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RAYLEIGH WAVES IN A NON-HOMOGENEOUS ISOTROPIC ELASTIC SEMI-SPACE (PART 1)[1]

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Introduction

Considerable progress has been made recently in the theory of surface waves in non-homogeneous and elastic solids. In a monograph by W. M. EWING, W. S. JARDETZKY, I. PRESS, [1], the reader will find a summary of recent contributions to these problems. What are called dynamic displacement equations have usually been used in attacking the foregoing problems (see, for example, H. KOLSKY, [2], A. SOMMERFELD, [3]). In this paper pure stress equations of motion method [4] is used to solve the problem of Rayleigh waves in a non-homogeneous, isotropic elastic semi-space, (plane-strain solution). It is shown that there exists a stress function which satisfies an ordinary, linear, fourth order differential equation for arbitrary non-homogeneity. Thus the equations of motion are separable in this case, and stresses can be expressed by one function, which satisfies the ordinary linear equation with variable coefficients. This equation can be treated by certain numerical methods. In the present paper only some classes of solutions in closed forms are obtained, and Rayleigh's velocity equations, which are new in the authors opinion, are established for the special non-trivial forms of non-homogeneity.

1. Two-Dimensional Stress Equations of Motion

Let a semi-space occupying the region $x_2 \geqslant 0$ be set in motion by forces applied at some distance from the free boundary $x_2 = 0$ of the solid. We suppose that the deformation is plane, [5], with $u_3 = 0$. We are then led to consider solutions of the two-dimensional stress equations of motion for the non-homogeneous elastic solid [4]

$$(1.1)^{2} \qquad \frac{1}{\mu(x)}\left[\ddot{\sigma}_{\alpha\beta}(x,\tau)-\frac{\lambda(x)\,\delta_{\alpha\beta}}{2\lambda(x)+2\mu(x)}\,\ddot{\sigma}_{\gamma\gamma}(x,\tau)\right]=\sigma_{\alpha\gamma,\gamma\beta}(x,\tau)+\sigma_{\beta\gamma,\gamma\alpha}(x,\tau)$$

[1] The investigation underlying this paper was carried out during the author's tenure of a fellowship from the Polish Academy of Sciences in the Division of Applied Mathematics, Brown University, Providence, Rhode Island, U.S.A., 1961/62.

[2] All Greek subscipts take on the values 1, 2, the single argument x stands for (x_1, x_2), and summation over repeated subscripts is implied. Moreover $\sigma_{\alpha\beta,\gamma} = \partial\sigma_{\alpha\beta}/\partial x_\gamma$.

in the region $x_2 > 0$, $|x_1| < \infty$, such that

$$\sigma_{22}(x_1, 0; \tau) = \sigma_{12}(x_1, 0; \tau) = 0, \quad 0 < \tau < \infty, \tag{1.2}$$
$$\sigma_{22}(x_1, \infty; \tau) = \sigma_{12}(x_1, \infty; \tau) = 0,$$

where $\sigma_{\alpha\beta}$ ($\alpha, \beta = 1, 2$) denotes the dimensionless stress tensor in the plane-strain solution and the following notations have been used

$$\sigma_{\alpha\beta} = \hat{\sigma}_{\alpha\beta}/\mu_0, \quad \mu = \hat{\mu}/\mu_0, \quad \lambda = \hat{\lambda}/\mu_0, \tag{1.3}$$
$$\tau = t\sqrt{\mu_0}/x_0\sqrt{\varrho_0}, \quad x_\alpha = \hat{x}_\alpha/x_0.$$

In (1.3) $\hat{\lambda} = \hat{\lambda}(\hat{x}_\alpha)$, $\hat{\mu} = \hat{\mu}(\hat{x}_\alpha)$ ($\alpha = 1, 2$) denote the Lamé coefficients, $\hat{x}_\alpha$ cartesian coordinates, t time, and ϱ_0 constant density of the medium. Moreover $\mu_0 = \hat{\mu}(x_\alpha^0)$ and x_α^0 denotes a fixed point of space and x_0 stands for a dimension coordinate, τ stands for a dimensionless time, and the usual indicial notations have been used in (1.1). The dot denotes the partial differentiation with respect to τ.

2. Stress Function for Rayleigh Waves in a Non–Homogeneous and Elastic Solid (Plane–Strain Solution)

Classical solutions of Rayleigh waves are based on the assumption that the amplitude of the waves dies off exponentially as one recedes from the free surface of the solid. We shall assume the stress solution of Eqs (1.1)-(1.3) in the form

$$\sigma_{11}(x, \tau) = \alpha(x_2) \exp[i(sx_1 - p\tau)],$$
$$\sigma_{22}(x, \tau) = \beta(x_2) \exp[i(sx_1 - p\tau)], \tag{2.1}$$
$$\sigma_{12}(x, \tau) = \gamma(x_2) \exp[i(sx_1 - p\tau)],$$

where $\alpha = \alpha(x_2)$, $\beta = \beta(x_2)$, $\gamma = \gamma(x_2)$ die off, not necessarily exponentially, as x_2 goes to infinity. The period of the waves in (2.1) is $2\pi/p$, the wave length is $2\pi/s$, and the velocity of propagation is $\hat{c}_0 = p/s$, $i^2 = -1$.

To determine the functions α, β, γ in (2.1), we insert from (2.1) into (1.1) and (1.2) and find that λ and μ must be the functions of x_2 only

$$\mu = \mu(x_2), \quad \lambda = \lambda(x_2), \tag{2.2}$$

and the following three equations must be satisfied:

$$-p^2[\alpha(x_2) + \beta(x_2)] = 2[\lambda(x_2) + \mu(x_2)][-s^2\alpha(x_2) + \beta''(x_2) + 2is\gamma'(x_2)],$$
$$-p^2[\alpha(x_2) - \beta(x_2)] = 2\mu(x_2)[-s^2\alpha(x_2) - \beta''(x_2)], \tag{2.3}$$
$$-p^2\gamma(x_2) = \mu(x_2)[is\alpha'(x_2) + is\beta'(x_2) + \gamma''(x_2) - s^2\gamma(x_2)].$$

The boundary conditions take the simple form:

$$\beta(0) = \gamma(0) = 0, \tag{2.4}$$
$$\beta(\infty) = \gamma(\infty) = 0.$$

The system (2.3) is equivalent to

$$\alpha(x_2) = -\frac{1}{s^2(2-\omega)}(s^2\omega + 2D^2)\beta\,,$$

(2.5)

$$2is\gamma(x_2) = \frac{1}{s^2(1-\omega)} D\left\{\frac{\omega}{2-\omega}\,\frac{1}{1-\varkappa}[D^2 - s^2(1-\varkappa\omega)]\beta - 4s^2\frac{1-\omega}{2-\omega}\beta\right\},$$

where

$$\left(\frac{1}{s^2}D\frac{1}{1-\omega}D - 1\right)\frac{1}{1-\varkappa}\,\frac{\omega}{2-\omega}[D^2 - s^2(1-\varkappa\omega)]\beta +$$

(2.6)

$$+4\left[\frac{1}{2-\omega}D^2 - D\frac{1}{1-\omega}D\frac{1-\omega}{2-\omega}\right]\beta = 0\,,$$

$$\omega \neq 1\,, \quad \omega \neq 2\,, \quad \varkappa \neq 1\,;$$

$\beta = \beta(x_2)$ plays the role of the stress function for Rayleigh waves in the non-homogeneous elastic solid (plane-strain solution)[3].

There are many procedures for finding the numerical solutions of (2.6), in the literature, and any type of non-homogeneity may be discussed in this way. In (2.5), (2.6) we have used the notations

$$\varkappa(x_2) = \frac{1-2\nu(x_2)}{2-2\nu(x_2)}\,, \qquad \nu(x_2) = \frac{1-2\varkappa(x_2)}{2-2\varkappa(x_2)}\,;$$

(2.7)

$$\omega(x_2) = \frac{\hat{c}_0^2}{\mu(x_2)}\,, \qquad D = \frac{d}{dx_2}\,;$$

$\nu = \nu(x_2)$ denotes Poisson's ratio.

3. A Class of Solutions in Closed Form

We assume $\mu(x_2) = \text{const}$, $\omega = \omega_0 < 1$. According to (2.6), we find:

(3.1) $$[D^2 - s^2(1-\omega_0)]\left\{\frac{1}{1-\varkappa}[D^2 - s^2(1-\varkappa\omega_0)]\beta\right\} = 0\,.$$

For arbitrary $0 < \varkappa(x_2) < 1$

(3.2) $$[D^2 - s^2(1-\varkappa\omega_0)]\beta = \beta_0\omega_0 s^2(1-\varkappa)\exp[-x_2 s\sqrt{1-\omega_0}]\,,$$

where β_0 denotes a constant, and we have assumed that the functions $\beta(x_2)$ and its second derivative vanish for $x_2 = \infty$, and $\varkappa = \varkappa(x_2)$ is bounded for $0 \leqslant x_2 < \infty$. We assume $\varkappa = \varkappa(x_2)$ in the form:

(3.3) $$\varkappa(x_2) = -(\varkappa_\infty - \varkappa_0)(1+\varepsilon x_2)^{-2} + \varkappa_\infty\,,$$

where $\varkappa_\infty = \varkappa(\infty)$, $\varkappa_0 = \varkappa(0)$, $\varepsilon > 0$.

[3] Here, the notion of stress function is used in the sense that it has served arbitrary non-homogeneity in the problem under consideration.

We shall consider the case $\varkappa_\infty - \varkappa_0 > 0$. Figure 1 displays the x_2 — dependence of the $\varkappa(x_2)$; $\varkappa'(x_2) > 0$, $\varkappa''(x_2) < 0$.

The function $\nu = \nu(x_2)$ is represented by:

(3.4) $$\nu(x_2) = 1 - (1 - \nu_\infty)\left[1 + \frac{\nu_0 - \nu_\infty}{1 - \nu_0}(1 + \varepsilon x_2)^{-2}\right]^{-1},$$

(3.5) $$\nu(0) = \frac{1 - 2\varkappa_0}{2 - 2\varkappa_0}, \quad \nu(\infty) = \frac{1 - 2\varkappa_\infty}{2 - 2\varkappa_\infty}.$$

$$\nu'(x_2) < 0, \quad \nu''(x_2) > 0.$$

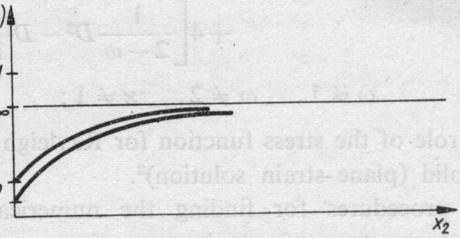

Fig. 1

Figure 2 shows the x_2 — dependence of $\bar{\nu}(x_2)$ for $\varepsilon = 1$; $\bar{\varkappa}_\infty = 1/3$, $\bar{\varkappa}_0 = 1/9$; $\bar{\nu}_0 = 0.4375$, $\bar{\nu}_\infty = 0.25$. The slope of $\nu(x_2)$ is determined by $\varepsilon > 0$, ν_0, ν_∞ and $\nu(x_2)$ is a monotonic decreasing function of x_2; the variable x_2 ranges over the infinite interval $0 \leqslant x_2 < \infty$. The case $\nu_0 < \nu_\infty$ corresponds to the monotonic increasing function $\nu = \nu(x_2)$ in the interval under consideration.

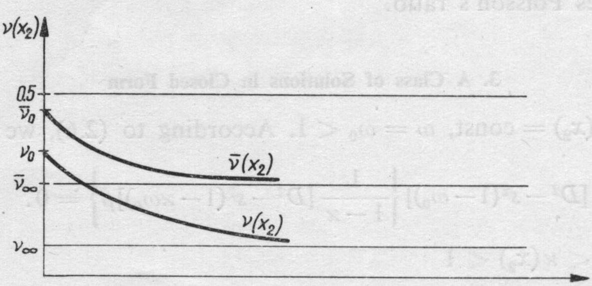

Fig. 2

The solution of Eq (3.2) vanishing at $x_2 = 0$ and $x_2 = \infty$ and particularized to the function $\nu = \nu(x_2)$ given by (3.4), takes the form

(3.6) $$\beta(x_2) = \beta_0 \left\{\sqrt{1 + \varepsilon x_2}\frac{K_n[(1 + \varepsilon x_2)m]}{K_n(m)} - \exp[-x_2 s\sqrt{1 - \omega_0}]\right\},$$

where $m = \dfrac{s}{\varepsilon}\sqrt{1 - \varkappa_\infty \omega_0}$, $\quad n = \sqrt{\dfrac{1}{4} + (\varkappa_\infty - \varkappa_0)\,\omega_0\left(\dfrac{s}{\varepsilon}\right)^2}$

and $K_n = K_n(x)$ is a modified Bessel function of the second kind. If $\varepsilon \to 0$, then $m \to \infty$, and we arrive at the approximate formula

$$K_n(m) \approx \sqrt{\frac{\pi}{2m}} \exp(-m), \tag{3.7}$$

and $\beta = \beta(x_2)$ becomes:

$$\bar{\beta} = \beta_0 \{\exp[-x_2 s \sqrt{1-\varkappa_\infty \omega_0}] - \exp[-x_2 s \sqrt{1-\omega_0}]\}. \tag{3.8}$$

According to (3.6), the function $\gamma = \gamma(x_2)$ takes the form

$$2is\gamma(x_2) = \frac{\beta_0}{(1-\omega_0)(2-\omega_0)} D\Big\{(\omega_0-2)^2 \exp[-x_2 s \sqrt{1-\omega_0}] - 4(1-\omega_0)\sqrt{1+\varepsilon x_2}\, \frac{K_n[m(1+\varepsilon x_2)]}{K_n(m)}\Big\}. \tag{3.9}$$

The boundary condition $\gamma(0) = 0$ becomes:

$$(2-\omega_0)^2 + 4\sqrt{1-\omega_0}\left[\frac{1}{2}\left(\frac{\varepsilon}{s}\right) + \sqrt{1-\varkappa_\infty \omega_0}\, \frac{K_n'(m)}{K_n(m)}\right] = 0. \tag{3.10}$$

If $\varepsilon \to 0$, $m \to \infty$, $K_n'(m)/K_n(m) \to -1$

and we arrive at the classical Rayleigh equation:

$$(2-\omega_0)^2 - 4\sqrt{1-\omega_0}\sqrt{1-\varkappa_\infty \omega_0} = 0. \tag{3.11}$$

From (3.10), it follows that ω_0 is dependent on the wave-length $2\pi/s$, and the parameter ε. If we assume $\varepsilon = s$, ω_0 is independent of s and there is no dispersion.

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References

[1] W. M. Ewing, W. S. Jardetzky, I. Press, *Elastic Waves in Layered Media*, New York 1957.

[2] H. Kolsky, *Stress Waves in Solids*, Oxford 1953.

[3] A. Sommerfeld, *Mechanics of Deformable Bodies*, New York 1950.

[4] J. Ignaczak, *A completeness problem for stress equations of motion in the linear elasticity theory*, Arch. Mech. Stos., 2, **15** (1963).

[5] I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, sec. ed., New York–Toronto–London 1956.

[6] N. V. Zwolinski, *Rayleigh waves in an inhomogeneous elastic half–space of special type*, in Russian, Izvest. Akad. Nauk. SSSR, Ser. Geograph. and Geophys., 3, **9**(1945), 261-278.

[7] A. G. Alenitsyn, *Rayleigh waves in nonhomogeneous elastic semi–space*, in Russian, Prik. Mat. Mech., 3, **27**(1963), 547-550.

Streszczenie

FALE RAYLEIGHA W NIEJEDNORODNEJ I IZOTROPOWEJ PÓŁPRZESTRZENI SPRĘŻYSTEJ (I)

Zastosowano metodę naprężeniowych równań ruchu w celu rozwiązania problemu fal Rayleigha w niejednorodnej i izotropowej półprzestrzeni sprężystej (płaski stan odkształcenia). Dowiedziono, że istnieje pewna funkcja naprężeń, która spełnia zwyczajne liniowe równanie różniczkowe rzędu czwartego dla dowolnej niejednorodności. Wynika stąd, że równania naprężeniowe ruchu można w tym przypadku rozdzielić. Naprężenia mogą być wyrażone przez jedną funkcję spełniającą zwyczajne równanie o zmiennych współczynnikach. Równanie to może być rozwiązywane właściwymi metodami numerycznymi. W pracy podano tylko pewne klasy rozwiązań w postaci zamkniętej. Stąd wyprowadzono równania prędkości fal Rayleigha dla szczególnych przypadków niejednorodności, które zdaniem autora dotychczas nie były rozważane.

Резюме

ВОЛНЫ РЭЛЕЯ В НЕОДНОРОДНОМ И ИЗОТРОПНОМ УПРУГОМ ПОЛУПРОСТРАНСТВЕ (I)

Применяются уравнения движения в напряжениях к решению задачи, касающейся волн Рэлея в неоднородном и изотропном упругом полупространстве (плоское деформированное состояние). Доказывается существование некоторой функции напряжений, удовлетворяющей обыкновенному, линейному дифференциальному уравнению четвертого порядка, для произвольной неоднородности. Отсюда следует, что уравнения движения в напряжениях можно, в этом случае, разделить. Напряжения можно выразить с помощью одной функции, удовлетворяющей обыкновенному уравнению с переменными коэффициентами. Это уравнение можно решить соотвествующим численным методом. В работе приводятся только некоторые классы решений в замкнутом виде. Отсюда выводится уравнение скорости волн Рэлея для частных случаев неоднородности, которые, по мнению автора, не были до сих пор рассмотрены.

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