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ON THE STRESS EQUATIONS OF MOTION IN THE LINEAR THERMOELASTICITY

JÓZEF IGNACZAK (WARSZAWA)

Introduction

It has been shown in [1] that if a dynamic problem of linear elasticity[1] is formulated in terms of stresses, it is sufficient to meet only one stress tensor equation[2] and suitable stress initial and boundary conditions. In this case, displacement vector is excluded from the considerations.

In [1] and [2] certain aspects of pure stress equations of motion method have been emphasized. From the theoretical point of view it may be interesting to ask if the dynamic stress problem can be reduced to a «natural stress–boundary value problem[3]». For example, can the system of plane stress equations of motion (three equations) be replaced by another system (two equations) in such a way that there will appear only such stress components as are prescribed on the boundary of the elastic body? In the last system the number of equations and unknown functions will be the same as in the displacement method of solution but the boundary conditions will take the simplest form.

One should not mix this question with the economy of solution methods in classical elastodynamics. In the present paper, we shall make neither any attemps to appraise the comparative merits of the alternative methods of solution of initial–boundary value problem nor prove that stress equations of motion method appears to be considerably more economical than the procedures previously used in this connection. Only some new aspects concerning direct determination of stresses from the stress equations of motion will be pointed out. The problem chosen for this purpose is that of the stress formulations in dynamic thermoelasticity. Anexample of the so–called« natural stress boundary value problem» will be also shown.

1. The Necessary and Sufficient Conditions for Tensor σ_{ij} to Belong to Linear, Dynamic Thermoelasticity

Let us consider a homogeneous, isotropic, elastic medium. We assume that the elastic moduli λ, μ and the coefficient of thermal expansion of the body α_t are constant and independent of the temperature $T = T(x, t)$.

[1] In [1] a linear non–homogeneous and anisotropic elastic solid has been considered.

[2] A system of six equations for six unknown components of the stress tensor σ_{ij}.

[3] The first initial–boundary value problem in linear elastodynamics (the displacement vector prescribed on the boundary for $t > 0$ and its value and velocity at initial moment $t = 0$) is naturally governed by the displacement equations of motion.

The following equations must be met by thermal stress tensor σ_{ij}, $(\sigma_{\alpha\beta})$, $i, j = 1, 2, 3$, $(\alpha, \beta = 1, 2)$ in any linear elastic body:

1. Three-dimensional space $x_i = (x_1, x_2, x_3)$, $t =$ time,

$$\text{(1.1)}\qquad \Box_2^2\sigma_{ij} + \frac{2\lambda+2\mu}{3\lambda+2\mu}\sigma_{kk,ij} + \left(\frac{1}{c_2^2} - \frac{1}{c_1^2}\right)\frac{\lambda\delta_{ij}}{3\lambda+2\mu}\ddot{\sigma}_{kk} + 2\mu\alpha_t\left(T_{,ij} + \frac{3\lambda+2\mu}{\lambda+2\mu}T_{,kk}\delta_{ij}\right) - \frac{5\lambda+4\mu}{\lambda+2\mu}\alpha_t\varrho\ddot{T}\delta_{ij} = 0$$

$$x \in V, \quad 0 < t < \infty,$$

$$\text{(1.2)}^4\qquad \Box_2^2\{\Box_1^2\sigma_{ij} - \vartheta_0[2\mu(T_{,ij} - T_{,ss}\delta_{ij}) + \varrho\ddot{T}\delta_{ij}]\} = 0,$$

$$\Box_\alpha^2 = \nabla^2 - \frac{1}{c_\alpha^2}\frac{\partial^2}{\partial t^2}, \quad \alpha = 1, 2,$$

$$\vartheta_0 = \frac{3\lambda+2\mu}{\lambda+2\mu}\alpha_t.$$

2. Two-dimensional case: $x_\alpha = (x_1, x_2)$.

a. Plane strain solution: $\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(x_\gamma, t)$.

$$\text{(1.3)}\qquad \Box_2^2\sigma_{\alpha\beta} + \sigma_{\gamma\gamma,\alpha\beta} + \left(\frac{1}{c_2^2} - \frac{1}{c_1^2}\right)\frac{\lambda\delta_{\alpha\beta}}{2\lambda+2\mu}\ddot{\sigma}_{\gamma\gamma} + 2\mu\vartheta_0\Box_2^2 T\delta_{\alpha\beta} = 0,$$

$$x \in D, \quad 0 < t < \infty,$$

$$\text{(1.4)}\qquad \Box_2^2\{\Box_1^2\sigma_{\alpha\beta} - \vartheta_0[2\mu(T_{,\alpha\beta} - T_{,\mu\mu}\delta_{\alpha\beta}) + \varrho\ddot{T}\delta_{\alpha\beta}]\} = 0.$$

b. Generalized plane-stress solution: $\bar{\sigma}_{\alpha\beta} = \bar{\sigma}_{\alpha\beta}(x_\mu, t)$.

$$\text{(1.5)}\qquad \Box_2^2\bar{\sigma}_{\alpha\beta} + \bar{\sigma}_{\gamma\gamma,\alpha\beta} + \left(\frac{1}{c_2^2} - \frac{1}{c_1^2}\right)\frac{\lambda\ddot{\bar{\sigma}}_{\gamma\gamma}}{3\lambda+2\mu}\delta_{\alpha\beta} + 2\mu\vartheta_0\Box_2^2 T\delta_{\alpha\beta} = 0,$$

$$x \in D, \quad 0 < t < \infty,$$

$$\text{(1.6)}\qquad \Box_2^2\{\bar{\Box}_1^2\bar{\sigma}_{\alpha\beta} - \bar{\vartheta}_0[2\mu(T_{,\alpha\beta} - T_{,\mu\mu}\delta_{\alpha\beta}) + \varrho\ddot{T}\delta_{\alpha\beta}]\} = 0, \quad x \in D, \quad 0 < t < \infty$$

$$\bar{\Box}_1^2 = \nabla^2 - \frac{1}{\bar{c}_1^2}\frac{\partial^2}{\partial t^2}, \quad \frac{1}{\bar{c}_1^2} = \frac{1}{c_2^2}\,\frac{\lambda+2\mu}{4\lambda+4\mu}, \quad \bar{\vartheta}_0 = \frac{3\lambda+2\mu}{2\lambda+2\mu}\alpha_t.$$

Proof of necessity of (1.1)-(1.2) has been given in [3]. To demonstrate (1.3)-(1.6) the definitions of plane strain and generalized plane stress solutions should be taken into account. See, for example, [4].

Equations (1.1)-(1.6) provide general suggestions about the stress tensor σ_{ij}. However, they are not sufficient to solve in general the stress boundary value problem even if suitable stress initial and boundary conditions are prescribed.

[4] For the notations that we have adopted, see [3].

The following stress initial-boundary value problems are formulated correctly:

I_σ: Find the tensor $\sigma_{ij} = \sigma_{ij}(x, t)$, x belongs to V, $0 < t < \infty$, such that:

(1.7) $$\sigma_{ij}(x, 0) = \dot{\sigma}_{ij}(x, 0) = 0, \quad x \in V,$$

(1.8) $$\sigma_{ij}(x, t) n_j(x) = 0, \quad x \in B, \quad 0 < t < \infty,$$

(1.9) $$\frac{1}{c_2^2}\left[\ddot{\sigma}_{ij}(x, t) - \frac{\lambda \delta_{ij}}{3\lambda + 2\mu}\ddot{\sigma}_{kk}(x, t)\right] + 2\alpha_t \varrho \ddot{T}\delta_{ij}$$
$$-\sigma_{ik,kj}(x, t) - \sigma_{jk,ki}(x, t) = 0, \quad x \in V, \quad 0 < t < \infty.$$

II_σ: Find the tensor $\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(x, t)$, $x \in D$, $0 < t < \infty$, that meets set:

(1.10) $$\sigma_{\alpha\beta}(x, 0) = \dot{\sigma}_{\alpha\beta}(x, 0) = 0, \quad x \in D,$$

(1.11) $$\sigma_{\alpha\beta}(x, t) n_\beta(x) = 0, \quad x \in \Omega, \quad 0 < t < \infty,$$

(1.12) $$\frac{1}{c_2^2}\left[\ddot{\sigma}_{\alpha\beta}(x, t) - \frac{\lambda \delta_{\alpha\beta}}{2\lambda + 2\mu}\ddot{\sigma}_{\gamma\gamma}(x, t)\right] + \frac{3\lambda + 2\mu}{\lambda + \mu}\alpha_t \varrho \ddot{T}\delta_{\alpha\beta}$$
$$-\sigma_{\alpha\gamma,\gamma\beta}(x, t) - \sigma_{\beta\gamma,\gamma\alpha}(x, t) = 0, \quad x \in D, \quad 0 < t < \infty.$$

III_σ: Find the tensor $\bar{\sigma}_{\alpha\beta} = \bar{\sigma}_{\alpha\beta}(x, t)$, $x \in D$, $0 < t < \infty$, such that:

(1.13) $$\bar{\sigma}_{\alpha\beta}(x, 0) = \dot{\bar{\sigma}}_{\alpha\beta}(x, 0) = 0, \quad x \in D,$$

(1.14) $$\bar{\sigma}_{\alpha\beta}(x, t) n_\beta(x) = 0, \quad x \in \Omega, \quad 0 < t < \infty,$$

(1.15) $$\frac{1}{c_2^2}\left[\ddot{\bar{\sigma}}_{\alpha\beta}(x, t) - \frac{\lambda \delta_{\alpha\beta}}{3\lambda + 2\mu}\ddot{\bar{\sigma}}_{\gamma\gamma}(x, t)\right] + 2\alpha_t \varrho \ddot{T}\delta_{\alpha\beta} - \bar{\sigma}_{\alpha\gamma,\gamma\beta}(x, t) - \bar{\sigma}_{\beta\gamma,\gamma\alpha}(x, t) = 0.$$
$$x \in D, \ 0 < t < \infty.$$

To prove the correctness of stress formulations I_σ, II_σ, III_σ see [1] and use Duhamel–Neumann strain–stress relations which account for temperature field.

The necessary conditions (1.1)–(1.6) can be useful to solve problems I_σ-III_σ, that we have just stated. According to (1.2), (1.4), (1.6), the following tensors can be accepted as the particular stress solutions of dynamic thermoelasticity:

(1.16) $$\sigma^p_{ij} = \vartheta_0 \Box_1^{-2}[2\mu(T_{,ij} - T_{,ss}\delta_{ij}) + \varrho \ddot{T}\delta_{ij}],$$

(1.17) $$\sigma^p_{\alpha\beta} = \vartheta_0 \Box_1^{-2}[2\mu(T_{,\alpha\beta} - T_{,\mu\mu}\delta_{\alpha\beta}) + \varrho \ddot{T}\delta_{\alpha\beta}],$$

(1.18) $$\bar{\sigma}^p_{\alpha\beta} = \bar{\vartheta}_0 \bar{\Box}_1^{-2}[2\mu(T_{,\alpha\beta} - T_{,\mu\mu}\delta_{\alpha\beta}) + \varrho \ddot{T}\delta_{\alpha\beta}],$$

where $\Box_1^{-2}$ and $\bar{\Box}_1^{-2}$ denote the inverse operator of $\Box_1^2$ and inverse operator of $\bar{\Box}_1^2$, respectively. We note that (1.16)-(1.18) display an explicit dependence of the stress tensor on the temperature function. Direct substitution of (1.16), (1.17) and (1.18) into (1.9), (1.12) and (1.15) respectively shows that σ^p_{ij}, $\sigma^p_{\alpha\beta}$, $\bar{\sigma}^p_{\alpha\beta}$, also meet sufficient conditions, consequently, they represent the particular solutions of dynamic thermoelasticity[5]. If, for example, I_σ problem has to be solved, we can write: $\sigma_{ij} = \sigma^p_{ij} + \sigma^q_{ij}$,

[5] The particular stress solutions (1.16)-(1.18) constitute a counterpart of potential solutions n thermoe lasticity. See, for example, [5] and [6].

where σ^g_{ij} meets a non-homogeneous boundary conditions and homogeneous stress Eq. (1.9). We shall not discuss here the economy of the last procedure that seems to be somewhat lengthier than the displacement method of analysis. We shall show an example concerning III_σ problem and show how this problem can be replaced by the so-called «natural stress boundary value problem»?

Example. Find the stress tensor $\bar{\sigma}_{\alpha\beta} = \bar{\sigma}_{\alpha\beta}(x, t)$ that meets III_σ, if D is a strip region $|x_1| < \infty$, $|x_2| < h < \infty$.

One of the most natural methods of solution in this case seems to be the integral transforms method [7]. Taking into account (1.18) as the particular stress solution, it is seen that the only system which will be left to discuss are three homogeneous Eqs. (1.15). Applying to (1.15) the Laplace transform and then Fourier transform defined by:

$$f^*(x_1, x_2; p) = \int_0^\infty e^{-pt} f(x_1, x_2; t)\, dt, \tag{1.19}$$

$$\hat{g}(\xi_1, x_2; p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi_1 x_1} g^*(x_1, x_2; p)\, dx_1, \tag{1.20}$$

we note that $\hat{\sigma}^g_{11}$ can be eliminated from the considerations and we arrive at two ordinary differential equations for $\hat{\sigma}^g_{22}$ and $\hat{\sigma}^g_{12}$.

$$\begin{gathered}[\varkappa_0 \hat{\omega} (D^2 - \mu_1^2) + (1 - \hat{\omega}) D^2] \hat{\sigma}^g_{22} - i\xi_1 D \hat{\sigma}^g_{12} = 0, \\ (D^2 - \mu_2^2)[(1 - \hat{\omega}) D \hat{\sigma}^g_{22} - i\xi_1 \hat{\sigma}^g_{12}] = 0.\end{gathered} \tag{1.21}$$

Boundary conditions in the transform domain take the simple form:

$$[\hat{\sigma}^g_{22} + \hat{\sigma}^p_{22}]_{x_2 = \pm h} = 0, \quad [\hat{\sigma}^g_{12} + \hat{\sigma}^p_{12}]_{x_2 = \pm h} = 0. \tag{1.22}$$

In (1.21) we have adopted the notations:

$$D = \frac{d}{dx_2}, \quad \bar{k}_1^2 = \frac{p^2}{c_1^2}, \quad k_2^2 = \frac{p^2}{c_2^2},$$

$$\mu_1 = \sqrt{\xi_1^2 + \bar{k}_1^2}, \quad \mu_2 = \sqrt{\xi_1^2 + k_2^2},$$

$$\hat{\omega} = \frac{k_2^2/2}{\xi_1^2 + k_2^2/2}, \quad \varkappa_0 = \frac{2\lambda + 2\mu}{3\lambda + 2\mu}.$$

The boundary value problem described by (1.21) and (1.22) constitutes the so-called «natural stress boundary value problem» for such stress components as are prescribed on the boundary.

We also draw attention to the fact that second equation of (1.21) is the third order equation with respect to x_2, a circumstance which does not appear in displacement treatment of the problem under consideration.

Clearly, the system (1.21) can be also used to solve general dynamic stress problem in the strip region, provided arbitrary surface tractions are prescribed on the boundary[6].

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[6] We reduce problem (1.21) to the form:

$$i\xi_1\mu_2^2\hat{\sigma}_{12}^g = D[\varkappa_0\hat{\omega}(D^2-\mu_1^2)+(1-\hat{\omega})\mu_2^2]\hat{\sigma}_{22}^g,$$

where

$$(D^2-\mu_1^2)(D^2-\mu_2^2)\hat{\sigma}_{22}^g = 0;$$

$\hat{\sigma}_{11}^g$ can be found from the relation:

$$(\xi_1^2+k_2^2/2)\hat{\sigma}_{\gamma\gamma}^g = -(D^2-\mu_2^2)\hat{\sigma}_{22}^g.$$

Streszczenie

O NAPRĘŻENIOWYCH RÓWNANIACH RUCHU W LINIOWEJ TERMOSPRĘŻYSTOŚCI

W pracy sformułowano warunki konieczne i dostateczne dla tensora naprężenia w liniowej, dynamicznej termosprężystości przy założeniu niezależności stałych materiałowych i cieplnych od temperatury. Pewien przykład redukcji naprężeniowego problemu dynamicznego do tzw. naturalnego naprężeniowego zagadnienia brzegowego jest podany dla pasma tarczowego, poddanego nieustalonemu polu temperatury.

Резюме

ОБ УРАВНЕНИЯХ ДВИЖЕНИЯ В НАПРЯЖЕНИЯХ В ЛИНЕЙНОЙ ТЕОРИИ ТЕРМОУПРУГОСТИ

Приводится формулировка необходимых и достаточных условий для тензора напряжений в линейной, динамической термоупругости, при предположении независимости постоянных материала и термических постоянных от температуры. Дается пример сведения динамической задачи в напряжениях к так называемой, естественной краевой задаче, для полосы подверженной действию нестационарного температурного поля.

DEPARTMENT OF MECHANICS OF CONTINUOUS MEDIA
IBTP POLISH ACADEMY OF SCIENCES

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