On corrected shape functions for six-node triangular elements applied to heat conduction problems

Paweł Jarzębski[1] **and Krzysztof Wiśniewski**[2]

[1] *IFTR, Polish Academy of Sciences, Pawińskiego 5B, 02-106 Warsaw, Poland e-mail: pjarz@ippt.pan.pl*

[2] *IFTR, Polish Academy of Sciences, Pawińskiego 5B, 02-106 Warsaw, Poland e-mail: kwisn@ippt.pan.pl*

Abstract

The paper concerns two-dimensional 6-node triangular elements for heat conduction problems. We modified the 6-node triangular element using the corrected shape functions proposed by Celia and Gray [1] instead of the isoparametric ones. The numerical tests indicate that, for distorted meshes, the solution errors are reduced compared to the errors for the standard shape functions.

Keywords: heat conduction, two-dimensional six-node triangular elements, corrected shape functions, shape distortions.

1. Introduction

The corrected shape functions were proposed by Celia and Gray [1] as a generalization of the standard concept of isoparametric elements. In [1], these functions were derived and tested for an 8-node (serendipity) quadrilateral element for the heat conduction equation integrated by the 4×4 rule.

The corrected shape functions have proven useful also in plane elasticity problems solved by 9-node elements based on two-level interpolations of strain (AS and MITC elements). It was shown in [4] that, besides the modified transformations, the corrected shape function were one of the essential improvements of the MITC9 element, which enabled passing the patch test for some irregular meshes. In [3], it was shown that several well known formulations of nine-node elements for plane elasticity can benefit from using the corrected shape functions.

In the current paper, we extend the range of applications of the concept of the corrected shape functions of [1] to the six-node triangular element for heat conduction, which requires

1. generalization of the concept of the corrected shape functions to the six-node triangular element, which is formulated in terms of the area (barycentric) coordinates. This is not so natural as in the case of quadrilaterals, for which the natural coordinates are used.
2. implementation of the corrected shape functions in the six-node element and passing it through a range of tests involving several types of mesh distortions. The purpose is to confirm passing the patch test, and to show their improved accuracy and reduced sensitivity to mesh distortions.

2. Corrected shape functions

2.1. *Corrected shape functions for one dimensional element*

Let us first present the corrected shape functions for a one-dimensional three-node element. The standard isoparametric shape functions of such an element are

$$\mathbf{N}_\xi \doteq [(1-\xi)(1-2\xi), \quad 4(1-\xi)\xi, \quad \xi(2\xi-1)], \tag{1}$$

where the coordinate $\xi \in [0,1]$. If, in derivation of these functions, we use a parameter α instead of zero as a coordinate of the middle node, then we obtain the corrected shape functions

$$\mathbf{N}_\xi = \left[\frac{(1-\xi)(1-2\xi+2\alpha)}{1+2\alpha}, \frac{4(1-\xi)\xi}{1-4\alpha^2}, \frac{\xi(2\xi-2\alpha-1)}{1-2\alpha}\right], \tag{2}$$

where the parameter $\alpha \in (-\frac{1}{2}, +\frac{1}{2})$ describes the position of the middle node in the natural coordinates. In other words, α is a shift of the middle node from $\xi = \frac{1}{2}$, and it can be calculated using distances of nodes in the physical space. For $\alpha = 0$, the corrected shape functions reduce to the standard ones.

2.2. *Standard shape functions for six-node element*

The standard shape functions for the six-node element are as follows:

$$\begin{aligned} N_1(\xi,\eta) &\doteq \xi(2\xi-1), & N_2(\xi,\eta) &\doteq \eta(2\eta-1), \\ N_3(\xi,\eta) &\doteq \zeta(2\zeta-1), & N_4(\xi,\eta) &\doteq 4\xi\eta, \\ N_5(\xi,\eta) &\doteq 4\eta\zeta, & N_6(\xi,\eta) &\doteq 4\xi\zeta, \end{aligned} \tag{3}$$

where $\zeta \doteq 1-\xi-\eta$ and the area (barycentric) coordinates $\xi, \eta, \zeta \in [0,1]$. The nodes are numbered as shown in Fig. 1, where 1, 2, 3 are the corner nodes, and 4, 5, 6 designate the side nodes.

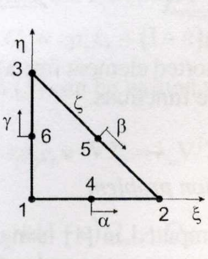

Figure 1: Six-node element. Numbering of nodes and positive values of distortion parameters α, β, γ.

2.3. *Corrected shape functions for six-node element*

For the six-node element, the shifts of the side nodes are described by 3 scalar parameters, $\alpha, \beta, \gamma \in (-\frac{1}{2}, +\frac{1}{2})$, see Fig.1. The corrected shape functions of the 6-node element (obtained in an analogous way as these for the one-dimensional element, see

[2]) are defined as

$$\begin{aligned}\bar{N}_1 &\doteq \frac{\xi\{(2\alpha-1)(2\gamma-1)\zeta+(2\gamma+1)[2\alpha\eta+(2\alpha-1)\xi+\eta]\}}{(2\alpha-1)(2\gamma+1)},\\ \bar{N}_2 &\doteq \frac{\eta\{(2\alpha+1)(2\beta+1)\zeta+(2\beta-1)[2\alpha\eta+(2\alpha-1)\xi+\eta]\}}{(2\alpha+1)(2\beta-1)},\\ \bar{N}_3 &\doteq \frac{\zeta[(2\beta+1)(2\gamma-1)\zeta+(2\beta-1)(2\gamma-1)\eta+(2\beta+1)(2\gamma+1)\xi]}{(2\beta+1)(2\gamma-1)},\\ \bar{N}_4 &\doteq \frac{4\eta\xi}{1-4\alpha^2}, \quad \bar{N}_5 \doteq \frac{4\zeta\eta}{1-4\beta^2}, \quad \bar{N}_6 \doteq \frac{4\zeta\xi}{1-4\gamma^2}.\end{aligned} \tag{4}$$

When the parameters α, β, γ are set to zero then the standard shape functions of eq. (3) are recovered.

The procedure of determining α, β, γ is as follows. Let us determine α for the side curve 1-4-2 given in the parametric form $X(\xi) = \bar{\mathbf{N}}_\xi \cdot [X_1, X_4, X_2]$ and $Y(\xi) = \bar{\mathbf{N}}_\xi \cdot [Y_1, Y_4, Y_2]$, where $\bar{\mathbf{N}}_\xi$ is defined in eq.(2). The fractional distance of node 4 relative to nodes 1 and 2 (along the boundary curve) is required to be identical in the physical space and in the local space,

$$\frac{L_0^{1/2+\alpha}}{L_0^1} = \frac{\int_0^{1/2+\alpha} d\xi}{\int_0^1 d\xi}, \tag{5}$$

where the arc-length of the curve in the physical space is

$$L_0^a \doteq \int_0^a \sqrt{(dX/d\xi)^2 + (dY/d\xi)^2}\, d\xi, \tag{6}$$

depends on some $a \in [0,1]$. Eq. (5) can be transformed to a single non-linear equation in α and solved using e.g. the Newton method. The initial value of α is selected as described in [4].

3. Numerical tests

We tested our six-node triangular element based on either the standard or the corrected shape functions for the Gauss integration rule involving 3 mid-side points. Regardless of the shape functions used, our element has a correct number of zero eigenvalues (1) and passes the patch test for the elements with straight and curved sides and with the side nodes displaced from the middle positions.

Figure 2 shows that different irregular shapes of the boundaries are obtained for the same node positions when the standard and the corrected shape functions are applied.

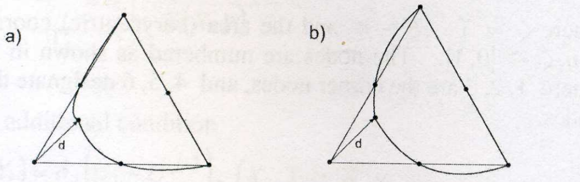

Figure 2: Shapes of distorted element for: a) standard shape functions, b) corrected shape functions.

3.1. *Angular conduction problem*

This example is computed in [1] using quadrilateral 8-node elements, while here we use our six-node triangular element with the standard and the corrected shape functions. The analyzed region, the finite element mesh and the boundary conditions are shown in Fig. 3. The analytical solution of this problem is

$$T_{ana} = (T_0 - T_1)\frac{2\theta}{\pi} + T_1, \qquad 0 \le \theta \le \frac{\pi}{2}. \tag{7}$$

Two analyzes were performed. The first one for the regular positions of side nodes and then the results are identical for both types of shape functions. For the second analysis, the side nodes designated by r_s and r'_s are moved radially in opposite directions, which distorts the mesh, see Fig. 3. The SRSS error is computed using the formula

$$e \doteq \left(\frac{1}{n}\sum_k^n e_k^2\right)^{1/2} \ge 0, \qquad e_k \doteq T_{ana} - T_k, \tag{8}$$

where e_k is an error at node k and n is the number of nodes where the solution is computed. Errors are shown in Fig .4 for varying value of the distortion.

We see that the use of the corrected shape function reduces the level of errors produced by the mesh distortion by up to two orders of magnitude. That is a similar reduction as in [1], where the error was reduced by up to three orders of magnitude.

Other examples of reduction of errors obtained by using the corrected shape functions are given in [2].

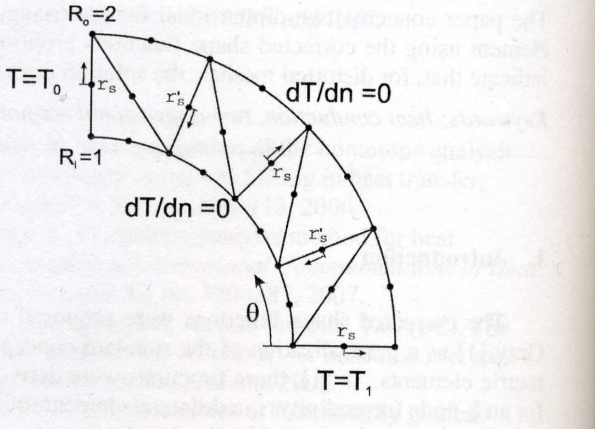

Figure 3: Finite element mesh and boundary conditions for the angular conduction. $T_0 = 0, T_1 = 8$.

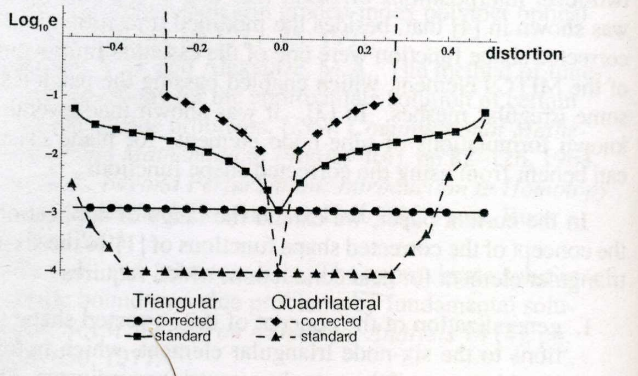

Figure 4: Angular conduction problem. Errors for distorted mesh using the corrected and the standard shape functions. Continuous lines - triangles, broken lines - quadrilaterals of [1].

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