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An Approximation to the Planar Harmonic Impedance for Interface Waves in Piezoelectric Body

Eugeniusz J. DANICKI

Institute of Fundamental Technological Research
Polish Academy of Sciences
Pawińskiego 5, 02-106 Warszawa, Poland
e-mail: edanicki@ippt.gov.pl

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It is well known that spatial-temporal Green function characterizes sufficiently the elastic media. For the case of elastic interface waves which propagate at the interface between two, perfectly mechanically contacting piezoelectric half-spaces, this function, or its scalar counterpart that is the planar harmonic impedance, provides full characterization of electric properties observed at the interface, which can be applied in analysis of interdigital transducers embedded there, for instance. This impedance however is not easy for evaluation as a function of complex wave-number in the most interesting domain near the cut-off wave-number of bulk waves. Here, a perturbation analysis is presented exploiting the Stroh matrix formulation for piezoelectrics which yields the analytical approximation to the investigated electric impedance at the crystal planar cross-section.

Keywords: surface acoustic waves, harmonic Green functions, waves in piezoelectrics.

1. Introduction

In the theory of surface acoustic wave (SAW) transducers and other distributed transducers of surface waves, there are certain problems with characterization of elastic substrates on which the transducers work. The theory of such transducers usually are quite complicated and application of the full formulation of the boundary-value problem for elastic body makes the theory too complicated and the results difficult for interpretation. It is better to apply an analytical characterization of the body having few most important parameters for the analyzed problem. Usually, such exact characterization, an example of which is the planar harmonic impedance describing the dependence of surface electric field on surface electric charge, does not exist and we need to find its approximation that is correct in most important albeit relatively narrow domain of the wave number of the analyzed surface waves. The presented analysis proposes a method for developing such approximation in the case of interfacial waves in piezoelectrics.

2. Wave motion in piezoelectric half-spaces

For harmonic wave-field f depending on the propagation direction x and vanishing in depth of the crystal half-space $z < 0$:

$$f = \exp(j\omega t)\exp(-jrx - jsz);$$

$$f_{,x} = -jrf, \qquad f_{,z} = -jsf,$$

where ω, r, s are temporal and spatial angular frequencies, the wave motion is governed by the following Stroh equations [1, 2] ($\mathbf{A}$ and $\mathbf{B}$ are 4×4 real matrices depending only on the material constants of the body):

$$-js\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{U} \\ \mathbf{T} \end{bmatrix} - jr\begin{bmatrix} \mathbf{B}^{\mathrm{T}} & -\mathbf{I} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{U} \\ \mathbf{T} \end{bmatrix} = 0, \tag{1}$$

for convenient variables describing the wave-field (superscript T means transposition):

$$\mathbf{U} = [-jru_i, -jr\varphi]^{\mathrm{T}}, \qquad \mathbf{T} = [T_{i3},\, D_3]^{\mathrm{T}},$$

where u, φ, T, D are displacements, electric potential, stress, and electric induction, respectively (index i takes values $1, 2, 3$; do not confuse it with $j = \sqrt{-1}$ appearing in entirely different circumstances); $\mathbf{I}$ and $\mathbf{0}$ are unitary and zero matrices. See the above-mentioned references for details. It is worth to note here that $\mathbf{A}$ and $\mathbf{C}$ are symmetric matrices, where only $\mathbf{C} = \mathrm{diag}\{[1, 1, 1, 0]\}\omega^2 g/r^2$ depends on the wave-motion parameter; g is the mass density of the substrate. Easy inversion of the first matrix of Eq. (1) transforms the above equations into the eigenvalue problem for matrix $\mathbf{H}$:

$$\mathbf{F}q = \mathbf{HF},$$

$$\mathbf{H} = \begin{bmatrix} -\mathbf{A}^{-1}\mathbf{B}^{\mathrm{T}} & \mathbf{A}^{-1} \\ \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^{\mathrm{T}} - \mathbf{C} & -\mathbf{B}\mathbf{A}^{-1} \end{bmatrix}, \tag{2}$$

$$\mathbf{F} = \begin{bmatrix} \mathbf{U} \\ \mathbf{T} \end{bmatrix}, \qquad q = s/r.$$

It can be checked by inspection that the left eigenvector $\mathbf{E}$ of the Stroh matrix $\mathbf{H}$ which is a real matrix for real r, can be obtained directly from its right one $\mathbf{F}$:

$$\mathbf{EH} = q\mathbf{E}, \qquad \mathbf{E} = [\mathbf{T}^{\mathrm{T}}, \mathbf{U}^{\mathrm{T}}]. \tag{3}$$

The matrix $\mathbf{H}$ has eight eigenvalue-eigenvector pairs $\{q_i, \mathbf{F}^{(i)}\}$ describing the wave-field on the surface $z = 0$, resulting from the wave-motion inside the body:

four pairs satisfying the radiation conditions at $z < 0$ and other four satisfying the conditions at $z > 0$ [2]. In the known Rayleigh–Lamb boundary-value problem for the half-space $z < 0$, three mechanical boundary conditions requiring that $T_{3i} = 0$ at $z = 0$ provides additional three equations, which allow us to evaluate the electric admittance which is the relation between the applied surface charge Q or normal induction D_3 and the surface electric field $E_1 = -\varphi_{,x}$. In the most interesting analyzed cases, the admittance exhibits a resonant dependence on the wave-number r in close vicinity of the cut-off wave-number of bulk waves r_c. Figure 1 presents an example computed directly from the above system of equations and from its analytical approximation derived by using the method presented in [2] (which is generalized in this paper); note the scale of $r - r_c$.

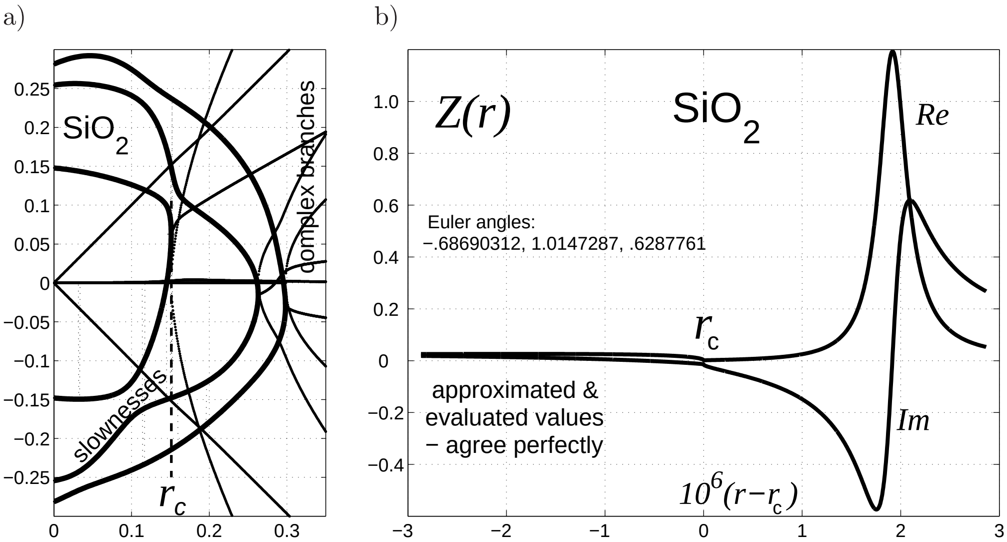

Fig. 1. a) Eigenvalues dependent on the wave-number r (horizontal axis) draw the slowness curves when they are real-valued; they are complex-valued above cut-off wave-numbers of bulk waves (the lowest one, r_c, is marked in the figure); b) Resonant behavior of the surface impedance $Z(r)$ in small domain around r_c; the approximated values evaluated using the method of [4] remain in perfect agreement within this important domain of wave-number r.

The approximation discussed below is developed for the case of interface waves which propagate at the contact plane of two piezoelectric half-spaces, $z < 0$ and $z > 0$. They are attached to each other with perfect mechanical contact (equal surface stress and displacement vectors on both sides of the interface). Certain charge distribution Q may exist at the contact plane causing the jump of the normal induction D_3 across the contact; the electric tangential field and potential ($E_1 = -\varphi_{,x}$) are naturally continuous and equal at both sides of the plane $z = 0$. The wave-field in both half-spaces must satisfy the radiation conditions at $z \to \pm\infty$. This requires the wave-fields in the domain $z < 0$ to be represented by the first subset of the eigenvalue-eigenvector pairs $\{q_i, \mathbf{F}^{(i)}\}$, $i = 1, \cdots, 4$ mentioned above, and the wave-fields in the other domain, $z > 0$, to be represented by the other subset including eigenvalues q_i^* and the corresponding wave-vectors denoted here by $\mathbf{F}^{(-i)}$.

3. Case of defective Stroh matrix

As discussed above, the matrix $\mathbf{H}(r)$ has eight eigenvalue-eigenvector pairs $(q_i, \mathbf{F}_i)$:

$$\mathbf{HF} = \mathbf{Fq}. \tag{4}$$

In the small domain at the cut-off wave-number r_c (see example in Fig. 1), most of them do not vary much and can be considered constant except the number of pairs where two or more eigenvectors melt into real one $\mathbf{F}_c$. This indicates that the matrix $\mathbf{H}$ is defective and that there are generalized eigenvectors $\mathbf{F}', \mathbf{F}'', \ldots$, orthogonal to $\mathbf{F}_c$ [3]. These generalized eigenvectors help us to evaluate the eigenvectors in a narrow domain around r_c by the matrix perturbation with respect to small variation of r around r_c ($\dot{\mathbf{H}} = \partial\mathbf{H}/\partial r$ at $r = r_c$), taking into account the normalization condition $\mathbf{F}\cdot\mathbf{F} = 1$ which requires that $\mathbf{F}_c \cdot \mathbf{F}' = 0$, for instance. In the case of second-order defective matrix, we have:

$$\begin{gathered}
\varepsilon = r - r_c = a\delta^2, \qquad q = q_c + \delta, \\
\mathbf{F} = \mathbf{F}_c + \delta\mathbf{F}' + \delta^2\ddot{\mathbf{F}} + \cdots, \\
\mathbf{H} = \mathbf{H}_c + \varepsilon\dot{\mathbf{H}}, \qquad \mathbf{H}_c\mathbf{F}_c = q_c\mathbf{F}_c, \\
(\mathbf{H}_c - q_c\mathbf{I})\mathbf{F}' = \mathbf{F}_c, \qquad \mathbf{F}_c\cdot\mathbf{F}' = 0, \\
(\mathbf{H}_c - q_o\mathbf{I})\ddot{\mathbf{F}} + a\dot{\mathbf{H}}\mathbf{F}_c = \mathbf{F}'.
\end{gathered} \tag{5}$$

It can be easily shown that in the above considered case of defective matrix,

$$\mathbf{E} = \mathbf{E}_c + \delta\mathbf{E}', \qquad a = (\mathbf{E}_c\mathbf{F}')/(\mathbf{E}_c\dot{\mathbf{H}}\mathbf{F}_c), \qquad \mathbf{E}_c\mathbf{F}_c = 0$$

(a is the slowness curvature). The detailed discussion presented below concerns the higher defective matrix, having three generalized eigenvalues.

In this case, the slowness curvature vanishes ($\mathbf{E}_c\mathbf{F}' = 0$) at the cut-off wave-number r_c, making its parabolic approximation, Eqs. (5), invalid. Instead, the approximation is [2–4]:

$$\begin{gathered}
\varepsilon = r - r_c = a\delta^4, \\
q = q_c + \delta, \qquad \delta_n = (\varepsilon/a)^{1/4}e^{-jn\pi/2}, \\
\mathbf{F} = \mathbf{F}_c + \delta\mathbf{F}' + \delta^2\mathbf{F}'' + \delta^3\mathbf{F}''', \\
a = \mathbf{E}_c\mathbf{F}'''/(\mathbf{E}_c\dot{\mathbf{H}}\mathbf{F}_c),
\end{gathered} \tag{6}$$

for this matrix defectiveness, where four eigenvectors melt into one ($\mathbf{F}_c$) at r_c. For $r \neq r_c$, the approximate eigenvectors and four real eigenvectors $\mathbf{F}^{(\pm i)}$ describing bulk waves span the eight-dimensional space of $\mathbf{H}$.

4. Interface waves

Here, we analyze the boundary-value problem for piezoelectric interface waves guided along the perfectly conducting or impedance plane embedded in a piezoelectric body at $z = 0$ [5, 6]; the wave-field decays on both sides of the plane. The eigenvectors of different subsets contribute to the wave-field on different sides of the plane, chosen to satisfy the radiation conditions at $z \to \pm\infty$ (corresponding eigenvalues are: δ and δ_1, or $-\delta, -\delta_1$). For typical convex slownesses, $a < 0$, the eigenvectors satisfying Eq. (4) are:

$$\mathbf{F}^{\pm} = [\mathbf{F}_c \pm \delta\mathbf{F}' + \delta^2\mathbf{F}'' \pm \delta^3\mathbf{F}''',\ \mathbf{F}_c \mp j\delta\mathbf{F}' - \delta^2\mathbf{F}'' \mp j\delta^3\mathbf{F}''',\ \mathbf{F}^{(\pm i)}], \tag{7}$$

where the eigenvectors $\mathbf{F}^{(\pm i)}$, $i = 3, 4$, are considered constant over the small domain of $r = r_c + a\delta^4$, where δ takes values

$$\delta = (\varepsilon/a)^{1/4} \quad \text{for} \quad \varepsilon/a > 0, \quad \text{or} \quad \delta = \exp(-j\pi/4)|\varepsilon/a|^{1/4} \quad \text{for} \quad \varepsilon/a < 0, \tag{8}$$

or $-j\delta$ as defined above (resulting from the fourth-root dependence of δ on ε), which case yields the second column of $\mathbf{F}^{\pm}$ in Eq. (7).

The field (displacement, stress and electric potential) continuity at $z = 0$ requires that

$$\begin{bmatrix} \mathbf{F}^+ & \mathbf{0} \\ \mathbf{0} & -\mathbf{F}^- \end{bmatrix} \begin{bmatrix} \mathbf{c}^+ \\ \mathbf{c}^- \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ Q \end{bmatrix}, \tag{9}$$

where $\mathbf{c}^{\pm}$ are unknown constants, from which electric potential φ can be evaluated for a given surface electric charge distribution Q on the plane $z = 0$ ($\mathbf{0}$ is the zero column matrix). Here, we are interested in evaluation of the interface electric impedance $Z = \varphi/(j\omega Q)$ dependent on r or, equivalently, on δ defined in Eqs. (8). The left-hand side of the above system of equations is (zeros omitted):

$$\left[\mathbf{F}_c, \mathbf{F}', \mathbf{F}'', \mathbf{F}''', \mathbf{F}^{(3)}, \mathbf{F}^{(-3)}, \mathbf{F}^{(4)}, \mathbf{F}^{(-4)}\right] \begin{bmatrix} 1 & -1 & 1 & -1 & & & & \\ \delta & \delta & -j\delta & -j\delta & & & & \\ \delta^2 & -\delta^2 & -\delta^2 & \delta^2 & & & & \\ \delta^3 & \delta^3 & j\delta^3 & j\delta^3 & & & & \\ & & & & 1 & & & \\ & & & & & -1 & & \\ & & & & & & 1 & \\ & & & & & & & -1 \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_1^- \\ c_2^+ \\ c_2^- \\ c_3^+ \\ c_3^- \\ c_4^+ \\ c_4^- \end{bmatrix}. \tag{10}$$

The upper left-hand cell of the second matrix can be written in the Smith canonical form

$$\mathrm{diag}\{[1,\delta,\delta^2,\delta^3]\}\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -j & -j \\ 1 & -1 & -1 & 1 \\ 1 & 1 & j & j \end{bmatrix}. \tag{11}$$

Multiplying matrix (10) by left eigenvectors and exploiting the known orthogonality relations for generalized eigenvectors which can be obtained from equations analogous to Eqs. (8) [2]: $\mathbf{E}_c\mathbf{F}_c = 0$, $\mathbf{E}_c\mathbf{F}' = 0$, $\mathbf{E}_c\mathbf{F}'' = \mathbf{E}'\mathbf{F}' = 0$, $\mathbf{E}_c\mathbf{F}''' = \mathbf{E}'\mathbf{F}'' = \mathbf{E}''\mathbf{F}'$, $\mathbf{E}'\mathbf{F}''' = \mathbf{E}''\mathbf{F}''$, helps us to solve Eq. (5). Note here that $\mathbf{E}[0,0,0,0,0,0,0,Q]^{\mathrm{T}} = F_4 Q$ due to the left eigenvector shape, Eq. (5) (F_4 is the fourth element of the vector $\mathbf{F}$). The resulting left matrix of Eq. (10):

$$\begin{bmatrix} \mathbf{d} & & \\ & \mathbf{E}^{(\pm 3)}\mathbf{F}^{(\pm 3)} & \\ & & \mathbf{E}^{(\pm 4)}\mathbf{F}^{(\pm 4)} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} & & & \mathbf{E}_c\mathbf{F}''' \\ & & \mathbf{E}_c\mathbf{F}''' & \mathbf{E}'\mathbf{F}''' \\ & \mathbf{E}_c\mathbf{F}''' & \mathbf{E}'\mathbf{F}''' & \mathbf{E}''\mathbf{F}''' \\ \mathbf{E}_c\mathbf{F}''' & \mathbf{E}'\mathbf{F}''' & \mathbf{E}''\mathbf{F}''' & \mathbf{E}'''\mathbf{F}''' \end{bmatrix} \tag{12}$$

can be easily inverted, as well as the second one, including the Smith matrix presented earlier, in order to evaluate $\mathbf{c}^{\pm}$, then φ, and finally Z:

$$2r\omega Z = \begin{bmatrix} \mathbf{F}_c \\ \mathbf{F}' \\ \mathbf{F}'' \\ \mathbf{F}''' \\ \mathbf{F}^{(\pm i)} \end{bmatrix}_4^{\mathrm{T}} \begin{bmatrix} 0 & j\delta^{-1} & 0 & \delta^{-3} & \\ \delta & 0 & j\delta^{-1} & 0 & \\ 0 & \delta & 0 & j\delta^{-1} & \\ j\delta^3 & 0 & \delta & 0 & \\ & & & & \mathbf{1} \end{bmatrix}$$
$$\times \begin{bmatrix} \dfrac{\mathbf{d}^{-1}}{1+j} & \\ & \dfrac{\pm 1}{\mathbf{E}^{(\pm i)}\mathbf{F}^{(\pm i)}} \end{bmatrix} \begin{bmatrix} \mathbf{F}_c \\ \mathbf{F}' \\ \mathbf{F}'' \\ \mathbf{F}''' \\ \mathbf{F}^{(\pm i)} \end{bmatrix}_4, \tag{13}$$

where $\mathbf{F}^{(\pm i)}$ are unperturbed eigenvectors ($i = 3, 4$; note the above shortened notation), and subscript 4 of column matrices points to fourth elements of the corresponding vectors $\mathbf{F}$; $\mathbf{1}$ is the fourth-order unitary matrix.

5. The impedance approximation

The resulting polynomial of δ has zeros at $\delta_n, n = 1, 2, \cdots, 6$:

$$2r\delta^3\omega Z = p_6\delta^6 + p_5\delta^5 + \cdots + p_0, \tag{14}$$

which can be rewritten in the form

$$Z = \frac{p_6 p_0}{2r\omega\delta^3}\frac{(1-\delta/\delta_1)(1-\delta/\delta_2)(1-\delta/\delta_3)}{(1+\delta/\delta_4)(1+\delta/\delta_5)(1+\delta/\delta_6)} \approx \frac{p_0 p_6}{2r\omega\delta^3}\frac{1-\delta/\delta_1}{1+\delta/\delta_4}, \tag{15}$$

where we have exploited the approximation $(1-\delta/\delta_n) = (1+\delta/\delta_n)^{-1}$ for small δ in order to obtain the expected functional dependence of Z on finite, albeit small δ. It is assumed in the above approximation that the values of δ_n other than δ_1, δ_4 are much larger, what was the case of our former computations. There is no rule however, how to order roots δ_n in order to obtain results which may be compared with direct calculation of $Z(r)$. Naturally, only the smallest δ_n, say, δ_1 and δ_4 are most important, possibly yielding the resonant behavior of $1/Z$ on small variation of r depending on δ. If the interface wave exists then zero of $1/Z$ would yield k_o, determined by the first zero of Z. The first pole, if it exists for small δ, would determine the wave-number of the wave propagating under condition of $Q = 0$. It would mean unperturbed crystal in the considered case, without any conducting plane embedded in it; no such interface wave can exist in this case. In the example of Fig. 2, the evaluated and approximated Z satisfactorily agree and indeed, only zero of Z exists for small δ.

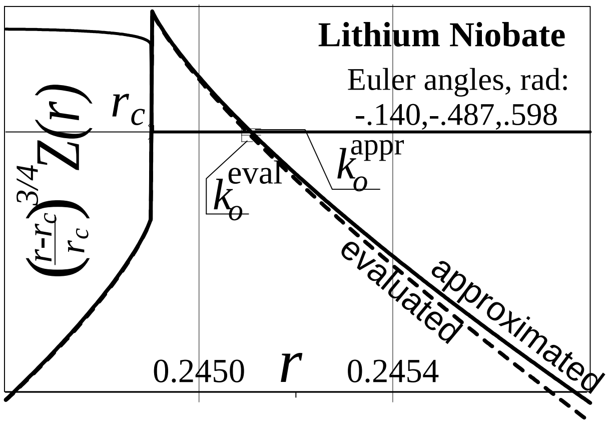

Fig. 2. The evaluated and approximated impedance for the 4-order defective Stroh matrix characterizing the rotated lithium niobate (the given Euler angles in radians determine the rotation). The applied multiplying function of r makes drawings of Z easier for comparison.

This confirms the fact that the interface impedance was evaluated with sufficient accuracy by perturbation analysis in the case of 4th-order defective Stroh matrix. Here, the trick in obtaining the valid approximation relies on the last careful factorization of the polynomial resulting from the perturbation analysis [4, 7].

In the theory of distributed SAW transducers mentioned in the Introduction, the generated surface waves are evaluated by Cauchy integral residual at the wave-number $k_o > r_c$. Its correct evaluation presented in Fig. 2 is thus the most important fact for the obtained approximation.

6. Conclusions

The analysis presented above shows how the to approximately characterize the arbitrarily anisotropic piezoelectric substrate in a quite complicated case of fourth-order defective Stroh matrix. In typical cases, the slowness curve can be approximated by a second-order curve what simplifies the analysis, as presented in [2, 4] and illustrated in Fig. 1. This paper shows that the approximation can be effectively derived even in more complicated cases.

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