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Hamiltonian Systems on Matrix Manifolds and Their Physical Applications

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Abstract. *Schrödinger equation as a self-adjoint differential equation of mathematical physics is discussed. For simplicity, a finite-level system is considered. A modified Schrödinger equation with the second time derivatives is described and some direct nonlinearity is admitted. The key of our idea is the assumption that the scalar product is not fixed once for all, but is a dynamical quantity mutually interacting with the state vector. We assume that the Lagrangian term describing its dynamics has the large symmetry group, the total complex linear group. This implies the strong essential nonlinearity. There is a hope that this geometrically implied nonlinearity may explain the decoherence and measurement paradoxes in quantum mechanics.*

1 Admitting second derivatives and direct nonlinearity

It is well-known that the Schrödinger equation is self-adjoint, i.e., derivable from the variational principle [9, 10, 11]. For simplicity we assume a finite-level quantum system, i.e., one with the finite-dimensional unitary space of states W. The more realistic infinite-dimensional case is ruled by a similar philosophy, although there are, of course, important differences in details. The sesquilinear hermitian form of the scalar product will be denoted by $\Gamma \in \overline{W^*} \otimes W^*$,

$$\Gamma(u, v) = \Gamma_{\overline{a}b}\overline{u}^{\overline{a}}v^{b}. \qquad (1)$$

Its hermicity,

$$\Gamma(u, v) = \overline{\Gamma(v, u)}, \qquad (2)$$

implies that the matrix $[\Gamma_{\overline{a}b}]$ is hermitian. As yet, Γ is an absolute element of the theory, later on it will become dynamical. Interactions are described by an appropriately chosen sesquilinear Hermitian form of the Hamiltonian, $\chi \in \overline{W^*} \otimes W^*$, or by its Γ-raised operator representation $H \in L(W)$,

$$H^{a}{}_{b} = \Gamma^{a\overline{c}}\chi_{\overline{c}b}\,, \quad \Gamma^{a\overline{c}}\Gamma_{\overline{c}b} = \delta^{a}{}_{b}. \qquad (3)$$

The Lagrangian describing Schrödinger dynamics may be given as:

$$L[1] = i\alpha\Gamma_{\overline{a}b}\left(\overline{\Psi}^{\overline{a}}\dot{\Psi}^{b} - \dot{\overline{\Psi}}^{\overline{a}}\Psi^{b}\right) - \gamma\chi_{\overline{a}b}\overline{\Psi}^{\overline{a}}\Psi^{b}, \tag{4}$$

where α, γ are real constants. This implies that $L[1]$ is a real-valued function of the complex quantities Ψ, $\dot{\Psi}$. Obviously, $\dot{\Psi}$ denotes the time derivative of Ψ. If, as usual in quantum mechanics, Γ is assumed to be positive, then there exists a basis in which $[\Gamma_{\overline{a}b}] = I_n$. Where I_n is the itentity matrix $n \times n$.

Obviously, the Euler-Lagrange equations for $L[1]$ have the form:

$$\alpha i \frac{d\Psi^{a}}{dt} = \frac{\gamma}{2} H^{a}{}_{b}\Psi^{b}. \tag{5}$$

When we put $\alpha = \hbar$, $\gamma = 2$, this becomes the usual finite-dimensional Schrödinger equation.

From now on, the Schrödinger equation will be treated formally as a "classical" self-adjoint equation of mathematical physics [11]. And our search of non-linear modifications will be based just on this treatment. Before doing any step towards introducing nonlinearity, we shall consider certain linear modifications suggested by the mentioned philosophy of analytical mechanics. First of all, the time-derivatives of Ψ enter the Lagrangian (4) linearly, although the coefficients depend algebraically on Ψ. Because of this, the Legendre transformation is non-invertible, it leads to (primary) constraints in the phase space, and the generalized Dirac procedure must be used if we wish to use a modified Hamiltonian formalism. But, let us remind, the primary attempts by Schrödinger were based on the second-order equation with is known today as the Klein-Gordon equation. Quite independently on the problems with statistical interpretation, the Klein-Gordon equation postulated by Schrödinger gave results for the atomic spectra incompatible with experiments. But the non-relativistic version with the first-order time derivatives worked perfectly. Today we known the reasons: there is no spin in Schrödinger equation, and the spin phenomena influence the atomic spectra just in the opposite direction to the relativistic effect of the velocity-dependence of mass. Yes, but there are still some reasons to expect the appearance of second derivatives, e.g., in connection with the conformal invariance [2, 5, 9, 11]. There is also some motivation from nanoscience [3, 6]. In any case the idea of admitting second derivatives in quantum-mechanical equations, i.e., of admitting terms quadratic in generalized velocities in Lagrangian, seems attractive [7, 8, 9, 10, 11]. The expression for $L[1]$ would be then replaced by

$$L[1,2] = i\alpha\Gamma_{\overline{a}b}\left(\overline{\Psi}^{\overline{a}}\dot{\Psi}^{b} - \dot{\overline{\Psi}}^{\overline{a}}\Psi^{b}\right) + \beta\Gamma_{\overline{a}b}\dot{\overline{\Psi}}^{\overline{a}}\dot{\Psi}^{b} - \gamma\chi_{\overline{a}b}\overline{\Psi}^{\overline{a}}\Psi^{b} \tag{6}$$

The resulting "Schrödinger equation" becomes then:

$$\alpha i \frac{d\Psi^{a}}{dt} - \frac{\beta}{2}\frac{d^{2}\Psi^{a}}{dt^{2}} = \frac{\gamma}{2} H^{a}{}_{b}\Psi^{b}. \tag{7}$$

It is important that the Legendre transformation is then non-degenerate, and the usual Hamiltonian formalism is applicable. The general structure of (7) is essentially analogous to the structure of nano-physical equations investigated by [3, 6].

The last modification of the first- and second-order Schrödinger equations is one concerning nonlinearity. The simplest way of introducing it is based on nonquadratic potentials V:

$$L[1;V] := L[1] + V \quad , \quad L[1,2;V] := L[1,2] + V. \tag{8}$$

The whole sophisticated art of playing with V- nonlinearity has been developed. Let us quote only some of them, e.g.,

$$V\left(\Psi, \overline{\Psi}\right) := f\left(\Gamma_{\overline{a}b}\overline{\Psi}^{\overline{a}}\Psi^{b}\right), \tag{9}$$

where f is an appropriately chosen real function of the one real variable. One of the frequently used models is based on the quartic form of Ψ:

$$f(x) = \varkappa(x-a)^2. \tag{10}$$

There are various arguments for this expression, taken both from the pure nonlinear science and from certain models of quantum field theory . Obviously, the nonlinear Schrödinger equations derived from (8), (9) will have the form:

$$\alpha i \frac{d\Psi^a}{dt} - \frac{\beta}{2}\frac{d^2\Psi^a}{dt^2} = \frac{\gamma}{2}H^a{}_b\Psi^b + \frac{1}{2}f'\Psi^a, \tag{11}$$

where the symbol f' denotes the usual derivative of f in the standard sense of differential calculus. Equation (11) exhausts, when playing with f, a large class of practically used nonlinearity models in quantum mechanics. Or, to be more honest: not equation (11), but rather, its infinite-level counterpart. Nonlinearity is contained in the last term in the right-hand side, because the multiplier $\frac{1}{2}f'$ depends on Ψ.

So, as said above, we pretend to forget for a moment about quantum problems and consider the above Schrödinger equations as some self-adjoint equations in classical analytical mechanics. So let us review the basic Poisson brackets and the ideas of the corresponding Hamiltonian mechanics. As usual, we formally consider Ψ^a and $\overline{\Psi}^{\overline{a}}$ as independent variables. Let us mention, one deals so also in classical field theory when using it as a primary step to the quantization procedure. Obviously, one can use as well the 2n-tuple of real generalized coordinates x^a, y^a, and their conjugate momenta u_a, v_a. But it is formally more convenient to use 2n-tuple of complex generalized coordinates Ψ^a, $\overline{\Psi}^{\overline{a}}$ and their conjugate momenta π_a, $\overline{\pi}_{\overline{b}}$. The relationship is given by

$$\Psi^a = \frac{1}{\sqrt{2}}\left(x^a + iy^a\right) \quad , \quad \overline{\Psi}^{\overline{a}} = \frac{1}{\sqrt{2}}\left(x^a - iy^a\right),$$
$$\pi_a = \frac{1}{\sqrt{2}}\left(u_a - iv_a\right) \quad , \quad \overline{\pi}_{\overline{a}} = \frac{1}{\sqrt{2}}\left(u_a + iv_a\right). \tag{12}$$

When doing so, the care must be taken to use real-valued Lagrangians and Hamiltonians. And then, deriving the equations of motion, it is sufficient to perform the variational procedure either with respect to $\overline{\Psi}$ or with respect to Ψ. The resulting equations of motion are evidently dependent, they are complex conjugations of each other. But it is more convenient to use the redundant 2n-dimensional complex language [11].

Equations (12) imply that the usual Poisson bracket,

$$\{f,g\} = \frac{\partial f}{\partial x^a}\frac{\partial g}{\partial u_a} + \frac{\partial f}{\partial y^a}\frac{\partial g}{\partial v_a} - \frac{\partial f}{\partial u_a}\frac{\partial g}{\partial x^a} - \frac{\partial f}{\partial v_a}\frac{\partial g}{\partial y^a}, \tag{13}$$

is transformed by the mentioned analytical continuation to

$$\{f,g\} = \frac{\partial f}{\partial \Psi^a}\frac{\partial g}{\partial \pi_a} + \frac{\partial f}{\partial \overline{\Psi}^{\bar{a}}}\frac{\partial g}{\partial \overline{\pi}_{\bar{a}}} - \frac{\partial f}{\partial \pi_a}\frac{\partial g}{\partial \Psi^a} - \frac{\partial f}{\partial \overline{\pi}_{\bar{a}}}\frac{\partial g}{\partial \overline{\Psi}^{\bar{a}}}. \tag{14}$$

The vector field generated by the Hamiltonian function F on the complex phase space $W \times \overline{W} \times W^* \times \overline{W}^*$ is given by

$$X_F = \frac{\partial F}{\partial \pi_a}\frac{\partial}{\partial \Psi^a} + \frac{\partial F}{\partial \overline{\pi}_{\bar{a}}}\frac{\partial}{\partial \overline{\Psi}^{\bar{a}}} - \frac{\partial F}{\partial \Psi^a}\frac{\partial}{\partial \pi_a} - \frac{\partial F}{\partial \overline{\Psi}^{\bar{a}}}\frac{\partial}{\partial \overline{\pi}_{\bar{a}}}. \tag{15}$$

One must remember that in this language of complex extension the quantities Ψ^a, $\overline{\Psi}^{\bar{a}}$ are treated as independent coordinates, and so are π_a, $\overline{\pi}_{\bar{b}}$.

For the model (6) of $L[1,2;V]$ regularized by the term quadratic in velocities the Legendre transformation [11],

$$\pi_a = i\alpha\overline{\Psi}^{\bar{b}}\Gamma_{\bar{b}a} + \beta\dot{\overline{\Psi}}^{\bar{b}}\Gamma_{\bar{b}a} \quad , \quad \overline{\pi}_{\bar{a}} = -i\alpha\Gamma_{\bar{a}b}\Psi^b + \beta\Gamma_{\bar{a}b}\dot{\Psi}^b, \tag{16}$$

is invertible,

$$\dot{\Psi}^a = \frac{1}{\beta}\Gamma^{a\bar{b}}\overline{\pi}_{\bar{b}} + \frac{\alpha i}{\beta}\Psi^a, \qquad \dot{\overline{\Psi}}^{\bar{a}} = \frac{1}{\beta}\pi_b\Gamma^{b\bar{a}} - \frac{\alpha i}{\beta}\overline{\Psi}^{\bar{a}}. \tag{17}$$

The "energy function" of analytical mechanics,

$$\mathcal{E} = \dot{\Psi}^a\frac{\partial L}{\partial \dot{\Psi}^a} + \dot{\Psi}^{\bar{a}}\frac{\partial L}{\partial \dot{\overline{\Psi}}^{\bar{a}}} - L, \tag{18}$$

becomes now as follows:

$$\mathcal{E} = \beta\Gamma_{\bar{a}b}\dot{\overline{\Psi}}^{\bar{a}}\dot{\Psi}^b + \gamma\chi_{\bar{a}b}\overline{\Psi}^{\bar{a}}\Psi^b + V\left(\Psi,\overline{\Psi}\right). \tag{19}$$

And one can show that the "Hamiltonian", i.e., "energy" expressed by (17) becomes [11]:

$$\begin{aligned}\mathcal{H} &= \frac{1}{\beta}\left(\Gamma^{a\bar{b}}\pi_a\overline{\pi}_{\bar{b}} + i\alpha\left[\pi_a\Psi^a - \overline{\pi}_{\bar{a}}\overline{\Psi}^{\bar{a}}\right]\right) \\ &+ \left(\frac{\alpha^2}{\beta}\Gamma_{\bar{a}b} + \gamma\chi_{\bar{a}b}\right)\overline{\Psi}^{\bar{a}}\Psi^b + V\left(\Psi,\overline{\Psi}\right).\end{aligned} \tag{20}$$

Everything is good here only if $\beta \neq 0$, i.e., when we really deal with the second-order Schrödinger equation. When $\beta \to 0$, then (17), (20) catastrophically become meaningless, and the Legendre transformation (16) leads to constraints in the phase space. For $\beta \neq 0$ the second-order Schrödinger equation is equivalent to the Hamilton equations:

$$\frac{d\Psi^a}{dt} = \frac{\partial \mathcal{H}}{\partial \pi_a} = \{\Psi^a, \mathcal{H}\}, \qquad \frac{d\pi_a}{dt} = -\frac{\partial \mathcal{H}}{\partial \Psi^a} = \{\pi_a, \mathcal{H}\}, \tag{21}$$

or to their complex conjugates. It is no longer the case when $\beta = 0$ and the Dirac procedure must be applied. But it is true, of course, that (19) behaves correctly when β approaches zero. We obtain then:

$$\mathcal{E} = \gamma \chi_{\overline{a}b} \overline{\Psi}^{\overline{a}} \Psi^b + V\left(\Psi, \overline{\Psi}\right), \tag{22}$$

i.e., an expression independent on generalized velocities. The expression (16) for the Legendre transformation is also continuous at $\beta = 0$ and becomes:

$$\pi_a = \alpha i \overline{\Psi}^{\overline{b}} \Gamma_{\overline{b}a}, \qquad \overline{\pi}_{\overline{a}} = -\alpha i \Gamma_{\overline{a}b} \Psi^b \tag{23}$$

This gives us the equations of primary constraints M in the phase space. They are very strong, because no arbitrariness of canonical momenta survives, at any configuration there is only one admissible momentum. The Dirac procedure must be used. Equations of primary constraints M have the form:

$$\Phi_a = \pi_a - i\alpha \Gamma_{\overline{b}a} \overline{\Psi}^{\overline{b}}, \qquad \overline{\Phi}_{\overline{a}} = \overline{\pi}_{\overline{a}} + i\alpha \Gamma_{\overline{a}b} \Psi^b. \tag{24}$$

The Hamilton function is postulated as:

$$\mathcal{H} = \mathcal{H}_0 + \lambda^a \Phi_a + \overline{\lambda}^{\overline{a}} \overline{\Phi}_{\overline{a}} \tag{25}$$

where $\mathcal{H}_0$ is simply given by (22)

$$\mathcal{H}_0 = \gamma \chi_{\overline{a}b} \overline{\Psi}^{\overline{a}} \Psi^b + V\left(\Psi, \overline{\Psi}\right). \tag{26}$$

The compatibility equation for $X_{\mathcal{H}}$ and M is uniquely solvable at every point of M, namely:

$$\lambda^a = -\frac{\gamma}{2\alpha} i H^a{}_b \Psi^b - \frac{i}{2\alpha} \Gamma^{a\overline{c}} \frac{\partial V}{\partial \overline{\Psi}^{\overline{c}}}, \qquad \overline{\lambda}^{\overline{a}} = \frac{\gamma}{2\alpha} i \overline{\psi}^{\overline{b}} H_{\overline{b}}{}^{\overline{a}} + \frac{i}{2\alpha} \frac{\partial V}{\partial \Psi^c} \Gamma^{c\overline{a}}, \tag{27}$$

where $H^a{}_b$ are given by (3), and $H_{\overline{b}}{}^{\overline{a}}$-by the corresponding Γ-shift of the indices. Therefore, the manifold of secondary constraints is identical with that of primary ones , $M^s = M$. The Dirac bracket is generated by the following basic rules for the functions on $M^s = M$:

$$\left\{\Psi^a, \Psi^b\right\}_M = 0, \qquad \left\{\overline{\Psi}^{\overline{a}}, \overline{\Psi}^{\overline{b}}\right\}_M = 0, \qquad \left\{\Psi^a, \overline{\Psi}^{\overline{b}}\right\}_M = \frac{1}{2\alpha i} \Gamma^{a\overline{b}}. \tag{28}$$

Therefore, on the manifold of Lagrangian constraints the complex conjugate vector of state becomes proportional to the canonical momentum conjugated to Ψ. The Schrödinger equation without second derivatives becomes formally identical with the Hamilton equation in the sense of Dirac bracket generated by (28):

$$\frac{d\Psi^a}{dt} = \{\Psi^a, \mathcal{H}\}_M \equiv i\hbar \frac{d\Psi^a}{dt} = H^a{}_b \psi^b + \frac{1}{2} \Gamma^{a\overline{b}} \frac{\partial V}{\partial \overline{\Psi}^{\overline{b}}} \tag{29}$$

(again we mean here the substitution $\alpha = \hbar$, $\gamma = 2$).

2 Krawietz-type metrics and geometric nonlinearity

In the above remarks we have treated very seriously the idea about quantum mechanics as ruled by a certain self-adjoint equation of mathematical physics. This equation was considered as an extremely over-simplified one, namely, in a finite-dimensional unitary space. Nevertheless, even in this very rough approach some important facts might be seen, e.g, the degeneracy of Lagrange-Hamilton structure of the corresponding variational principle. The Dirac procedure of dynamical constraints had to be used. This also enabled one to suppose that perhaps the term with second time derivatives in Schrödinger equation should be admitted, although it opens as well some questions far from being solved. There are also some nano-physical arguments showing that such a modifications may be just desirable. Quite other argument comes from the gauge-conformal attempts of reformulating gravitation theory. And finally, the idea of self-adjoint analytical interpretation of Schrödinger equation opens a wide field of looking for nonlinear corrections to quantum mechanics. Let us remind that such corrections are sought for many years in connection with the well-known quantum paradoxes concerning the measurement theory and decoherence [1]. In all these problems it is linearity of quantum mechanics that seems to be guilty. Nevertheless, the majority of nonlinearities suggested according to the above scheme seems to be rather naive and introduced "by hand". Now we would like to present another possibility, much more geometric, based on the invariance groups, and because of this, we mean, incomparatively deeper. This is the method of dynamical scalar product. Let us remind that the sesquilinear Hermitian scalar product $\Gamma \in \overline{W}^* \otimes W^*$ is fixed once for all in any quantum-mechanical model. One can ask why really. An interesting motivation against this pattern comes even from the generally-relativistic gravitation theory. Traditional Maxwell electrodynamics without changes is linear, although there are also nonlinear modifications like Born-Infeld theory. Other field theories, e.g., gauge treatments of physical phenomena are nonlinear, nevertheless, they are, in a sense, minimally-nonlinear, just only to the extent one is forced by gauge invariance. But everything changes drastically when gravitation is introduced into the treatment. It is well-known that the Hilbert action functional for gravitation is essentially nonlinear and it introduces this nonlinearity to any system of fields mutually interacting with gravitation. This nonlinearity is strongly connected with the demand of general covariance, i.e., invariance of any fundamental theory with respect to the space-time group of diffeomorphisms. The invariance demand with respect to this huge group introduces the essential dynamical nonlinearity for any systems of fields mutually interacting with gravitation.

Let us stress that from some point of view quantum mechanics resembles rather specially-relativistic than the generally-relativist physics. In special relativity the dynamics of fields and particles is controlled by Minkowskian metric tensor fixed once for all as an absolute object of the theory. In general relativity the metric tensor is a dynamical field subject to equations of motion on equal footing with all other fields; its physical meaning is just the gravitation. Analogy is obvious:

perhaps the scalar product in quantum mechanics is something similar to the space-time metric and should be rather a dynamical field, not a fixed absolute object. But what has this all to do with nonlinearity, decoherence, etc.? The answer is simple and has also to do with the physical demand of invariance under a "large" symmetry group. Let us begin with some introductory examples from a rather long time ago.

One example comes from the mechanics of continuous media, to be more precise from plasticity. Namely, rather a long time ago A. Krawietz [4] discussed the problem of natural Riemann metrics on a manifold of Euclidean metric tensors on the real linear space V. Obviously, the trivial possibility would be to fix some particular metric tensor g_0 and to define the arc element by

$$ds_0^2 = Tr\left(\left(g_0^{-1}dg\right)^2\right). \tag{30}$$

This is unnatural, very artificial in distinguishing the standard reference metric g_0 . The metric corresponding to (30) is evidently flat. But it is possible to define another, very natural metric which does not distinguish anything. It is given by

$$ds^2 = Tr\left(\left(g^{-1}dg\right)^2\right) = g^{bc}g^{da}dg_{ab}dg_{cd}, \tag{31}$$

where, of course, the upper-case quantity is the contravariant inverse of g. This is the very natural Riemann metric in the manifold of Euclidean metrics on V. It naturallty resembles that of the Killing metric tensor on the Lie group. Let us observe that no reference metric g_0 is chosen and the Riemann metric corresponding to (31) is invariant under the total $GL(V)$ acting in a natural way on the manifold of Euclidean metrics on V. Unlike this, (30) is invariant only under the action of $O(V, g_0)$, the subgroup of isometries of g_0. Let us observe that, as a matter of fact (31) might be replaced by the metric

$$ds^2 = \lambda g^{bc}g^{da}dg_{ab}dg_{cd} + \mu g^{ba}g^{dc}dg_{ab}dg_{cd} \tag{32}$$

which has the same properties like (31). Obviously, the quantities λ, μ are arbitrary real constants. The λ-term is the main constituents whereas the singular μ-term is merely a secondary correction.

The metric tensors (31), (32) are used in plasticity theory, first of all in its incremental methods. One can ask what would be the counterparts of those objects when instead of symmetric metric tensors in a real vector space V we considered the manifold of sesquilinear Hermitian forms on a complex linear space W. Let us assume that the forms are antilinear in the first index and linear in the second one, so we use the analytical symbols $g_{\overline{a}b}$ and $g^{a\overline{b}}$ for their contravariant inverses:

$$g^{a\overline{c}}g_{\overline{c}b} = \delta^a{}_b. \tag{33}$$

Obviously, the counterpart of (32) will be

$$ds^2 = \lambda g^{b\overline{c}}g^{d\overline{a}}dg_{\overline{a}b}dg_{\overline{c}d} + \mu g^{b\overline{a}}g^{d\overline{c}}dg_{\overline{a}b}dg_{\overline{c}d}, \tag{34}$$

where again the second term is merely a correction to the first, main one. This arc element is also $GL(W)$-invariant and does not assume a choice of anything in W or $\overline{W}^{*} \otimes W^{*}$.

But if so, one can try to think in the following way: perhaps in quantum dynamics two objects participate on the almost equal footing, i.e.: the "wave function" $\Psi \in W$ and scalar product $G \in \overline{W}^{*} \otimes W^{*}$. Their dynamics in the process of mutual interaction should be ruled by Lagrangian depending on the pair of variables $(\Psi, G) \in W \times (\overline{W}^{*} \otimes W^{*})$ and given by

$$\begin{aligned} L[\Psi, G] &= L[1,2;V] + T[G] = \\ &= i\alpha\Gamma_{\overline{a}b}\left(\overline{\Psi}^{\overline{a}}\dot{\Psi}^{b} - \dot{\overline{\Psi}}^{\overline{a}}\Psi^{b}\right) + \beta\Gamma_{\overline{a}b}\dot{\overline{\Psi}}^{\overline{a}}\dot{\Psi}^{b} - \gamma\chi_{\overline{a}b}\overline{\Psi}^{\overline{a}}\Psi^{b} + V(\Psi, \overline{\Psi}) \\ &\quad + \frac{A}{2}\Gamma^{b\overline{c}}\Gamma^{d\overline{a}}\dot{\Gamma}_{\overline{a}b}\dot{\Gamma}_{\overline{c}d} + \frac{B}{2}\Gamma^{b\overline{a}}\Gamma^{d\overline{c}}\dot{\Gamma}_{\overline{a}b}\dot{\Gamma}_{\overline{c}d}. \end{aligned} \tag{35}$$

Now all the quantities occurring as arguments of $L[\Psi, G]$ are dynamical variables subject to the variational procedure. It is easily seen that the total Lagrangian (35) corresponds to an effectively nonlinear theory even if the directly nonlinear term $V(\Psi, \overline{\Psi})$ is completely absent. Moreover, the scheme with $V(\Psi, \overline{\Psi}) = 0$ is much more natural and geometric. The first three terms in (35) correspond to the usual linear theory and its only unusuality is the second term which leads to the linear appearance of second time derivatives of Ψ. The main nonlinearity of the theory based on (35) is predicted by the last two terms, quadratic in $\frac{d}{dt}\Gamma$, but with Γ-dependent coefficients. This structure is very similar to that of generally-relativistic field theories where the main non-linear term comes from the Hilbert Lagrangian of the metric field. The metric tensor on the total configuration space, i.e., one responsible for non-linearity, is just (34) with Γ substituted for g.

The attempts of fighting for understanding of the decoherence and measurement paradoxes are our main motives for introducing this kind of nonlinearity to the theory. Nevertheless, even if those attempts fail, the structure of dynamical models following from (35) is interesting in itself. Let us mention, this structure of affinely-invariant dynamics was discovered by us in early seventies in mechanics of so-called affinely-rigid bodies, i.e., ones rigid in the sense of affine geometry [12, 13]. Let us remind the basic ideas. The configuration space of affinely-rigid body moving in the affine space M with the translation space V may be identified with $Q = M \times F(V)$, where $F(V)$ denotes the manifold of all linear frames in V. In certain applications one replaces $F(V)$ by the connected manifold $F(V)^{+}$ consisting of frames positively oriented with respect to some fixed orientation. In any case, configurations are given by pairs $(m; e_1, \ldots, e_n)$, where $m \in M$ represents the translational position of the body in M, and $e = (e_1, \ldots, e_n)$, or equivalently the dual frame $e^{-1} = (e^1, \ldots, e^n)$, represents the configuration of internal degrees of freedom. Analytically, configurations are described by generalized coordinates

$$\left(\ldots, x^{i}, \ldots; \ldots, e^{i}{}_{A}, \ldots\right) \text{ or } \left(\ldots, x^{i}, \ldots; \ldots, e^{A}{}_{i}, \ldots\right), \tag{36}$$

where, obviously,

$$e^{A}{}_{i} e^{i}{}_{B} = \delta^{A}{}_{B} \tag{37}$$

and x^i are coordinates of translational degrees of freedom. The usual formula for the kinetic energy is

$$T = \frac{m}{2} g_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{1}{2} g_{ij} \left(\frac{d}{dt} e^i{}_A\right)\left(\frac{d}{dt} e^j{}_B\right) J^{AB} = T_{tr} + T_{int} \tag{38}$$

where m is the mass, J^{AB} is symmetric and positively definite constant tensor of inertial momentum, and g_{ij} are components of the spatial metric tensor. Without any use of variational principle, equations of motion are derived via the d'Alembert procedure as follows

$$e^i{}_A \frac{d^2}{dt^2} e^j{}_B J^{AB} = N^{ij}, \tag{39}$$

where N^{ij} is the tensor of affine torques. Its skew-symmetric part is the usual torque of forces. The contributions T_{tr}, T_{int} refer to translational and internal kinetic energies. Unfortunately, neither (38) nor (39) is invariant under the left-acting (spatial, Eulerian) and right-acting (material, Lagrangian) affine transformations. In this sense the analogy with the rigid-body mechanics, when

$$g_{ij} e^i{}_A e^j{}_B = \delta_{AB}, \tag{40}$$

is broken and the system is not an invariant (either left- or right-) system on the affine or linear group. At the same time, in certain applications, like the collective affine model of the nuclear droplet, neither the d'Alembert procedure nor the model (38) seems to be adequate. On the contrary, it seems that affinely-invariant models of the dynamics are more adequate. To introduce them we have to use so-called affine velocities, i.e., affine generalization of angular velocity tensor. Affine velocity in the spatial (Euler) representation is the mixed tensor given by

$$\Omega^i{}_j = \frac{de^i{}_A}{dt} e^A{}_j, \tag{41}$$

its co-moving (Lagrange) representation is obviously given by

$$\widehat{\Omega}^A{}_B = e^A{}_i \frac{d}{dt} e^i{}_B = e^A{}_i \Omega^i{}_j e^j{}_B. \tag{42}$$

They either transform respectively under the adjoint rule or are invariant under the action of $GL(V)$ and $GL(n, \mathbb{R})$:

$$\begin{gathered}
[e^i{}_A] \to [L^i{}_j e^j{}_B] \;:\; [\Omega^i{}_j] \to [L^i{}_k \Omega^k{}_m L^{-1\,m}{}_j],\ [\widehat{\Omega}^A{}_B] \to [\widehat{\Omega}^A{}_B], \\
[e^i{}_A] \to [e^i{}_B K^B{}_A] \;:\; [\Omega^i{}_j] \to [\Omega^i{}_j],\ [\widehat{\Omega}^A{}_B] \to [K^{-1\,A}{}_C \widehat{\Omega}^C{}_D K^D{}_B].
\end{gathered} \tag{43}$$

Because of this the only possible kinetic energy form of internal degrees of freedom, invariant under all left- and right-acting linear mappings is a combination of two second-order Casimirs:

$$T_{int} = \frac{A}{2} Tr\left(\Omega^2\right) + \frac{B}{2}\left(Tr\ \Omega\right)^2 = \frac{A}{2} Tr\left(\widehat{\Omega}^2\right) + \frac{B}{2}\left(Tr\ \widehat{\Omega}\right)^2. \tag{44}$$

Let us mention that due to the lack of semi-simplicity in the affine group, there is neither translational nor total kinetic energy invariant simultaneously under left and right affine group. There are only models left-affinely and right-metrically invariant, and conversely, left-metrically and right-affinely invariant. They are given respectively by:

$$T = \frac{m}{2} C_{ij} v^i v^j + \frac{A}{2} Tr\left(\Omega^2\right) + \frac{B}{2}\left(Tr\ \Omega\right)^2 = \frac{m}{2}\eta_{AB}\widehat{v}^A \widehat{v}^B + \frac{A}{2} Tr\left(\widehat{\Omega}^2\right) + \frac{B}{2}\left(Tr\ \widehat{\Omega}\right)^2, \tag{45}$$

$$T = \frac{m}{2} g_{ij} v^i v^j + \frac{A}{2} Tr(\Omega^2) + \frac{B}{2}(Tr\Omega)^2 = \frac{m}{2} G_{AB}\hat{v}^A \hat{v}^B + \frac{A}{2} Tr(\Omega^2) + \frac{B}{2}(Tr\Omega)^2, \tag{46}$$

where C_{ij}, G_{AB} denote respectively the Cauchy and Green deformation tensors and $\widehat{v}^A$ are co-moving components of v.

It is interesting that the internal terms are not positively definite, besides of "centrifugal repulsion" between deformation invariants, they contain also their "centrifugal attraction". Because of this even in purely geodetic models there exists an open subset of bounded geodetic vibrations and an open subset of "above-threshold" geodetic scattering motion [12, 13]. This application resembles the Maupertuis principle, where the dynamics is described by the metric tensor of the configuration space. If there is no potential, those solutions may be expressed by the matrix exponential function.

Something similar may be done for our "classical-quantum" model. From the analytical point of view it is also a model of dynamical systems on a matrix manifold. There are, of course, certain differences between it and the model of affinely-rigid body. Matrices may describe various geometric objects: mixed tensors (the case of affinely-rigid body), twice covariant tensors, e.g., sesquilinear-Hermitian (just the model described here) and twice contravariant tensors. The geometric structure of our model is very nice, but it is rather difficult to find any solution. The reason is its very strong, geometrically-implied non-perturbative nonlinearity. But some academic, very particular estimations are possible. If we fix the scalar product to some constant value, then everything reduces to the Schrödinger equation, perhaps with non-linearity and second derivatives. In principle those are rather touchable things. And let us try to think about the opposite possibility: just the dynamics for the last two terms (35) as a Lagrangian. It turns out that in spite of all differences, also then there exist some matrix-exponential solutions. Namely, one can easily show that there exist for the time-dependence of G solutions of the form:

$$\Gamma_{\overline{a}b}(t) = G_{\overline{a}c}\exp(Et)^c{}_b, \tag{47}$$

where $E \in L(W) \simeq W \otimes W^*$ and the initial value $\Gamma(0) = G \in Herm(\overline{W}^* \otimes W^*)$ is a Hermitian form. Then one can show that $\Gamma(t)$ remains Hermitian for any time value $t \in \mathbb{R}$ if the linear mapping

$${}_G E_{\overline{a}b} = G_{\overline{a}c} E^c{}_b \tag{48}$$

is also Hermitian, of course, in the absolute G-independent sense. Let us observe that if E has negative eigenvalues, then ${}_\Gamma E$ tends to zero when $t \to \infty$. And it tends to infinity when the eigenvalues are positive. One can suspect this to be something like the academic model of decoherence and measurement, at least in some sense. Of course, as usual in the nonlinear interaction problems, there is no superposition of particular solutions. So, the problem of solving equations following from the total Lagrangian (35) may be very difficult. Nevertheless, we hope that the high symmetry $GL(W)$ may be helpful in finding at least some physically interesting solutions. Let us also state that the last two terms in (35) correspond to the simplest possibility. The total class of $GL(W)$-invariant Lagrangians for (Ψ, G) is much wider [11]. And let us stress that (48) are solutions of equations obtained by variation of Γ-Lagrangian given by the last two terms of (35):

$$\frac{A}{2}\Gamma^{b\bar{c}}\Gamma^{d\bar{a}}\dot{\Gamma}_{\bar{a}b}\dot{\Gamma}_{\bar{c}d} + \frac{B}{2}\Gamma^{b\bar{a}}\Gamma^{d\bar{c}}\dot{\Gamma}_{\bar{a}b}\dot{\Gamma}_{\bar{c}d}. \qquad (49)$$

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