ABSTRACT BOOK VOL. 2

8th International Congress on

# THERMAL STRESSES 2009

JUNE 1–4, 2009

University of Illinois at
Urbana-Champaign,
Illinois, USA

Clarkson University

Proceedings of the 8th International Congress on Thermal Stresses, TS2009
1–4 June, University of Illinois at Urbana-Champaign

# MODELING HEAT TRANSFER IN METAL FILMS BY A THIRD-ORDER DERIVATIVE-IN-TIME DISSIPATIVE AND DISPERSIVE WAVE EQUATION

**Józef Ignaczak**

*Institute of Fundamental Technological Research,*
*Polish Academy of Sciences, Warsaw, Poland*

The paper deals with harmonic solutions to the third-order derivative-in-time temperature equation

$$\left[\frac{\partial^3}{\partial t^3}+2\frac{\partial^2}{\partial t^2}+2\frac{\partial}{\partial t}-\left(1+\alpha\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial x^2}\right]\theta=0 \qquad (*)$$

in which $\theta=\theta(t,x)$ represents a temperature field of a two-temperature model of ballistic heat transport in a metal film, and $\alpha$ is a nonnegative parameter; $t$ and $x$ denote the time and space variables, respectively. It is shown that for a suitably restricted range of the wave numbers $k$ and of the parameter $\alpha$ there are three real-valued harmonic solutions $\theta_i\,(i=1,2,3)$ of equation (*) such that $\theta_1$ represents a dissipative standing wave with a damping $h_1=h_1(k)>0$, while $\theta_2$ and $\theta_3$ are both dispersive and dissipative waves moving with the same damping $h_2=h_2(k)>0$ and a velocity $c=c(k)>0$ in the directions of decreasing and increasing $x$, respectively.
Also, use of the harmonic solutions is made to obtain a closed-form transient-in-time solution to a Cauchy problem for equation (*) in which $|x|<\infty$ and $\alpha\geq 3/2$. Numerical results are included.

Key Words**:** *Dispersion; Dissipation; Heat Transfer; Metal Films; Modeling; Third-Order Derivative-In-Time Heat Conduction Equation; Wave Motion.*

## 1. Introduction

D.Y. Tzou has shown in [1] that a central equation of the one-dimensional theory of a two-temperature model of ballistic heat transport in the electronic gas of a metal film subject to a laser heat, takes the form

$$\left[\frac{\partial^3}{\partial t^3}+2\frac{\partial^2}{\partial t^2}+2\frac{\partial}{\partial t}-\left(1+\alpha\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial x^2}\right]\theta=0\,, \qquad (1)$$

where $\theta=\theta(t,x)$ represents a dimensionless metal lattice temperature; $t$ and $x$ denote the dimensionless time and space variables, respectively; and $\alpha$ is a nonnegative parameter. [Equation (1) is equivalent to eq (10.21) from [1] if we let $\alpha=B$, $t=\beta$, and $\delta=x\sqrt{2}$ ; where $B,\beta$ and $\delta$ are the notations from [1] ] .

In the present paper we are to show that the third-order derivative-in-time temperature equation (1) contains two types of energy loss: absorption and dispersion. This will be shown by looking for the harmonic solutions of eq. (1) in the form

$$\hat{\theta}(t,x)=\theta_0\exp[i(kx-\omega t)]\,, \qquad (2)$$

where $i=\sqrt{-1}$, $k$ is a real-valued wave number, $\omega$ is a complex-valued variable and $\theta_0$ is a complex-valued constant. In particular, we are to show that for a suitably restricted range of the parameters $\alpha$ and $k$ there are three real-valued solutions to (1) of the form

$$\theta_i(t,x)=\mathrm{Re}\{\hat{\theta}_i(t,x)\},\quad i=1,2,3\,, \qquad (3)$$

such that $\theta_1$ represents a standing wave that is strongly attenuated on the time axis; while $\theta_2$ and $\theta_3$ are the dispersive strongly attenuated on the time axis waves moving with a velocity $c>0$ in the directions of decreasing and increasing $x$, respectively. In Section 2 a dispersion relation for the propagation of harmonic waves governed by eq. (1) is obtained. In Section 3 the existence of the three real-valued harmonic solutions to eq. (1), in the form of four theorems, is proved. In Section 4, use of the harmonic solutions of Section 3 is made to obtain a closed-form solution to a Cauchy problem for eq. (1) in which $|x| < \infty$ and $\alpha \geq 3/2$. Finally, in Section 5 results and conclusions are summarized.

## 2. Dispersion Relation

By substituting $\theta = \theta(t,x)$ from (2) into (1), the dispersion relation for the propagation of harmonic waves governed by eq. (1) is obtained

$$\omega^3 + 2i\omega^2 - (2 + \alpha k^2)\omega - ik^2 = 0\,. \qquad (4)$$

By letting

$$\omega = \Omega - \frac{2}{3}i \qquad (5)$$

and substituting $\omega$ from (5) into (4) we obtain

$$\Omega^3 - 3\psi(k)\,\Omega - 2i\varphi(k) = 0\,, \qquad (6)$$

where

$$\psi(k) = \frac{1}{3}\left(\frac{2}{3} + \alpha k^2\right), \qquad (7)$$

$$\varphi(k) = \frac{1}{2}\left[\left(1 - \frac{2}{3}\alpha\right)k^2 - k_0^2\right], \qquad (8)$$

and

$$k_0 = \frac{2}{3}\sqrt{\frac{5}{3}}\,. \qquad (9)$$

If we let

$$\Omega = iY\,, \qquad (10)$$

and substitute $\Omega$ from (10) into (6), the third-order algebraic equation with real-valued coefficients for $Y$ is obtained

$$Y^3 + 3\psi(k)Y + 2\varphi(k) = 0\,, \qquad (11)$$

The following theorem holds true.

**Theorem 1.** There is a real-valued root $Y_1$ and two complex conjugate roots $Y_2$ and $Y_3$ of eq. (11) given by

$$Y_1(k) = a(k) - b(k)\,, \qquad (12)$$

$$Y_{2.3}(k) = -\frac{1}{2}[a(k) - b(k)] \pm i\frac{\sqrt{3}}{2}[a(k) + b(k)]\,, \qquad (13)$$

where

$$a(k) = \{[\varphi^2(k) + \psi^3(k)]^{1/2} - \varphi(k)\}^{1/3} \qquad (14)$$

and

$$b(k) = \{[\varphi^2(k) + \psi^3(k)]^{1/2} + \varphi(k)\}^{1/3}\,. \qquad (15)$$

In eq. (13) the roots $Y_2$ and $Y_3$ correspond to the plus and minus signs, respectively.

**Proof.** First, by substituting (12) into (11) we prove that $Y_1$ is a solution to (11). Next, we show that

$$Y^3 + 3\psi Y + 2\varphi = (Y - Y_1)(Y - Y_2)(Y - \overline{Y}_2)\,, \qquad (16)$$

where $\overline{Y}_2 = Y_3$; and this completes the proof.

Theorem 1 plays a central role in formulating the existence theorems for harmonic solutions to eq. (1).

## 3. Existence Theorems

The following theorem holds true.

**Theorem 2.** For the wave numbers $k$ satisfying the inequality

$$k \geq \left(1 - \frac{2}{3}\alpha\right)^{-1/2} k_0\,, \qquad (17)$$

where

$$\frac{1}{2} < \alpha < \frac{3}{2}\,, \qquad (18)$$

there are three real-valued solutions to eq. (1) of the form

$$\theta_1(t,x) = \theta_1^{(0)} \exp[-h_1(k)\,t] \sin kx\,, \qquad (19)$$

$$\theta_2(t,x) = \theta_2^{(0)} \exp[-h_2(k)\,t] \sin k[x + c(k)\,t]\,, \qquad (20)$$

$$\theta_3(t,x) = \theta_3^{(0)} \exp[-h_2(k)\,t] \sin k[x - c(k)\,t]\,, \qquad (21)$$

where

$$h_1(k) = \left\{\frac{2}{3} - [a(k) - b(k)]\right\} > 0\,, \qquad (22)$$

$$h_2(k) = \left\{\frac{2}{3} + \frac{1}{2}[a(k) - b(k)]\right\} > 0\,, \qquad (23)$$

$$c(k) = \frac{1}{k}\frac{\sqrt{3}}{2}[a(k) + b(k)] > 0\,, \qquad (24)$$

and $\theta_i^{(0)}$ $(i = 1,2,3)$ are real-valued constants. The function $\theta_1$ represents a standing wave that is strongly attenuated on the time-axis, while $\theta_2$ and $\theta_3$ are strongly attenuated on the time-axis dispersive waves propagating with the velocity $c = c(k)$ in the negative and positive directions of the x-axis, respectively.

**Proof.** First, we note that the inequality (17) together with (7)-(9), (14), and (15) imply that

$$b(k) - a(k) \geq 0 \qquad \forall k \geq \hat{k}_0\,, \qquad (25)$$

where

$$\hat{k}_0 = \left(1 - \frac{2}{3}\alpha\right)^{-1/2} k_0\,. \qquad (26)$$

Next, it follows from Theorem 1 as well as from (5) and (10) that there are three complex-valued frequencies $\omega_k$ $(k = 1,2,3)$ such that

$$i\,(kx-\omega_1\,t) = i\,kx - \left[\frac{2}{3}-(a-b)\right]t\,, \qquad (27)$$

$$i\,(kx-\omega_2\,t) = i\left[kx+\frac{\sqrt{3}}{2}(a+b)\,t\right]-\left[\frac{2}{3}+\frac{1}{2}(a-b)\right]t \qquad (28)$$

$$i(kx-\omega_3\,t) = i\left[kx-\frac{\sqrt{3}}{2}(a+b)\,t\right]-\left[\frac{2}{3}+\frac{1}{2}(a-b)\right]t \qquad (29)$$

Therefore, substituting (27), (28), and (29), respectively, into

$$\theta_j(t,x) = \theta_0^{(j)}\exp[i(kx-\omega_j\,t)] \qquad (30)$$

for $j$ = 1, 2, and 3; and taking the real parts of the resulting equations, we obtain eqs. (19)-(24), except for the inequality $h_2(k) > 0$ [see Eq. (23)]. As a result, to complete proof of Theorem 2 we need to show that

$$0 \le [b(k)-a(k)] < \frac{4}{3} \qquad \forall k \ge \hat{k}_0\,. \qquad (31)$$

To this end, we rewrite (31) in the equivalent form

$$0 \le (b-a)^3 < \left(\frac{4}{3}\right)^3 \qquad \forall k \ge \hat{k}_0\,, \qquad (31a)$$

or, using (11) and (12), in the form

$$2\varphi + 3\psi(a-b) < \left(\frac{4}{3}\right)^3 \qquad \forall k \ge \hat{k}_0\,, \qquad (32)$$

or, using (7)-(9), in the form

$$\left(1-\frac{2}{3}\alpha\right)k^2 - k_0^2 < \left(\frac{4}{3}\right)^3 + \left(\frac{2}{3}+\alpha k^2\right)(b-a) \qquad (33)$$
$$\forall k \ge \hat{k}_0\,.$$

Since the inequalities (31a) and (33) are equivalent, it is sufficient to prove that the inequality (33) in which $(b-a)$ on RHS of (33) is replaced by 4/3, holds true. To this end, we replace $(b-a)$ in (33) by 4/3 and obtain

$$\left(1-\frac{2}{3}\alpha\right)k^2 - k_0^2 < \left(\frac{4}{3}\right)^3 + \frac{4}{3}\left(\frac{2}{3}+\alpha\,k^2\right) \qquad (34)$$
$$\forall k \ge \hat{k}_0\,,$$

or, equivalently,

$$(1-2\alpha)k^2 - 4 < 0 \qquad \forall k \ge \hat{k}_0 > 0\,. \qquad (35)$$

Since, by (18), $1-2\alpha < 0$, therefore (35) holds true; and , as a result, (31) holds true. This completes proof of Theorem 2.

The physical interpretation of the solutions $\theta_i = \theta_i(t,x)$ $(i=1,2,3)$ given by (19) – (21) is well motivated by the nonlinear attenuations coefficients $h_i = h_i(k)$ $(i=1,2)$, and the nonlinear velocity $c = c(k)$ on the k-axis; in particular, the waves $\theta_2$ and $\theta_3$ reveal two types of energy loss: absorption (time decay) and dispersion (velocity is a function of the wave number); while the wave $\theta_1$ reveals a nonlinear absorption only. Note that since a wavelength $\lambda = 2\pi/k$, the hypothesis (17) implies that the harmonic waves $\theta_k$ $(k=1,2,3)$ are restricted to the wavelengths $\lambda$ from the interval

$$0 < \lambda \le \hat{\lambda}_0\,, \quad \hat{\lambda}_0 = 7.3\left(1-\frac{2}{3}\alpha\right)^{1/2}, \qquad (36)$$

and they represent short waves for $\alpha \to 3/2-\varepsilon$ as $\varepsilon \to 0+0$. The function $\hat{\lambda}_0 = \hat{\lambda}_0(a)$ on (1/2, 3/2) represents an upper bound on the wavelengths at which the harmonic solutions $\theta_k$ do exist.

Also note that the inequality $\alpha > 1/2$ that implies stability of $\theta_k$ has also been postulated by R. Quintanilla [2] in connection with a study of stability of a one-dimensional dual-phase-lag thermoelastic model.

Finally, for a finite value of $\alpha \in (1/2, 3/2)$ we obtain

$$h_1(k) \to \frac{1}{\alpha}, \quad h_2(k) \to 1-\frac{1}{2\alpha} > 0, \qquad (37)$$
$$\text{and}\;\; c(k) \to \sqrt{\alpha} \;\;\text{as}\;\; k \to \infty\,.$$

By extending range of $\alpha$ to the interval

$$\alpha \ge \frac{3}{2}\,, \qquad (38)$$

we arrive at the following theorem.

**Theorem 3.** For any positive wave number $(k > 0)$ and for $\alpha$ satisfying the inequality (38) there are tree real-valued solutions to eq. (1) of the form (19)-(21) with the physical interpretation similar to that of Theorem2.

**Proof** of Theorem 3 is omitted.

The existence results on the propagation of harmonic waves governed by eq. (1) include also the following theorem.

**Theorem 4.** If $\alpha \in [0, 1/2]$ and

$$\left(1-\frac{2}{3}\alpha\right)^{-1/2} k_0 \le k < 2(1-2\alpha)^{-1/2}\,, \qquad (39)$$

there are three real-valued solutions to eq. (1) of the form (19) – (21).

**Proof** of Theorem 4 is omitted.

A result similar to that of Thm. 4 is described by the following theorem.

**Theorem 5.** If $0 < \alpha < 3/2$ and

$$0 < k < \left(1 - \frac{2}{3}\alpha\right)^{-1/2} k_0, \qquad (40)$$

then there are three real-valued solutions to eq. (1) described by (19) – (21).

**Proof** of Theorem 5 is omitted.

It follows from Theorems 2 – 5 that eq. (1) describes dispersive and dissipative waves over a wavelength range that depends strongly on the parameter $\alpha$ ; in particular, for $\alpha \geq 3/2$ equation (1) is dissipative and dispersive for any finite wavelength, while for $\alpha \in [0, 1/2]$ equation is dissipative and dispersive only if the wavelength belongs to a finite $\alpha$-dependent strip.

## 4. Cauchy Problem

In this Section we are to find a transient-in-time solution $\theta = \theta(t, x)$ of the equation

$$\left[\frac{\partial^3}{\partial t^3} + 2\frac{\partial^2}{\partial t^2} + 2\frac{\partial}{\partial t} - \left(1 + \alpha\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial x^2}\right]\theta = 0 \qquad (41)$$

$$\text{for } |x| < \infty, \quad t > 0$$

subject to the initial conditions

$$\theta(0, x) = \delta(x), \quad \frac{\partial}{\partial t}\theta(0, x) = \frac{\partial^2}{\partial t^2}\theta(0, x) = 0 \qquad (42)$$

$$\text{for } |x| < \infty$$

where $\delta = \delta(x)$ is the Dirac delta function. In addition, we restrict the range $\alpha$ to the interval

$$\alpha \geq \frac{3}{2}. \qquad (43)$$

The problem of finding $\theta$ that satisfies eq. (41) and the initial conditions (42) will be called a Cauchy Problem for eq. (1). Using the results of Section 3 we arrive at the following theorem

**Theorem 6.** There is a unique solution to the Cauchy Problem described by eqs. (41) – (42) subject to the condition $\alpha \geq 3/2$, and this solution takes the form

$$\theta(t, x) = \frac{1}{\pi}\int_0^\infty A(k) e^{-h_1 t} \cos kx \, dk$$
$$- \frac{1}{2\pi}\int_0^\infty e^{-h_2 t}\left[B(k)\cos k(x - ct) - C(k)\sin k(x - ct)\right]dk$$
$$- \frac{1}{2\pi}\int_0^\infty e^{-h_2 t}\left[B(k)\cos k(x + ct) + C(k)\sin k(x + ct)\right]dk, \qquad (44)$$

where

$$A(k) = \frac{h_2^2 + k^2 c^2}{[(h_1 - h_2)^2 + k^2 c^2]},$$

$$B(k) = \frac{h_1(2h_2 - h_1)}{[(h_1 - h_2)^2 + k^2 c^2]}, \qquad (45)$$

$$C(k) = \frac{h_1}{kc}\frac{(h_2 - h_1)h_2 - k^2 c^2}{[(h_1 - h_2)^2 + k^2 c^2]},$$

and the dampings $h_1 = h_1(k)$ and $h_2 = h_2(k)$; and the velocity $c = c(k)$ are given by eqs. (22) – (24).

**Proof** of Theorem 6 is omitted.

Note that the first integral on RHS of (44) represents a packet of harmonic time-decaying standing waves, while the second and third integrals on RHS of (44 represent the packets of harmonic time-decaying waves traveling with the group velocity $c = c(k)$ in the positive and negative directions of the *x*-axis, respectively

## 5. Results and Conclusions

(a) Three harmonic solutions to a third-order derivative-in-time dissipative and dispersive wave equation of the theory of ballistic heat transport in the electronic gas of a metal film subject to a laser heat, are obtained in a closed-form. One of the harmonic solutions is shown to be a dissipative standing wave, while the two other solutions represent the dissipative and dispersive traveling waves.

(b) A Cauchy problem for the wave equation is formulated, and its solution in the form of a sum of three packets of harmonic waves, is obtained; one of the three packets consists of the harmonic dissipative standing waves, while the two remaining packets are made of the harmonic dissipative and dispersive waves traveling with the same group velocity but in opposite directions.

(c) The results obtained should prove useful to those modeling heat transfer in metal films as well as to researchers trying to understand macro- to micro-scale heat transfer processes.

## 6. References


[1] Tzou, D.Y., Macro- to Microscale Heat Transfer; The Lagging Behavior. Taylor and Francis**,** Washington, 1997.

[2] Quintanilla, R., A Condition on the Delay Parameters in One-Dimensional Dual-Phase-Lag Thermoelastic Theory. J. Thermal Stresses, vol. 26, 713-721, 2003.