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Limit Condition for Anisotropic Materials with Asymmetric Elastic Range

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In the paper a new proposition of an energy-based hypothesis of material effort is introduced. It is based on the concept of influence functions introduced by Burzyński [3] and on the concept of decomposition of elastic energy density introduced by Rychlewski [18]. A new proposition enables description of a wide class of linearly elastic materials of arbitrary symmetry exhibiting strength differential effect.

Key words: linear elasticity, anisotropy, material effort hypotheses, limit state criteria.

1. Introduction

The aim of the paper is to introduce a new proposition of a limit condition for anisotropic bodies. In Sec. 2 the foundations of the energy-based hypothesis of material effort proposed by Rychlewski is briefly discussed, while the further parts of the paper are devoted to presentation of the own proposition of the authors. In Sec. 3 a new proposition of the energy-based hypothesis of material effort is presented. The introduction of the influence functions plays essential role by accounting for the assymetry of elastic range. Then the detail discussion of failure criteria specified for some chosen elastic symmetries is provided. The important from practical point of view case of plane orthotropy is studied and the possibilities of specification of the yield criterion in this case are discussed.

Finally, the case of isotropic solid is studied. The specification of influence functions reveals that the earlier discussed case of the criterion accounting for the influence of the Lode angle or in particular the classical Burzyński criterion can be obtained.

Our approach to the problem of the formulation of a limit condition yields from the energy-based concepts of Burzyński [3] and Rychlewski [18]. It distinguishes, however in accounting for the asymmetry of the elastic range, which manifests itself in the difference of the values of yield strength in tension and compression performed with use of the specimen cut out of anisotropic material in any direction. This is the so-called strength differential effect, discussed e.g. by Drucker [4] or Spitzig, Sober and Richmond [20]. Among other earlier formulations of the limit criteria accounting for anisotropy (Mises [11], Hill [7]), strength differential effect (Drucker–Prager [5], Bigoni–Piccolroaz [2]) or both of those features (Hoffman [8], Tsai–Wu [24], Theocaris [23]) the criteria derived from the hypothesis that the measure of material effort is the density of elastic energy accumulated in an anisotropic solid have the following advantages:

- physical interpretation as a combination of energy densities connected with certain energetically independent stress states,
- general treatment of the linear elastic anisotropy due to application of the spectral decomposition of elasticity tensors.

2. Theoretical foundations of energy-based approach

In the linear theory of elasticity an important role is played by certain fourth order tensors, namely compliance tensor $\mathbf{C}$, stiffness tensor $\mathbf{S}$ and limit state tensor $\mathbf{H}$. First two tensors appear in the generalized Hooke's law as a linear operators mapping the space of symmetric second order tensors into itself $\mathscr{S} \to \mathscr{S}$

$$(2.1)\quad \begin{cases} \boldsymbol{\sigma} = \mathbf{S}\cdot\boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} = \mathbf{C}\cdot\boldsymbol{\sigma} \\ \mathbf{C}:\mathbf{S} = \mathbf{S}:\mathbf{C} = \mathbf{I}^S \end{cases} \Rightarrow \begin{cases} \sigma_{ij} = S_{ijkl}\varepsilon_{kl} \\ \varepsilon_{ij} = C_{ijkl}\sigma_{kl} \\ C_{ijkl}S_{klmn} = S_{ijkl}C_{klmn} = \dfrac{1}{2}\left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right), \end{cases}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}$ is the symmetric part of the gradient of small displacements (infinitesimal strain tensor) and $\mathbf{I}^S$ is an identity operator in the space of symmetric second order tensors. The limit state tensor $\mathbf{H}$ appears in the quadratic form of a limit state condition, which constitutes a constraint on the range of stresses for which the Hooke's law is valid

$$(2.2)\quad \boldsymbol{\sigma}\cdot\mathbf{H}\cdot\boldsymbol{\sigma} \le 1.$$

If we consider Levy-Mises flow rule associated with the limit condition of the type as shown above then the constitutive relations between an increment of plastic strain and stress state is expressed by the limit state tensor $\mathbf{H}$ which acts as an linear operator on $\mathscr{S}$:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\mathbf{H} \cdot \boldsymbol{\sigma}. \tag{2.3}$$

Finally the stiffness and compliance tensors appear in the expression for the elastic energy density as a quadratic forms

$$\Phi = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{C} \cdot \boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\varepsilon} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}. \tag{2.4}$$

One can see that each of these three tensors $\mathbf{C}, \mathbf{S}$ and $\mathbf{H}$ can be treated both as a linear operator and a quadratic form. Unless none of the so-called "*locked*" stress or strain states as well as the safe stress states are taken into consideration all quadratic forms $\boldsymbol{\sigma} \cdot \mathbf{C} \cdot \boldsymbol{\sigma}$, $\boldsymbol{\varepsilon} \cdot \mathbf{S} \cdot \boldsymbol{\varepsilon}$, $\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma}$ are positive definite and symmetric.

Tensors $\mathbf{C}, \mathbf{S}$ and $\mathbf{H}$ have internal symmetries characterized by the symmetry group

$$\widetilde{\sigma}_4 = \{\langle 1,2,3,4\rangle, \langle 2,1,3,4\rangle, \langle 1,2,4,3\rangle, \langle 3,4,1,2\rangle\}\,. \tag{2.5}$$

which ensures existance of real eigenvalues of those operators. According to the classical theorem on the spectral decomposition of a linear operator each of the considered fourth rank tensors can be represented in the following form [17, 19]:

$$\begin{gathered}
\mathbf{S} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \ldots + \lambda_\rho \mathbf{P}_\rho = \lambda_I (\boldsymbol{\omega}_I \otimes \boldsymbol{\omega}_I) + \ldots + \lambda_{VI} (\boldsymbol{\omega}_{VI} \otimes \boldsymbol{\omega}_{VI}), \\
\mathbf{C} = \frac{1}{\lambda_1}\mathbf{P}_1 + \frac{1}{\lambda_2}\mathbf{P}_2 + \ldots + \frac{1}{\lambda_\rho}\mathbf{P}_\rho = \frac{1}{\lambda_I}(\boldsymbol{\omega}_I \otimes \boldsymbol{\omega}_I) + \ldots + \frac{1}{\lambda_{VI}}(\boldsymbol{\omega}_{VI} \otimes \boldsymbol{\omega}_{VI}), \\
\mathbf{H} = \frac{1}{h_1}\mathbf{R}_1 + \frac{1}{h_2}\mathbf{R}_2 + \ldots + \frac{1}{h_\chi}\mathbf{R}_\chi = \frac{1}{h_I}(\boldsymbol{h}_I \otimes \boldsymbol{h}_I) + \ldots + \frac{1}{h_{VI}}(\boldsymbol{h}_{VI} \otimes \boldsymbol{h}_{VI}), \\
\rho, \chi \le 6,
\end{gathered} \tag{2.6}$$

where $\boldsymbol{\omega}_K$ and $\mathbf{h}_K$ are the second order tensors representating the eigenstates corresponding with the K-th eigenvalue of the considered operators and $\mathbf{P}_K$ and $\mathbf{R}_K$ are orthogonal projectors on the corresponding eigensubspaces. The expression of the linear operators $\mathbf{C}$, $\mathbf{S}$, $\mathbf{H}$ as a linear combination of orthogonal projectors is unique. It is not so in case of the decomposition into the scaled sum of dyads of the eigenstates. In case of multidimensional eigensubspaces the basis of the eigenstates in such subspace can be done arbitrary in an infinite number of ways.

If the elasticity tensors and the limit state tensor are coaxial then they have the same eigensubspaces and thus the same orthogonal projectors, however even then they may still have different eigenvalues. If any eigensubspace of one of those tensors is not an eigensubspace of the other one but it is still a direct sum of eigenspaces of this second tensor then we call those tensors as being compatible. Even in case when elasticity tensors and limit state tensor are not coaxial but they are still compatible, there exists such a basis in which each of those tensors can be expressed as a linear combination of dyads of the same set of eigenstates. It can be also shown that if two tensors are compatible and all of their eigensubspaces are one-dimensional, then they are coaxial.

According to the theorems of algebra for any quadratic form, there exists a bilinear form which is polar to the considered quadratic form [6]. The linear space of the symmetric second order tensors becomes an euclidean space when a scalar product is defined in it. Rychlewski [18] has used the theorem on the simultaneous reduction of two quadratic forms $\boldsymbol{\sigma} \cdot \mathbf{C} \cdot \boldsymbol{\sigma}$ and $\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma}$ to their canonical forms, assuming that the scalar product is defined as:

$$\begin{gathered} \boldsymbol{\sigma_1} \bullet \boldsymbol{\sigma_2} = \boldsymbol{\sigma_1} \cdot \mathbf{C} \cdot \boldsymbol{\sigma_2} = \boldsymbol{\sigma_2} \cdot \mathbf{C} \cdot \boldsymbol{\sigma_1} \\ \boldsymbol{\sigma_1} \dot{\perp} \boldsymbol{\sigma_2} \quad \Leftrightarrow \quad \boldsymbol{\sigma_1} \bullet \boldsymbol{\sigma_2} = 0 \end{gathered} \tag{2.7}$$

to formulate the following theorem:

Theorem 1: Rychlewski's theorem [18].

For every elastic material defined by its compliance tensor $\mathbf{C}$ *and limit state tensor* $\mathbf{H}$, *there exist exactly one energetically orthogonal decomposition of the linear space of symmetric second order tensors* $\mathscr{S}$*:*

$$\begin{gathered} \mathscr{S} = \mathscr{H}_1 \oplus ... \oplus \mathscr{H}_\chi, \quad \chi \le 6, \\ \mathscr{H}_\alpha \dot{\perp} \mathscr{H}_\beta \quad \text{for} \quad \alpha \neq \beta \end{gathered} \tag{2.8}$$

and exactly one set of pairwise unequal constants

$$h_1, ..., h_\chi, \qquad h_\alpha \neq h_\beta \quad \text{for} \quad \alpha \neq \beta \tag{2.9}$$

such that, for an arbitrary stress state $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + ... + \boldsymbol{\sigma}_\chi, \qquad \boldsymbol{\sigma}_\alpha \in \mathscr{H}_\alpha \tag{2.10}$$

the measure of material effort given by formula (2.2) *is equal*

$$\boldsymbol{\sigma}\mathbf{H}\boldsymbol{\sigma} = \frac{1}{h_1}\Phi(\boldsymbol{\sigma}_1) + ... + \frac{1}{h_\chi}\Phi(\boldsymbol{\sigma}_\chi) \tag{2.11}$$

where

$$\Phi(\boldsymbol{\sigma}_1) + ... + \Phi(\boldsymbol{\sigma}_\chi) = \Phi(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{C} \cdot \boldsymbol{\sigma} \tag{2.12}$$

is the total elastic energy density.

Let us remind that none of the so-called "*locked*" stress or strain states (corresponding with the zero Kelvin modulus) are allowed in this case since the operator used for the definition of the scalar product must be positive definite. We will call the limit condition of form (2.11) the *Rychlewski limit condition.* If only a single stress state component $\boldsymbol{\sigma}_\alpha$ (from the decomposition (2.10)) occurs in the limit condition (2.11) then it can be rewritten in the following form:

$$\Phi(\boldsymbol{\sigma}_\alpha) = h_\alpha. \tag{2.13}$$

The quantity h_α can be interpreted as a limit value of the energy density corresponding to the specified stress state $\boldsymbol{\sigma}_\alpha$.

Rychlewski has interpreted the scalar product defined in (2.7) in terms of energy - one can note that in case of any two stress states which are orthogonal in the sense of the considered scalar product, the work performed by one of the stress state through strains respective for the other one are equal zero:

$$\boldsymbol{\sigma}_1 \bullet \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_1 \cdot \underbrace{\mathbf{C} \cdot \boldsymbol{\sigma}_2}_{\boldsymbol{\varepsilon}_2} = 0 \quad \Rightarrow \quad \mathcal{L} = \frac{1}{2}\boldsymbol{\varepsilon}_1 \cdot \boldsymbol{\sigma}_2 = \frac{1}{2}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\varepsilon}_2 = 0. \tag{2.14}$$

In general, tensors $\mathbf{C}$ and $\mathbf{H}$ are independent – in [9] an energy-based limit condition for solids of cubic elasticity and orthotropic limit state is discussed. If $\mathbf{C}$ and $\mathbf{H}$ are coaxial (they have the same eigensubspaces) then the decomposition of $\mathscr{S}$ into eigensubspaces of each of those tensors is the same in both cases and the stress states $\boldsymbol{\sigma}_\alpha$ are the eigenstates of the stiffness and compliance tensors as well as of the limit state tensor. Then decomposition of elastic energy density (2.12) takes the following form:

$$\Phi(\boldsymbol{\sigma}) = \Phi(\boldsymbol{\sigma}_1) + ... + \Phi(\boldsymbol{\sigma}_\rho) \qquad \rho \leq 6, \tag{2.15}$$

where stress states $\boldsymbol{\sigma}_1, ..., \boldsymbol{\sigma}_\rho$ are both orthogonal and energetically orthogonal. It is the only such decomposition of the energy density in which mutually energetically orthogonal states are also mutually orthogonal in sense of classical definition of the scalar product. We call it the *main decomposition* of the elastic energy density. Examples of the limit criteria based on the main decomposition of elastic energy density for cubic symmetry and transversal isotropy can be found in [12].

3. New proposition of an energy-based hypothesis of material effort

Authors' proposition of a limit condition for pressure-sensitive materials of arbitrary symmetry, exhibiting strength differential effect will be now introduced. Making use of the innovative idea of Burzyński we can modify Rychlewski's yield condition (2.11) in a way similar to the one in which Burzyński modified classical yield condition by Maxwell–Huber [18], namely by taking into account only parts of the specified energy densities, defining their contribution to the total measure of material effort by multiplying their values by a proper functions of the stress state in the corresponding subspace – let's call them *influence functions* η_α

$$\eta_1 \Phi(\boldsymbol{\sigma}_1) + ... + \eta_\chi \Phi(\boldsymbol{\sigma}_\chi) = 1, \qquad \boldsymbol{\sigma}_\alpha \in \mathscr{H}_\alpha, \qquad \chi \leq 6, \tag{3.1}$$

where

$$\Phi(\boldsymbol{\sigma}_1) + ... + \Phi(\boldsymbol{\sigma}_\chi) = \Phi \tag{3.2}$$

is the total elastic energy density and

$$\begin{gathered} \boldsymbol{\sigma}_1 + ... + \boldsymbol{\sigma}_\chi = \boldsymbol{\sigma}, \qquad \boldsymbol{\sigma}_\alpha \in \mathscr{H}_\alpha, \\ \mathscr{H}_1 \oplus ... \oplus \mathscr{H}_\chi = \mathscr{S} \end{gathered} \tag{3.3}$$

is any decomposition of the strain and stress state space $\mathscr{S}$ into direct sum of mutually energetically orthogonal tensor subspaces $\mathscr{H}_\alpha$. The introduced influence functions should be interpreted as a scaling parameters (weights) describing the contribution of each term of energy density into the measure of material effort according to the current stress state. The clue difference between the newly introduced proposition and the Rychlewski's criterion (2.11) is that ***the coefficients of the linear combination of the energy densities are not constant (only material dependent) parameters but they are also functions of the current stress state***. In this way they take into account various modes of the stress states belonging to the corresponding subspace. The influence functions play then the role of the *stress mode indicators*.

In particular one can consider a special case in which the decomposition (3.3) coincide with the decomposition of $\mathscr{S}$ into eigensubspaces of the elasticity tensors. If the influence functions are constant scalar parameters then the proposed limit condition is equivalent to the generalized quadratic limit condition (2.2) and the special case mentioned above occurs when the limit tensor $\mathbf{H}$ is coaxial with the elasticity tensors. Such choice of the decomposition of $\mathscr{S}$ seems to be the most natural one of all possible energetically orthogonal decompositions of $\mathscr{S}$ since it is the only one which is both energetically orthogonal and orthogonal.

3.1. Assumptions on influence functions

Following assumptions are made for the influence functions.

Interpretation of the value of influence functions at the limit state
If, under certain load, only a single term of energy occurs in the limit condition, then the influence function defines the limit value of the elastic energy density corresponding with the considered stress and strain state:

$$\Phi_\alpha^{\lim} = \frac{1}{\eta_\alpha} \tag{3.4}$$

Domain of the influence functions

It seems natural that, to keep mutual independence of all terms of the condition, each influence function η_α should depend only on the projection of a stress state $\boldsymbol{\sigma}$ on the tensor subspace $\mathscr{H}_\alpha$, this means on $\boldsymbol{\sigma}_\alpha$.

$$\eta_\alpha = \eta_\alpha(\boldsymbol{\sigma}_\alpha) \qquad \text{(no summation!)} \tag{3.5}$$

Isotropy of the influence functions in their domains

Since η_α is a scalar function of a tensor argument one should expect that in practical calculations it is expressed in terms of components or invariants of $\boldsymbol{\sigma}_\alpha$. To define it by components of $\boldsymbol{\sigma}_\alpha$ one should chose certain basis in the corresponding subspace which in case of multidimensional subspaces can be done arbitrary in the infinite number of ways. This purely mathematical operation distinguishes certain stress states (basis states) among an infinite number of eigenstates belonging to that subspace and it has no physical sense. This is why functions η_α are assumed to be isotropic in the subspace in which they are defined – according to the theorem on the representation of the scalar isotropic functions, they can be expressed in terms of invariants of corresponding stress projection

$$\eta_\alpha(\boldsymbol{\sigma}_\alpha) = \eta_\alpha\left(I_1(\boldsymbol{\sigma}_\alpha); I_2(\boldsymbol{\sigma}_\alpha); I_3(\boldsymbol{\sigma}_\alpha)\right). \tag{3.6}$$

The arguments of influence functions could be also any other invariants of $\boldsymbol{\sigma}_\alpha$ – i.e. its principal values, its norm etc. If the considered space is one-dimensional then each invariant is proportional to the measure of projection (or its power) of the stress state onto the considered space – thus this measure should be the only argument of the influence function.

Influence functions in subspaces of deviators

If the considered eigensubspace is a space of deviators then $I_1(\boldsymbol{\sigma}_\alpha) = 0$ and $I_2(\boldsymbol{\sigma}_\alpha)$ is proportional to the corresponding energy density. This indicates that in fact it is the third invariant of stress tensor deviator which makes the qualitative distinction between various deviators belonging to the same subspace.

It is strictly connected with abstract angles in multidimensional subspace of deviators – a kind of curvilinear coordinates in the considered subspace. Lode angle is an example of such parameter in case of isotropy. In case of one-dimensional deviatoric subspaces influence function is a constant parameter which is equal the inversion of the limit value of the energy density respective for this state.

3.2. Failure criterion specification for chosen elastic symmetries

Let us now present a few examples of the general specification of the discussed proposition of a limit condition. It is assumed that the considered energetically orthogonal decompositon of the linear space of symmetric second order tensors is the one respective for the spectral decomposition of the sitffness and compliance tensor – then the proposed limit condition is a combination of terms of the main decomposition of the elastic energy density with unequal, stress state dependent weights.

3.2.1. Plane orthotropy. Energetic character of the considered hypothesis makes it easy to formulate the failure criterion in case of plane stress/strain state. Omitting description of a total plane anisotropy (lack of any symmetry) we will now discuss general plane orthotropy. Spectral decomposition of the plane orthotropic elasticity tensors gives us three orthogonal eigensubspaces:

- one-dimensional subspace of the states with non-zero hydrostatic component

$$\lambda_1 = \frac{2E_x E_y}{(E_x + E_y) + \sqrt{(E_x - E_y)^2 + 4\nu^2 E_x^2}}, \qquad \boldsymbol{\omega}_1 \cong \begin{bmatrix} \cos\aleph & 0 \\ 0 & \sin\aleph \end{bmatrix},$$

- one-dimensional subspace of the states with non-zero hydrostatic component

$$\lambda_2 = \frac{2E_x E_y}{(E_x + E_y) - \sqrt{(E_x - E_y)^2 + 4\nu^2 E_x^2}}, \qquad \boldsymbol{\omega}_2 \cong \begin{bmatrix} -\sin\aleph & 0 \\ 0 & \cos\aleph \end{bmatrix},$$

where

$$\tan\aleph = -\frac{E_x}{2\nu}\left[\frac{1}{E_y} - \frac{1}{E_x} + \sqrt{\left(\frac{1}{E_y} - \frac{1}{E_x}\right)^2 + 4\left(\frac{\nu}{E_x}\right)^2}\right],$$

- one dimensional subspace of pure shears

$$\lambda_3 = 2G_{xy}, \qquad \boldsymbol{\omega}_3 \cong \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where E_x E_y are the Young moduli and G_{xy} is the Kirchhoff modulus, each measured in the directions of the plane orthotropy, and ν is the Poisson's ratio at tension/compression along the x direction. Failure criterion (3.1) takes form

$$\Phi_1 \widetilde{\eta}_1 + \Phi_2 \widetilde{\eta}_2 + \Phi_3 \frac{1}{h_3} = 1, \tag{3.7}$$

where h_3 is the limit value of the elastic energy density corresponding with the shearing eigenstate. Since all eigensubspaces of the plane orthotropic elasticity tensor are one-dimensional all invariants of each projection of the stress state onto every subspace are proportional to the measure of this projection or its power. The limit condition may be thus rewritten in the following form

$$\eta_1(\sigma_1) \cdot \sigma_1^2 + \eta_2(\sigma_2) \cdot \sigma_2^2 + \frac{\sigma_3^2}{2k_s} = 1, \tag{3.8}$$

where projections on proper eigensubspaces (not to be mistaken with principal stresses):

$$\sigma_1 = \sigma_{xx} \cos \aleph + \sigma_{yy} \sin \aleph,$$

$$\sigma_2 = -\sigma_{xx} \sin \aleph + \sigma_{yy} \cos \aleph,$$

$$\sigma_3 = \sqrt{2} \tau_{xy}.$$

The parameters η_1 and η_2 are unknown influence functions and k_s is the limit shear stress in the directions parallel and perpendicular to the symmetry axes of the material. The proposed general limit condition in a very special case of plane stress state together with its specification for chosen plane symmetries is discussed in details in [22].

3.2.2. Plane symmetry of square. Special case of orthotropy in which elastic properties of the material are identical in two perpendicular directions and different than in any other pair of perpendicular directions is called the symmetry of square. It can be considered as the plane orthotropy for which $E_x = E_y = E$ what corresponds with the value of the parameter $\tan \aleph = -1$. Spectral decomposition of the plane elasticity tensors characterized by the symmetry of square gives us three orthogonal eigensubspaces:

- one-dimensional subspace of plane hydrostatic stress states

$$\lambda_1 = \frac{E}{1-\nu}, \qquad \boldsymbol{\omega}_1 \cong \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

- one-dimensional subspace of pure shears in directions at angle 45° referring to the symmetry axes

$$\lambda_2 = \frac{E}{1+\nu}, \qquad \boldsymbol{\omega}_2 \cong \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

- one dimensional subspace of pure shears in directions parallel and perpendicular to the symmetry axes

$$\lambda_3 = 2G_{xy}, \qquad \boldsymbol{\omega}_3 \cong \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

The limit condition can be rewritten in the following form:

$$\eta_v(p)\cdot p^2 + \frac{\tau_{45}^2}{k_{s45}^2} + \frac{\tau^2}{2k_s^2} = 1 \tag{3.9}$$

where $p = \dfrac{1}{2}(\sigma_{xx}+\sigma_{yy})$, $\tau_{45} = \dfrac{1}{\sqrt{2}}(\sigma_{yy}-\sigma_{xx})$, $\tau = \sqrt{2}\tau_{xy}$ and k_s and k_{s45} are the limit shear stresses at pure shear along symmetry axes and at angle 45° to the symmetry axes respectively. Parameters k_s and k_{s45} can be found during shearing properly oriented samples. It is worth noting that the sole term in the limit condition which is still unknown is only pressure dependent. Since uniaxial stress state has non-zero hydrostatic component, which in turn changes as the orientation of the uniaxial load referring to the symmetry axes changes, so the values of the pressure influence function can be explicitly found during simple tension/compression test at various orientations of the specimen:

$$\eta_v\left(\frac{1}{2}k_\varphi\right) = \frac{4}{k_\varphi^2}\cdot\left[1-\left(\frac{(\tau_{45}(\varphi))^2}{k_{s45}^2}+\frac{(\tau(\varphi))^2}{2k_s^2}\right)\right], \tag{3.10}$$

where k_φ is the limit normal stress at tension / compression in direction at angle φ referring to the symmetry axes and

$$\begin{aligned} \tau_{45}(\varphi) &= \frac{k_\varphi}{\sqrt{2}}\left(\cos^2\varphi - \sin^2\varphi\right), \\ \tau(\varphi) &= k_\varphi\sqrt{2}\cos\varphi\sin\varphi. \end{aligned} \tag{3.11}$$

If the Burzyński's [3] pressure influence function is assumed then $\eta_p(p)$ takes the following form:

$$\eta_p(p) = \left[\frac{4}{k_c k_r} - \frac{1}{k_{s45}^2}\right] + \frac{2}{p}\cdot\frac{(k_c-k_r)}{k_c k_r}, \tag{3.12}$$

where k_c and k_r denote limit uniaxial stress along the symmetry axes at compression and at tension respectively.

3.2.3. Case of isotropy. In case of isotropy criterion (3.1) can be written as follows:

$$\widetilde{\eta}_f(J_2, J_3)\Phi_f + \widetilde{\eta}_v(I_1)\Phi_v = 1, \tag{3.13}$$

where I_1 is the first stress tensor invariant, and J_2, J_3 are second and third stress deviator invariants respectively. Using principal stresses (which is allowed in case of isotropy without further assumptions on the orientation of coordinate system) it can be rewritten in the following form:

$$\eta_f(\theta)q^2 + \eta_p(p) = 1, \tag{3.14}$$

where

$$p = \frac{1}{3}I_1 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}),$$

$$q = \sqrt{2J_2} = \sqrt{\frac{1}{3}\left[(\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + (\sigma_1 - \sigma_2)^2\right]}$$
$$= \sqrt{\frac{1}{3}\left[(\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + (\sigma_{xx} - \sigma_{yy})^2 + 6(\tau_{yz}^2 + \tau_{zx}^2 + \tau_{xy}^2)\right]},$$

$$\theta = \frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right) \text{ – Lode angle.}$$

The only limitation for the form of the Lode angle influence function is that it has to be periodic with the period equal 120°. It is often assumed that the function decsribing the influence of the Lode angle is in fact a function of a variable $y = \cos(3\theta)$. Specific form of the Lode angle influence function can be chosen among many propositions available in the literature [1], e.g.:

- two-parameter power function by Raniecki and Mróz [16]

$$\eta_f(\theta) = [1 + \alpha y]^\beta,$$

- two-parameter exponential function by Raniecki and Mróz [16]

$$\eta_f(\theta) = 1 + \alpha\left[1 - e^{-\beta(1+y)}\right],$$

- one-parameter trigonometric function by Lexcellent *et al.* [10]

$$\eta_f(\theta) = \cos\left[\frac{1}{3}\arccos\left[1 - \alpha(1 - y)\right]\right],$$

- two-parameter trigonometric function by Podgórski [15] (see also Bigoni and Piccolroaz [2])

$$\eta_f(\theta) = \frac{1}{\cos(30^\circ - \beta)} \cos\left[\frac{1}{3}\arccos\left(\alpha \cdot y\right) - \beta\right].$$

Isotropic case of the presented proposition was discussed in details in [13] and [21]. Its specification according to the experimental data available in the literature was presented in [14].
It is worth mentioning that after substituting:

$$\widetilde{\eta}_v(I_1) = \begin{cases} -\dfrac{2K}{p^2} \cdot M p_c \sqrt{\left(F - F^m\right)\left[2(1-\alpha)F + \alpha\right]} & \text{if } \; F \in [0,1], \\ +\infty & \text{if } \; F \notin [0,1], \end{cases} \tag{3.15}$$

$$\widetilde{\eta}_f(J_2, J_3) = \frac{2\sqrt{6}G}{q} \cos\left[\beta\frac{\pi}{6} - \frac{1}{3}\arccos(\gamma\cos(3\theta))\right], \tag{3.16}$$

where K is the bulk modulus, G is the shear modulus, F is defined as:

$$F = \frac{-p + c}{p_c + c} \tag{3.17}$$

and p_c, c, m, M, α, β, γ are certain constant material parameters, then the presented general limit condition for isotropy (3.13) is equivalent to the one proposed and precisely analyzed in various aspects by Bigoni and Piccolroaz in [2].

4. Summary and conclusions

The new proposition of an energy-based hypothesis of material effort for anisotropic materials exhibiting strength differential effect was introduced. General statement derived from Burzyński's idea of influence functions and Rychlewski's theorems on the orthogonal and energetically orthogonal decompositions of the space of symmetric second rank tensors was presented. Particular assumptions on the form and properties of the influence functions were formulated.

It was stated in the second section that the studied limit condition should be applied only in case of proportionality limit state due to assumption of validity of Hooke's law used in its derivation. However it seems that the mathematical form of this condition could be well used also in case of e.g. yield limit. It also seems reasonable to use it as a plastic potential in an associated flow rule, however it might need some modifications e.g. due to requirement of material's incompressibility. Some applications of the newly introduced limit condition for Inconel 718 alloy according to the experimental results available in the literature are presented in [14].

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References

1. Bardet J.P., *Lode dependences for isotropic pressure sensitive materials*, J. Appl. Mech., **57**, 498–506, 1990.
2. Bigoni D., Piccolroaz A., *Yield criteria for quasibrittle and frictional materials*, Int. J. Solids Structures, **41**, 2855–2878, 2004.
3. Burzyński W., *Studium nad hipotezami wytężenia*, Akademia Nauk Technicznych, Lwów, 1928; see also: *Selected passages from Włodzimierz Burzyński's doctoral dissertation "Study on material effort hypotheses"*, Engng. Trans., **57**, 3–4, 185–215, 2009.
4. Drucker D.C., *Plasticity Theory, Strength-Differential (SD)Phenomenon, and Volume Expansion in Metals and Plastics*, Metall. Trans., **4**, 667–673, 1973.
5. Drucker D.C., Prager W., *Soil mechanics and plastic analysis for limit design*, Quart. Appl. Math., **10**, 2, 157-–165, 1952.
6. Gelfand I.M., *Lectures on Linear Algebra* [in Russian: *Lekhtsii po linyeynoy algebre*], Nauka, Moscow, 1966.
7. Hill R., *A theory of the yielding and plastic flow of anisotropic metals*, Proc. Roy. Soc. London, 193, 281–297, 1948.
8. Hoffman O., *The brittle strength of orthotropic materials*, J. Comp. Mater., **1**, 200–206, 1967.
9. Kowalczyk K., Ostrowska-Maciejewska J., Pęcherski R.B., *An energy-based yield criterion for solids of cubic elasticity and orthotropic limit state*, Arch. Mech., **55**, 5–6, 431–448, 2003.
10. Lexcellent C., Vivet A., Bouvet C., Calloch S., Blanc P., *Experimental and numerical determinations of the initial surface of phase transformation under biaxial loading in some polycrystalline shape-memory alloys*, J. Mech. Phys. Sol., **50**, 2717–2735, 2002.
11. Mises R. von, *Mechanik der plastischen Formanderung von Kristallen*, Z. Angew. Math. u. Mech., **8**, 161–185, 1928.
12. Ostrowska-Maciejewska J., Rychlewski J., *Plane elastic and limit states in anisotropic solids*, Arch. Mech., **40**, 4, 379–386, 1988.

13. Nowak M., Ostrowska-Maciejewska J., Pęcherski R.B., Szeptyński P., *Yield criterion accounting for the third invariant of stress tensor deviator. Part I. Derivation of the yield condition basing on the concepts of energy-based hypotheses of Rychlewski and Burzyński.*, Engng. Trans., 5**9**, 4, 273–281, 2011.

14. Pęcherski R.B., Szeptyński P., Nowak M., *An extension of Burzyński hypothesis of material effort accounting for the third invariant of stress tensor*, Arch. Metall. Mat., **56**, 2, 503–508, 2011.

15. Podgórski J., *Limit state condition and the dissipation function for isotropic materials*, Arch. Mech., **36**, 3, 323–342, 1984.

16. Raniecki B., Mróz Z., *Yield or martensitic phase transformation conditions and dissipation functions for isotropic, pressure-insensitive alloys exhibiting SD effect*, Acta. Mech., **195**, 81–102, 2008.

17. Rychlewski J., *„CEIIINOSSSTTUV" Mathematical structure of elastic bodies* [in Russian], Preprint 217, IPM AN SSSR, IPPT PAN, Moscow 1983.

18. Rychlewski J., *Elastic energy decomposition and limit criteria* [in Russian], Advances in Mechanics (Uspekhi mekhaniki), 7, 51–80, 1984; see also: *Elastic energy decomposition and limit criteria*, Engng. Trans., **59**, 1, 31–63, 2011.

19. Rychlewski J., *On Hooke's law* [in Russian], Prikl. Mat. Mekh., 48, 420–435, 1984; see also: J. Appl. Math. Mech., **48**, 303–314, 1984.

20. Spitzig W.A., Sober R.J., Richmond O., *The effect of hydrostatic pressure on the deformation behavior of maraging and HY-80 steels and its implications for plasticity theory*, Metall. Trans. A, 7A, 1703–1710, 1976.

21. Szeptyński P., *Yield criterion accounting for the influence of the third invariant of stress tensor deviator. Part II: Analysis of convexity condition of the yield surface*, Engng. Trans., **59**, 4, 283–297, 2011.

22. Szeptyński P., Pęcherski R.B., *Proposition of a new yield criterion for orthotropic metal sheets accounting for asymmetry of elastic range* [in Polish], Rudy i Metale Nieżelazne, **57**, 4, 243–250, 2012.

23. Theocaris P.S., *The elliptic paraboloid failure criterion for cellular solids and brittle foams*, Acta Mech., **89**, 93–121, 1991.

24. Tsai S.W., Wu E.M., *A general theory of strength for anisotropic materials*, J. Comp. Mater., **5**, 58–80, 1971.

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