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On the second-order work in plasticity(*)

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THE CLASSICAL concept of the second-order work of deformation is extended to indirect strain paths whose complexity is preserved as their length tends to zero. Explicit expressions valid for arbitrarily circuitous paths are derived for elastic-plastic materials with a discrete set of internal plastic deformation mechanisms which obey the normality and symmetry postulates. Classical elastoplastic solids are included as a special case. For a general time-independent and incrementally nonlinear material, a constitutive inequality is formulated in terms of the second-order work corresponding to direct and certain indirect paths leading to the same final strain increment. It is shown that fundamental qualitative properties of a macroscopic constitutive law can be deduced from a second-order work inequality established or postulated at a micro-level.

Klasyczne pojęcie pracy drugiego rzędu rozszerzono na ścieżki nieproporcjonalnego odkształcania, których złożoność pozostaje zachowana gdy ich długość dąży do zera. Dla materiałów sprężysto-plastycznych z dyskretnym zbiorem wewnętrznych mechanizmów deformacji plastycznych spełniających postulaty stowarzyszoności i symetrii, wyprowadzono jawne wyrażenia na pracę drugiego rzędu słuszne dla dowolnie skomplikowanych dróg odkształcenia. Klasyczne materiały sprężysto-plastyczne uwzględniono jako szczególny przypadek. Dla ogólnego, przyrostowo-nieliniowego materiału niewrażliwego na zmiany skali czasu sformułowano nierówność konstytutywną, porównując pracę drugiego rzędu dla prostych i pewnych złożonych dróg prowadzących do tego samego odkształcenia końcowego. Pokazano, że podstawowe cechy jakościowe makroskopowego związku konstytutywnego można wyprowadzić z takiej nierówności obowiązującej lub postulowanej na poziomie mikroskopowym.

Классическое понятие работы второго порядка расширено на пути непропорционального деформирования, сложность которых сохраняется, когда их длина стремится к нулю. Для упруго-пластических материалов с дискретным множеством внутренних механизмов пластической деформации, удовлетворяющих постулатам ассоциирования и симметрии, выведены явные выражения для работы второго порядка справедливые для произвольно сложных путей деформирования. Классические упруго-пластические материалы учтены как частный случай. Для общего, нелинейного в приростах материала нечувствительного на изменения шкалы времени, сформулировано определяющее неравенство, сравнивая работы второго порядка для простых и некоторых сложных путей, приводящих к этой самой конечной деформации. Показано, что основные качественные свойства макроскопического определяющего соотношения можно вывести из такого же самого неравенства, которое обязывает или постулируется на микроскопическом уровне.

1. Introduction

THE SECOND-ORDER WORK of deformation is a classical concept in the theory of plasticity. In the so-called small strain theory, or more precisely, when

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geometry changes are disregarded, the second-order work per unit volume during *proportional* application of a small increment $\delta\varepsilon_{ij}$ in strain components is, by definition, equal to $\frac{1}{2}\,\delta\sigma_{ij}\,\delta\varepsilon_{ij}$ (or $\frac{1}{2}\,\delta\boldsymbol{\sigma}\cdot\delta\boldsymbol{\varepsilon}$ in the symbolic notation), where $\delta\sigma_{ij}$ are the respective small increments of the stress components; the summation convention is used for repeated subscripts. That expression plays a fundamental role in DRUCKER'S [2, 3] definition of work-hardening, interpreted as a postulate of stability of the material in a restricted sense. A similar expression, integrated over the body volume, appears in HILL'S [4, 5] condition for stability of equilibrium of an inelastic continuous body under dead loading, with geometry changes taken into account. In the most concise form of Hill's condition, the second-order work is written down precisely in terms of increments of the *nominal* stress and deformation gradient. It is nowadays well known that the actual changes in geometry of the material element generally make the value of an expression of the type: (stress increment × strain increment) sensitive to the choice of stress and strain measures, even if the measures are restricted to be work-conjugate. That effect can be of the same order of magnitude as the expression value itself, irrespective of whether the strains theselves are „small" or not. However, we will be primarily concerned here with the comparison of the second-order work corresponding to distinct routes leading to *the same* final strain, and the conclusions drawn from such a comparison will be shown to be measure-invariant.

The principal difference between the present analysis and the classical approach is that the work of deformation is evaluated here not only along direct paths of proportional straining, but also along more complicated paths whose total length remains small. The „second-order" term can be specified by scaling down a given path in strain space such that its length tends to zero while its complexity is preserved. The idea of examining a path-dependent *second-order* work in inelastic solids appears to be not explored in the literature, especially for a general, incrementally nonlinear material response, e.g. at a yield-surface vertex whose formation is predicted by micromechanical theories of plasticity [7]. Te present discussion is a continuation of the analysis made in the previous paper by the author [18] where certain qualitative properties of elastic-plastic models of metal crystals or polycrystals were also investigated but without reference to the work of deformation. It will be shown below that the comparison of the second-order work on direct and certain indirect paths leading to the same final increment of strain yields a new and more physical formulation of the constitutive inequality derived in [18]. Since the inequality has far-reaching implications for constitutive modeling as well as e.g. for the bifurcation theory, this result is of more than academic interest. Validity of the constitutive inequality was recently examined in [16] for solids characterized by existence of a thermodynamic potential and by the maximum dissipation principle.

In Sect. 2 the concept of the second-order work is introduced for a general class of deformation paths and an arbitrary material. In Sect. 3 explicit expressions for the second-order work are derived for elastic-plastic materials with a discrete set of internal plastic deformation mechanisms (e.g. single crystals deformed by multisilip) which obey the normality and symmetry postulates. A general connection between the constitutive inequality mentioned above and a second-order work inequality is established in Sect. 4. Certain illustrative examples are presented in the last section.

2. The second-order work

As a starting point, an infinitesimal increment of the work of deformation under current tractions on the material element is expressed in the form [9, 10, 11]

(2.1) $$d\omega = t_{ij}\, de_{ij} = \mathbf{t} \cdot d\mathbf{e} \quad \text{per unit reference volume,}$$

where $d\mathbf{e}$ is an infinitesimal increment of an arbitrarily chosen strain measure $\mathbf{e}$ of Lagrangian type and $\mathbf{t}$ is the conjugate stress; for simplicity, the tensor components are taken on a fixed rectangular basis. The reference configuration is arbitrary but fixed, so that a finite work increment per unit reference volume is obtained by integration of Eq. (2.1) along a given directed continuous line (path) in the strain space, provided the (path-dependent) stress variations along the path are defined, at least in principle. We shall examine arbitrarily circuitous but piecewise smooth strain paths $\mathscr{P}$, leading from a given initial strain $\bar{\mathbf{e}}$ to a neighboring final strain $\bar{\bar{\mathbf{e}}} = \bar{\mathbf{e}} + \hat{\mathbf{e}}$. A path $\mathscr{P}$ is parameterized by the length variable $\theta = \int |d\mathbf{e}|$, where $|d\mathbf{e}| = (de_{ij}\, de_{ij})^{1/2}$. The symbols $(-)$, $(=)$ and $(\hat{\ })$ over a quantity denote its initial and final values on a path $\mathscr{P}$ and their difference, respectively; by definition, $\bar{\theta} = 0$ and $\bar{\bar{\theta}} = \hat{\theta}$. One could employ the usual prefix δ instead of the symbol $(\hat{\ })$ to indicate a small increment, but it seems preferable to use another symbol for increments which need not be reached on a direct path.

We introduce a natural assumption (which, however, excludes rigid-plastic solids) that a stress increment along any strain path is (at the most) of order of the path length increment, viz.

(2.2) $$|\mathbf{t}(\theta_2) - \mathbf{t}(\theta_1)| \leqslant C\,|\theta_2 - \theta_1| \quad \text{along any path } \mathscr{P},$$

where C is a constant number, the same for all paths.

Contrary to a typical asymptotic expansion along a given path, we do *not* require a strain path to be fixed as $\bar{\bar{\mathbf{e}}} \to \bar{\mathbf{e}}$ and $\hat{\theta} \to 0$. Paths under consideration may thus differ substantially from a smooth arc no matter how small

$\hat{\theta}$ is. Nevertheless, integration of Eq. (2.1) followed by substitution of (2.2) and by the limit transition still yields the standard first-order formula

$$\hat{\omega} = \bar{\mathbf{t}} \cdot \hat{\mathbf{e}} + o(\hat{\theta}) \tag{2.3}$$

which remains valid for any sequence of paths $\mathscr{P}$ whose length $\hat{\theta}$ tends to zero. Here and in the following the classical order symbol $o(\hat{\theta}^n)$ is used for indicating a scalar quantity ε of order higher than $\hat{\theta}^n$, i.e. such that $\varepsilon/\hat{\theta}^n \to 0$ as $\hat{\theta} \to 0$. We may rewrite (2.3) as

$$\hat{\omega} = \bar{\mathbf{t}} \cdot \hat{\mathbf{e}} + \Delta_2\omega + o(\hat{\theta}^2). \tag{2.4}$$

The quantity $\Delta_2\omega$, if defined, is of order $\hat{\theta}^2$ (or vanishes) on account of Eq. (2.2) and is thus called the second-order work. Henceforth the convention is introduced that all equations or inequalities involving $\Delta_2\omega$ are valid to that order only, i.e. can be violated by terms of order $o(\hat{\theta}^2)$. For incrementally nonlinear and path-dependent solids, the actual expression for $\Delta_2\omega$ may depend on the class of paths $\mathscr{P}$ which are taken into account as the path length $\hat{\theta}$ tends to zero. Typically, $\Delta_2\omega$ will involve $\hat{\mathbf{e}}$ and $\hat{\mathbf{t}}$ as well as other characteristics of the considered paths; examples will be discussed in the subsequent sections.

On integrating Eq. (2.1) along a path $\mathscr{P}$ and rearranging, we obtain

$$\begin{aligned} \hat{\omega} &= \bar{\mathbf{t}} \cdot \hat{\mathbf{e}} + \int_0^{\hat{\theta}} e'_{ij}(\theta) \left(\int_0^{\theta} t'_{ij}(s)\, ds \right) d\theta \\ &= \bar{\mathbf{t}} \cdot \hat{\mathbf{e}} + \frac{1}{2} \hat{\mathbf{t}} \cdot \hat{\mathbf{e}} + \int_0^{\hat{\theta}} e'_{ij}(\theta) \left(\int_0^{\theta} \{ t'_{ij}(s) - \hat{t}_{ij}/\hat{\theta} \}\, ds \right) d\theta \\ &\quad + (\hat{t}_{ij}/\hat{\theta}) \int_0^{\hat{\theta}} \{ e'_{ij}(\theta) - \hat{e}_{ij}/\hat{\theta} \}\, \theta\, d\theta, \end{aligned} \tag{2.5}$$

where a prime denotes differentiation by the function argument along a fixed path. A special but important case arises when $\Delta_2\,\omega$ is evaluated for a class $\mathscr{D}$ of *direct* paths along which, by definition, increments of the *derivatives* (with respect to the path length) of strain and stress are, at the most, of order of the path length increment(¹), viz.

$$\begin{aligned} |\mathbf{e}'(\theta_2) - \mathbf{e}'(\theta_1)| &\leqslant C_1\, |\theta_2 - \theta_1|, \\ |\mathbf{t}'(\theta_2) - \mathbf{t}'(\theta_1)| &\leqslant C_2\, |\theta_2 - \theta_1|, \end{aligned} \tag{2.6}$$

along any path $\mathscr{P} \in \mathscr{D}$, where C_1 and C_2 are positive constants, the same for all paths from $\mathscr{D}$.(²) No matter how a path from $\mathscr{D}$ varies as $\hat{\theta} \to 0$, the expressions

(¹) Validity of Eq. (2.7) can be proved under a weaker assumption, namely, that $|\mathbf{e}'(\theta) - \mathbf{e}'(0)| = o(\theta^0)$ and $|\mathbf{t}'(\theta) - \mathbf{t}'(0)| = o(\theta^0)$ for all paths from $\mathscr{D}$.

(²) It is clear that Eq. (2.6) can be violated for a sequence of smooth paths, e.g. if their "curvature" tends to infinity.

in curly brackets in Eq. (2.5) are of order $\hat{\theta}$, by the Taylor formula, and the integrals over $[0, \hat{\theta}]$ are of order $\hat{\theta}^3$. By comparison with Eq. (2.4) we arrive at the result that within a class $\mathscr{D}$ of direct paths, the second-order work is defined by the familiar formula

$$\text{(2.7)} \qquad \Delta_2 \omega = \frac{1}{2} \hat{\mathbf{t}} \cdot \hat{\mathbf{e}} \quad \text{if} \quad \mathscr{P} \in \mathscr{D};$$

it is recalled that $\hat{\mathbf{e}}$ and $\hat{\mathbf{t}}$ are the final increments of stress and strain reached along a variable path. If a path is fixed when $\hat{\theta} \to 0$ then Eq. (2.7) reduces to the „trapezoid rule of quadrature" [10, 11]; cf. also [17].

The above considerations might seem to be elementary, in view of simplicity of the formulae. However, this is not quite so since a nonstandard limit transition for both the variable path and its length is performed. A detailed specification of the assumptions was thus necessary.

Following HILL [9, 11], we discuss now briefly the question of transformation to another work-conjugate pair of stress and strain measures, say $(\mathbf{t}^*, \mathbf{e}^*)$, where $\mathbf{e}^*$ is a sufficiently smooth and invertible function of $\mathbf{e}$ alone (possibly associated with another reference configuration). Components of a strain increment defined by geometric variables transform according to the standard formula

$$\text{(2.8)} \qquad \hat{e}^*_{ij} = \left(\frac{\partial e^*_{ij}}{\partial e_{kl}}\right)_{\bar{\mathbf{e}}} \hat{e}_{kl} + \frac{1}{2} \left(\frac{\partial^2 e^*_{ij}}{\partial e_{kl}\, \partial e_{pq}}\right)_{\bar{\mathbf{e}}} \hat{e}_{kl}\, \hat{e}_{pq} + o\left(|\hat{\mathbf{e}}|^2\right).$$

The total work per unit *mass* must be invariant, so that $\hat{\omega}$ transforms to $\hat{\omega}^* = (\rho^*/\rho)\, \hat{\omega}$ and

$$\text{(2.9)} \qquad \left(\frac{\partial e^*_{ij}}{\partial e_{kl}}\right)_{\bar{\mathbf{e}}} \bar{t}^*_{ij} = (\rho^*/\rho)\, \bar{t}_{kl},$$

where ρ and ρ^* stand for the material density in the respective reference configurations. On substituting Eqs. (2.8) and (2.9) in Eq. (2.4) it can be seen that decomposition of the work into the first- and second-order terms is itself *not* an invariant concept. Explicitly, the second-order work per unit mass is *not* measure-invariant but transforms according to the formula

$$\text{(2.10)} \qquad \frac{1}{\rho^*} \Delta_2 \omega^* = \frac{1}{\rho} \Delta_2 \omega - \frac{1}{2\rho^*} \bar{t}^*_{ij} \left(\frac{\partial^2 e^*_{ij}}{\partial e_{kl}\, \partial e_{pq}}\right)_{\bar{\mathbf{e}}} \hat{e}_{kl}\, \hat{e}_{pq};$$

this is a straightforward extension of the transformation formula obtained by HILL [11] for an expression of the type (2.7) above. In particular, it follows that the sign of $\Delta_2 \omega$ can depend on a subjective choice of the strain measure. However, it still makes sense to compare the second-order work on different routes leading to *the same* strain increment since the contributions from the last expression in Eq. (2.10) then cancel and the conclusions are measure invariant.

3. Elastic-plastic materials with a discrete set of internal plastic deformation mechanisms

3.1. Constitutive framework

In this Section we assume the known „normality structure" of constitutive rate equations for crystals deformed plastically by multislip, or more generally, for solids with a finite number N of internal mechanisms of rate-independent inelastic deformation. The constitutive framework for such materials at finite strain was given by HILL and RICE [12] and SEWELL [19], generalizing earlier theories of KOITER [13], MANDEL [14] and HILL [6]. Within that framework, we adopt here the set of assumptions as in [18], viz.

$$\dot{\gamma}_K \geqslant 0, \quad f_K \leqslant 0, \quad f_K \dot{\gamma}_K = 0 \quad \text{(no sum)}, \tag{3.1}$$

$$\boldsymbol{\lambda}_K = \partial f_K / \partial \mathbf{e}, \tag{3.2}$$

$$\mathbf{t} = \mathbf{E} \cdot \dot{\mathbf{e}} - \boldsymbol{\lambda}_K \dot{\gamma}_K, \tag{3.3}$$

$$\dot{f}_K = \boldsymbol{\lambda}_K \cdot \dot{\mathbf{e}} - g_{KL} \dot{\gamma}_L, \tag{3.4}$$

$$E_{ijkl} = E_{klij}, \tag{3.5}$$

$$g_{KL} = g_{LK}, \tag{3.6}$$

where a dot over a symbol denotes the *right-hand* rate of change with respect to a time-like parameter t.

The reader is referred to the papers cited above for a detailed characterization of the above assumptions. Briefly, upper case lower indices varying from 1 to N refer to quantities related to a specific plastic deformation mechanism; the summation convention is adopted here for those indices except when an indication „no sum" appears. $\dot{\gamma}_K$ denotes a scalar measure of the rate of activity of the K-th mechanism and f_K is the respective smooth yield function (in strain space), functionally dependent on the deformation history. For instance, in crystals $\dot{\gamma}_K$ may stand for the rate of shearing on K-th slip system, with $f_K = \tau_K - \tau_K^c$, where τ_K is the *generalized* resolved shear stress on that system and τ_K^c is its critical value. If $f_K = 0$ then Eq. (3.1) implies $\dot{f}_K \leqslant 0$ and $\dot{f}_K \dot{\gamma}_K = 0$ (no sum) while $\boldsymbol{\lambda}_K$ defined by Eq. (3.2) represents a normal to the K-th yield surface in strain space, directed outward from the elastic domain. In short, we may say that Eqs. (3.1) ÷ (3.6) characterize elastic-plastic solids obeying the postulates of normality (cf. Eq. (3.3)) and symmetry (cf. Eqs. (3.5) and (3.6)). The form of evolution equations for elastic moduli E_{ijkl}, parameters

g_{KL} and λ_K need not be specified here; it is only assumed that the rates of change of these quantities along a strain path are bounded provided $\dot{\mathbf{e}}$ and $\dot{\gamma}_K$ are bounded.

Since the matrix (g_{KL}) is not required to be positive definite, $\dot{\gamma}_K(t)$ need not be uniquely determined *from the above equations* even if $\dot{\mathbf{e}}(t)$ is given [12]; in such cases a path $\mathscr{P}$ is understood as a path in strain space specified jointly with *some* $\dot{\gamma}_K(t)$ compatible with Eqs. (3.1) ÷ (3.6) at every instant. To satisfy Eq. (2.2), we assume that all $\dot{\gamma}_K$ are bounded by

$$\dot{\gamma}_K \leqslant C_\gamma \left|\dot{\mathbf{e}}\right|, \tag{3.7}$$

where C_γ is a positive constant. This is a weaker restriction than positive definiteness of (g_{KL}); for instance, Eq. (3.7) is implied by $g_{KL} \geqslant 0$ with strict inequality for $L = K$. In particular, Eq. (3.7) ensures that no internal plastic rearrangement in the material is possible if $\dot{\mathbf{e}} = \mathbf{0}$.

From Eq. (3.7) and the assumption on bounded rates of λ_K, E_{ijkl} and g_{KL} it follows that increments of these quantities along a strain path are of order of the path length increment; this property will be essential below.

3.2. The second-order work

For a material element as specified above, consider a class of arbitrarily circuitous strain paths $\mathscr{P}$. We may identify t with the path length variable θ and replace the time derivative symbol $(\dot{})$ by a prime; a parameter $\gamma_K(\theta)$ is formally defined as a time integral of $\dot{\gamma}_K$ with an initial value $\gamma_K(0) = 0$. On substituting (3.3) and (3.5) into the former work expression in (2.5) and taking into account that variations of E_{ijkl} of order $\hat{\theta}$ cannot affect the first- and second-order terms, we obtain

$$\hat{\omega} = \bar{\mathbf{t}} \cdot \hat{\mathbf{e}} + \frac{1}{2} \hat{\mathbf{e}} \cdot \bar{\mathbf{E}} \cdot \hat{\mathbf{e}} - \int_0^{\hat{\theta}} e'_{ij}(\theta) \left(\int_0^{\theta} (\lambda_K)_{ij}(s)\, \gamma'_K(s)\, ds \right) d\theta + o(\hat{\theta}^2). \tag{3.8}$$

Similarly, by using Eq. (3.7) it can be shown that variations of λ_K and g_{KL} along $\mathscr{P}$ may be disregarded in Eq. (3.8), to terms of second-order in $\hat{\theta}$. Substitution of Eq. (3.4) into Eq. (3.8) thus yields

$$\hat{\omega} = \bar{\mathbf{t}} \cdot \hat{\mathbf{e}} + \frac{1}{2} \hat{\mathbf{e}} \cdot \bar{\mathbf{E}} \cdot \hat{\mathbf{e}} - \int_0^{\hat{\theta}} (f'_K(\theta) + \bar{g}_{KL}\gamma'_L(\theta))\, \gamma_K(\theta)\, d\theta + o(\hat{\theta}^2). \tag{3.9}$$

On integrating Eq. (3.9) by parts with the use of Eqs. $(3.1)_3$ and (3.6), we obtain

$$\hat{\omega} = \bar{\mathbf{t}} \cdot \hat{\mathbf{e}} + \frac{1}{2} \hat{\mathbf{e}} \cdot \bar{\mathbf{E}} \cdot \hat{\mathbf{e}} - \frac{1}{2} \bar{g}_{KL} \hat{\gamma}_K \hat{\gamma}_L - \bar{f}_K \hat{\gamma}_K + o(\hat{\theta}^2), \tag{3.10}$$

where $\hat{\gamma}_K = \gamma_K(\hat{\theta}) \geqslant 0$. From Eq. $(3.1)_3$ it follows that for $\hat{\theta}$ sufficiently small we have $\hat{\gamma}_K = 0$ unless $K \in \bar{A}$, where the set $\bar{A} = \{K : \bar{f}_K = 0\}$ contains $\bar{N}$ indices of the so-called potentially active mechanisms at the initial state of all paths $\mathcal{P}$ under consideration. The summation in Eq. (3.10) needs thus to be carried out only for $K, L \in \bar{A}$, and the final value $\bar{f}_{,K}$ in Eq. (3.10) can be replaced by the respective increment $\hat{f}_K$. Henceforth $(\bar{g}_{KL})$ with $K, L \in \bar{A}$ will denote the $(\bar{N} \times \bar{N})$ submatrix of (g_{KL}) which corresponds to the potentially active systems at $\theta = 0$.

By comparing Eq. (3.10) with Eq. (2.4), we arrive at the following result:

For a material characterized by Eqs. (3.1) ÷ (3.7) and for arbitrary strain paths $\mathcal{P}$, the second-order work is given by

$$\Delta_2 \omega = \frac{1}{2} \hat{\mathbf{e}} \cdot \bar{\mathbf{E}} \cdot \hat{\mathbf{e}} - \frac{1}{2} \bar{g}_{KL} \hat{\gamma}_K \hat{\gamma}_L - \hat{f}_K \hat{\gamma}_K. \tag{3.11}$$

Although the underlying constitutive framework is well-known, this formula and its consequences discussed below appear to be new.

Since variations of $\bar{\boldsymbol{\lambda}}_K$ and g_{KL} along any $\mathcal{P}$ are of order $\hat{\theta}$, Eq. (3.4) implies

$$\hat{f}_K = \bar{\boldsymbol{\lambda}}_K \cdot \hat{\mathbf{e}} - \bar{g}_{KL} \hat{\gamma}_L + o(\hat{\theta}), \tag{3.12}$$

no matter how the path $\mathcal{P}$ varies as $\hat{\theta} \to 0$; validity of the analogous first-order formula for the stress increment $\hat{\mathbf{t}}$ obtained from Eq. (3.3) is also evident. On substituting those formulae into Eq. (3.11), we obtain alternative expressions for $\Delta_2 \omega$, viz.

$$\Delta_2 \omega = \frac{1}{2} \hat{\mathbf{e}} \cdot \bar{\mathbf{E}} \cdot \hat{\mathbf{e}} + \frac{1}{2} \bar{g}_{KL} \hat{\gamma}_K \hat{\gamma}_L - \hat{\gamma}_K \bar{\boldsymbol{\lambda}}_K \cdot \hat{\mathbf{e}}, \tag{3.13}$$

$$\Delta_2 \omega = \frac{1}{2} \hat{\mathbf{t}} \cdot \hat{\mathbf{e}} - \frac{1}{2} \hat{f}_K \hat{\gamma}_K, \tag{3.14}$$

equivalent to each other (to second order terms in $\hat{\theta}$).

The formula (3.13) shows limited path-independence of the second-order work for the materials under consideration: only the *final* increments of $\mathbf{e}$ and γ_K are relevant, and not how they are reached. From Eq. (3.14) we obtain that in a general case the expression in Eq. (2.7) is incorrect for indirect paths (e.g. if a zigzag path is scaled down proportionally as $\hat{\theta} \to 0$) and underestimates the actual second-order work. For, if $\hat{\gamma}_K > 0$ for $\hat{\theta}$ arbitrarily small then, as remarked above, we must have $\bar{f}_K = 0$ which in turn implies $\hat{f}_K \leqslant 0$ on account of Eq. $(3.1)_2$; the latter term in Eq. (3.14) is thus non-negative. In particular, the second-order work on *closed* paths ($\hat{\mathbf{e}} = \mathbf{0}$) is non-negative for the considered materials.

3.3. Lower bound to the second-order work

Consider now the difference between the second-order work on arbitrary paths $\mathscr{P}$ and on direct paths from $\mathscr{D}$ which lead to the same final strain increment. Quantities associated with direct paths are distinguished by a superscript (D); variations of the derivatives $e_{ij}^{D\prime}(\theta)$ and $\gamma_K^{D\prime}(\theta)$ along a direct paths are taken to be, at the most, of order $\hat{\theta}^D$ to satisfy Eq. (2.6). Since for direct paths we readily have $\hat{\theta}^D \leqslant \hat{\theta}$ and $\hat{f}_K^D\, \hat{\gamma}_K^D = 0$ (both valid at least to order $\hat{\theta}^2$), from Eqs. (3.13) and (3.11) with $\hat{\mathbf{e}} = \hat{\mathbf{e}}^D$ we obtain

$$(3.15)\qquad \Delta_2\omega - \Delta_2\omega^D = \frac{1}{2}\,\bar{g}_{KL}\,\hat{\gamma}_K\hat{\gamma}_L - \hat{\gamma}_K\,\bar{\boldsymbol{\lambda}}_K\cdot\hat{\mathbf{e}} + \frac{1}{2}\,\bar{g}_{KL}\,\hat{\gamma}_K^D\,\hat{\gamma}_L^D.$$

By using Eqs. (3.6) and (3.12), we can rearrange this as follows

$$(3.16)\qquad \Delta_2\omega - \Delta_2\omega^D = \frac{1}{2}\,\bar{g}_{KL}\,(\hat{\gamma}_K - \hat{\gamma}_K^D)\,(\hat{\gamma}_L - \hat{\gamma}_L^D) - \hat{f}_K^D\,\hat{\gamma}_K.$$

The latter term is nonnegative by the same argument as above, and the former will be nonnegative if the submatrix $(\bar{g}_{KL})$ is positive definite or at least positive semidefinite. We have thus proved the following theorem:

THEOREM 1. *For a material characterized by Eqs.* (3.1) $\div$ (3.7) *and among all paths* $\mathscr{P}$ *initiated at a given state and leading to the same final strain, the second-order work is minimized on direct paths, viz.*

$$(3.17)\qquad \Delta_2\omega \geqslant \Delta_2\omega^D \quad \textit{if} \quad \hat{\mathbf{e}}^D = \hat{\mathbf{e}}$$

provided that

$$(3.18)\qquad \bar{g}_{KL}\,a_K\,a_L \geqslant 0 \quad \textit{for all } a_K.$$

It is recalled that by the convention introduced in Sect. 2, the inequality (3.17) concerns the second-order terms only. More precisely, (3.17) is understo-od to hold in the limit as $\hat{\theta} \to 0$ when both sides of the inequality are divided beforehand by $\hat{\theta}^2$.

If the submatrix $(\bar{g}_{KL})$ which corresponds to the potentially active mechanisms at $\theta = 0$ is positive definite (i.e (3.18) holds with strict inequality unless all a_K are zero) then (3.7) need not be assumed separately. For, that submatrix is then also positive definite for θ sufficiently small, by the continuity argument. $\dot{\gamma}_K$ is then uniquely defined by $\dot{\mathbf{e}}$ [12] and (3.7) is ensured along any $\mathscr{P}$ for sufficiently small $\hat{\theta}$. In that case, by using (3.12) and rearranging, (3.11) can be transformed to

$$(3.19)\quad \Delta_2\omega = \frac{1}{2}\,\hat{\mathbf{e}}\cdot\bar{\mathbf{L}}^0\cdot\hat{\mathbf{e}} + \frac{1}{2}\,\bar{g}^{-1}{}_{KL}\,\hat{f}_K\hat{f}_L, \qquad \bar{\mathbf{L}}^0 = \bar{\mathbf{E}} - \bar{g}^{-1}{}_{KL}\,\bar{\boldsymbol{\lambda}}_K\otimes\bar{\boldsymbol{\lambda}}_L,$$

where $\bar{g}_{KL}^{-1}$ denote elements of the $(\bar{N}\times\bar{N})$ matrix inverse to $(\bar{g}_{KL})$ and $\bar{L}_{ijkl}^0$ are the so-called „total loading" moduli which coincide with the tangent moduli along a deformation path associated with $\dot{\gamma}_K > 0$ for all $K \in \bar{A}$.

The inequality (3.17) derived above is of interest for the theory of stability of equilibrium. For, as pointed out by HILL [4, 5], to prove sufficiency of his stability condition one needs a lower bound to the deformation work done on arbitrarily circuitous paths in a neighborhood of the equilibrium state (see also [15]). Te above theorem indicates a fairly wide class of solids for which such a bound to the second-order work is available.

4. The constitutive inequality

4.1. Preliminaries

It has been shown in [18] that, under the constitutive assumptions (3.1) ÷ (3.6) adopted *at a micro-level* of a heterogeneous aggregate (cf. [10]), the inequality

$$\int \dot{\mathbf{e}}^* \cdot d\mathbf{t} \geqslant \int \dot{\mathbf{t}}^* \cdot d\mathbf{e} \tag{4.1}$$

holds for all segments of every piecewise smooth path $\mathscr{E}$ of macroscopically uniform strain $\mathbf{e}(t)$ and the associated macroscopic stress path $\mathbf{t}(t)$, while the starred rates correspond to a macroscopically uniform *virtual* deformation mode which may vary arbitrarily along the path. It is understood that at each instant $\dot{\mathbf{t}}^*$ is related to $\dot{\mathbf{e}}^*$, similarly as $d\mathbf{t}$ to $d\mathbf{e}$, by the currently valid incremental constitutive law which varies along the considered path in a manner unaffected in any way by the starred mode.

Let a (macroscopic) incremental constitutive law for a time-independent solid at a given state be written down symbolically as

$$\dot{\mathbf{t}} = \boldsymbol{\eta}(\dot{\mathbf{e}}) = \mathbf{L}(\dot{\mathbf{e}}) \cdot \dot{\mathbf{e}}, \quad \mathbf{L} = \frac{\partial \boldsymbol{\eta}}{\partial \dot{\mathbf{e}}} \tag{4.2}$$

$$\left(\text{or} \quad \dot{t}_{ij} = \eta_{ij}(\dot{\mathbf{e}}) = L_{ijkl}(\dot{\mathbf{e}})\, \dot{e}_{kl}, \quad L_{ijkl} = \partial \eta_{ij}/\partial \dot{e}_{kl}\right).$$

The constitutive function $\boldsymbol{\eta}$ is single-valued, continuous and positively homogeneous of degree one; the instantaneous „stiffness" moduli L_{ijkl} can depend in a piecewise-smooth but possibly discontinuous manner on the strain-rate direction. The function $\boldsymbol{\eta}$ itself varies along a strain path with possible discontinuous changes at a discrete set of points (e.g. at an unloading point). However, it is natural to assume that the *tangent* moduli, which relate the actual infinitesimal *forward* increments of stress and strain along a given path, vary along any path in a *right-hand* continuous manner whenever they are well defined. The above assumptions are treated as a part of the constitutive law (4.2). It can be seen that the assumptions are not physically restrictive so that a very general class of time-independent solids is considered here.

For our present purposes we shall introduce in subsection 4.2 an additional regularity assumption (4.6) which is likewise plausible.

The inequality (4.1) may be used as a micromechanically-based restriction on a phenomenological constitutive law (4.2), and has then far-reaching implications [18]. It has been shown that (4.1) implies the symmetry property $L_{ijkl} = L_{klij}$ and hence existence of a strain-rate potential W such that

$$\dot{\mathbf{t}} = \frac{\partial W}{\partial \dot{\mathbf{e}}}, \qquad W(\dot{\mathbf{e}}) = \frac{1}{2}\,\dot{\mathbf{e}} \cdot \boldsymbol{\eta}\,(\dot{\mathbf{e}}). \tag{4.3}$$

Under a reasonable continuity restriction imposed on the so-called unloading cone, (4.1) has been shown to imply also the (generalized) normality flow rule. Another consequence of (4.1) is that at a *regular* point on a strain path (at which, by definition, the actual stress and strain rates, say $(\dot{\mathbf{t}}^0, \dot{\mathbf{e}}^0)$, and the constitutive function $\boldsymbol{\eta}$ do not change discontinuously), we have

$$\dot{\mathbf{t}}^0 \cdot \dot{\mathbf{e}}^* - \dot{\mathbf{t}}^* \cdot \dot{\mathbf{e}}^0 \geqslant 0 \qquad \text{for every } \dot{\mathbf{e}}^*, \tag{4.4}$$

where $\dot{\mathbf{t}}^* = \boldsymbol{\eta}\,(\dot{\mathbf{e}}^*)$. The inequality (4.4), which has the interpretation that no abrupt unloading at a micro-level is associated with the actual deformation mode, has been discussed in [18] with particular reference to the question of uniqueness of a solution to the first-order rate boundary value problem. It has been shown that (4.4) provides justification for determining the primary bifurcation point on a regular deformation path from the *linearized* bifurcation problem formulated for the tangent moduli, although the actual relationship (4.2) may be highly nonlinear and even need not be specified.

4.2. The second-order work for direct and certain indirect paths

Consider a regular point P on a given piecewise smooth strain path $\mathscr{E}$ as an initial point of a class of short strain paths $\mathscr{P}$ discussed in Sect. 2. Two particular families of paths $\mathscr{P}$ are considered (Fig. 1). The first family $\mathscr{D}$ contains straight (direct) paths $\mathscr{S}^D$ leading from P to a point Q chosen arbitrarily in a neighborhood of P in the strain space. The paths from the second family $\mathscr{B}$ lead from P to the same final strain at Q but on a special indirect route $P \to R \to Q$ where a straight(³) segment $\mathscr{S}^* : R \to Q$ is preceded by a smooth segment $\mathscr{S}^0 : P \to R$ of the path $\mathscr{E}$. Of course, the $\mathscr{D} \subset \mathscr{B}$ since the segment $\mathscr{S}^0$ may be of zero length. The final increments of strain and conjugate stress (evaluated at a fixed reference configuration) on the segments $\mathscr{S}^D$, $\mathscr{S}^0$ and $\mathscr{S}^*$ are denoted by $(\hat{\mathbf{e}}, \hat{\mathbf{t}}^D)$ $(\hat{\mathbf{e}}^0, \hat{\mathbf{t}}^0)$ and $(\hat{\mathbf{e}}^*, \hat{\mathbf{t}}^*)$, respectively.

(³) Straight segments $P \to Q$ and $R \to Q$ can be replaced by smoothly curved segments satisfying $(2.6)_1$, with no influence on the following considerations; that change is generated e.g. by transformation to another strain measure.

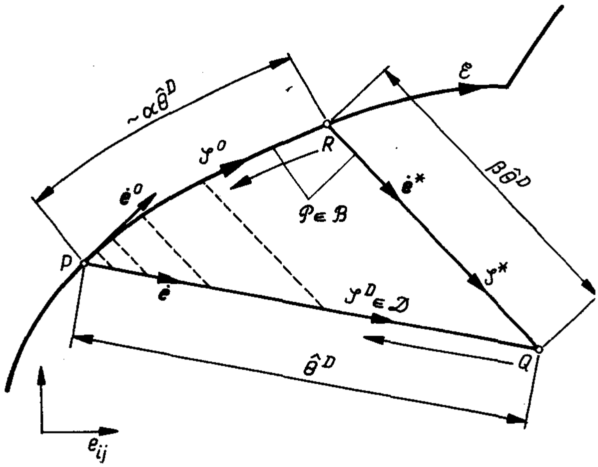

FIG. 1. Schematic representation of indirect strain paths from the family $\mathscr{B}$.

A time-like parameter t is indetified with the path length variable θ; the total length of a path from the family $\mathscr{B}$ is denoted by $\hat{\theta}$ while $\hat{\theta}^D$ stands for the length of a segment $\mathscr{S}^D$. In the limit as $\hat{\theta} \to 0$, the points Q and R are taken to approach P in such a manner that the quotients $\hat{e}/\hat{\theta}^D$, $\hat{e}^0/\hat{\theta}^D$, $\hat{e}^*/\hat{\theta}^D$ and $\hat{\theta}/\hat{\theta}^D$ tend to well-defined limits $\dot{e}$, $\alpha\dot{e}^0$, $\beta\dot{e}^*$ and $\alpha+\beta$, respectively, where $\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha+\beta \geqslant 1$, and $|\dot{e}| = |\dot{e}^0| = |\dot{e}^*| = 1$. It is evident that the above construction can be performed for any choice of $\dot{e}$ or $\dot{e}^*$ provided that

$$\dot{e} = \alpha\dot{e}^0 + \beta\dot{e}^*. \tag{4.5}$$

We introduce now the assumption that for $\hat{\theta}$ sufficiently small we have

$$|\mathfrak{t}'(\theta_2) - \mathfrak{t}'(\theta_1)| \leqslant C_2|\theta_2 - \theta_1| \quad \text{along each of } \mathscr{S}^D, \mathscr{S}^0, \mathscr{S}^* \text{ separately,} \tag{4.6}$$

where C_2 is a positive constant, the same for all such segments (cf. $(2.6)_2$).(4) It follows that those segments constitute direct paths to which the formula (2.4) with $\Delta_2\omega$ given by (2.7) can *separately* be applied. By adding the work expressions for the segments $\mathscr{S}^0$ and $\mathscr{S}^*$ and taking into account that the initial stress for a segment $\mathscr{S}^*$ differs from the stress $\mathfrak{t}$ at P by $\hat{\mathfrak{t}}^0$, we obtain

$$\hat{\omega} = \mathfrak{t}\cdot\hat{e}^0 + \frac{1}{2}\hat{\mathfrak{t}}^0\cdot\hat{e}^0 + (\mathfrak{t}+\hat{\mathfrak{t}}^0)\cdot\hat{e}^* + \frac{1}{2}\hat{\mathfrak{t}}^*\cdot\hat{e}^* + o(\hat{\theta}^2) \quad \text{for } \mathscr{P}\in\mathscr{B}. \tag{4.7}$$

(4) This may be regarded as a natural regularity restriction on (4.2), related also to smoothness of the path $\mathscr{E}$ in vicinity of P. If P were not a regular point on $\mathscr{E}$ then validity of (4.6) could be questioned for the segments $\mathscr{S}^*$ whose initial state would approach a singular point.

The second-order work on paths from the family $\mathscr{B}$ thus reads

$$\Delta_2\omega^B = \frac{1}{2}\mathfrak{t}^0\cdot\hat{\mathrm{e}}^0 + \mathfrak{t}^0\cdot\hat{\mathrm{e}}^* + \frac{1}{2}\mathfrak{t}^*\cdot\hat{\mathrm{e}}^*, \tag{4.8}$$

while the second-order work on direct paths from $\mathscr{D}$ is $\Delta_2\omega^D = \frac{1}{2}\mathfrak{t}^D\cdot\hat{\mathrm{e}}$. By using the assumed smoothness properties of the segments, the second-order work formulae can be expressed in terms of the limit strain-rates $\dot{\mathrm{e}}$, $\dot{\mathrm{e}}^0$ and $\dot{\mathrm{e}}^*$ and the respective stress-rates $\dot{\mathfrak{t}}$, $\dot{\mathfrak{t}}^0$ and $\dot{\mathfrak{t}}^*$ related by (4.2) at the initial points of the segments $\mathscr{S}^D$, $\mathscr{S}^0$ and $\mathscr{S}^*$, respectively. We obtain

$$\Delta_2\omega^D = \frac{1}{2}\dot{\mathfrak{t}}\cdot\dot{\mathrm{e}}(\hat{\theta}^D)^2, \tag{4.9}$$

$$\Delta_2\omega^B = \left(\frac{1}{2}\alpha^2\,\dot{\mathfrak{t}}^0\cdot\dot{\mathrm{e}}^0 + \alpha\beta\,\dot{\mathfrak{t}}^0\cdot\dot{\mathrm{e}}^* + \frac{1}{2}\beta^2\,\dot{\mathfrak{t}}^*\cdot\dot{\mathrm{e}}^*\right)(\hat{\theta}^D)^2, \tag{4.10}$$

for paths from the families $\mathscr{D}$ and $\mathscr{B}$, respectively. By the definition of a regular point on a path $\mathscr{E}$, we have

$$\dot{t}^*_{ij} \equiv \eta^R_{ij}(\dot{\mathrm{e}}^*) = \bar{\eta}_{ij}(\dot{\mathrm{e}}^*) + o(\hat{\theta}^0) \tag{4.11}$$

so that both the expressions (4.9) and (4.10) can be evaluated to second order by using the constitutive rate equation (4.2) with the function $\eta = \bar{\eta}$ taken precisely at the point P (η^R denotes η at a point R). The assumption that P is a regular point is essential here and not only formal; for instance, if P were a corner point on a strain path then (4.11) would not be true in general.

4.3. The second-order work inequality

We can now formulate and prove our basic result.

THEOREM 2. *Under the regularity assumption (4.6) imposed on a constitutive law (4.2), the following two conditions are equivalent:*

(i) The inequality (4.1) holds for all segments of every piecewise smooth strain path $\mathscr{E}$;

(ii) At any regular point P on every piecewise smooth strain path $\mathscr{E}$ and among all indirect paths $\mathscr{P}\in\mathscr{B}$ initiated at P and leading to the same final strain, the second-order work is minimized on direct paths, viz.

$$\Delta_2\omega^B \geqslant \Delta_2\omega^D. \tag{4.12}$$

The proof of Theorem 2 is given in Appendix.

If the constitutive inequality (4.1) is assumed or derived from the given constitutive relations then Theorem 2 provides interpretation for (4.1) in terms of the second-order work([5]). Conversely, the second-order work inequality (4.12) can be adopted as a postulate (which has a physical meaning) or verified for some materials, and then (4.1) with all its implications demostrated in [18] and briefly recapitulated in the Subsect. 4.1 are obtained as a consequence. This is an attractive way of deriving such qualitative properties of a constitutive law as symmetry of instantaneous stiffness moduli, the normality flow rule or the inequality (4.4). Moreover, the inequality (4.1) transmits from a micro-level to the macro-level of constitutive description for a heterogeneous continuum [10, 18]. It follows that *the fundamental qualitative properties of a macroscopic constitutive law (4.2) can be deduced from the second-order work inequality (4.12) established or postulated at a micro-level.*

The inequalities (4.1) and (4.12) hold for an elastic-plastic material (e.g. a model of a metal crystal deformed by mutislip) with a discrete set of internal plastic deformation mechanisms characterized by (3.1) $\div$ (3.6) if the submatrix $(\bar{g}_{KL})$ for potentially active systems at any P is positive definite([6]). In fact, a stronger property has been proved in Sect. 3 for such materials (cf. (3.17)) that the second-order work is minimized on direct paths among *all* indirect paths which lead to the same strain increment. As a particular case, (4.1) and (4.12) hold for the classical elastoplastic solids, discussed below.

5. Illustrations

5.1. Classical elastoplasticity

Consider a particular case of the material described by the constitutive relations (3.1) $\div$ (3.6) when only one internal mechanism of plastic deformation is distinguished, i.e. when $N = 1$ so that all upper case lower indices can simply be omitted (so that (3.6) is trivial). Moreover, assume that the elastic moduli tensor **E** is invertible (if regarded as a linear operator in the space of symmetric tensors) and that $g > 0$. Then the classical elastoplasticity equations (extended to finite strain) for a material with a smooth yield surface and the normality flow rule are recovered, viz. (cf. [4, 8])

([5]) One can compare the above interpretation with that for (4.4) obtained in terms of the work done on a *virtual cycle* of strain [10].

([6]) In [18] it has been shown that the assumption of a positive definite submatrix $(\bar{g}_{KL})$ is not needed for validity of (4.1). However, it is this condition which ensures that the constitutive relation (4.2) corresponding to the assumptions (3.1) $\div$ (3.6) is single-valued [12]; the latter property has been used in the proof of Theorem 2. Note also that the transmissibility of the inequality (4.1) between micro- and macro-levels need not imply the same for the property (4.12); for, uniqueness of the response of a heterogeneous material may be lost after transition to another scale of observation.

$$\dot{\mathbf{e}} = \dot{\mathbf{e}}^p + \dot{\mathbf{e}}^e, \quad \dot{\mathbf{e}}^e = \mathbf{M}\cdot\dot{\mathbf{t}}, \quad \dot{\mathbf{e}}^p = \dot{\gamma}\,\partial f/\partial\mathbf{t}, \tag{5.1}$$

where

$$\dot{\gamma} = \begin{cases} (\dot{\mathbf{e}}\cdot\boldsymbol{\lambda})/g & \text{if} \quad f = 0 \quad \text{and} \quad \dot{\mathbf{e}}\cdot\boldsymbol{\lambda} > 0, \\ 0 & \text{elsewhere,} \end{cases} \tag{5.2}$$

$$\mathbf{M} = \mathbf{E}^{-1}, \quad M_{ijkl} = M_{klij}, \quad \partial f/\partial\mathbf{t} = \boldsymbol{\lambda}\cdot\mathbf{M}. \tag{5.3}$$

We need not make any distinction here between "softening" and "hardening" which themselves are not measure-invariant concepts [8]. If the "hardening" parameter $h = g - \boldsymbol{\lambda}\cdot\mathbf{M}\cdot\boldsymbol{\lambda}$ is positive then the loading condition at $f = 0$ reduces to the more familiar condition $\dot{\mathbf{t}}\cdot(\partial f/\partial\mathbf{t}) > 0$.

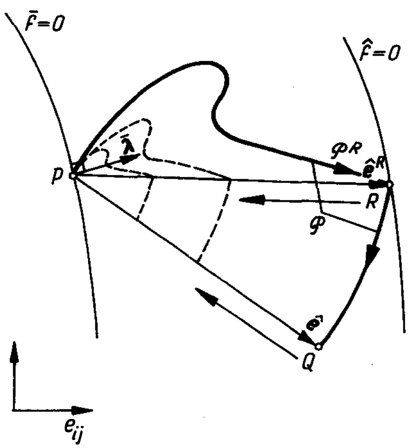

FIG. 2. Decomposition of an arbitrarily circuitous strain path $\mathscr{P}$ for a classical elastic-plastic material.

Consider a point P on the current yield surface and a class of strain paths $\mathscr{P}$ of final length $\hat{\theta}$, initiated at that point (Fig. 2). Let us discuss first a path $\mathscr{P}^R$ such that its final point R lies on the final yield surface; respective increments are distinguished by a superscript (R). From (5.2) it can be deduced that (3.7) is satisfied and that the final increment of γ along $\mathscr{P}^R$ is given by the following first-order formula

$$\hat{\gamma}^R = \hat{\mathbf{e}}^R\cdot\bar{\boldsymbol{\lambda}}/\bar{g} + o(\hat{\theta}^R) \tag{5.4}$$

for arbitrarily circuitous paths $\mathscr{P}^R$, corresponding possibly to temporary unloading. Since $\hat{f}^R = 0$, from (3.14) or (3.11) we find that

$$(5.5)\qquad \Delta_2\omega^R = \frac{1}{2}\mathbf{t}^R\cdot\hat{\mathbf{e}}^R = \frac{1}{2}\hat{\mathbf{e}}^R\cdot\bar{\mathbf{L}}^p\cdot\hat{\mathbf{e}}^R, \qquad \bar{L}^p_{ijkl} = \bar{E}_{ijkl} - \bar{\lambda}_{ij}\bar{\lambda}_{kl}/\bar{g},$$

where L^p_{ijkl} are the (elastic-plastic) moduli from the loading branch. This means that within the class of paths terminating on the yield surface, the second-order work is path-independent.

Generally, any path $\mathscr{P}$ can be decomposed into two segments: a segment $\mathscr{P}^R$ just discussed followed by a purely elastic segment $R \to Q$ (Fig. 2); one of those segments may be absent. From (5.5) and in analogy to (4.8), the second-order work for arbitrary paths $\mathscr{P}$ is expressed by the formula

$$(5.6)\qquad \Delta_2\omega = \frac{1}{2}\mathbf{t}^R\cdot\hat{\mathbf{e}}^R + \mathbf{t}^R\cdot(\hat{\mathbf{e}}-\hat{\mathbf{e}}^R) + \frac{1}{2}(\hat{\mathbf{e}}-\hat{\mathbf{e}}^R)\cdot\bar{\mathbf{E}}\cdot(\hat{\mathbf{e}}-\hat{\mathbf{e}}^R),$$

which can be rearranged as follows:

$$(5.7)\qquad \Delta_2\omega = \frac{1}{2}\,\hat{\mathbf{e}}\cdot\bar{\mathbf{L}}^p\cdot\hat{\mathbf{e}} + \frac{1}{2}\,((\hat{\mathbf{e}}-\hat{\mathbf{e}}^R)\cdot\bar{\boldsymbol{\lambda}})^2/\bar{g}$$
$$= \frac{1}{2}\,\hat{\mathbf{e}}\cdot\bar{\mathbf{E}}\cdot\hat{\mathbf{e}} + \frac{1}{2}\,\bar{g}\,(\hat{\gamma}^R)^2 - (\hat{\mathbf{e}}\cdot\bar{\boldsymbol{\lambda}})\,\hat{\gamma}^R.$$

This formula can be understood in the following asymptotic sense. As the total length $\hat{\theta}$ of a path $P \to R \to Q$ tends to zero, let the points R and Q lie on and inside the final yield surface, respectively, and approach a fixed initial point P in such a way that the triangles PQR are similar to each other without rotation (Fig. 2). Then the work of deformation on the path $P \to R \to Q$ is expressed with accuracy to second-order terms by the formula (2.4) with $\Delta_2\omega$ given by (5.7).

From (2.7) and the constitutive assumptions it follows that the second-order work on a direct path resulting in a final strain increment $\hat{\mathbf{e}}$ is expressed by

$$(5.8)\qquad \Delta_2\omega^D = \frac{1}{2}\,\hat{\mathbf{e}}\cdot\bar{\mathbf{L}}\cdot\hat{\mathbf{e}}, \qquad \bar{\mathbf{L}} = \begin{cases} \bar{\mathbf{L}}^p & \text{if} \quad \hat{\mathbf{e}}\cdot\bar{\boldsymbol{\lambda}} \geqslant 0, \\ \bar{\mathbf{E}} & \text{if} \quad \hat{\mathbf{e}}\cdot\bar{\boldsymbol{\lambda}} \leqslant 0. \end{cases}$$

If $\hat{\mathbf{e}}$ in (5.7) and (5.8) is the same then

$$(5.9)\qquad \Delta_2\omega - \Delta_2\omega^D = \begin{cases} \dfrac{1}{2}((\hat{\mathbf{e}}-\hat{\mathbf{e}}^R)\cdot\bar{\boldsymbol{\lambda}})^2/\bar{g} & \text{if} \quad \hat{\mathbf{e}}\cdot\bar{\boldsymbol{\lambda}} \geqslant 0, \\[2ex] \dfrac{1}{2}\bar{g}\,(\hat{\gamma}^R)^2 - (\hat{\mathbf{e}}\cdot\bar{\boldsymbol{\lambda}})\,\hat{\gamma}^R & \text{if} \quad \hat{\mathbf{e}}\cdot\bar{\boldsymbol{\lambda}} \leqslant 0. \end{cases}$$

Since $\bar{g}$ has been assumed positive and $\hat{\gamma}^R$ is nonnegative, it is clear that the above second-order work difference is always nonnegative. This conclusion provides an illustration to Theorem 1 proved in Sect. 3 under more general assumptions. In a particular case when $\mathscr{P}^R$ coincides with a smooth segment $\mathscr{P}^0$ of a loading path, this demonstrates validity of the inequality (4.12) for the classical elastic-plastic solids. From Theorem 2 from the preceding section it follows that the inequality (4.1) holds for those materials; this result is a special case (for $N = 1$) of that obtained in [18] for the materials obeying (3.1) ÷ (3.6) with N arbitrary.

5.2. Two mechanisms of plastic deformation

Suppose now that the yield surface is not everywhere smooth and discuss the second-order work for strain paths initiated at a point P which lies on an edge formed by intersection of two smooth yield-surface sections (Fig. 3).

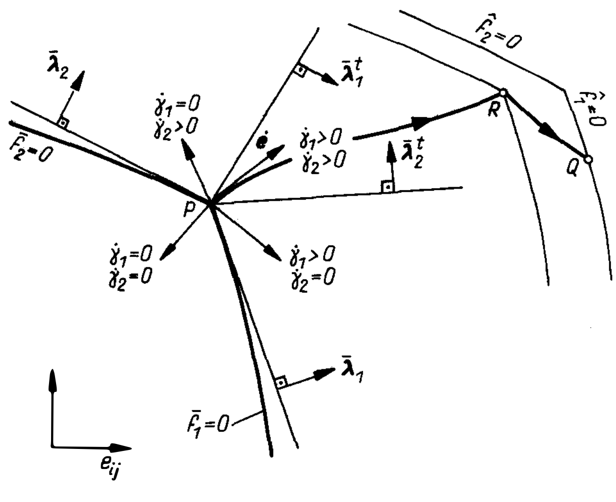

FIG. 3. Four ranges of the incremental constitutive response of a material with two mechanisms of plastic deformation.

Formation of such an edge corresponds to simultaneous activation of two different mechanisms of plastic deformation, as e.g. in the double-slip model of a single crystal discussed by ASARO [1]. The material response in vicinity of the corner point P is assumed to obey the relations (3.1) ÷ (3.6) with $N = 2$ under the additional restriction that

$$(g_{KL}) \text{ is positive definite.} \tag{5.10}$$

This ensures that the actual rate form of the constitutive law can be written down as (4.2) with a single-valued function $\eta(\cdot)$; the function $\eta(\cdot)$ need not be invertible unless the matrix of "hardening" coefficients $h_{KL} = g_{KL} - \boldsymbol{\lambda}_L \cdot \mathbf{M} \cdot \boldsymbol{\lambda}_K$ is positive definite [12, 11].

The strain-rate space is decomposed into four wedge-shaped sectors corresponding to ($\dot{\gamma}_1 = 0$, $\dot{\gamma}_2 = 0$), ($\dot{\gamma}_1 > 0$, $\dot{\gamma}_2 = 0$), ($\dot{\gamma}_1 = 0$, $\dot{\gamma}_2 > 0$) or ($\dot{\gamma}_1 > 0$, $\dot{\gamma}_2 > 0$) as illustrated in the figure. The normals $\bar{\boldsymbol{\lambda}}_1^t$, $\bar{\boldsymbol{\lambda}}_2^t$ to hyperplanes bounding the "total loading" ($\dot{\gamma}_1 > 0$, $\dot{\gamma}_2 > 0$) sector at the point P generally do not coincide with the yield-surface normals $\bar{\boldsymbol{\lambda}}_1$, $\bar{\boldsymbol{\lambda}}_2$ but from the compatibility conditions $\dot{f}_1 = 0$, $\dot{f}_2 = 0$ are found to be

$$\bar{\boldsymbol{\lambda}}_1^t = \bar{\boldsymbol{\lambda}}_1 - \bar{\boldsymbol{\lambda}}_2\, \bar{g}_{12}/\bar{g}_{22}, \quad \bar{\boldsymbol{\lambda}}_2^t = \bar{\boldsymbol{\lambda}}_2 - \bar{\boldsymbol{\lambda}}_1\, \bar{g}_{21}/\bar{g}_{11}. \tag{5.11}$$

By using (3.11) and applying a similar argument as in the Subsect. 5.1, it is not difficult to show that within the class of paths $P \to R$ which terminate at the actual (shifted) corner point on the yield surface, the second-order work is path-independent and reads

$$\Delta_2 \omega^R = \frac{1}{2}\, \hat{\mathbf{t}}^R \cdot \hat{\mathbf{e}}^R = \frac{1}{2}\, \hat{\mathbf{e}}^R \cdot \bar{\mathbf{L}}^0 \cdot \hat{\mathbf{e}}^R, \quad \bar{\mathbf{L}}^0 = \bar{\mathbf{E}} - \bar{g}^{-1}{}_{KL}\, \bar{\boldsymbol{\lambda}}_K \otimes \bar{\boldsymbol{\lambda}}_L; \tag{5.12}$$

note that symmetry and invertibility of $(\bar{g}_{KL})$ are essential in obtaining this result. L^0_{ijkl} are the "total loading" moduli which coincide with the tangent moduli along a deformation path associated with $\dot{\gamma}_1 > 0$, $\dot{\gamma}_2 > 0$.

For other paths $\mathscr{P}$ we can directly apply any of the formulae (3.11), (3.13), (3.14), (3.19). For instance, within the class of paths $P \to R \to Q$ terminating on the yield-surface section corresponding to $f_1 = 0$ (Fig. 3), we have the formula

$$\Delta_2 \omega = \frac{1}{2}\, \hat{\mathbf{e}} \cdot \bar{\mathbf{L}}^0 \cdot \hat{\mathbf{e}} + \frac{1}{2}\, \bar{g}^{-1}{}_{22} \hat{f}_2^2 \quad \text{if} \quad \hat{f}_1 = 0. \tag{5.14}$$

By the theorem proved in Sect. 3, this expression is not smaller in value (to second order) than the expression in (2.7) evaluated for direct paths which lead to the same strain increment $\hat{\mathbf{e}}$ (but to another value of $\hat{f}_2$). This can be seen directly from (5.14) if $\hat{\mathbf{e}}/\hat{\theta}$ falls (in the limit as $\hat{\theta} \to 0$) into the sector corresponding to ($\dot{\gamma}_1 > 0$, $\dot{\gamma}_2 > 0$) or to ($\dot{\gamma}_1 = 0$, $\dot{\gamma}_2 = 0$). Finally, it may be remarked that the symmetry conditions (3.5) and (3.6) as well as the normality rule implied by (3.3) are necessary for the second-order work to be always minimized on direct paths. For, these conditions are necessary for existence of a potential (4.3) which, in turn, is a consequence of (4.12) as follows from a corollary of Theorem 2.

Appendix

Proof of Theorem 2

Suppose first that (4.1) is satisfield for all segments of every piecewise smooth path $\mathscr{E}$. Then, as shown in [18], the constitutive rate equation (4.2) admits a potential (4.3) and the inequality (4.4) holds at every regular point P on $\mathscr{E}$. Let a regular point P on a path $\mathscr{E}$ and the strain-rate $\dot{\mathbf{e}}$ be fixed. $(\beta\dot{\mathbf{e}}^*)$ and $(\beta\dot{\mathbf{t}}^*)$ can then be found from (4.5) and from the *homogeneous* equation (4.2) as functions of α so that $\Delta_2\omega^B$ becomes a function of α and $\hat{\theta}^D$ only. From (4.10), (4.5), (4.3), (4.12) and the chain rule of differentation, we obtain

$$(\text{A.1})\qquad (\partial/\partial\alpha)\Delta_2\omega^B = (\alpha\,\dot{\mathbf{t}}^0\cdot\dot{\mathbf{e}}^0 + \alpha\,\dot{\mathbf{t}}^0\cdot(-\dot{\mathbf{e}}^0) + \beta\,\dot{\mathbf{t}}^0\cdot\dot{\mathbf{e}}^* + (-\dot{\mathbf{e}}^0)\cdot\bar{\boldsymbol{\eta}}(\beta\dot{\mathbf{e}}^*))(\hat{\theta}^D)^2 = \beta\,(\dot{\mathbf{t}}^0\cdot\dot{\mathbf{e}}^* - \dot{\mathbf{t}}^*\cdot\dot{\mathbf{e}}^0)\,(\hat{\theta}^D)^2.$$

From (A.1), (4.4) and $\beta \geqslant 0$ it follows that the second-order work $\Delta_2\omega^B$ does not decrease with increasing α when $\dot{\mathbf{e}}$ and $\hat{\theta}^D$ are prescribed. This implies (4.12) for any $\alpha \geqslant 0$ on account of $\dot{\mathbf{e}}^* = \dot{\mathbf{e}}$ and $\Delta_2\omega^B = \Delta_2\omega^D$ at $\alpha = 0$. Since the path $\mathscr{E}$, the point P and the strain-rate $\dot{\mathbf{e}}$ can be chosen arbitrarily, we have shown that (i) implies (ii) as stated in the theorem.

Conversely, suppose that (4.12) holds at every regular point P on $\mathscr{E}$. We shall prove the following lemma: (4.3) *is necessary for* (4.12) *to be valid for* β *arbitrarily small.* Let the point P and $\dot{\mathbf{e}}^*$ be fixed, and denote $p = \dot{\mathbf{e}}^0\cdot\dot{\mathbf{e}}^*$. Since (4.5) implies that $1 = \alpha^2 + \beta^2 + 2\,\alpha\,\beta\,p$, it follows that α becomes a well-defined function of β (for $\beta < 1$) such that $\alpha \to 1$ and $\partial\alpha/\partial\beta \to -p$ as β decreases to zero. From (4.5) we can thus determine $\dot{\mathbf{e}}$ as a function of β, so that $\Delta_2\omega^D$ and $\Delta_2\omega^B$ become, after substituting (4.2) with $\boldsymbol{\eta} = \bar{\boldsymbol{\eta}}$, functions of β and $\hat{\theta}^D$ only. From (4.9), (4.5), (4.3), (4.12) and the chain rule of differentiation, we obtain

$$\frac{\partial}{\partial\beta}\Delta_2\omega^D\Big|_{\beta=0^+} = \frac{1}{2}\,(\dot{\mathbf{e}}^0\cdot\bar{\mathbf{L}}^0\cdot(-p\,\dot{\mathbf{e}}^0 + \dot{\mathbf{e}}^*) + \dot{\mathbf{t}}^0\cdot(-p\,\dot{\mathbf{e}}^0 + \dot{\mathbf{e}}^*))\,(\hat{\theta}^D)^2$$

$$= (-p\,\dot{t}^0_{ij}\,\dot{e}^0_{ij} + \frac{1}{2}\,(\bar{L}^0_{klij}\,\dot{e}^0_{kl} + \dot{t}^0_{ij})\,\dot{e}^*_{ij})\,(\hat{\theta}^D)^2$$

provided the moduli $\bar{L}^0_{ijkl} \equiv (\partial\bar{\eta}_{ij}/\partial\dot{e}_{kl})(\dot{\mathbf{e}}^0)$ which satisfy $\dot{t}^0_{ij} = \bar{L}^0_{ijkl}\,\dot{e}^0_{kl}$ are well-defined. Similarly, from (4.10) we obtain

$$\frac{\partial}{\partial\beta}\Delta_2\omega^B\Big|_{\beta=0^+} = (-p\,\dot{t}^0_{ij}\,\dot{e}^0_{ij} + \dot{t}^0_{ij}\,\dot{e}^*_{ij})(\hat{\theta}^D)^2.$$

Since for $\beta = 0$ we have $\Delta_2\omega^B = \Delta_2\omega^D$, for (4.12) to be valid for arbitrarily small β it is necessary that

$$\frac{\partial}{\partial\beta}(\Delta_2\omega^B - \Delta_2\omega^D)\Big|_{\beta=0^+} = \frac{1}{2}(\dot{t}^0_{ij} - \bar{L}^0_{klij}\,\dot{e}^0_{kl})\,\dot{e}^*_{ij}\,(\hat{\theta}^D)^2 \geqslant 0.$$

This holds for arbitrary $\dot{\mathbf{e}}^*$ if and only if the expression vanishes identically, i.e. $\dot{t}^0_{ij} = \bar{L}^0_{klij}\dot{e}^0_{kl}$. By the assumed right-hand continuity of the tangent moduli along a path, the latter equality must be valid, by the limit transition, also at a singular point on $\mathscr{E}$ provided the tangent moduli at such a singular point are themselves well-defined. Since the path $\mathscr{E}$ is arbitrary, the *right-hand* rate of strain at a singular point is also arbitrary; it follows that (4.12) implies

$$\dot{t}_{ij} = L_{klij}(\dot{\mathbf{e}})\,\dot{e}_{kl}$$

whenever the moduli $L_{klij}(\dot{\mathbf{e}})$ are well-defined. As shown in [18], this does not contradict (4.2) if and only if $L_{ijkl} = L_{klij}$ so that the incremental constitutive law can be written down in a potential form (4.3). The lemma has been proved.

Now, let a regular point P be fixed simultaneously with the strain-rate $\dot{\mathbf{e}}$. Existence of a potential (4.3) just proved implies, as shown above, validity of the formula (A.1). From $\Delta_2\omega^B = \Delta_2\omega^D$ at $\alpha = 0$ and from (4.12) it follows that the expression in (A.1) must be nonnegative at $\alpha = 0^+$; this is nothing else than (4.4) with $\dot{\mathbf{e}}^* = \dot{\mathbf{e}}$. Hence, (4.4) is valid at any regular point, and thus at almost every point on $\mathscr{E}$. By integrating (4.4) along a piecewise smooth path $\mathscr{E}$ we obtain (4.1) since (4.4) can be violated only at a discrete set of singular points on $\mathscr{E}$ which make no contribution to the integral. We have thus shown that (ii) implies (i) which completes the proof of Theorem 2.

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