f_\vartheta(\vartheta)}{d\vartheta^2} + \coth\vartheta\frac{d f_\vartheta(\vartheta)}{d\vartheta} - \left(\frac{n^2+l^2-2nl\cosh\vartheta}{\sinh^2\vartheta} - \frac{2I}{\hbar^2}(E - V(\vartheta))\right) f_\vartheta(\vartheta) = 0, \tag{54}$$

the situation is not clear, because in the geodetic problem $\gamma = 0$ one deals with the continuous spectrum, where the Sommerfeld polynomial method is not literally applicable.

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