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Mathematical Foundations of the Classical Maxwell-Lorentz Electrodynamic Models in the Canonical Lagrangian and Hamiltonian Formalisms

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Abstract We present new mathematical foundations of classical Maxwell–Lorentz electrodynamic models and related charged particles interaction-radiation problems, and analyze the fundamental least action principles via canonical Lagrangian and Hamiltonian formalisms. The corresponding electrodynamic vacuum field theory aspects of the classical Maxwell–Lorentz theory are analyzed in detail. Electrodynamic models of charged point particle dynamics based on a Maxwell type vacuum field medium description are described, and new field theory concepts related to the mass particle paradigms are discussed. We also revisit and reanalyze the mathematical structure of the classical Lorentz force expression with respect to arbitrary inertial reference frames and present new interpretations of some classical special relativity theory relationships.

Keywords Lagrangian and Hamiltonian Formalism, Least Action Principle, Vacuum Field Theory, Lorentz Force Problem, Feynman Approach, Maxwell Equations, Lorentz Constraint, Electron Radiation Theory

1 Introduction

Our main goal in this investigation is a rather ambitious one: namely, to employ modern mathematical innovations to rigorously reformulate some of the foundations of classical electrodynamics in a way that provides solutions, partial solutions or at least new insights into problems that have persisted for many decades. More specifically, our investigation shall be focused on four areas within the field of electrodynamics. First, we apply, in the context of Lagrangian and Hamiltonian formalisms, a new variational approach to charged particle interaction radiation problems. Next, we undertake a rather novel detailed mathematical analysis of Maxwell–Lorentz electrodynamics from a vacuum field theory perspective. This approach is shown to clarify and simplify several aspects of the theory. Then, we employ a Maxwell type vacuum field medium to obtain a more complete picture of charged particle dynamics and also to lead to certain novel concepts in the realm of mass particle paradigms. Finally, we revisit the topic of the Lorentz force - which has continued to have its perplexing aspects since its very introduction - with a fresh mathematical approach. The analysis we use provides a more complete understanding of how to express the Lorentz force with respect to arbitrary inertial frames and also leads to novel interpretations of certain relationships in special relativity theory.

2 Classical relativistic electrodynamics via least action principles revisited: Lagrangian and Hamiltonian analysis

2.1 Introductory setting

Classical Maxwell–Lorentz electrodynamics is nowadays considered [56, 68] as the most fundamental physical theory, largely owing to the depth of its theoretical foundations and wealth of experimental verifications. In this work we present the main mathematical structures in the foundations of modern classical electrodynamics, based on new least action principle approaches to the classical Maxwell-Lorentz electromagnetic theory, taking into account a vacuum field medium model interacting with charged Material objects. We reanalyze in detail some of the important modern electrodynamics problems related to the description of charged point particle dynamics under an external electromagnetic field with respect to arbitrary inertial reference frames. We remark here that by a *"a charged point particle"* we as usual understand an elementary material charged particle whose internal spatial structure is assumed to be unimportant and is not taken into account, if the contrary is not specified.

The important physical principles, characterizing the related electrodynamic vacuum field structures, are from the mathematical point of view based on the least action principle, which we discuss subject to different charged point particle dynamics. In particular, the main classical special relativity relationships, characterizing charged point particle dynamics, we obtain by means of the least action principle within the original Feynman approach to the Maxwell electromagnetic equations and the Lorentz type force derivation. Moreover, for each least action principle constructed, we describe the corresponding Hamiltonian pictures and present the related energy conservation laws. Making use of the modified least action approach, a classical hadronic string model is analyzed in detail.

As the classical Lorentz force expression with respect to an arbitrary inertial reference frame is related to many theoretical and experimental controversies, such as the relativistic potential energy impact on the charged point particle mass, the Aharonov-Bohm effect [2] and the Abraham–Lorentz–Dirac radiation force [26, 50] expression, the analysis of its structure subject to the assumed vacuum field medium structure is a very interesting and important problem, which has been discussed by many physicists including E. Fermi, G. Schott, R. Feynman, F. Dyson [32, 33, 36, 37, 43, 83] and many others. To describe the essence of the electrodynamic problems related to the description of a charged point particle dynamics under external electromagnetic field, let us begin by analyzing the classical Lorentz force expression

$$dp/dt = F_\xi := \xi E + \xi u \times B, \tag{1}$$

where $\xi \in \mathbb{R}$ is a particle electric charge, $u \in T(\mathbb{R}^3)$ is its velocity [1, 13] vector, expressed here in the light speed c units,

$$E := -\partial A/\partial t - \nabla \varphi \tag{2}$$

is the corresponding external electric field and

$$B := \nabla \times A \tag{3}$$

is the corresponding external magnetic field, acting on the charged particle, expressed in terms of suitable vector $A : M^4 \to \mathbb{E}^3$ and scalar $\varphi : M^4 \to \mathbb{R}$ potentials. Here "∇"is the standard gradient operator with respect to the spatial variable $r \in \mathbb{E}^3$, "$\times$"is the usual vector product in three-dimensional Euclidean vector space $\mathbb{E}^3 := (\mathbb{R}^3, <\cdot,\cdot>)$, which is naturally endowed with the classical scalar product $<\cdot,\cdot>$. These potentials are defined on the Minkowski space $M^4 \simeq \mathbb{R} \times \mathbb{E}^3$, which models a chosen laboratory reference frame $\mathcal{K}$. Now, it is a well-known fact [37, 56, 68, 91] that the force expression (1) does not take into account the dual influence of the charged particle on the electromagnetic field and should be considered valid only if the particle charge $\xi \to 0$. This also means that expression (1) cannot be used for studying the interaction between two different moving charged point particles, as was pedagogically demonstrated in classical manuals [37, 56].

Other questionable inferences, which strongly motivated the analysis here, are related both to an alternative interpretation of the well-known *Lorentz condition*, imposed on the four-vector of electromagnetic potentials $(\varphi, A) : M^4 \to M^4$ and the classical Lagrangian formulation [56] of charged particle dynamics under an external electromagnetic field. The Lagrangian approach is strongly dependent on an important Einsteinian notion of the rest (proper) reference frame $\mathcal{K}_\tau$ and the related least action principle. Therefore, before explaining it in more detail, we first analyze the classical Maxwell electromagnetic theory from a strictly dynamical point of view.

Let us consider with respect to a laboratory reference frame $\mathcal{K}$ the additional *Lorentz condition*

$$\partial\varphi/\partial t + <\nabla, A> = 0, \tag{4}$$

a priori assuming the Lorentz invariant wave scalar field equation

$$\partial^2\varphi/\partial t^2 - \nabla^2\varphi = \rho \tag{5}$$

and the charge continuity equation

$$\partial\rho/\partial t + <\nabla, J> = 0, \tag{6}$$

where $\rho : M^4 \to \mathbb{R}$ and $J : M^4 \to \mathbb{E}^3$ are, respectively, the charge and current densities of the ambient matter. Then one can derive [75, 15] the Lorentz invariant wave equation

$$\partial^2 A/\partial t^2 - \nabla^2 A = J \tag{7}$$

and the classical electromagnetic Maxwell field equations [50, 56, 37, 68, 91]

$$\begin{aligned} &\nabla \times E + \partial B/\partial t = 0, \qquad <\nabla, E> = \rho, \\ &\nabla \times B - \partial E/\partial t = J, \qquad <\nabla, B> = 0, \end{aligned} \tag{8}$$

hold for all $(t, r) \in M^4$ with respect to the chosen laboratory reference frame $\mathcal{K}$.

Notice here that, inversely, Maxwell's equations (8) do not directly reduce, via definitions (2) and (3), to the wave field equations (5) and (7) without the Lorentz condition (4). This fact is very important and suggests that when it comes to a choice of governing equations, it may be reasonable to replace Maxwell's equations (8) with the Lorentz condition (4) and the charge continuity equation (6). To make the equivalence statement, claimed above, more transparent we formulate it as the following proposition.

Proposition 2.1. *The Lorentz invariant wave equation (5) together with the Lorentz condition (4) for the observable potentials $(\varphi, A) : M^4 \to T^*(M^4)$ and the charge continuity relationship (6) are completely equivalent to the Maxwell field equations (8).*

Proof. Substituting (4), into (5), one easily obtains

$$\partial^2\varphi/\partial t^2 = -<\nabla, \partial A/\partial t> = <\nabla, \nabla\varphi> + \rho, \tag{9}$$

which implies the gradient expression

$$< \nabla, -\partial A/\partial t - \nabla\varphi >= \rho. \tag{10}$$

Taking into account the electric field definition (2), expression (10) reduces to

$$< \nabla, E >= \rho, \tag{11}$$

which is the second of the first pair of Maxwell's equations (8).

Now upon applying $\nabla\times$ to definition (2), we find, owing to definition (3), that

$$\nabla \times E + \partial B/\partial t = 0, \tag{12}$$

which is the first pair of the Maxwell equations (8). Differentiating with respect to the temporal variable $t \in \mathbb{R}$ the equation (5) and taking into account the charge continuity equation (6), one finds that

$$< \nabla, \partial^2 A/\partial t^2 - \nabla^2 A - J >= 0. \tag{13}$$

The latter is equivalent to the wave equation (7) if one observes that the vector potential $A : M^4 \to \mathbb{E}^3$ is defined by means of the Lorentz condition (4) up to a vector function $\nabla \times S : M^4 \to \mathbb{E}^3$. Now applying operation $\nabla\times$ to the definition (3), owing to the wave equation (7) one obtains

$$\begin{aligned}\nabla \times B = \nabla \times (\nabla \times A) = \nabla < \nabla, A > -\nabla^2 A = \\ = -\nabla(\partial\varphi/\partial t) - \partial^2 A/\partial t^2 + (\partial^2 A/\partial t^2 - \nabla^2 A) \\ = \frac{\partial}{\partial t}(-\nabla\varphi - \partial A/\partial t) + J = \partial E/\partial t + J,\end{aligned} \tag{14}$$

which leads directly to

$$\nabla \times B = \partial E/\partial t + J,$$

which is the first of the second pair of the Maxwell equations (8). The final "*no magnetic charge*" equation

$$< \nabla, B >=< \nabla, \nabla \times A >= 0,$$

in (8) follows directly from the elementary identity $< \nabla, \nabla\times >= 0$, thereby completing the proof. □

This proposition allows us to consider the potential functions $(\varphi, A) : M^4 \to T^*(M^4)$ as fundamental ingredients of the ambient *vacuum field medium*, by means of which we can try to describe the related physical behavior of charged point particles imbedded in space-time M^4. The following observation provides strong support for this approach:

Observation. *The Lorentz condition (4) actually means that the scalar potential field $\varphi : M^4 \to \mathbb{R}$ continuity relationship, whose origin lies in some new field conservation law, characterizes the deep intrinsic structure of the vacuum field medium.*

To make this observation more transparent and precise, let us recall the definition [37, 56, 68, 91] of the electric current $J : M^4 \to \mathbb{E}^3$ in the dynamical form

$$J := \rho u, \tag{15}$$

where the vector $u \in T(\mathbb{R}^3)$ is the corresponding charge velocity. Thus, the following continuity relationship

$$\partial\rho/\partial t+ < \nabla, \rho u >= 0 \tag{16}$$

holds, which can easily be rewritten [61] as the integral conservation law

$$\frac{d}{dt}\int_{\Omega_t} \rho d^3 r = 0 \tag{17}$$

for the charge inside any bounded domain $\Omega_t \subset \mathbb{E}^3$, moving in the space-time M^4 with respect to the natural evolution equation

$$dr/dt := u. \tag{18}$$

Following the above reasoning, we are led to the following result.

Proposition 2.2. *The Lorentz condition (4) is equivalent to the integral conservation law*

$$\frac{d}{dt}\int_{\Omega_t} \varphi d^3 r = 0, \tag{19}$$

where $\Omega_t \subset \mathbb{E}^3$ is any bounded domain, moving with respect to the evolution equation

$$dr/dt := v, \tag{20}$$

which represents the velocity vector of local potential field changes propagating in the Minkowski space-time M^4.

Proof. Consider first the corresponding solutions to potential field equations (5), taking into account condition (15). Owing to the results from [37, 56], one finds that

$$A = \varphi v, \tag{21}$$

which gives rise to the following form of the Lorentz condition (4):

$$\partial\varphi/\partial t+ < \nabla, \varphi v >= 0. \tag{22}$$

This obviously can be rewritten [61] as the integral conservation law (19), so the proof is complete. □

The above proposition suggests a physically motivated interpretation of electrodynamic phenomena in terms of what should naturally be called *the vacuum potential field*, which determines the observable interactions among charged point particles.

Remark. It is important to remark here that in the devised vacuum field theory approach the well-known [50, 56] gauge invariance of the potential functions $(\varphi, A) : M^4 \to T^*(M^4)$ breaks down owing to the strongly fixed analytical form of the induced vector potential $A = u\varphi \in \mathbb{E}^3$, attributed to the vacuum field medium, and $A = \varphi u \in \mathbb{E}^3$, attributed to the electromagnetic field generated by a moving point charged particle. The latter is in complete agreement both with the Lienard–Wiechert potentials structure and with well-known experimental evidence [2, 26, 38, 94] from superconductivity theory.

In the work we are using the term "vacuum field medium" instead of the classical term "ether" as the latter possesses many different contexts. For instance, A. Einstein wrote [34] that "...according to the general theory of relativity space is endowed with physical quantities; in this sense, therefore, there exists an ether. According to the general theory of relativity space without ether is unthinkable..."Simultaneously, H.A. Lorentz in [60] wrote, that "Indeed, one of the most important of our fundamental assumptions must be that the ether not only occupies all space between molecules, atoms or electrons, but that it pervades all these particles. We shall add the hypothesis that, though the particles may move, the ether always remains at rest. We can reconcile ourselves with this, at first sight, somewhat startling idea, by thinking of the particles of matter as of some local modifications in the state of the ether. These modifications may of course very well travel onward while the volume-elements of the medium in which they exist remain at rest."

More precisely, we can *a priori* endow the ambient vacuum medium with a scalar potential field function $W := \xi\varphi : M^4 \to \mathbb{R}$, satisfying the governing *vacuum field equations*

$$\partial^2 W/\partial t^2 - \nabla^2 W = \rho, \quad \partial W/\partial t + <\nabla, Wv>= 0, \\ \partial\rho/\partial t + <\nabla, \rho u>= 0, \tag{23}$$

taking into account that there are no external sources besides material charged particles, which possess only a virtual capability for disturbing the vacuum field medium. Moreover, this vacuum potential field function $W : M^4 \to \mathbb{R}$ allows the natural potential energy interpretation, whose origin should be assigned not only to the charged interacting medium, but also to any other medium possessing interaction capabilities, including for instance, material particles, interacting through the gravity.

This leads naturally to the next important step, consisting in deriving the equation governing the corresponding potential field $\bar{W} : M^4 \to \mathbb{R}$, assigned to a charged point particle moving in the vacuum field medium with velocity $u \in T(\mathbb{R}^3)$ and located at point $r(t) = R(t) \in \mathbb{E}^3$ at time $t \in \mathbb{R}$. As can be readily shown [75, 77, 80], the corresponding evolution equation governing the related potential field function $\bar{W} : M^4 \to \mathbb{R}$, assigned to a freely moving in the Euclidean space $\mathbb{E}^3$ charged particle ξ, has the form

$$\frac{d}{dt}(-\bar{W}u) = -\nabla\bar{W}, \tag{24}$$

where $\bar{W} := W(r,t)|_{r\to R(t)}$, $u(t) := dR(t)/dt$ at point particle location $(t, R(t)) \in M^4$.

Similarly, if there are two interacting charged point particles, located at points $r(t) = R(t)$ and $r_f(t) = R_f(t) \in \mathbb{E}^3$ at time $t \in \mathbb{R}$ and moving, respectively, with velocities $u := dR(t)/dt$ and $u_f := dR_f(t)/dt$, the corresponding potential field function $\bar{W}' : M^4 \to \mathbb{R}$, considered with respect to the reference frame $\mathcal{K}'$ specified by Euclidean coordinates $(t', r - r_f) \in \mathbb{E}^4$ and moving with the velocity $u_f \in T(\mathbb{R}^3)$ subject to the laboratory reference frame $\mathcal{K}$, should satisfy [75, 77] the dynamical equality

$$\frac{d}{dt'}[-\bar{W}'(u' - u'_f)] = -\nabla\bar{W}'. \tag{25}$$

Here, by definition, we have denoted the velocity vectors $u' := dr/dt', u'_f := dr_f/dt' \in T(\mathbb{R}^3)$. The dynamical potential field equations (24) and (25) appear to have important properties and can be used to represent classical electrodynamic phenomena. Consequently, we shall proceed to investigate their physical properties in more detail and compare them with classical results for Lorentz type forces arising in the electrodynamics of moving charged point particles in an external electromagnetic field.

In this investigation, we were in part inspired by very interesting investigations [24, 25, 40, 41, 42, 97] and studies [22, 23] devoted to solving the classical problem of reconciling gravitational and electrodynamic charges in the context of the Mach-Einstein ether paradigm. First, we will revisit the classical Mach–Einstein relativistic electrodynamics of a moving charged point particle, and second, we study the resulting electrodynamic theories associated with our vacuum potential field dynamical equations (24) and (25), making use of the fundamental Lagrangian and Hamiltonian formalisms which were devised in [15, 76]. Concerening the modern theoretical and mathematical treatments of the related electrodynamics problems, one can also recommend [26, 38, 47, 69, 88].

2.1.1 Classical relativistic electrodynamics revisited

The classical relativistic electrodynamics of a freely moving charged point particle in the Minkowski spacetime $M^4 \simeq \mathbb{R} \times \mathbb{E}^3$ is based on the Lagrangian approach [26, 37, 56, 68, 91] with Lagrangian function

$$\mathcal{L}_0 := -m_0(1 - |u|^2)^{1/2}, \tag{26}$$

where $m_0 \in \mathbb{R}_+$ is the so-called particle rest mass parameter and $u \in T(\mathbb{R}^3)$ is its spatial velocity in the Euclidean space $\mathbb{E}^3$, expressed here and in the sequel in light speed units (with light speed $c = 1$). The least action principle in the form

$$\delta S = 0, \quad S := -\int_{t_1}^{t_2} m_0(1 - |u|^2)^{1/2} dt \tag{27}$$

for any fixed temporal interval $[t_1, t_2] \subset \mathbb{R}$ gives rise to the well-known relativistic relationships for the mass of the particle

$$m = m_0(1 - |u|^2)^{-1/2}, \tag{28}$$

and momentum of the particle

$$p := mu = m_0 u(1 - |u|^2)^{-1/2} \tag{29}$$

and the energy of the particle

$$\mathcal{E}_0 = m = m_0(1 - |u|^2)^{-1/2}. \tag{30}$$

It follows from [56, 68], that the origin of the Lagrangian (26) can be extracted from the action

$$S := -\int_{t_1}^{t_2} m_0(1 - |u|^2)^{1/2} dt = -\int_{\tau_1}^{\tau_2} m_0 d\tau, \tag{31}$$

on the suitable temporal interval $[\tau_1, \tau_2] \subset \mathbb{R}$. Here $m_0 \in \mathbb{R}_+$ is considered as a constant positive parameter *a priori* attributed to the point particle,

$$d\tau := dt(1 - |u|^2)^{1/2} \tag{32}$$

and $\tau \in \mathbb{R}$ is the so-called, proper temporal parameter assigned to a freely moving particle with respect to the rest reference frame $\mathcal{K}_\tau$. The action (31) is rather questionable from the dynamical point of view, since it is physically defined with respect to the rest reference frame $\mathcal{K}_\tau$, giving rise to the constant action $S = -m_0(\tau_2 - \tau_1)$, as the limits of integrations $\tau_1 < \tau_2 \in \mathbb{R}$ were taken to be fixed from the very beginning. Moreover, considering this particle to have a charge $\xi \in \mathbb{R}$ and be moving in the Minkowski space-time M^4 under the action of an external electromagnetic field $(\varphi, A) \in M^4$, the corresponding classical (relativistic) action functional is chosen (see [9, 76, 37, 56, 68, 15, 91]) with respect to the *rest reference system* $\mathcal{K}_\tau$ as follows:

$$S := \int_{\tau_1}^{\tau_2} [-m_0 \ + \xi < A, \dot{r} > \ -\xi\varphi(1 + |\dot{r}|^2)] d\tau. \qquad (33)$$

It is parameterized by the *Euclidean* space-time variables $(\tau, r) \in \mathbb{E}^4$ satisfying the infinitesimal relationship $d\tau^2 + |dr|^2 = dt^2$, where we have denoted $\dot{r} := dr/d\tau$ in contrast to the definition $u := dr/dt$. The action (33) can be rewritten with respect to the laboratory reference frame $\mathcal{K}$ as

$$S = \int_{t_1}^{t_2} \mathcal{L} dt, \ \mathcal{L} := -m_0(1 - |u|^2)^{1/2} + \xi < A, u > -\xi\varphi, \qquad (34)$$

defined on the suitable temporal interval $[t_1, t_2] \subset \mathbb{R}$. The action function (34) contains two physically incompatible sub-integral parts - the first one $-m_0(1 - |u|^2)^{1/2} dt = -m_0 d\tau$, having sense with respect to the rest reference frame $\mathcal{K}_\tau$ and the second one $\xi < A, dr > -\xi\varphi dt$, having sense with respect to the laboratory reference frame $\mathcal{K}$. Nevertheless, the least action principle applied to the functional (34) gives rise to the following [37, 56, 68, 91] dynamical equation

$$dP/dt = -\nabla(\xi\varphi - < \xi A, u >), \qquad (35)$$

where, by definition, the generalized particle-field momentum

$$P = p + \xi A, \qquad (36)$$

the particle momentum

$$p = mu \ = m_0 u(1 - |u|^2)^{-1/2} \qquad (37)$$

and its so called "*inertial*" mass

$$m = m_0(1 - |u|^2)^{-1/2}. \qquad (38)$$

The corresponding particle conserved energy equals

$$\mathcal{E} \ = (m_0^2 + |p|^2)^{1/2} + \xi\varphi, \qquad (39)$$

that is

$$d\mathcal{E}\ /dt = 0 = d\mathcal{E}\ /d\tau \qquad (40)$$

with respect to both the laboratory reference frame $\mathcal{K}$ and the rest reference frame $\mathcal{K}_\tau$.

The above expression (39) for the particle energy $\mathcal{E} \in \mathbb{R}$ appears to be open to question, since the electrical potential energy $\xi\varphi$, entering additively, has no effect on the relativistic particle mass $m = m_0(1 - |u|^2)^{-1/2}$, contradicting the experimental facts [37, 50] that some part of the observable charged particle mass is of electromagnetic origin. This fact was also underlined by L. Brillouin [21], who remarked that the fact that the potential energy has no effect on the particle mass tells us that "*... any possibility of existence of a particle mass related with an external potential energy, is completely excluded*". Moreover, it is necessary to stress here that the least action principle, based on the action functional (34) and formulated with respect to the laboratory reference frame $\mathcal{K}$ time parameter $t \in \mathbb{R}$, appears to be logically inadequate, for there is a strong physical inconsistency with other time parameters of the Lorentz equivalent laboratory reference frames depending simultaneously both on the spatial and temporal coordinates. This was first mentioned by R. Feynman in [37] in his efforts to rewrite the Lorentz force expression with respect to the rest reference frame $\mathcal{K}_\tau$. This and other special relativity theory and electrodynamics problems stimulated many prominent physicists of the past [19, 21, 37, 68, 95] and present [22, 24, 25, 45, 58, 59, 64, 65, 67, 80, 97] to try to develop alternative relativity theories based on completely different space-time and matter structure principles.

There also is another controversial inference from the action expression (34) and resulting dynamical equation (35): the force $F_\xi = dP/dt$, exerted by the external electromagnetic field on the *particle-field cluster* [37] carrying the momentum $P = p + \xi A$, appears to be the standard gradient expression

$$F_\xi = -\nabla W_\xi, \qquad (41)$$

where the generalized "*potential energy*"

$$W_\xi := \xi\varphi - < \xi A, u > . \qquad (42)$$

Its first part $\xi\varphi \in \mathbb{R}$ equals the classical [37, 50] electrical potential energy, but its second part $- < \xi A, u >$ is strictly related to the magnetic vector potential $A \in \mathbb{E}^3$ and has nowadays no reasonable physical explanation. As one can easily show [13, 37, 56, 68, 91] from (35), the corresponding expression for *the classical Lorentz force* is given as

$$dp/dt = F := \xi E + \xi u \times B, \qquad (43)$$

where we have defined, as before,

$$E := -\partial A/\partial t - \nabla\varphi \qquad (44)$$

for the corresponding electric field and

$$B := \nabla \times A \qquad (45)$$

for the related magnetic field, acting on the point particle with the electric charge $\xi \in \mathbb{R}$. The expression (43) means, in particular, that the Lorentz force (43) depends linearly on the particle velocity vector $u \in T(\mathbb{R}^3)$, and so there is a strong dependence on the reference frame with respect to which the charged point particle ξ moves. Attempts to reconcile this and some related controversies [21, 37, 80, 52] forced Einstein to devise his special relativity theory and proceed further to creating his general relativity theory trying to explain the gravity by means of geometrization of space-time and matter in the Universe.

Here we once again mention that the classical Lagrangian function $\mathcal{L} : T(M^4) \to \mathbb{R}$ in (34) is simultaneously written as a combination of terms incompatibly expressed from the physical point of view by means of both the Euclidean rest reference frame variables $(\tau, r) \in \mathbb{E}^4$, naturally attributed to the charged point particle, and arbitrarily chosen Minkowski reference frame variables $(t, r) \in M^4$. It is therefore worth relating this with similar ideas suggested in [40, 41, 42], where a canonical proper-time approach to relativistic mechanics and classical electrodynamics was devised. It also provides a physically complete classical background for a new approach to relativistic quantum theory. It was demonstrated that there are two versions of Maxwell's equations - the new one fixes the clock of the field source for all inertial observers. This implies that the effective speed of light is no longer an invariant for all observers, but depends on the motion of the source. This approach allows us to account for radiation reaction without the Lorentz–Dirac equation, divergent self-energy, advanced potentials or any assumptions about the structure of the source. The theory also provides a new invariance group which, in general, is a nonlinear and nonlocal representation of the Lorentz group. In addition, this approach provides a natural (and unique) definition of simultaneity for all observers.

Some of these problems were recently analyzed using a completely different "*no-geometry*" approach [75, 77], where new dynamical equations were derived, which were free of the controversial elements mentioned above. Moreover, this approach avoided the introduction of the well-known Lorentz transformations of the space-time reference frames with respect to which the action functional (34) is invariant. From this point of view, there are interesting conclusions worthy of discussion in [6, 7, 8, 46, 82], and where some electrodynamic models, possessing intrinsic Galilean and Poincaré–Lorentz symmetries, are reanalyzed from diverse geometrical points of view. Subject to a possible geometric space-type structure and the related vacuum field background exerting the decisive influence on the particle dynamics, we need to mention here recent results [3, 87] and the closely related classical articles [51, 71]. Next, we shall revisit the results obtained recently in [13, 75, 77] from the classical Lagrangian and Hamiltonian formalisms [15] in order to shed new light on the physical underpinnings of the vacuum field theory approach to the study of combined electromagnetic and, eventually, also gravitational effects.

2.2 Vacuum field theory electrodynamics: Lagrangian analysis

2.2.1 Point particle moving in vacuo - an alternative electrodynamic model

In the vacuum field theory approach to electromagnetism devised in [75, 77], the main vacuum potential field function $\bar{W} : M^4 \to \mathbb{R}$, related to a charged point particle ξ, satisfies the differential evolution equation (24), namely

$$\frac{d}{dt}(-\bar{W}u) = -\nabla \bar{W}, \tag{46}$$

in the case when all of the external charged particles are at rest. In particular $\partial \bar{W}/\partial t = 0$, and as above, $u := dr/dt$ is the particle velocity with respect to some laboratory reference system $\mathcal{K}$, specified by the Minkowski coordinates $(t, r) \in M^4$.

To analyze the dynamical equation (46) from the Lagrangian point of view, we write the corresponding action functional as

$$S := -\int_{t_1}^{t_2} \bar{W} dt = -\int_{\tau_1}^{\tau_2} \bar{W}(1 + |\dot{r}|^2)^{1/2}\, d\tau, \tag{47}$$

expressed with respect to the rest reference frame $\mathcal{K}_\tau$, specified by the Euclidean coordinates $(\tau, r) \in \mathbb{E}^4$. Fixing the proper temporal parameters $\tau_1 < \tau_2 \in \mathbb{R}$, one finds from the least action principle $\delta S = 0$ that

$$\begin{aligned} p &:= \partial \mathcal{L}/\partial \dot{r} = -\bar{W}\dot{r}(1 + |\dot{r}|^2)^{-1/2} = -\bar{W}u, \\ \dot{p} &:= dp/d\tau = \partial \mathcal{L}/\partial r = -\nabla \bar{W}(1 + |\dot{r}|^2)^{1/2}, \end{aligned} \tag{48}$$

where, owing to (47), the corresponding Lagrangian function is

$$\mathcal{L} := -\bar{W}(1 + |\dot{r}|^2)^{1/2}. \tag{49}$$

Recalling now the definition of the particle "*inertial*" mass

$$m := -\bar{W} \tag{50}$$

and the relationships

$$d\tau = dt(1-|u|^2)^{1/2} = dt(1+|\dot{r}|^2)^{-1/2},\ \dot{r}d\tau = udt, \tag{51}$$

from (48) we easily obtain the classical dynamical equation exactly coinciding with (46):

$$dp/dt = -\nabla \bar{W}. \tag{52}$$

Moreover, one now readily finds that the corresponding dynamical mass, defined by (50), is given as

$$m = m_0(1 - |u|^2)^{-1/2}, \quad m_0 := -\bar{W}(R(t_0)), \tag{53}$$

where $u(t)|_{t=t_0} = 0$ at the spatial point $r = R(t_0) \in \mathbb{E}^3$, and which completely coincides with expression (28) of the preceding section. Now one can formulate the following proposition using the results obtained above.

Proposition 2.3. *The alternative freely moving point particle electrodynamic model (46) allows the physically reasonable least action formulation based on the action functional (47) with respect to the "rest" reference frame variables, where the Lagrangian function is given by expression (49). The related electrodynamics is completely equivalent to that of a classical relativistic freely moving point particle, described in Subsection 2.1.*

2.2.2 Interacting charge particle pair moving in vacuo - an alternative electrodynamic model

We proceed now to the case when our charged point particle ξ moves in the space-time with velocity vector $u \in T(\mathbb{R}^3)$ and interacts with another external charged point particle ξ_f, moving with velocity vector $u_f \in T(\mathbb{R}^3)$ with respect to a common reference frame $\mathcal{K}$. As was shown in [75, 77], the corresponding modified dynamical equation for the vacuum potential field function $\bar{W}' : M^4 \to \mathbb{R}$ subject to the moving reference frame $\mathcal{K}'$ is given by the equality (25), or

$$\frac{d}{dt'}[-\bar{W}'(u' - u'_f)] = -\nabla \bar{W}'. \qquad (54)$$

Here, as before, the velocity vectors $u' := dr/dt', u'_f := dr_f/dt' \in T(\mathbb{R}^3)$. Since the external charged particle ξ_f moves in the space-time M^4, it generates the related magnetic field $B := \nabla \times A$, whose magnetic vector potential $A : M^4 \to \mathbb{E}^3$ is defined, owing to the results of [75, 77, 80], as

$$\xi A := \bar{W} u_f. \qquad (55)$$

Whence, taking into account that the field potential

$$\bar{W} = \bar{W}'(1 - |u_f|^2)^{-1/2} \qquad (56)$$

and the particle momentum $p' = -\bar{W}'u' = -\bar{W}u$, equality (54) becomes equivalent to

$$\frac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}', \qquad (57)$$

if considered with respect to the moving reference frame $\mathcal{K}'$, or to the Lorentz force equality

$$\frac{d}{dt}(p + \xi A) = -\nabla \bar{W}(1 - |u_f|^2), \qquad (58)$$

if considered with respect to the laboratory reference frame $\mathcal{K}$, owing to the classical Lorentz invariance relationship (56), as the corresponding magnetic vector potential, generated by the external charged point test particle ξ_f with respect to the reference frame $\mathcal{K}'$, is identically equal to zero. To imbed the dynamical equation (58) into the classical Lagrangian formalism, we start from the following action functional, which naturally generalizes the functional (47):

$$S := -\int_{\tau_1}^{\tau_2} \bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2}\, d\tau. \qquad (59)$$

Here, as before, $\bar{W}'$ is the associated calculated vacuum field potential $\bar{W}$ subject to the moving reference frame $\mathcal{K}'$, $\dot{r} = u'dt'/d\tau, \dot{r}_f = u'_f dt'/d\tau$, $d\tau = dt'(1 - |u' - u'_f|^2)^{1/2}$, which takes into account the relative velocity of the charged point particle ξ subject to the reference frame $\mathcal{K}'$, specified by the Euclidean coordinates $(t', r - r_f) \in \mathbb{E}^4$, and moving simultaneously with velocity vector $u_f \in T(\mathbb{R}^3)$ with respect to the laboratory reference frame $\mathcal{K}$, specified by the Minkowski coordinates $(t, r) \in M^4$. It is also related to those of the reference frame $\mathcal{K}'$ and $\mathcal{K}_\tau$ by means of the following infinitesimal relationships:

$$dt^2 = (dt')^2 + |dr_f|^2, \quad (dt')^2 = d\tau^2 + |dr - dr_f|^2. \qquad (60)$$

So, it is clear in this case that our charged point particle ξ moves with the velocity vector $u' - u'_f \in T(\mathbb{E}^3)$ with respect to the reference frame $\mathcal{K}'$ in which the external charged particle ξ_f is at rest. Consequently, we have reduced the problem of deriving the charged point particle ξ dynamical equation to that solved in Subsection 2.2.1.

Now we can compute the least action variational condition $\delta S = 0$, taking into account that, owing to (59), the corresponding Lagrangian function with respect to the rest reference frame $\mathcal{K}_\tau$ is given as

$$\mathcal{L} := -\bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2}. \qquad (61)$$

Simple calculations show that the generalized momentum of the charged particle ξ equals

$$\begin{aligned} P &:= \partial\mathcal{L}/\partial\dot{r} = -\bar{W}'(\dot{r} - \dot{r}_f)(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} \\ &= -\bar{W}'\dot{r}(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} + \bar{W}'\dot{r}_f(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} \\ &= mu' + \xi A' := p' + \xi A' = p + \xi A, \end{aligned} \qquad (62)$$

where, owing to (56) the vectors $p' := -\bar{W}'u' = -\bar{W}u = p \in \mathbb{E}^3$, $A' = \bar{W}'u'_f = \bar{W}u_f = A \in \mathbb{E}^3$, and giving rise to the dynamical equality

$$\frac{d}{d\tau}(p' + \xi A') = -\nabla \bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2} \qquad (63)$$

with respect to the rest reference frame $\mathcal{K}_\tau$. As $dt' = d\tau(1 + |\dot{r} - \dot{r}_f|^2)^{1/2}$ and $(1 + |\dot{r} - \dot{r}_f|^2)^{1/2} = (1 - |u' - u'_f|^2)^{-1/2}$, we obtain from (63) the equality

$$\frac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}', \qquad (64)$$

coinciding with equality (57) subject to the moving reference frame $\mathcal{K}'$. Now, making use of expressions (60) and (56), one can rewrite (64) as that with respect to the laboratory reference frame $\mathcal{K}$:

$$\begin{aligned} &\tfrac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}' \Rightarrow \\ &\Rightarrow \tfrac{d}{dt'}\Big(\tfrac{-\bar{W}u'}{(1+|u'_f|^2)^{1/2}} + \tfrac{\xi \bar{W} u'_f}{(1+|u'_f|^2)^{1/2}}\Big) = -\tfrac{\nabla \bar{W}}{(1+|u'_f|^2)^{1/2}} \Rightarrow \\ &\Rightarrow \tfrac{d}{dt'}\Big(\tfrac{-\bar{W}dr}{(1+|u'_f|^2)^{1/2}dt'} + \tfrac{\xi \bar{W} dr_f/}{(1+|u'_f|^2)^{1/2}}\Big) = -\tfrac{\nabla \bar{W}}{(1+|u'_f|^2)^{1/2}} \Rightarrow \\ &\Rightarrow \tfrac{d}{dt}(-\bar{W}\tfrac{dr}{dt} + \xi \bar{W}\tfrac{dr_f}{dt}) = -\nabla \bar{W}(1 - |u_f|^2), \end{aligned} \qquad (65)$$

which is identical with (58):

$$\frac{d}{dt}(p + \xi A) = -\nabla \bar{W}(1 - |u_f|^2). \qquad (66)$$

Remark The equation (66) allows to infer the following important and physically reasonable phenomenon: if the test charged point particle velocity $u_f \in T(\mathbb{R}^3)$ tends to the light velocity $c = 1$, the corresponding acceleration force $F_{ac} := -\nabla \bar{W}(1 - |u_f|^2)$ tends to vanish. Therefore, the electromagnetic fields generated by such rapidly moving charged point particles, have no influence on the dynamics of charged objects if observed with respect to an arbitrarily chosen laboratory reference frame $\mathcal{K}$.

Equation (66) can be easily rewritten as

$$\begin{aligned} dp/dt &= -\nabla \bar{W} - \xi dA/dt + \nabla \bar{W}|u_f|^2 \\ &= \xi(-\xi^{-1}\nabla \bar{W} - \partial A/\partial t) \\ &- \xi < u, \nabla > A + \xi \nabla < A, u_f >, \end{aligned} \qquad (67)$$

or, using the well-known [56] identity

$$\nabla < a,b >=< a,\nabla > b+ < b,\nabla > a + b\times(\nabla\times a) + a\times(\nabla\times b), \quad (68)$$

where $a,b \in \mathbb{E}^3$ are arbitrary vector functions, in the standard Lorentz form

$$dp/dt = \xi E + \xi u\times B - \ \nabla < \xi A, u-u_f > . \quad (69)$$

The result (69), which was found with respect to the moving reference frame $\mathcal{K}'$ in [75, 77, 80] and in a slightly different form in [62], leads directly to the following important result.

Proposition 2.4. *The alternative classical relativistic electrodynamic model (57) allows the least action formulation based on the action functional (59) with respect to the rest reference frame $\mathcal{K}_\tau$, where the Lagrangian function is given by expression (61). The resulting Lorentz force expression equals (69), as modified by the additional force component $F_c := -\nabla < \xi A, u-u_f >$, important for explaining [2, 18, 92] the well-known Aharonov-Bohm effect.*

2.2.3 Moving charged particle formulation dual to the classical alternative electrodynamic model

It is easy to see that the action functional (59) is written using the standard classical Lorentz transformations of reference frames. If we now consider the action functional (47) for a charged point particle moving with respect to the rest reference frame $\mathcal{K}_\tau$, and take into account its interaction with an external magnetic field generated by the vector potential $A: M^4 \to \mathbb{E}^3$, it can be naturally generalized as

$$S := \int_{t_1}^{t_2} (-\bar{W}dt + \xi < A, dr >) = \int_{\tau_1}^{\tau_2} (-\bar{W}(1+|\dot{r}|^2)^{1/2} + \xi < A, \dot{r} >)d\tau, \quad (70)$$

where $d\tau = dt(1-|u|^2)^{1/2}$. The chosen form of functional (70) can be explained by means of the following physically motivated reasoning. Consider an action functional like (70) and calculate its value along any smooth arbitrarily chosen and dynamically admissible closed path $l \subset M^4$, which should be naturally defined to be zero:

$$0 = \oint_l (-\bar{W}dt + \xi < A, dr >). \quad (71)$$

Applying this to the right-hand side of (71), one finds from Stokes' theorem [1] that

$$\oint_l (-\bar{W}dt + \xi < A, dr >) = \int_{S(l)} (- < \nabla\bar{W}, dr\wedge dt > - < \xi\partial A/\partial t, dr\wedge dt) = \int_{S(l)} < -\nabla\bar{W} - \xi\partial A/\partial t, dr\wedge dt >$$
$$= \iint_{S(l)} < \xi E, dr\wedge dt >= \iint_{S(l)} < F_\xi \ , dr\wedge dt > \quad (72)$$
$$= -\iint_{S(l)} < d\mathcal{E}\wedge dt >= -\iint_{S(l)} d\ (\mathcal{E}dt) = -\oint_l \mathcal{E}dt = 0,$$

if and only if the charged point particle energy $\mathcal{E} \in \mathbb{R}$ is conserved along this arbitrarily chosen and admissible path $l \subset M^4$. As a simple consequence of (71), the work done by the electromagnetic force F_ξ depends only on the electric field $E \in \mathbb{E}^3$, and not on the related magnetic field $B = \nabla\times A \in \mathbb{E}^3$. Thus, having assumed that the corresponding charged point particle dynamical equations conform to the energy conservation condition above, the action functional (70) can be deemed physically reasonable.

Remark It is also interesting to remark that $\oint_l \mathcal{L}dt = 0$, similar to (71), calculated for the Lagrangian $\mathcal{L} = m|\dot{r}|^2/2 - \bar{W}$ in the classical mechanics of a point particle with mass $m \in \mathbb{R}_+$, moving under an external potential $\bar{W}: \mathbb{R}^3 \to \mathbb{R}$, gives rise to classical Newtonian mechanics

$$\oint_l (\frac{m|\dot{r}|^2}{2} - \bar{W})dt = \oint_l (\frac{m}{2} < \dot{r}, \dot{r} > \ -\bar{W})dt =$$
$$= \int_{S(l)} (< m\dot{r}, d\dot{r}\wedge dt > - < \nabla\bar{W}, dr\wedge dt >)$$
$$= \int_{S(l)} (- < mdr, \wedge d\dot{r} \quad > - < \nabla\bar{W}, dr\wedge dt >)$$
$$= \int_{S(l)} (- < dr, \wedge m\ddot{r}dt \quad > - < \nabla\bar{W}, dr\wedge dt >)$$
$$= -\int_{S(l)} (< m\ddot{r}, dr\wedge dt \quad > + < \nabla\bar{W}, dr\wedge dt >)$$
$$= -\int_{S(l)} < m\ddot{r} + \nabla\bar{W}, dr\wedge dt \quad >= 0, \quad (73)$$

if and only if Newton's equation

$$m\ddot{r} = -\nabla\bar{W} \quad (74)$$

holds.

The least action condition $\delta S = 0$, as calculated with respect to the rest reference frame $\mathcal{K}_\tau$, states in the spirit of Feynman [37] that the charged point particle ξ chooses in the Minkowski space-time M^4 a trajectory which realizes the least action value of the functional (70), calculated with respect to its own rest reference time parameter $\tau \in \mathbb{R}$, which is *a unique physically sensible quantity attributed to the charged point particle dynamics*. Actually, as stressed by R. Feynman [37], the least action principle, as applied to the functional (70) with respect to the laboratory reference frame time parameter $t \in \mathbb{R}$, gives rise to a senseless expression, whose value is both ambiguous and not physically well-defined. Thus, the corresponding common generalized particle-field momentum takes the form

$$P := \partial\mathcal{L}/\partial\dot{r} = -\bar{W}\dot{r}(1+|\dot{r}|^2)^{-1/2} + \xi A = mu + \xi A := p + \xi A, \quad (75)$$

and satisfies the equation

$$\begin{aligned}\dot{P} &:= dP/d\tau = \partial\mathcal{L}/\partial r = -\nabla\bar{W}(1+|\dot{r}|^2)^{1/2} \\ &\quad + \xi\nabla < A, \dot{r} > \qquad (76)\\ &= -\nabla\bar{W}(1-|u|^2)^{-1/2} + \xi\nabla < A, u > (1-|u|^2)^{-1/2},\end{aligned}$$

where

$$\mathcal{L} := -\bar{W}(1+|\dot{r}|^2)^{1/2} + \xi < A, \dot{r} > \qquad (77)$$

is the associated Lagrangian function. Since $d\tau = dt(1-|u|^2)^{1/2}$, one easily finds from (76) that

$$dP/dt = -\nabla(\bar{W} - < \xi A, u >). \qquad (78)$$

Upon substituting (75) into (78) and making use of the identity (68), we obtain the classical expression for the Lorentz force F_ξ, acting on the moving charged point particle ξ :

$$dp/dt := F_\xi = \xi E + \xi u \times B, \qquad (79)$$

where, by definition,

$$E := -\xi^{-1}\nabla\bar{W} - \partial A/\partial t \qquad (80)$$

is its associated electric field and

$$B := \nabla \times A \qquad (81)$$

is the corresponding magnetic field. This questionable result can be summarized as follows.

Proposition 2.5. *The classical relativistic Lorentz force (79) allows the least action formulation based on the action functional (70) with respect to the rest reference frame $\mathcal{K}_\tau$, where the Lagrangian function is given by formula (77).*

Concerning the related electrodynamics of a charged point particle ξ, described by the dual classical Lorentz force (79), we need to state that it is *not equivalent* to that of the classical Lorentz force (43). Moreover, one can easily observe that the classical Lorentz force $F_\xi = \xi E + \xi u \times B$, exerted on the charged point particle ξ by an external charged point test particle ξ_f is not *a priori* vanishing as it should follow from relativistic physics. The details of these aspects will be analyzed in more details in the next section.

Comparing the above Lorentz forces expressions (79) and (69), differing by the gradient term $F_c := -\xi\nabla < A, u - u_f >$, which reconciles the dual Lorentz force acting on a moving charged point particle ξ with respect to an arbitrarily chosen laboratory reference frames $\mathcal{K}$ and, as shall be shown in the sequel, is responsible [2, 18, 92] for the Aharonov-Bohm effect. This fact is important for our vacuum field theory approach since it uses no special geometry and makes it possible to analyze electromagnetic and, under some conditions, also gravitational fields simultaneously by employing the new definition of the dynamical mass expressed by (50).

2.3 Vacuum field theory electrodynamics: Hamiltonian analysis

Any Lagrangian theory has an equivalent canonical Hamiltonian representation via the classical Legendre transformation [1, 5, 13, 73, 91]. As we have already formulated our vacuum field theory of a moving particle with a charge $\xi \in \mathbb{R}$ in Lagrangian form, we proceed now to its Hamiltonian analysis making use of the action functionals (47), (61) and (70).

Take, first, the Lagrangian function (49) and the momentum expression (48) for defining the corresponding Hamiltonian function

$$\begin{aligned}H &:= < p, \dot{r} > - \mathcal{L} \\ &= -|p|^2\bar{W}^{-1}(1-|p|^2/\bar{W}^2)^{-1/2} + \bar{W}(1-|p|^2/\bar{W}^2)^{-1/2} \\ &= -|p|^2\bar{W}^{-1}(1-|p|^2/\bar{W}^2)^{-1/2} \\ &\quad + \bar{W}^2\bar{W}^{-1}(1-|p|^2/\bar{W}^2)^{-1/2} \qquad (82)\\ &= -(\bar{W}^2-|p|^2)(\bar{W}^2-|p|^2)^{-1/2} = -(\bar{W}^2-|p|^2)^{1/2}.\end{aligned}$$

Consequently, it is easy to show [1, 5, 73, 91] that the Hamiltonian function (82) is a conservation law of the dynamical field equation (46); that is, for all $\tau, t \in \mathbb{R}$

$$dH/dt = 0 = dH/d\tau, \qquad (83)$$

which naturally leads to an energy interpretation of H. Thus, we can represent the particle energy as

$$\mathcal{E} = (\bar{W}^2 - |p|^2)^{1/2}. \qquad (84)$$

The corresponding Hamiltonian system equivalent to the vacuum field equation (46) can be written as

$$\begin{aligned}\dot{r} &:= dr/d\tau = \partial H/\partial p = p(\bar{W}^2-|p|^2)^{-1/2} \qquad (85)\\ \dot{p} &:= dp/d\tau = -\partial H/\partial r = \bar{W}\nabla\bar{W}(\bar{W}^2-|p|^2)^{-1/2},\end{aligned}$$

and we have the following result.

Proposition 2.6. *The alternative freely moving point particle electrodynamic model, based on the action functional (47), allows the canonical Hamiltonian formulation (85) with respect to the rest reference frame $\mathcal{K}$, where the Hamiltonian function is given by (82).*

As for the charged point particle electrodynamics, based on the dynamical equations (85), it is completely equivalent to the classical relativistic freely moving point particle electrodynamics described in Subsection 2.1.

In an analogous manner, one can now use the Lagrangian (61) and equation (76) to construct the Hamiltonian function for the dynamical field equation (58), describing the motion of a charged point particle ξ in an external electromagnetic field as

$$\dot{r} := dr/d\tau = \partial H/\partial P,\ \dot{P} := dP/d\tau = -\partial H/\partial r, \qquad (86)$$

where

$$\begin{aligned}H &:= < P, \dot{r} > - \mathcal{L} \\ &= < P, \dot{r}_f - P\bar{W}'^{-1}(1-|P|^2/\bar{W}'^2)^{-1/2} > \\ &\quad + \bar{W}'[\bar{W}'^2(\bar{W}'^2-|P|^2)^{-1}]^{1/2} = < P, \dot{r}_f > \\ &\quad + |P|^2(\bar{W}'^2-|P|^2)^{-1/2} - \bar{W}'^2(\bar{W}'^2-|P|^2)^{-1/2} \\ &= -(\bar{W}'^2-|P|^2)(\bar{W}'^2-|P|^2)^{-1/2} + < P, \dot{r}_f > \qquad (87)\\ &= -(\bar{W}'^2-|P|^2)^{1/2} - < \xi A, P > (\bar{W}'^2-|P|^2)^{-1/2}.\end{aligned}$$

Here we took into account that, owing to definitions (55) and (62),

$$\begin{aligned}\xi_f A' &:= \bar{W}' u'_f = \bar{W}' dr_f/dt' = \bar{W} u_f = \xi_f A = \quad (88)\\ &= \bar{W}' \frac{dr_f}{d\tau} \cdot \frac{d\tau}{dt'} = \bar{W} \dot{r}_f (1 - |u' - u'_f|)^{1/2}\\ &= \bar{W}' \dot{r}_f (1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} == -\bar{W}' \dot{r}_f\\ &(\bar{W}'^2 - |P|^2)^{1/2} \bar{W}'^{-1} = -\dot{r}_f (\bar{W}'^2 - |P|^2)^{1/2},\end{aligned}$$

or

$$\dot{r}_f = -\xi_f A \; (\bar{W}'^2 - |P|^2)^{-1/2}, \qquad (89)$$

where $A : M^4 \to \mathbb{R}^3$ is the related magnetic vector potential generated by the moving external charged particle ξ_f with respect to the laboratory reference frame $\mathcal{K}$. Equations (86) can be easily rewritten with respect to the laboratory reference frame $\mathcal{K}$ in the form

$$\begin{aligned}&dr/dt = u,\\ &dp/dt = \xi E + \xi u \times B - \nabla < \xi A, u - u_f >, \qquad (90)\end{aligned}$$

which coincides with the result (69).

Whence, we see that the Hamiltonian function (87) satisfies the energy conservation conditions

$$dH/dt = 0 = dH/d\tau, \qquad (91)$$

for all $\tau, t \in \mathbb{R}$, and the suitable energy expression, owing to (56), is

$$\begin{aligned}\mathcal{E} = \left(\bar{W}^2 - |\xi A|^2 - |P|^2\right)^{1/2} + &< \xi A, P >\\ &\left(\bar{W}^2 - |\xi A|^2 - |P|^2\right)^{-1/2}, \qquad (92)\end{aligned}$$

where the generalized momentum $P = p + \xi A$. The result (92) differs essentially from that obtained in [56], which is strongly based on the Einsteinian Lagrangian for a moving charged point particle ξ in an external electromagnetic field, generated by a charged point test particle ξ_f, moving with velocity $u_f \in T(\mathbb{R}^3)$ with respect to a laboratory reference frame $\mathcal{K}$. Thus, we have the following proposition,

Proposition 2.7. *The alternative classical relativistic electrodynamic model (90), which is intrinsically compatible with the classical Maxwell equations (6), allows the Hamiltonian formulation (86) with respect to the rest reference frame $\mathcal{K}_\tau$, where the Hamiltonian function is given by expression (87).*

The inference above is a natural candidate for experimental validation of our vacuum field theory. It is strongly motivated by the following remark.

Remark It is necessary to mention here that the Lorentz force expression (90) uses the particle momentum $p = mu$, where the dynamical "mass" $m := -\bar{W}$ satisfies condition (92). This gives rise to the following crucial relationship between the particle energy $\mathcal{E}_0$ and its rest mass $m_0 = -\bar{W}_0$ (for the velocity $u = 0$ at the initial time moment $t = 0$) :

$$\mathcal{E}_0 = m_0 \frac{(1 - |\xi A_0/m_0|^2)}{(1 - 2|\xi A_0/m_0|^2)^{1/2}}, \qquad (93)$$

or, equivalently, under the condition $|\xi A_0/m_0|^2 < 1/2$

$$m_0 = \mathcal{E}_0 \left(\frac{1}{2} + |\xi A_0/\mathcal{E}_0|^2 \pm \frac{1}{2}\sqrt{1 - 4|\xi A_0/\mathcal{E}_0|^2}\right)^{1/2}, \qquad (94)$$

where $A_0 := A|_{t=0} \in \mathbb{E}^3$, which differs markedly from the classical expression $m_0 = \mathcal{E}_0 - \xi\varphi_0$, following from (39) and is does not a priori on the external potential energy $\xi\varphi_0$. As the quantity $|\xi A_0/\mathcal{E}_0| \to 0$, the following asymptotic mass values follow from (94):

$$m_0^{(+)} \simeq \mathcal{E}_0, \qquad m_0^{(-)} \simeq \pm\sqrt{2}|\xi A_0|. \qquad (95)$$

The first mass value $m_0^{(+)} \simeq \mathcal{E}_0$ is physically correct, giving rise to the bounded charged particle energy $\mathcal{E}_0$, but the second mass value $m_0^{(-)} \simeq \pm\sqrt{2}|\xi A_0|$ is not physical, as it gives rise to the vanishing denominator $(1 - 2|\xi A_0/m_0^{(-)}|^2)^{1/2} \simeq 0$ in (93), which is equivalent to the unboundedness of the charged particle energy $\mathcal{E}_0$.

To make this difference more transparent, we now analyze the dual classical Lorentz force (79) from the Hamiltonian point of view, based upon the Lagrangian function (77). Thus, we readily find that the corresponding Hamiltonian function is

$$\begin{aligned}&H :=< P, \dot{r} > -\mathcal{L} =< P, \dot{r} > +\bar{W}(1 + |\dot{r}|^2)^{1/2}\\ &- \xi < A, \dot{r} >=< P - \xi A, \dot{r} > +\bar{W}(1 + |\dot{r}|^2)^{1/2} \qquad (96)\\ &= -|p|^2 \bar{W}^{-1}(1 - |p|^2/\bar{W}^2)^{-1/2} + \bar{W}(1 - |p|^2/\bar{W}^2)^{-1/2}\\ &= -(\bar{W}^2 - |p|^2)(\bar{W}^2 - |p|^2)^{-1/2} = -(\bar{W}^2 - |p|^2)^{1/2}.\end{aligned}$$

Since $p = P - \xi A$, the expression (96) assumes the final "*no interaction*" [55, 56, 68, 78] form

$$H = -(\bar{W}^2 - |P - \xi A|^2)^{1/2}, \qquad (97)$$

which is conserved with respect to the evolution equations (75) and (76), that is

$$dH/dt = 0 = dH/d\tau \qquad (98)$$

for all $\tau, t \in \mathbb{R}$. These equations are equivalent to the following Hamiltonian system

$$\begin{aligned}&\dot{r} = \partial H/\partial P = (P - \xi A)(\bar{W}^2 - |P - \xi A|^2)^{-1/2}, \quad (99)\\ &\dot{P} = -\partial H/\partial r = (\bar{W}\nabla\bar{W} - \nabla < \xi A, (P - \xi A) >)\\ &\qquad (\bar{W}^2 - |P - \xi A|^2)^{-1/2},\end{aligned}$$

as one can readily check by direct calculations. Actually, the first equation

$$\begin{aligned}\dot{r} &= (P - \xi A)(\bar{W}^2 - |P - \xi A|^2)^{-1/2} = p(\bar{W}^2 - |p|^2)^{-1/2}\\ &= mu(\bar{W}^2 - |p|^2)^{-1/2} \qquad (100)\\ &= -\bar{W}u(\bar{W}^2 - |p|^2)^{-1/2} = u(1 - |u|^2)^{-1/2},\end{aligned}$$

holds, owing to the condition $d\tau = dt(1 - |u|^2)^{1/2}$ and definitions $p := mu$, $m = -\bar{W}$, postulated from the very beginning. Similarly, we obtain

$$\begin{aligned}\dot{P} &= -\nabla\bar{W}(1 - |p|^2/\bar{W}^2)^{-1/2}\\ &\quad + \nabla < \xi A, u > (1 - |p|^2/\bar{W}^2)^{-1/2} \qquad (101)\\ &= -\nabla\bar{W}(1 - |u|^2)^{-1/2} + \nabla < \xi A, u > (1 - |u|^2)^{-1/2},\end{aligned}$$

or equivalently, the dual Lorentz dynamical expression

$$dp/dt = \xi E + \xi u \times B, \tag{102}$$

exactly coinciding with that of equation (79). This result can be reformulated as follows.

Proposition 2.8. *The dual to the classical relativistic electrodynamic model (79) allows the canonical Hamiltonian formulation (99) with respect to the rest reference frame $\mathcal{K}_\tau$, where the Hamiltonian function is given by expression (97). Moreover, this formulation circumvents the "mass-potential"energy controversy associated to the classical electrodynamic model based on the classical action functional (34).*

The classical Lorentz force expression (102) and the related conserved energy relationship

$$\mathcal{E} = (\bar{W}^2 - |P - \xi A|^2)^{1/2} \tag{103}$$

are characterized by the following remark.

Remark If we make use of the modified relativistic Lorentz force expression (102) as an alternative to the classical one of (43), the corresponding charged particle energy relationship (103) gives rise to a different energy expression (for the velocity $u = 0$ at the initial time moment $t = 0$). Namely, one naturally obtains the physically reasonable Einsteinian mass-energy relationship $\mathcal{E}_0 = m_0$ instead of the senseless classical expression $\mathcal{E}_0 = m_0 + \xi\varphi_0$ following from (39), where $\varphi_0 := \varphi|_{t=0}$ and where the mass parameter m_0 is a constant parameter independent of the external electromagnetic field.

2.4 Comments

All of the field equations discussed above are canonical Hamiltonian systems with respect to the corresponding physically proper rest reference frames $\mathcal{K}_\tau$, parameterized by the Euclidean coordinates $(\tau, r) \in \mathbb{E}^4$. Upon passing to the basic laboratory reference frame $\mathcal{K}$, naturally parameterized by the Minkowski coordinates $(t, r) \in M^4$, the related Hamiltonian structure is lost, giving rise to a suitably altered interpretation of the real particle motion. Namely, as was demonstrated above, a least action principle for a charged point particle dynamics makes sense only with respect to the proper rest reference frame $\mathcal{K}_\tau$ as, otherwise, it becomes completely senseless with respect to all other laboratory reference frames. As for the Hamiltonian expressions (82), (87) and (97), one observes that they all depend strongly on the vacuum potential field function $\bar{W} : M^4 \to \mathbb{R}$, thereby avoiding the mass problem related with the well-known classical energy expression and pointed out by L. Brillouin [21].

Some comments are also in order concerning the classical relativity principle and how it is applied to real physical phenomena. We have obtained our results using the standard Lorentz transformations of reference frames - relying only on the natural notion of the rest reference frame $\mathcal{K}_\tau$ and its suitable parametrization with respect to any other laboratory reference frame $\mathcal{K}$. It seems physically reasonable that the true state evolution of a moving charged particle ξ is exactly realized only with respect to its proper rest reference system $\mathcal{K}_\tau$.

Thus, the only remaining question would be about the physical justification of the corresponding relationship between time parameters of the corresponding laboratory and rest reference frames. The relationship between these reference frames that we have used through is simply expressed as

$$d\tau = dt(1 - |u|^2)^{1/2}, \tag{104}$$

where $u := dr/dt \in \mathbb{E}^3$ is the velocity vector with which the rest reference frame $\mathcal{K}_\tau$ moves with respect to another arbitrarily chosen reference frame $\mathcal{K}$. Expression (104) implies, in particular, that

$$dt^2 - |dr|^2 = d\tau^2, \tag{105}$$

which is evidently identical to the classical infinitesimal Lorentz invariant. This is not a coincidence, since all our dynamical vacuum field equations were derived in turn [75, 77] from the governing equations of the vacuum potential field function $W : M^4 \to \mathbb{R}$ in the form

$$\begin{gathered}\partial^2 W/\partial t^2 - \nabla^2 W = \rho,\ \partial W/\partial t + <\nabla, vW> = 0,\\ \partial\rho/\partial t + <\nabla, v\rho> = 0,\end{gathered} \tag{106}$$

which is *a priori* Lorentz invariant. Here $\rho \in \mathbb{R}$ is the charge density and $v := dr/dt$ the associated local velocity of the vacuum field potential evolution. Consequently, the dynamical infinitesimal Lorentz invariant (105) reflects this intrinsic structure of equations (106). If it is rewritten in the following slightly nonstandard Euclidean form:

$$dt^2 = d\tau^2 + |dr|^2 \tag{107}$$

it gives rise to a completely different relationship between the reference frames $\mathcal{K}$ and $\mathcal{K}_\tau$, namely

$$dt = d\tau(1 + |\dot{r}|^2)^{1/2}, \tag{108}$$

where $\dot{r} := dr/d\tau$ is the related particle velocity with respect to the rest reference system $\mathcal{K}_\tau$. Thus, we observe that all our Lagrangian analysis is strongly related to the functional expressions written in these "Euclidean"space-time coordinates and with respect to which the least action principle was applied. So we see that there are two alternatives - the first one is to apply the least action principle to the corresponding Lagrangian functions, expressed in the Minkowski space-time variables with respect to an arbitrarily chosen reference frame $\mathcal{K}$, and the second one is to apply the least action principle to the corresponding Lagrangian functions expressed in the Euclidean space-time variables with respect to the rest reference frame $\mathcal{K}_\tau$. But, as it was demonstrated above, the second alternative appeared to be physically reasonable in contrast to the first one, which gives rise to different physically senseless controversies.

The above discussion leads to a slightly amusing but thought-provoking observation: It follows that all of the results of classical special relativity related to the electrodynamics of charged point particles can be obtained (but not in a one-to-one correspondence) using our new reasonable definitions of the dynamical particle mass and the physically motivated least action principle, calculated with respect to the related Euclidean space-time variables specifying the rest reference frame $\mathcal{K}_\tau$.

3 Maxwell's equations and the Lorentz force - Feynman's legacy

3.1 Problem setting

In 1948 R. Feynman presented but did not publish [32, 33] a very interesting, in some respects "heretical" , quantum-mechanical derivation of the classical Lorentz force acting on a charged particle under influence of an external electromagnetic field. His result was analyzed by many authors [4, 20, 31, 35, 48, 49, 57, 85, 93] from different points of view, including its relativistic generalization [89]. As this problem is completely classical, we reanalyze Feynman's derivation from the classical Hamiltonian dynamics point of view on the coadjoint space $T^*(N), N \subset \mathbb{R}^3$, and construct its nontrivial generalization compatible with results [15, 72, 75] of Section 1, based on a recently devised vacuum field theory approach [76, 77]. Having also obtained the classical Maxwell electromagnetic equations, we supply the complete legacy of Feynman's approach to the Lorentz force derivation and demonstrate its compatibility with the relativistic generalization, presented in Section 2.

Consider a charged point particle moving under an electromagnetic field. For its description, it is convenient to introduce a trivial fiber bundle structure π: $M \to N, M = N \times G,\ N \subset \mathbb{R}^3$, with the abelian structure group $G := \mathbb{R}\backslash\{0\}$, equivariantly acting on the canonically symplectic coadjoint space $T^*(M)$, and to endow it with a connection one-form $\mathcal{A} : M \to T^*(M) \times \mathcal{G}$ as

$$\mathcal{A}(q, g) := g^{-1}(d + \alpha^{(1)}(q))\ g \qquad (109)$$

on the phase space M, where $d : \Lambda(M) \to \Lambda(M)$ is the usual exterior derivative, $\alpha^{(1)}$: $M \to \Lambda^1(N)\otimes\mathcal{G}$ is some smooth mapping, $q \in N$ and $g \in G$. If $l : T^*(M) \to \mathcal{G}^*$ is the related momentum mapping, one can construct the reduced phase space $\bar{\mathcal{M}}_\xi := l^{-1}(\xi)/G \simeq T^*(N)$, where $\xi \in \mathcal{G}^* \simeq \mathbb{R}$ is taken to be fixed and have the reduced symplectic structure

$$\bar{\omega}^{(2)}_\xi(q, p) =< dp, \wedge dq > +d < \xi, \alpha^{(1)}(q) >_{\mathcal{G}} . \qquad (110)$$

From (110)one readily computes the corresponding Poisson brackets on $T^*(N)$:

$$\{q^i, q^j\}_{\omega^{(2)}_\xi} = 0,\ \{p_j, q^i\}_{\omega^{(2)}_\xi} = \delta^i_j,$$
$$\{p_i, p_j\}_{\omega^{(2)}_\xi} = \xi F_{ji}(q) \qquad (111)$$

for $i, j = \overline{1, 3}$ with respect to the reference frame $\mathcal{K}(t, q)$, characterized by the phase space coordinates $(q, p) \in T^*(N)$. If one introduces a new momentum variable $\tilde{p} := p + \xi A(q)$ on $T^*(N) \ni (q, p)$, where $\alpha^{(1)}(q) :=< A(q), dq > \in T^*_q(N)$, it is easy to verify that $\bar{\omega}^{(2)}_\xi \to \tilde{\omega}^{(2)}_\xi :=< d\tilde{p}, \wedge dq >$, giving rise to the following "minimal interaction" canonical Poisson brackets:

$$\{q^i, q^j\}_{\tilde{\omega}^{(2)}_\xi} = 0,\ \{\tilde{p}_j, q^i\}_{\tilde{\omega}^{(2)}_\xi} = \delta^i_j,\ \{\tilde{p}_i, \tilde{p}_j\}_{\tilde{\omega}^{(2)}_\xi} = 0 \qquad (112)$$

for $i, j = \overline{1, 3}$ with respect to the reference frame $\mathcal{K}_f(t, q - q_f)$, characterized by the phase space coordinates $(q, \tilde{p}) \in T^*(N)$, if and only if the Maxwell field equations

$$\partial F_{ij}/\partial q_k + \partial F_{jk}/\partial q_i + \partial F_{ki}/\partial q_j = 0 \qquad (113)$$

are satisfied on N for all $i, j, k = \overline{1, 3}$ with the curvature tensor $F_{ij}(q) := \partial A_j/\partial q^i - \partial A_i/\partial q^j,\ i, j = \overline{1, 3}, q \in N$.

3.2 Lorentz force and Maxwell's equations: Lagrangian analysis

The Poisson structure (112) makes it possible to describe a charged particle $\xi \in \mathbb{R}$ at point $q \in N \subset \mathbb{R}^3$moving with velocity $dq/dt := u \in T_q(N)$ with respect to the laboratory reference frame $\mathcal{K}(t, q)$, specified by coordinates $(t, q) \in M^4$, under the electromagnetic influence of an external charged particle $\xi_f \in \mathbb{R}$ at point $q_f \in N \subset \mathbb{R}^3$ and moving with respect to the same reference frame $\mathcal{K}(t, q)$ with velocity $dq_f/dt := u_f \in T_{q_f}(N)$. Consider a new shifted reference frame $\mathcal{K}'_f(t', q-q_f)$ moving with respect to the reference frame $\mathcal{K}(t, q)$ with velocity u_f. With respect to the reference frame $\mathcal{K}'_f(t', q-q_f)$, specified by coordinates $(t', q-q_f) \in M^4$, the charged point particle ξ moves with velocity $u' - u'_f := dr/dt' - dr_f/dt' \in T_{q-q_f}(N)$ and the charged particle ξ_f remains rest. Then one can express the standard *classical Lagrangian function* of the charged particle ξ with a mass $m' \in \mathbb{R}_+$ subject to the reference frame $\mathcal{K}'_f(t', q - q_f)$ as

$$\mathcal{L}_f(q, u') = \frac{m'}{2}|u' - u'_f|^2 - \xi\varphi', \qquad (114)$$

and the suitably Lorentz transformed scalar potential $\varphi' = \varphi/(1 + |u'_f|^2) \in C^2(N; \mathbb{R})$ is the corresponding potential energy with respect to the reference frame $\mathcal{K}'_f(t', q - q_f)$. On the other hand, owing to (114) and the Poisson brackets (112) the following equality for the charged particle ξ *canonical momentum* with respect to the reference frame $\mathcal{K}'_f(t', q - q_f)$ holds:

$$\tilde{p}' := p' + \xi A'(q) = \partial\mathcal{L}_f(q, u')/\partial u', \qquad (115)$$

or, equivalently,

$$p' + \xi A'(q) = m'(u' - u'_f), \qquad (116)$$

expressed in the units when the light speed $c = 1$. Taking into account that the charged particle ξ momentum with respect to the reference frame $\mathcal{K}(t, q)$ is $p' := m'u' \in T_q(N)$, one can easily obtain from (116) the important relationship

$$\xi A'(q) = -m'u'_f \qquad (117)$$

for the vector potential $A \in C^2(N; \mathbb{E}^3)$, which was obtained in [76, 77, 80] and described before in Section 2. Now from (114) and (117) one finds the following Lagrangian equation:

$$\frac{d}{dt'}[p' + \xi A'(q)] = \partial\mathcal{L}_f(q, u')/\partial q = -\xi\nabla\varphi', \qquad (118)$$

obtained with respect to the shifted reference frame $\mathcal{K}'_f(t', q - q_f)$ in [76, 77] and giving rise, as the result of the obvious relationships $p' = p, A' = A$, to the charged point particle ξ dynamics

$$\begin{aligned} dp/dt &= -\xi\partial A/\partial t - \xi\nabla\varphi(1-|u_f|^2) - \xi < u, \nabla > A \\ &= -\xi\partial A/\partial t - \xi\nabla\varphi - \xi < u, \nabla > A + \xi\nabla < u, A > \\ &- \xi\nabla < u - u_f, A >= -\xi(\partial A/\partial t + \nabla\varphi) \\ &+ \xi u \times (\nabla \times A) - \xi\nabla < u - u_f, A > \end{aligned} \tag{119}$$

with respect to the laboratory reference frame $\mathcal{K}(t,q)$. Using (119), we now obtain the modified Lorentz type force

$$dp/dt = \xi E + \xi u \times B - \xi\nabla < u - u_f, A >, \tag{120}$$

where

$$E := -\partial A/\partial t - \nabla\varphi, \quad B := \nabla \times A, \tag{121}$$

and we have used the slightly modified from the classical [50, 26, 37, 56] Lorentz force expression

$$dp/dt = \xi E + \xi u \times B \tag{122}$$

differing by the gradient component

$$F_c := -\xi\nabla < u - u_f, A > . \tag{123}$$

Observe that the modified Lorentz force expression (120) can be naturally generalized to the relativistic case by taking into account that the standard Lorentz condition

$$\partial\varphi/\partial t+ < \nabla, A >= 0 \tag{124}$$

is imposed on the electromagnetic potential $(\varphi, A) \in C^2(N; M^4)$.

More specifically, from (121) one sees that the Lorentz invariant field equation

$$\partial^2\varphi/\partial t^2 - \Delta\varphi = \rho_f, \tag{125}$$

where $\rho_f : N \to \mathcal{D}'(N)$ is a generalized density function of the external charge distribution ξ_f. Now it follows directly from [76, 77] that we can easily find from (125) and the charge conservation law

$$\partial\rho_f/\partial t+ < \nabla, J_f >= 0 \tag{126}$$

the next *Lorentz invariant* equation for the vector potential $A \in C^2(N; \mathbb{E}^3)$:

$$\partial^2 A/\partial t^2 - \Delta A = J_f. \tag{127}$$

Moreover, relationships (121),(125) and (127) imply that the true classical Maxwell equations

$$\begin{aligned} \nabla \times E &= -\partial B/\partial t, \ \nabla \times B = \partial E/\partial t + J_f, \\ < &\nabla, E >= \rho_f, \quad < \nabla, B >= 0 \end{aligned} \tag{128}$$

on the electromagnetic field $(E, B) \in C^2(N; \mathbb{E}^3 \times \mathbb{E}^3)$.

Consider now the Lorentz condition (124) and observe that it is equivalent to the local conservation law

$$\frac{d}{dt}\int_{\Omega_t} W d^3q = 0, \tag{129}$$

which gives rise to the important relationship for the magnetic potential $A \in C^2(N; \mathbb{E}^3)$

$$A = u_f\varphi \tag{130}$$

with respect to the laboratory reference frame $\mathcal{K}(t,q)$, where $\Omega_t \subset N$ is any open domain with smooth boundary $\partial\Omega_t$, moving together with the charge distribution ξ_f in the domain $N \subset \mathbb{R}^3$ with the corresponding velocity u'_f. Taking into account (117), one can find the expression for our charged particle ξ "*inertial*" mass:

$$m = -\bar{W}, \quad \bar{W} := \xi\varphi, \tag{131}$$

coinciding with that obtained in [76, 77, 80], where we denoted by $\bar{W} \in C^2(N; \mathbb{R})$ the corresponding potential energy of the charged point particle ξ.

3.3 Modified least action principle: Hamiltonian analysis

Using the representations (130) and (131) one can rewrite the determining Lagrangian equation (118) with respect to the shifted reference frame $\mathcal{K}'_f(t', q_f)$ as

$$\frac{d}{dt'}[-\bar{W}'(u' - u'_f)] = -\nabla\bar{W}', \tag{132}$$

which is reduced to the Lorentz force expression (120) calculated with respect to the laboratory reference frame $\mathcal{K}(t,q)$:

$$dp/dt = \xi E + \xi u \times B - \xi\nabla < u - u_f, A >, \tag{133}$$

where, as before,

$$E := -\partial A/\partial t - \nabla\varphi, \quad B := \nabla \times A. \tag{134}$$

Remark It is interesting to note here that equation (133) does not allow the Lagrangian representation with respect to the reference frame $\mathcal{K}(t,q)$ in contrast to that of equation (132) which is equivalent to (118).

The remark above is a challenging source of our further analysis concerning the direct relativistic generalization of the modified Lorentz force (120). Namely, the following proposition holds.

Proposition 3.1. *The Lorentz force (120) in the case when the charged point particle ξ momentum is defined, owing to (131), as $p = -\bar{W}u$. And it is the exact relativistic expression allowing the Lagrangian representation of the charged particle ξ dynamics with respect to the rest reference frame $\mathcal{K}_\tau(\tau, q - q_f)$, related to the shifted reference frame $\mathcal{K}'_f(t', q - q_f)$ by means of the classical relativistic proper time infinitesimal transformation:*

$$dt' = d\tau(1 + |u' - u'_f|^2)^{1/2}, \tag{135}$$

where $\tau \in \mathbb{R}$ is the proper time parameter in the rest reference frame $\mathcal{K}_\tau(\tau, q - q_f)$.

Proof. Take the following action functional with respect to the charged point particle ξ rest reference frame $\mathcal{K}_\tau(\tau, q - q_f)$:

$$S^{(\tau)} := -\int_{t_1(\tau_1)}^{t_2(\tau_2)} \bar{W}'dt' = \int_{\tau_1}^{\tau_2} \bar{W}'(1 + |u' - u'_f|^2)^{1/2}d\tau, \tag{136}$$

where the proper temporal values $\tau_1, \tau_2 \in \mathbb{R}$ are considered, in the spirit of Feynman [37], to be fixed in contrast to the temporal parameters $t_2(\tau_2), t_2(\tau_2) \in \mathbb{R}$ depending, owing to (135), on the charged particle ξ trajectory in the phase space M^4. The least action condition

$$\delta S^{(\tau)} = 0, \delta q(\tau_1) = 0 = \delta q(\tau_2), \tag{137}$$

applied to (136) yields the dynamical equation (132), which is simultaneously equivalent to the relativistic Lorentz force expression (120) with respect to the laboratory reference frame $\mathcal{K}(t, q)$. This completes the proof. □

Making use of the relationships between the reference frames $\mathcal{K}(t, q)$ and $\mathcal{K}_\tau(\tau, q-q_f)$ when the external charge particle velocity $u_f = 0$, we can easily derive the following corollary.

Corollary 3.2. *Let the external charge point e_f be in rest, that is the velocity $u_f = 0$. Then equation (132) reduces to*

$$\frac{d}{dt}(-\bar{W}u)] = -\nabla \bar{W}, \tag{138}$$

allowing the following conservation law:

$$H_0 = \bar{W}(1 - |u|^2)^{1/2} = -(\bar{W}^2 - |p|^2)^{1/2}. \tag{139}$$

Moreover, equation (138) is Hamiltonian with respect to the canonical Poisson structure (112), with Hamiltonian function (139) and the rest reference frame $\mathcal{K}_r(\tau, q)$:

$$dq/d\tau := \partial H_0/\partial p = p(\bar{W}^2 - |p|^2)^{-1/2}$$
$$dp/d\tau := -\partial H_0/\partial q = -\bar{W}(\bar{W}^2 - |p|^2)^{-1/2}\nabla \bar{W},$$

which implies

$$dq/dt = -p\bar{W}^{-1},$$
$$dp/dt = -\nabla \bar{W}. \tag{140}$$

In addition, if we define the rest particle mass $m_0 := -H_0|_{u=0}$, the "inertial"particle mass quantity $m \in \mathbb{R}$ takes the well-known classical relativistic form

$$m = -\bar{W} = m_0(1 - |u|^2)^{-1/2}, \tag{141}$$

depending on the particle velocity $u \in \mathbb{R}^3$.

For the general case of equation (132), results analogous to the above hold, such as those described in part in Section 2. We need only mention that the induced Hamiltonian structure of the general equation (132) results naturally from its least action representation (136) and (137) with respect to the rest reference frame $\mathcal{K}_\tau(\tau, q)$.

3.3.1 Comments

In Section 3 we have demonstrated the complete legacy of the Feynman's approach to the Lorentz force based derivation of Maxwell's electromagnetic field equations. Moreover, we succeeded in finding the exact relationship between Feynman's approach and the vacuum field approach of Section 2, devised in [76, 77]. Thus, the results obtained present a strong argument for the vacuum field theory approach, based upon which one can simultaneously describe the physical phenomena both of electromagnetic and gravity origins. The latter is physically based on the particle "inertial"mass expression (131), naturally following from Feynman's approach to the Lorentz force derivation and from vacuum field theory.

4 Modified Lorentz force and charge radiation: vacuum field theory approach

4.1 Introductory setting

Maxwell's equations is one of the fundamental theories of physics known to allow two main forms of representations: either by means of the electric and magnetic fields or potentials. The latter were mainly considered as a mathematically motivated representation useful for different applications but having no physical significance.

That the situation is not so simple and the evidence that the magnetic potential possesses physical properties was doubtless, was understood by the physics community when Y. Aharonov and D. Bohm [2] formulated their "paradox"concerning the measurement of a magnetic field outside a separated region where it vanishes. Later similar effects were also revealed in the superconductivity theory of Josephson media. As the existence of any electromagnetic field in an ambient space can be tested only through its interaction with electric charges, their dynamical behavior, being of great importance, was studied at length by M. Faraday, A. Ampère and H. Lorentz subject to its classical second Newton law form. Namely, the classical Lorentz force

$$dp/dt = \xi E + \xi\frac{u}{c} \times B \tag{142}$$

was derived, where E and $B \in \mathbb{E}^3$ are, respectively, electric and magnetic fields, acting on a point charged particle $\xi \in \mathbb{R}$, possessing the momentum $p = mu$, where $m \in \mathbb{R}_+$ is the observed particle mass and $u \in T(\mathbb{R}^3)$ is its velocity, measured with respect to a suitably chosen laboratory reference frame $\mathcal{K}$.

That the Lorentz force (142) is not a completely satisfactory expression was known to Lorentz himself. The nonuniform Maxwell equations also describe the electromagnetic fields radiated by any accelerated charged particle, easily seen from expressions for the Lienard–Wiechert electromagnetic four-potential $(\varphi, A) : M^4 \to T^*(M^4)$, related to the electromagnetic fields by means of the well-known [56, 50, 26] relationships

$$E := -\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t}, B := \nabla \times A. \tag{143}$$

This fact had inspired many physicists to "improve "the classical Lorentz force expression (142) and its modification was then suggested by G.A. Shott [83] and later by M. Abraham and P.A.M. Dirac (see [26, 50]), who found the so-called classical "radiation reaction "force due to the self-interaction of a charged particle with charge $\xi \in \mathbb{R}$ equals

$$\frac{dp}{dt} = \xi E + \xi\frac{u}{c} \times B + \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}. \tag{144}$$

The additional self-reaction force expression

$$F_r := \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}, \tag{145}$$

depending on the particle acceleration led right away to many questions concerning its physical meaning, since for instance, a uniformly accelerated charged particle, owing to the expression (144), experiences no radiation reaction, contradicting the fact that any accelerated charged particle always radiates electromagnetic waves. This "paradox"was a challenging problem during the twentieth century [17, 27, 50, 70, 83] and still remains to be completely explained [62, 63, 81]. As there exist different approaches to explaining this reaction radiation phenomenon, we mention here only such popular ones as the Wheeler–Feynman [96] "absorber radiation"theory, based on a very sophisticated elaboration of the retarded and advanced solutions to the nonuniform Maxwell equations, the vacuum Casimir effect approach devised in [66, 84], and the construction of Teitelboim [90] which extensively exploits the intrinsic structure of the electromagnetic energy tensor subject to the advanced and retarded solutions to the nonuniform Maxwell equations.

It is also worth of mentioning here the deep development of Teitelboim's theory devised recently in [53, 86] and applied to the non-abelian Yang–Mills equations, naturally generalizing the classical Maxwell equations. Nevertheless, all of these explanations prove to be somewhat unsatisfactory from the modern physics of view. In view of this, we will reanalyze once more the structure of the "radiative" Lorentz type force (144) from the vacuum field theory approach of Section 2 and show that this force allows some natural slight modification.

4.2 Radiation reaction force: vacuum-field theory approach

Here, we develop further our vacuum field theory approach devised in [15, 76, 75, 77] for the electromagnetic Maxwell and Lorentz electron theories and show that it is in complete agreement with the classical results and even more; it allows some nontrivial generalizations, which may have important physical applications. It will be also shown that the closely related electron mass problem can be satisfactorily explained in the context of this vacuum field theory approach and the spatial electron structure assumption.

The modified Lorentz force acting on a particle of charge $\xi \in \mathbb{R}$ exerted by a charged particle $\xi_f \in \mathbb{R}$ moving with velocity $u_f \in T(\mathbb{E}^3)$ was derived in Section 2 and equals

$$dp/dt = -\xi E + \xi \frac{u}{c} \times B - \nabla < A, u - u_f >, \tag{146}$$

where $(\varphi, A) \in T^*(M^4)$ is the external electromagnetic potential calculated with respect to a fixed laboratory reference frame $\mathcal{K}$. To take into account the self-interaction of this particle, we make use of a spatially distributed charge density $\rho : M^4 \to \mathbb{R}$ satisfying the condition

$$\xi = \int_{\mathbb{R}^3} \rho(t,r) d^3 r \tag{147}$$

for all $t \in \mathbb{R}$ for this laboratory reference frame $\mathcal{K}$ with coordinates $(t,r) \in M^4$. Then, owing to 146 and the reasoning from Section 2, the self-interacting force of this spatially structured charge $\xi \in \mathbb{R}$ can be expressed with respect to this laboratory reference frame $\mathcal{K}$ in the following equivalent form:

$$\begin{aligned} dp/dt := F_s = &- \tfrac{1}{c}\tfrac{d}{dt}\left[\textstyle\int_{\mathbb{E}^3} d^3 r \rho(t,r) A_s(t,r)\right] \\ &- \textstyle\int_{\mathbb{E}^3} d^3 r \rho(t,r) \nabla \varphi_s(t,r)\ (1 - |u/c|^2), \end{aligned} \tag{148}$$

where

$$\begin{aligned} \varphi_s(t,r) &= \int_{\mathbb{E}^3} \frac{\rho(t',r')|_{ret} d^3 r'}{|r-r'|}, \\ A_s(t,r) &= \frac{1}{c}\int_{\mathbb{E}^3} \frac{u(t')\rho(t',r')|_{ret} d^3 r'}{|r-r'|}, \end{aligned} \tag{149}$$

are the well-known retarded Lienard-Wiechert potentials, which should be calculated at the retarded time parameter $t' := t - |r - r'|/c \in \mathbb{R}$. Also taking into account the continuity equation

$$\partial\rho/\partial t + < \nabla, J >= 0 \tag{150}$$

for the spatially distributed charge density $\rho : M^4 \to \mathbb{R}$ and current $J = \rho u : M^4 \to \mathbb{E}^3$ and the Taylor expansions for retarded potentials (149)

$$\begin{aligned} \varphi_s(t,r) &= \sum_{n \in \mathbb{Z}_+} \frac{\partial^n}{\partial t^n} \int_{\mathbb{E}^3} \frac{(-|r-r'|)^n}{c^n n!} \frac{\rho(t,r') d^3 r'}{|r-r'|}, \\ A_s(t,r) &= \sum_{n \in \mathbb{Z}_+} \frac{\partial^n}{\partial t^n} \int_{\mathbb{E}^3} \frac{(-|r-r'|)^n}{c^n n!} \frac{J(t,r') d^3 r'}{|r-r'|}, \end{aligned} \tag{151}$$

from (148) and (151), assuming a spherical charge distribution, $|u|/c \ll 1$ and small acceleration, followed by calculations similar to those of [50, 62], one obtains

$$\begin{aligned} F_s = &\textstyle\sum_{n \in \mathbb{Z}_+} \frac{(-1)^n}{n! c^{n+2}} (1 - |u/c|^2) \int_{\mathbb{E}^3} d^3 r \rho(t,r) \\ &\textstyle\int_{\mathbb{E}^3} d^3 r' |r-r'|^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} \left(\frac{J(t,r')}{n+2} + \frac{(n-1) J(t,r')}{3(n+2)} \right) \\ &\textstyle+ \frac{d}{dt} [\sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m! c^m} \int_{\mathbb{E}^3} d^3 r \rho(t,r) \\ &\textstyle\int_{\mathbb{E}^3} d^3 r' \frac{|r-r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} J(t,r')] \\ &\textstyle= \sum_{n \in \mathbb{Z}_+} \frac{(-1)^{n+1}}{n! c^{n+2}} (1 - |u/c|^2) \int_{\mathbb{E}^3} d^3 r \rho(t,r) \\ &\textstyle\int_{\mathbb{E}^3} d^3 r' |r-r'|^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} J(t,r') \\ &\textstyle+ \frac{d}{dt} [\sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m! c^m} \int_{\mathbb{E}^3} d^3 r \rho(t,r) \\ &\textstyle\int_{\mathbb{E}^3} d^3 r' \frac{|r-r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} J(t,r')] \\ &\textstyle= \sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m! c^{m+2}} \int_{\mathbb{E}^3} d^3 r \rho(t,r) (1 - |u/c|^2) \\ &\textstyle\int_{\mathbb{E}^3} d^3 r' |r-r'|^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} J(t,r') \\ &\textstyle+ \frac{d}{dt} [\sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m! c^m} \int_{\mathbb{E}^3} d^3 r \rho(t,r) \\ &\textstyle\int_{\mathbb{E}^3} d^3 r' \frac{|r-r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} J(t,r')] \\ &\textstyle= \frac{d}{dt} [\sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m! c^m} \int_{\mathbb{E}^3} d^3 r \rho(t,r) \\ &\textstyle\int_{\mathbb{E}^3} d^3 r' \frac{|r-r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} [J(t,r') - \frac{1}{3} J(t,r')] \\ &\textstyle- \sum_{n \in \mathbb{Z}_+} \frac{(-1)^m}{3 m! c^{m+2}} (1 - |u/c|^2) \int_{\mathbb{E}^3} d^3 r \frac{\partial \rho(t,r)}{\partial t} \\ &\textstyle\int_{\mathbb{E}^3} d^3 r' |r-r'|^{m-1} \frac{\partial^{m+1}}{\partial t^{m+1}} J(t,r') \\ &\textstyle= \frac{d}{dt} [\sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m! c^m} \int_{\mathbb{E}^3} d^3 r \rho(t,r) \\ &\textstyle\int_{\mathbb{E}^3} d^3 r' \frac{|r-r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} [J(t,r') - \frac{1}{3} J(t,r')], \end{aligned} \tag{152}$$

The relationship above can be rewritten, owing to the charge continuity equation (150), gives rise to the radiation force expression

$$F_s = \sum_{n\in\mathbb{Z}_+} \tfrac{(-1)^n}{n!c^{n+2}}(1-|u/c|^2)\int_{\mathbb{E}^3} d^3r\rho(t,r)$$
$$\int_{\mathbb{E}^3} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}\left(\tfrac{J(t,r')}{n+2}+\tfrac{n-1}{n+2}\tfrac{<r-r',J(t,r')>(r-r')}{|r-r'|^2}\right)$$
$$+\tfrac{d}{dt}[\sum_{m\in\mathbb{Z}_+}\tfrac{(-1)^{m+1}}{m!c^m}\int_{\mathbb{E}^3} d^3r\rho(t,r)$$
$$\int_{\mathbb{E}^3} d^3r'\tfrac{|r-r'|^{m+1}}{c^2}\tfrac{\partial^m}{\partial t^m}J(t,r')]$$
$$=\sum_{n\in\mathbb{Z}_+}\tfrac{(-1)^{n+1}}{n!c^{n+2}}(1-|u/c|^2)\int_{\mathbb{E}^3} d^3r\rho(t,r)$$
$$\int_{\mathbb{E}^3} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}\left(\tfrac{J(t,r')}{n+2}+\tfrac{n-1}{n+2}\tfrac{|r-r',u|^2J(t,r')}{|r-r'|^2|u|^2}\right)$$
$$+\tfrac{d}{dt}[\sum_{m\in\mathbb{Z}_+}\tfrac{(-1)^{m+1}}{m!c^m}\int_{\mathbb{E}^3} d^3r\rho(t,r)$$
$$\int_{\mathbb{E}^3} d^3r'\tfrac{|r-r'|^{m+1}}{c^2}\tfrac{\partial^m}{\partial t^m}J(t,r')]. \tag{153}$$

Applying to (4.2) the rotational symmetry property for calculation of the internal integral, one easily obtains

$$F_s = \sum_{n\in\mathbb{Z}_+}\tfrac{(-1)^n}{n!c^{n+2}}(1-|u/c|^2)\int_{\mathbb{E}^3} d^3r\rho(t,r)$$
$$\int_{\mathbb{E}^3} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}\left(\tfrac{J(t,r')}{n+2}+\tfrac{(n-1)J(t,r')}{3(n+2)}\right)$$
$$+\tfrac{d}{dt}[\sum_{m\in\mathbb{Z}_+}\tfrac{(-1)^{m+1}}{m!c^m}\int_{\mathbb{E}^3} d^3r\rho(t,r)$$
$$\int_{\mathbb{E}^3} d^3r'\tfrac{|r-r'|^{m+1}}{c^2}\tfrac{\partial^m}{\partial t^m}J(t,r')]$$
$$=\sum_{n\in\mathbb{Z}_+}\tfrac{(-1)^n}{3n!c^{n+2}}(1-|u|^2)\int_{\mathbb{E}^3} d^3r\rho(t,r)$$
$$\int_{\mathbb{E}^3} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}J(t,r')$$
$$+\tfrac{d}{dt}[\sum_{m\in\mathbb{Z}_+}\tfrac{(-1)^{m+1}}{m!c^m}\int_{\mathbb{E}^3} d^3r\rho(t,r)$$
$$\int_{\mathbb{E}^3} d^3r'\tfrac{|r-r'|^{m+1}}{c^2}\tfrac{\partial^m}{\partial t^m}J(t,r')]$$

$$=\sum_{n\in\mathbb{Z}_+}\tfrac{(-1)^n}{3n!c^{n+2}}(1-|u/c|^2)\int_{\mathbb{E}^3} d^3r\rho(t,r)$$
$$\int_{\mathbb{E}^3} d^3r'|r-r'|^{n-1}\tfrac{\partial^{n+1}}{\partial t^{n+1}}J(t,r')$$
$$+\tfrac{d}{dt}[\sum_{m\in\mathbb{Z}_+}\tfrac{(-1)^{m+1}}{m!c^m}\int_{\mathbb{E}^3} d^3r\rho(t,r)$$
$$\int_{\mathbb{E}^3} d^3r'\tfrac{|r-r'|^{m+1}}{c^2}\tfrac{\partial^m}{\partial t^m}J(t,r')] \tag{154}$$
$$=\tfrac{d}{dt}[\sum_{m\in\mathbb{Z}_+}\tfrac{(-1)^{m+1}}{m!c^m}\int_{\mathbb{E}^3} d^3r\rho(t,r)$$
$$\int_{\mathbb{E}^3} d^3r'\tfrac{|r-r'|^{m+1}}{c^2}\tfrac{\partial^m}{\partial t^m}[J(t,r')-\tfrac{1}{3}J(t,r')]$$
$$-\sum_{n\in\mathbb{Z}_+}\tfrac{(-1)^m}{3m!c^{m+2}}(1-|u/c|^2)\int_{\mathbb{E}^3} d^3r\tfrac{\partial\rho(t,r)}{\partial t}$$
$$\int_{\mathbb{E}^3} d^3r'|r-r'|^{m-1}\tfrac{\partial^{m+1}}{\partial t^{m+1}}J(t,r')$$

$$=\tfrac{d}{dt}[\sum_{m\in\mathbb{Z}_+}\tfrac{(-1)^{m+1}}{m!c^m}\int_{\mathbb{E}^3} d^3r\rho(t,r)$$
$$\int_{\mathbb{E}^3} d^3r'\tfrac{|r-r'|^{m+1}}{c^2}\tfrac{\partial^m}{\partial t^m}[J(t,r')-\tfrac{1}{3}J(t,r')]$$
$$-\sum_{n\in\mathbb{Z}_+}\tfrac{(-1)^m}{3m!c^{m+2}}(1-|u/c|^2)\int_{\mathbb{E}^3} d^3r\tfrac{\partial\rho(t,r)}{\partial t}$$
$$\int_{\mathbb{E}^3} d^3r'|r-r'|^{m-1}\tfrac{\partial^{m+1}}{\partial t^{m+1}}J(t,r')$$
$$=\tfrac{d}{dt}[\sum_{m\in\mathbb{Z}_+}\tfrac{(-1)^{m+1}}{m!c^m}\int_{\mathbb{E}^3} d^3r\rho(t,r)$$
$$\int_{\mathbb{E}^3} d^3r'\tfrac{|r-r'|^{m+1}}{c^2}\tfrac{\partial^m}{\partial t^m}[J(t,r')-\tfrac{1}{3}J(t,r')]$$
$$+\sum_{n\in\mathbb{Z}_+}\tfrac{(-1)^m}{3m!c^{m+2}}(1-|u/c|^2)\int_{\mathbb{E}^3} d^3r<\nabla,J(t,r')>$$
$$\int_{\mathbb{E}^3} d^3r'|r-r'|^{m-1}\tfrac{\partial^{m+1}}{\partial t^{m+1}}J(t,r')$$
$$=\tfrac{d}{dt}[\sum_{m\in\mathbb{Z}_+}\tfrac{(-1)^{m+1}}{m!c^m}\int_{\mathbb{E}^3} d^3r\rho(t,r)$$
$$\int_{\mathbb{E}^3} d^3r'\tfrac{|r-r'|^{m+1}}{c^2}\tfrac{\partial^m}{\partial t^m}[J(t,r')-\tfrac{1}{3}J(t,r')],$$

where we took into account [50] that in case of the spherical charge distribution the following equalities

$$\int_{\mathbb{E}^3} d^3r\int_{\mathbb{E}^3} d^3r'\rho(t,r)\rho(t,r')\tfrac{|<r-r',u(t)>|^2}{|r-r'|^2|u(t)|^2}=\tfrac{1}{3}\xi^2,$$
$$\int_{\mathbb{E}^3} d^3r<\nabla,J(t,r')>$$
$$\int_{\mathbb{E}^3} d^3r'|r-r'|^{m-1}\tfrac{\partial^{m+1}}{\partial t^{m+1}}J(t,r')=0,$$
$$\int_{\mathbb{E}^3} d^3r\int_{\mathbb{E}^3} d^3r\rho(t,r)\rho(t,r')\tfrac{(r-r')}{|r-r'|^3}=0 \tag{155}$$

hold. Thus, from (152) one easily finds up to the $O(1/c^4)$ accuracy the following radiation reaction force expression:

$$\tfrac{dp}{dt}=F_s=-\tfrac{d}{dt}\left(\tfrac{4\mathcal{E}_{es}}{3c^2}u(t)\right)-\tfrac{d}{dt}\left(\tfrac{2\mathcal{E}_{es}}{3c^2}|u/c|^2\,u(t)\right)$$
$$+\tfrac{2\xi^2}{3c^3}\tfrac{d^2u}{dt^2}+O(1/c^4)$$
$$=-\tfrac{d}{dt}\left(\tfrac{4}{3}m_0^{(es)}(1+\tfrac{|u/c|^2}{2})u(t)\right)+\tfrac{2\xi^2}{3c^3}\tfrac{d^2u}{dt^2}+O(1/c^4)$$
$$=-\tfrac{d}{dt}\left(\tfrac{4}{3}\tfrac{m_0^{(es)}u(t)}{(1-|u/c|^2)^{1/2}}\right)+\tfrac{2\xi^2}{3c^3}\tfrac{d^2u}{dt^2}+O(1/c^4)$$
$$=-\tfrac{d}{dt}\left(\tfrac{4}{3}m^{(es)}u(t)\right)+\tfrac{2\xi^2}{3c^3}\tfrac{d^2u}{dt^2}+O(1/c^4), \tag{156}$$

where we defined the electrostatic self-interaction repulsive energy as

$$\mathcal{E}_{es}:=\frac{1}{2}\int_{\mathbb{R}^3}d^3r\int_{\mathbb{R}^3}d^3r'\frac{\rho(t,r)\rho(t,r')}{|r-r'|}, \tag{157}$$

and the electromagnetic charged particle rest and inertial masses, respectively, as

$$m_0^{(es)}:=\frac{\mathcal{E}_{es}}{c^2},\qquad m^{(es)}:=\frac{m_0^{(es)}}{(1-|u/c|^2)^{1/2}}. \tag{158}$$

Now it follows from (146) that

$$\frac{d}{dt}\left[(m_g+m^{(es)})u\right]=\frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}+O(1/c^4), \tag{159}$$

where we made use of the inertial mass definition

$$m_g:=-\bar{W}_g/c^2,\quad \nabla\bar{W}_g\simeq 0,\ m_s:=W_s/c^2, \tag{160}$$

following from the vacuum field theory approach, where the $m_g\in\mathbb{R}$ is the corresponding gravitational mass of the charged particle ξ, generated by the vacuum field potential $\bar{W}_g$. The corresponding radiation force

$$F_r=\frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}+O(1/c^4), \tag{161}$$

coincides exactly with the classical Abraham-Lorentz-Dirac results. From (159) it follows that the observable physical charged particle mass $m_{ph}\simeq m_g+m^{(es)}$ consists of two impacts: the electromagnetic and gravitational components, giving rise to the final force expression

$$\frac{d}{dt}(m_{ph}u)=\frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}+O(1/c^4), \tag{162}$$

where $m_{ph}\simeq m_g+m^{(es)}$ is the physically observed charged particle mass. It means, in particular, that the physically observed "inertial" mass m_{ph} of a real electron strongly depends on the external physical interaction with the ambient vacuum medium, as it was recently demonstrated via different approaches in [66, 84] based on vacuum Casimir effect considerations. Moreover, the assumed boundedness of the electrostatic self-energy $\mathcal{E}_{es}$ appears to be completely equivalent to the existence of so-called intrinsic Poincaré type "tensions", analyzed in [17, 39, 66], and to the existence of a special compensating Coulomb "pressure", suggested in [84], guaranteeing the observable electron stability.

4.3 Comments

The charged particle radiation problem, revisited in this section, allows to conceive the following explanation of the charged particle mass as that of a compact and stable object which should be influenced by a vacuum field interaction energy potential $\bar{W} \in \mathbb{R}^3$ of negative sign as follows from (160). This can be satisfied if and only if the expression (159) holds, thereby imposing on the intrinsic charged particle structure [63] certain nontrivial geometrical constraints. Moreover, as follows from the physically observed particle mass expressions (160), the electrostatic potential energy, being of the repulsive force origin, does contribute to the full mass as its main energy component.

There also exist different relativistic generalizations of the force expression (159), which suffer the same common physical inconsistency related to the no radiation effect of a charged particle in uniform motion.

Another problem closely related to the radiation reaction force analyzed above, is the search for an explanation of the Wheeler and Feynman reaction radiation mechanism, called the absorption radiation theory, based on the Mach type interaction of a charged particle with the ambient vacuum electromagnetic medium. There are observable relationships between this problem and the ones investigated here using the vacuum field theory approach, but this needs a more detailed and extended analysis.

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