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BURZYŃSKI YIELD CONDITION VIS-À-VIS THE RELATED STUDIES REPORTED IN THE LITERATURE

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The paper is written in accompaniment of the publication of English translation of W. Burzyński's paper [1], which deals with the yield criterion for materials revealing the sensitivity of yield strength to pressure derived by W. Burzyński during preparation of his doctor thesis in 1927 [2]. More recently the dependence of yield strength on pressure is related to the so-called strength differential (SD) effect, i.e. asymmetry of elastic range, cf. e.g. [3, 4]. Therefore, the original Burzyński's formulation of yield condition remains actual and acquires increasing significance. The position of Burzyński's energy-based approach in the literature is reported and his main achievement in this field concerning the recent studies is discussed.

Key words: Burzyński yield condition (criterion), strength differential effect, asymmetry of elastic range, hypotheses of material effort.

1. Introduction

The aim of the paper is to show that the original results of W. Burzyński presented in his doctoral thesis [2] are of fundamental significance and remain important also for recent studies related with modelling of yield and failure of solids characterized by asymmetry of elastic range and possessing, in general, anisotropic properties. It concerns, in particular, soils and rocks, e.g. applications in modelling of interaction of a cutting-tool with geological settings [5], as well as modern materials, e.g.: polymers [6], different kinds of composites and cellular or porous solids [7, 8], high-strength steels or, in general, ultra-fine grained alloys and nano-metals [9]. It is worthy to mention that the Burzyński criterion is cited in the aforementioned papers [5–9]. Therefore, we have decided to publish English translation of the paper of Burzyński [1] that contains not only the main results of his doctoral thesis [2], which appeared on January 1928 as a comprehensive monograph, but presents also his matured view on the state of the art of yield conditions and failure criteria at that time, which ripened during the nine months long post-doctoral study travel to Germany and Switzerland, connected with seminars and discussions with leading specialists in the field, cf.

the biographical note [10]. From December 1928 until March 1929, W. Burzyński visited the University of Göttingen, and from April until August 1929 he visited the Confederate Material Testing Laboratory (*Eidgenössische Materialprüfung Anstalt – EMPA*) at the Zurich Polytechnic (*ETH – Eidgenössische Technische Hochschule*) in Zürich, where, among others, on the 1st of June 1929 he took part in the 26th Conference of the Swiss Association of Material Testing for Technology (*26. Diskussionstag des Schweizerischen Verbandes für die Materialprüfungen der Technik*) and delivered the lecture, published in [11]. Unfortunately, W. Burzyński was unable to disseminate his knowledge and defend his views since 1949, when serious illness terminated his scientific carrier. Therefore, we would like to share his scientific legacy with the research community, in particular now, when his concepts concerning the yield conditions have been confirmed and rediscovered many times independently by many researchers.

2. Burzyński yield condition reported in the literature

The concept of Burzyński yield condition was presented in detail and compared with several later independent propositions by M. Życzkowski [12, 13], J.J. Skrzypek [14] and M. Jirásek and Z.P. Bažant [15], as well as by G.S. Pisarenko and A.A. Lebedev [16] and V.V. Bozhidarnik, G.T. Sulym [17]. It has been also discussed in recent works on strength theory [18] and plasticity [19]. The first foreign references can be found in the papers of G.D. Sandel (1930) [20], H. Geiringer and W. Prager (1934) [21], M. Roš and A. Eichinger (1949) [22], cf. also the comprehensive discussion on the impact of the Burzyński's results on the development of yield criteria by A. Becchi [23], in his historical essay concerning the hundred years of studies on the yield criteria. The strong critics of the previous proposition of G.D. Sandel in [24] by W. Burzyński in [11], as well as in papers in Polish [1, 2], awoke the vivid polemics and exchange of letters of the both authors with the editor [25, 26] and [27].

At that time, the yield criterion proposed by M.T. Huber (1904), R. v. Mises (1913) and H. Hencky (1924) [28–30], for isotropic solids characterized by equal magnitude of yield stress in tension and compression, was well established and confirmed experimentally, cf. e.g. [1]. The open question remained, however, in the subject of yield criteria for isotropic materials revealing different magnitudes of the yield stress in tension and compression, the so-called strength differential (SD) effect leading to the asymmetry of elastic range. Also the formulation of yield criteria for anisotropic solids was an open question at that time; as said in [2], p. 127: *"it's still a thing of a distant future"*. Nevertheless, the first work on anisotropic yield criteria was published by R. v. Mises in 1928 [31]. Quite independently, an energy-based approach to the description of yield criterion of orthotropic and transversally isotropic materials was presented in depth by W. Burzyński in [2], as a consistent development of the energy-based Huber's approach to isotropic solids [28], cf. also the discussion in [1], p. 291–292, and the concise remark in [11], p. 261. The contribution of W. Burzyński was discussed in [12], p. 111: *"The generalization of Huber-Mises-Hencky yield condition in the case of anisotropic bodies may also be achieved by using energy considerations ... In the most general case of anisotropy, the elastic energy cannot be decomposed into the energy of volume change and energy of shape change. This problem was first investigated by* W. Burzyński *[2], who proved that the existence of such a decomposition results in five relations between the elastic moduli, and thus only 16 moduli remain independent".* (In [12], p. 69, the reference to the known since the publication of Origins of Clerk Maxwell's Electric Ideas, Cambridge, 1937, the first proposition of elastic energy of distortion as a measure of material effort by J.C. Maxwell in 1856 is also mentioned). The Burzyński yield criterion for orthotropic solids and its relation with the condition proposed twenty years later by R. Hill [32] for materials with symmetric elastic range, as well as for orthotropic solids revealing the SD effect studied by P.S. Theocaris [33–35] will be discussed independently in the forthcoming paper [36].

According to the comprehensive analysis of existing criteria in [1] and [2], the first problem was undertaken already in the Coulomb criterion, which can be expressed by the following equivalent relations, cf. [2]:

$$\begin{gathered}\frac{\sigma_1-\sigma_3}{2}+\frac{k_c-k_t}{k_c+k_t}\frac{\sigma_1+\sigma_3}{2}=k_s, \qquad k_c\sigma_1-k_t\sigma_3=k_ck_t\\ \text{or} \qquad \pm\tau+\frac{k_c-k_t}{2\sqrt{k_ck_t}}\sigma-\frac{1}{2}\sqrt{k_ck_t}=0,\end{gathered}\tag{2.1}$$

where k_c and k_t are the magnitudes of the yield stress at compression and tension, respectively, while k_s is the yield strength in shear, given by the relation $k_s\dfrac{k_tk_c}{k_t+k_c}$, and τ, σ are the shear and normal stresses acting in the plane of shear. Another approach was related to the Duguet–Mohr hypothesis, which reads [2]:

$$\begin{gathered}(\sigma_1-\sigma_3)^2+(k_c-k_t)(\sigma_1+\sigma_3)=k_ck_t,\\ \tau^2+\frac{k_c-k_t}{2}\sigma-\frac{(k_c+k_t)^2}{16}=0.\end{gathered}\tag{2.2}$$

These conditions have been, however, completely rejected by the researches at that time because of the just detected large discrepancies with experimental results, which were related, *inter alia*, with the lack of influence of the intermediate principal stress, [1, 2, 11]. This problem was also studied by G.D. Sandel [24], who assumed that the measure of material effort is the shear strain that depends linearly on volumetric change of strain, which led to the following equivalent relations:

$$(2.3) \qquad (n+1)\sigma_1 + n\sigma_2 + (n-1)\sigma_3 \leq 2k_s, \qquad k_c\sigma_1 + \frac{k_c k_t}{2}\sigma_2 - k_t\sigma_3 = k_c k_t,$$

where $n = \dfrac{k_c - k_t}{k_c + k_t}$, $k_s = \dfrac{k_t k_c}{k_t + k_c}$.

Then the problem was undertaken, within the framework of energy-based approach, by F. Schleicher (1900–1957), cf. the short biographical note in [38], who presented his results during the application lecture delivered on the 8th of May 1925 at the Technische Hochschule Karlsruhe, in the summary of the presentation at the autumn 1925 GAMM conference in Danzig (Gdańsk) [39] and published as a full paper in [40]. Also R. v. Mises mentioned the possibility of accounting for the SD effect assuming that the yield strength depends on pressure, in the editorial note to the paper of F. Schleicher [40], p. 199: *"Eine mit der hier entwickelten wesentlich gleichlautende Plastizitätsbedingung ist von mir in einem Vortrage im Ausschuß für Technische Mechanik des Berliner Bezirksvereines deutscher Ingeniueure am 17. Juli 1925 mitgeteilt worden. Ich habe dabei namentlich gezeigt, wie die neue Hypothese, die eine konsequente Erweiterung der von mir im Jahre 1913 eingeführten darstellt, durch die neuen Versuche von Lode notwendig gemacht und durch sie voll bestätigt wird. Die Bezeichnung "Energiekriterium" lehne ich ab, da der in Frage kommende Ausdruck für den plastischen Körper kein Mass der Energie bildet. R. v. Mises".*

(*One of developed here, essentially similar yield condition has been announced by myself in a lecture delivered during the meeting of the Commission of Technical Mechanics of the Berlin Branch of the German Association of Engineers on the 17th of July 1925. I have particularly shown by that how the new hypothesis, which is a consistent extension of that one introduced by myself in the year 1913, appeared necessary due to the new experimental investigations by Lode and also gained full confirmation by these experiments. I am rejecting the notion "energy-based criterion" because the pertinent relation for the case of plastic bodies gives no measure of energy*).

F. Schleicher proposed in [40] an energy-based hypothesis, in which the equivalent stress reads:

$$(2.4) \qquad \sigma_{vf} = \sqrt{2E\Phi} = f(p),$$

where E is the Young modulus, Φ is the total elastic energy density and p is the pressure $p = \dfrac{\sigma_1 + \sigma_2 + \sigma_3}{3}$ and f is a certain function which, according to the assumption of F. Schleicher, can be linear or parabolic with respect to pressure p. In particular, the following relation can be obtained in the space of principal stresses:

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) + (k_c - k_t)(\sigma_2 + \sigma_2 + \sigma_3) = k_c k_t, \tag{2.5}$$

where μ is the Poisson ratio. The Schleicher's application of the total elastic energy density as a measure of material effort was strongly criticized by W. Burzyński in his paper published in German [11], as well as in his earlier doctoral thesis [2] and the later paper written in Polish [1]. The discrepancy with the experimental data discussed in [40], the unrealistic transition to the Beltrami criterion for $k_t = k_c$ and the presence in the yield condition (2.5) of the Poisson ratio μ, were mostly criticized. In contrast to G.D. Sandel, F. Schleicher neither answered to the Burzyński's critics nor referred to any of his papers. Nevertheless, he changed his view on the measure of material effort and in the next paper, published on the 13th April 1928 in [41], he replaced in (2.4) the total elastic energy Φ by the density of elastic energy of distortion Φ_f:

$$\sigma_g = \sqrt{6G\Phi_f} = f(p). \tag{2.6}$$

Discussing the possible applications of the general form (2.6), F. Schleicher suggested the application of linear dependence of the equivalent stress on pressure to certain brittle materials. In such a way, he is arriving at the cone in the coordinates (σ_g, p), what corresponds also with a certain special linear form of Burzyński criterion, cf. [1], p. 289, and to the similar condition derived later by Drucker and Prager [3]. Considering (2.6) in an equivalent form

$$\frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2} = F(p), \tag{2.7}$$

one obtains the relation proposed in [2], p. 183, as a certain generalization of Mohr's criterion. Similar generalization was considered later by some authors, e.g. A. Nadai [42], p. 225–228.

3. Conclusions

The more careful study of the discussed above problem leads to the conclusion that as a matter of fact there is not the dependency of the yield strength on pressure that is essential for the adequate formulation of yield condition, but it is rather the proper relation between the densities of energy of distortion Φ_f and volumetric change Φ_v for different materials under varying states of stress. The physically justified interplay of the both parts of elastic energy at the elastic limit, which defines in a proper way the material effort at a given state of stress, is a key point of the formulation of an adequate yield function and yield condition. This problem was underlined and discussed in recent papers of R.M. Christensen [43, 44]. One of main achievements of W. Burzyński was, according to our opinion, that he had solved this crucial question in an original way proposing the following formulation for the hypothesis of variable-volumetric-distortional limit energy, cf. [1], p. 288:

$$\Phi_f + \eta(p)\Phi_v = K, \tag{3.1}$$

where a particular form of the pressure dependency of the function $\eta(p)$ is assumed, $\eta = \omega + \dfrac{\delta}{p}$. The core of Burzyński's idea is the exchange of three material parameters: ω, δ, K, appearing in (3.1) with the triplet of material constants, k_t, k_c, k_s known from experiments of tension, compression and simple shear. The other form of the function $\eta(p)$ could be also considered in order to account for the ductile-brittle transition under the tri-axial states of stress in a considered material. The Eq. (3.1) leads to one of possible formulations of the W. Burzyński yield condition

$$\begin{aligned}\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\lambda(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)& \\ + (k_c - k_t)(\sigma_2 + \sigma_2 + \sigma_3) &= k_c k_t,\end{aligned} \tag{3.2}$$

where $\lambda = \dfrac{k_c k_t}{2k_s^2} - 1$ and, depending on the sign of λ and the relation between material constants k_c, k_t and k_s, the Eq. (3.2) can represent in the axes of principal stresses a paraboloid, ellipsoid or a cone of revolution, cf. the discussion in [1], p. 289. Similar formulations were repeated independently during the last eighty years over and over by many researches, often without the clarity of the in-depth analysis and physical foundations of Burzyński's work.

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References

1. W. Burzyński, *Teoretyczne podstawy hipotez wytężenia*, Czasopismo Techniczne, 1929, **47**, 1–41, Lwów [in Polish]; reprinted in: Włodzimierz Burzyński: *Dzieła Wybrane*, tom I, Polska Akademia Nauk, PWN Warszawa, 1982, 264–303 [in Polish]; English translation: *Theoretical foundations of the hypotheses of material effort*, Engng. Trans., **56**, 3, 269–305, 2008.

2. W. Burzyński, *Studjum nad Hipotezami Wytężenia*, (*Study on Material Effort Hypotheses*), Nakładem Akademji Nauk Technicznych (issued by the Academy of Technical Sciences), Lwów, January 7, 1928, 1–192 [in Polish]; reprinted in: Włodzimierz Burzyński: *Dzieła Wybrane*, tom I, Polska Akademia Nauk, PWN Warszawa, 1982, 67–258 [in Polish].

3. D. C. Drucker, *Plasticity theory, strength-differential (SD) phenomenon, and volume expansion in metals and plastics*, Met. Trans., **4**, 667–673, 1973.

4. C. D. Wilson, *A critical re-examination of classical metal plasticity*, J. Appl. Mech., **69**, 63–68, 2002.

5. A. Nardin, G. Zavarise, B. A. Schrefler, *Modelling of cutting tool – soil interaction – Part I: contact behaviour*, Computational Mechanics, **31**, 327–339, 2003.

6. V. A. Kolupaev, S. Kolling, A. Bolchoun, M. Moneke, *A limit surface formulation for plastically compressible polymers*, Mechanics of Composite Materials, **43**, 245–258, 2007.

7. P. Kordzikowski, M. Janus–Michalska, R. B. Pęcherski, *Specification of energy-based criterion of elastic limit states for cellular materials*, Archives of Metallurgy and Materials, **50**, 621–634, 2005.

8. V. A. Kolupaev, A. Kraatz, M. Moneke, A. Bolchoun, *Beschreibung der mehraxialen Kriechphänomene bei Hartschaumstoffen*, KGK. Kautschuk, Gummi, Kunststoffe, **59**, 17–26, 2006.

9. R. B. Pęcherski, P. Kordzikowski, *Phenomenological description of mechanical properties and modelling of inelastic deformation of nano-structured materials*, Archives of Metallurgy and Materials, 2009 – submitted for publication.

10. Z. S. Olesiak, *Włodzimierz Stanisław Trzywdar Burzyński*, Engng. Trans., **56**, 4, 377–382, 2008.

11. W. Burzyński, *Ueber die Anstrengungshypothesen,* Schweizerische Bauzeitung, **94**, Nr. 21, 23. November 1929, 259–262; reprinted in Włodzimierz Burzyński: *Dzieła Wybrane*, tom I, Polska Akademia Nauk, PWN Warszawa, 1982, 259–262.

12. M. Życzkowski, *Combined Loadings in the Theory of Plasticity*, PWN-Polish Scientific Publishers, Warszawa, 1981.

13. M. Życzkowski, *Discontinuous bifurcations in the case of the Burzyński-Torre yield condition*, Acta Mechanica, **132**, 19–35, 1999.

14. J. J. Skrzypek, *Plasticity and Creep. Theory, Examples, and Problems*, R. B. Hetnarski (English Edition Editor), CRC Press, Boca Raton, 1993.

15. M. Jirásek. Z. P. Bažant, *Inelastic Analysis of Structures*, J. Wiley, Chichester, 2002.

16. G. S. Pisarenko, A. A. Lebedev, *The Deformation and Strength of Materials under Complex State of Stress* [in Russian], Naukova Dumka, Kiev, 1973.

17. V. V. Bozhidarnik, G. T. Sulym, *Elements of the Theory of Plasticity and Fracture* [in Ukrainian], CWIT, Lviv, 1999.

18. M.-H. Yu, *Unified Strength Theory and its Applications*, Springer-Verlag, Berlin, Heidelberg, 2004.

19. M.-H. Yu, G.-W. Ma, H.-F.Qiang, Y.-Q. Zhang, *Generalized Plasticity*, Springer-Verlag, Berlin, Heidelberg, 2006.

20. G. D. Sandel, *Die Anstrengungsfrage*, Schweizerische Bauzeitung, **95**, Nr. 26, 28. Juni 1930, 335–338.

21. H. Geiringer, W. Prager, *Mechanik isotroper Körper im plastischen Zustand*, Ergebnisse der exakten Naturwissenschaften, **13**, 314–363, 1934.

22. M. Roš, A. Eichinger, *Die Bruchgefahr fester Körper bei ruhender – statischer – Beanspruchung. Ergebnisse der an der Eidg. Materialprüfungs- und Versuchsanstalt für Industrie, Bauwesen und Gewerbe von 1925–1949 durchgeführten Untersuchungen*, Bericht Nr. 172, Zürich, September 1949.

23. A. Becchi, *I Criteri di Plasticità: Cento Anni di Dibattito* (1864–1964), Doctor Thesis, Firenze, 1994.

24. G. D. Sandel, *Ueber die Festigkeitsbedingungen*, Dissertation, T.H. Stuttgart, 1919; Jänecke Verlagsbuchhandlung, Leipzig, 1925.

25. G. D. Sandel, Letter to the editor, Schweizerische Bauzeitung, **95**, 15. Februar, Nr. 7, 87, 1930.

26. W. Burzyński, Letter to the editor, Schweizerische Bauzeitung, **95**, 15. Februar, Nr. 7, 87, 1930.

27. G. D. Sandel, Letter to the editor, Schweizerische Bauzeitung, **95**, 1 März, Nr. 9, 123, 1930.

28. M. T. Huber, *Właściwa praca odkształcenia jako miara wytężenia materjału*, Czasopismo Techniczne, Lwów, 1904 (see also the English translation: *Specific work of strain as a measure of material effort*, Arch. Mech., **56**, 173–190, 2004).

29. R. v. Mises, *Mechanik der festen Körper im plastisch-deformablen Zustand*, Göttingen Nachrichten, Math.-Phys. Klasse, Z. 4 (1), 582–592, 1913.

30. H. Hencky, *Zur Theorie plastischer Deformationen und der hierdurch im Material hervorgerufenen Nachspannungen*, ZAMM, **4**, 323–334, 1924.

31. R. v. Mises, *Mechanik der plastischen Formänderung von Kristallen*, ZAMM, **8**, 161–185, 1928.

32. R. Hill, *A theory of the yielding and plastic flow of anisotropic metals*, Proc. Roy. Soc. London, **A193**, 281–297, 1948; also *The Mathematical Theory of Plasticity*, Oxford, Clarendon Press, 1950, p. 318–340.

33. P. S. Theocaris, T. P. Philippidis, *The paraboloidal failure surface of initially anisotropic elastic solids*, J. Reinf. Plastics Compos., **6**, 378–395, 1987.

34. P. S. Theocaris, *The parabolid failure surface for the general orthotropic material*, Acta Mech., **79**, 53–79, 1989.

35. P. S. Theocaris, *Variances in failure tensor polynomials for anisotropic bodies*, Eng. Fracture Mech., **51**, 707–733, 1995.

36. R. B. Pęcherski, T. Frąś, *Study of the yield condtion for anisotropic solids revealing asymmetry of elastic range with comparison of the criteria proposed by Burzyński*, Hill and Theocaris – forthcoming paper.

37. G. D. Sandel, *Über die Festigkeitsbedingungen*, Dissertation, T. H. Stutgart, 1919.

38. Karl-Eugen Kurrer, *The History of The Theory of Structures: From Arch Analysis to Computational Mechanis*, J. Wiley, 2008; *Geschichte von Baustatik*, Ernst and Sohn Verlag, Berlin, 2002.

39. F. Schleicher, *Die Energiegrenze der Elastizität (Plastizitätbedingung)*, *Vorträge der Danziger Tagung*, ZAMM, **5**, 478–479, 1925.

40. F. Schleicher, *Der Spannungszustand an der Fließgrenze (Plastizitätsbedingung)*, ZAMM, **6**, 199–216, 1926.

41. F. Schleicher, *Über die Sicherheit gegen Überschreiten der Fliessgrenze bei statischer Beanspruchung*, Der Bauingenieur, **9**, Heft 15, 253–261, 13 April 1928.

42. A. Nadai, *Theory of Flow and Fracture of Solids*, Vol. I, second edition, McGraw Hill, New York, 1950.

43. R. M. Christensen, *A two-property yield, failure (fracture) criterion for homogeneous, isotropic materials*, J. Eng. Mater. Technol., **126**, 45–52, 2004.

44. R. M. Christensen, *A comparative evaluation of three isotropic, two property failure theories*, J. Appl. Mech., **73**, 852–859, September 2006.

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