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# SIMPLE SHEAR TEST IN IDENTIFICATION OF CONSTITUTIVE BEHAVIOUR OF MATERIALS SUBMITTED TO LARGE DEFORMATIONS – HYPERELASTIC MATERIALS CASE

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The present work is directed at evaluation of the simple shear test for identification of constitutive behaviour of materials submitted to large deformations. For that purpose, actual experimental conditions together with theoretical background of the test are analyzed on the example of two hyperelastic material models. Advantages and disadvantages of various strain and stress measures used for presentation of simple shear test (SST) results are analyzed. The most often presented as the only result of "standard" SST proof chart, i.e. shear nominal stress $\leftrightarrow$ shear nominal strain ($\sigma_{12}^{(N)} \leftrightarrow \gamma/2$), characterizes the material energetically in the sense that it reveals its capacity for elastic energy storage $dW/V_0 = \sigma_{12}^{(N)} d\gamma$. However, it characterizes the constitutive behaviour of the material only partially, since it is equivalent to shear II Piola Krichoff stress $\leftrightarrow$ shear Green–Lagrange strain ($\sigma_{21}^{(2)} \leftrightarrow E_{21}^{(2)}$) chart, within the large deformations context. This data alone does not even allow to reconstruct the shear Cauchy stress $\leftrightarrow$ shear spatial Hencky strain ($\sigma_{12}^{(0)} \leftrightarrow e_{12}^{(0)}$) chart for the tested material. In order to take full advantage of the constitutive information available from simple shear test, it is highly recommended to extend the experimental methodology of "standard" SST proof in such a way as to determine simultaneously two components (shear and normal) of nominal stress tensor in the same SST proof. Such experimental information allows for subsequent recalculation of non-symmetric nominal stress tensor components into Cauchy stress components.

**Key words:** simple shear, large deformations, hyperelasticity, identification of constitutive behaviour.

## NOTATIONS

| | |
|---|---|
| $B_o$, $B_t$ | undeformed configuration (initial; reference) and deformed configuration (actual), |
| $dA$, $da$ | initial and actual cross-sectional area with direction vectors $\underline{\mathbf{N}}$ and $\underline{\mathbf{n}}$, respectively, |
| $d\underline{\mathbf{f}}^t$, $d\underline{\mathbf{F}}^t \equiv \underline{\mathbf{F}}^{-1} d\underline{\mathbf{f}}^t$ | actual force operating on cross-sectional area $\underline{\mathbf{n}}\, da$, force "pulled back" to reference configuration, |
| $\underline{\mathbf{N}}$, $\underline{\mathbf{n}}$ | unit direction vectors of initial and actual cross-sectional areas, |
| $\underline{\mathbf{x}}$ | position vector of material point in actual configuration, |

| | |
|---|---|
| $\underline{\mathbf{X}}$ | position vector of material point in reference (initial) configuration, |
| $\underline{\mathbf{u}} = \underline{\mathbf{x}} - \underline{\mathbf{X}}$ | displacement vector, |
| $\{\underline{\mathbf{e}}_\alpha\}$, $\{\underline{\mathbf{E}}_i\}$ | fixed orthonormal coordinate frames defined in actual and reference configuration, |
| $\underline{\mathbf{F}} \equiv \partial x_i/\partial X_j \; \underline{\mathbf{e}}_i \otimes \underline{\mathbf{E}}_j$ | deformation gradient, |
| $\underline{\mathbf{R}}(\underline{\mathbf{R}}^T\underline{\mathbf{R}} = \underline{\mathbf{I}})$ | material rotation tensor, |
| $\underline{\mathbf{U}} = (\underline{\mathbf{F}}^T\underline{\mathbf{F}})^{1/2} = \sum \lambda_i \underline{\mathbf{u}}_i \otimes \underline{\mathbf{u}}_i$ | right stretch tensor, |
| $\underline{\mathbf{V}} = (\underline{\mathbf{F}}\,\underline{\mathbf{F}}^T))^{1/2} = \sum \lambda_\alpha \underline{\mathbf{v}}_\alpha \otimes \underline{\mathbf{v}}_\alpha$ | left stretch tensor, |
| $\gamma$ | deformation parameter, |
| $J = \det(\underline{\mathbf{F}}) = dv/dV = \rho_0/\rho$ | determinant of deformation gradient, |
| $\lambda_i$; $i = 1,2,3$ | eigenvalues of stretch tensors (principal stretches), |
| $\underline{\mathbf{u}}_i$; $i = 1,2,3$ | material principal directions, |
| $\underline{\mathbf{v}}_\alpha$; $\alpha = 1,2,3$ | spatial principal directions |
| $\theta_L$ | instantaneous angle between $\underline{\mathbf{v}}_i$ and $\underline{\mathbf{E}}_i$, |
| $\theta_E$ | instantaneous angle between $\underline{\mathbf{v}}_\alpha$ and $\underline{\mathbf{e}}_\alpha$, |
| $\dot{\underline{\mathbf{F}}}$ | deformation gradient velocity tensor, |
| $\underline{\mathbf{L}} \equiv \partial\underline{\mathbf{v}}/\partial\underline{\mathbf{x}} = \dot{\underline{\mathbf{F}}}\underline{\mathbf{F}}^{-1}$ | velocity gradient tensor, |
| $\underline{\mathbf{D}} \equiv 0.5(\underline{\mathbf{L}} + \underline{\mathbf{L}}^T)$ | deformation velocity tensor, |
| $dW/V_0$ | work increment per unit reference volume of undeformed material, |
| $\underline{\mathbf{E}}^{(0)} \equiv \ln(\underline{\mathbf{U}})$ | Hencky strain tensor, |
| $\underline{\mathbf{E}}^{(1)} = \underline{\mathbf{U}} - \underline{\mathbf{I}}$ | Biot strain tensor, |
| $\underline{\mathbf{E}}^{(2)} = 0.5(\underline{\mathbf{C}} - \underline{\mathbf{I}})$ | Green Lagrange strain tensor ($\underline{\mathbf{C}} = \underline{\mathbf{U}}^2$), |
| $\underline{\mathbf{E}}^{(N)} = \underline{\mathbf{F}} - \underline{\mathbf{I}}$ | "nominal" strain tensor, |
| $\underline{\mathbf{E}}^{(s)} \equiv 0.5(\underline{\mathbf{F}} + \underline{\mathbf{F}}^T - 2\,\underline{\mathbf{I}})$ | small strains tensor, |
| $\underline{\mathbf{e}}^{(0)} = \underline{\mathbf{R}}\,\underline{\mathbf{E}}^{(0)}\,\underline{\mathbf{R}}^T$ | spatial (rotated) Hencky strain, |
| $\underline{\mathbf{e}}^{(2)} = \underline{\mathbf{R}}\,\underline{\mathbf{E}}^{(2)}\,\underline{\mathbf{R}}^T$ | spatial (rotated) Green Lagrange strain, |
| $\underline{\boldsymbol{\sigma}}^{(0)}$, $\underline{\boldsymbol{\sigma}}^{(1)}$, $\underline{\boldsymbol{\sigma}}^{(2)}$, $\underline{\boldsymbol{\sigma}}^{(N)}$, $\underline{\boldsymbol{\sigma}}^{(IPK)}$ | Cauchy, Biot, II Piola–Kirchoff, Nominal, I Piola–Kirchoff stress tensors, $\underline{\boldsymbol{\sigma}}^{(0)} \equiv d\underline{\mathbf{f}}^t/\underline{\mathbf{n}}\,da\;(\sigma_{ij}^{(0)} \equiv df_{ij}^t/da_i)$, $\underline{\boldsymbol{\sigma}}^{(N)} \equiv d\underline{\mathbf{f}}^t/\underline{\mathbf{N}}\,dA\;(\sigma_{ji}^{(N)} = \sigma_{ij}^{(IPK)} \equiv df_{ij}^t/dA_i)$, $\underline{\boldsymbol{\sigma}}^{(2)} \equiv d\underline{\mathbf{F}}^t/\underline{\mathbf{N}}\,dA$, |
| $df_{ij}^t$ | actual force component in direction $\underline{\mathbf{e}}_j$ operating on cross-sectional area with direction vector $\underline{\mathbf{e}}_i$, |
| $da_i$ | actual cross-sectional area with direction vector $\underline{\mathbf{e}}_i$, |
| $dA_i = \text{const}$ | initial cross-sectional area with direction vector $\underline{\mathbf{E}}_i$, |
| $p \equiv -\text{tr}(\underline{\boldsymbol{\sigma}}^{(*)})/3$ | pressure, |
| $e_v \equiv \text{tr}(\underline{\mathbf{e}}^{(*)})/3$ | volumetric part of strain, |
| $W(\lambda_1, \lambda_2, \lambda_3)$ | energy function of elastic material, |
| $\hat{W}(\lambda_1, \lambda_2)$ | energy function of incompressible elastic material, |
| $L_0$ | length of deformation path, |
| $H_0$ | width of deformation path, |
| $G_0$ | thickness of deformation path, |
| SST | simple shear test. |

## 1. Introduction

Experimental setup of the simple shear test (SST) with two symmetric deformation paths is sketched in Fig. 1.

A thin sheet of specimen piece is fixed in a rigid jig. The specimen is specially designed to deliver two symmetric deformation paths. In earlier experimental literature, asymmetric layout with only one deformation path dominates. However, symmetric layout ensures better precision of measurements, results in smaller asymmetric loadings in the testing machine and in this way, it allows for its prolonged longevity. It also allows for applying less stiff and in this way, less massive grips in comparison to traditional layout with one deformation path, which is advantageous in dynamic tests. The deformation scheme is outlined in the inset. The loading force $F_1(t)$ and deformation parameter $\gamma(t)$ are registered in a "standard" simple shear test.

Simple Shear Test (SST) is more and more commonly applied for experimental identification of constants and functions of constitutive models describing the behaviour of rubberlike materials [7], elastic-plastic materials – metals [6], polymers [4] and recently also biological tissues [3]. The SST is used in quasi-static and dynamic proofs at small and large deformations imposed during the proof [6].

The SST proof growing popularity results from the fact that in a relatively simple technical way it allows to investigate the response of the material sub-

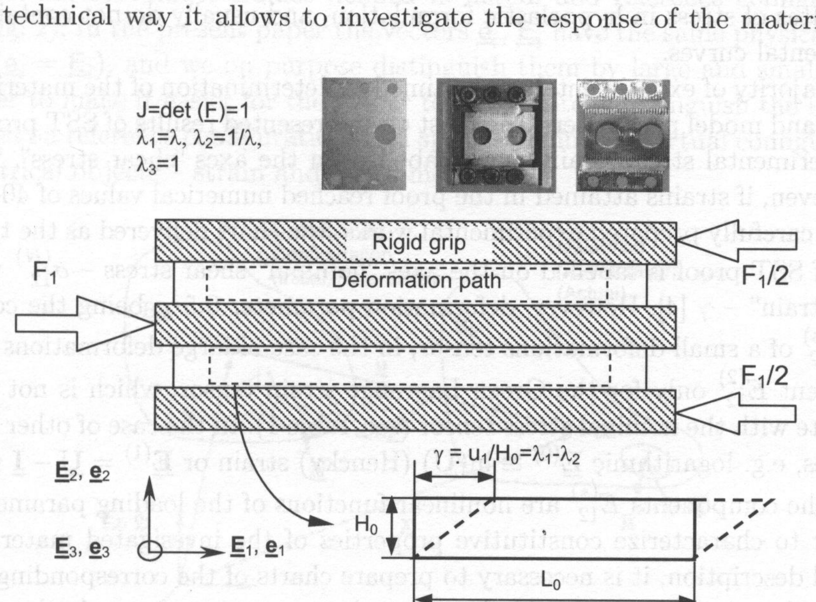



FIG. 1. Simple Shear Test – scheme of experimental setup with two symmetric deformation paths.

mitted to multiaxial, non-proportional in the stress space loading. The SST kinematical scheme of deformation results in continuous rotation of principal axes of the stress tensor. The SST is especially well suited for industrial investigation of material properties of "flat" semi-products in the form of sheets, tapes, etc. submitted to strictly defined series of production technological operations. The reason is that test specimen preparation for a SST proof is limited only to cutting of the prescribed shape, without any additional handlings (thermomechanical treatments) not taking place in a standard technological production cycle. When necessary for the test specimen preparation, such additional operations can easily influence e.g. microstructure of the specimen material and in the effect, they can lead to erroneous values of material properties determined on the basis of tests in which such specially handled specimen were used.

The simple shear test, besides its growing popularity, seems to be insufficiently analyzed in view of effective identification of the material properties. There exists a rich theoretical literature (see e.g. [2], recently [1]) devoted to investigation of hypoelastic formulation of constitutive relations and its compatibility with hyperelastic formulation for materials submitted to large deformations, in which the SST results are recalled. The knowledge of the costitutive relations governing the material behaviour is assumed to be known in advance in these works. They in principle contain only a comparison of theoretical curves obtained for various assumed constitutive model formulations – various corotational rates of stress in hypoelastic formulation, and usually do not contain any experimental curves.

In majority of experimental works aimed at determination of the material behaviour and model parameters, the most often presented results of SST proof are the experimental stress-strain charts labelled on the axes "shear stress", "shear strain" even, if strains attained in the proof reached numerical values of 40–50%. In more carefully prepared experimental works, the chart delivered as the typical result of SST proof is labelled on the axes "nominal" shear stress – $\sigma_{12}^{(N)}$ versus "shear strain" – $\gamma$ [4]. However, deformation parameter $0.5\,\gamma$, being the component $E_{12}^{(s)}$ of a small deformations tensor, in the case of large deformations is the component $E_{12}^{(2)}$ only for the Green Lagrange strain tensor, which is not work-conjugate with the nominal stress tensor (see Table 1). In the case of other strain measures, e.g. logarithmic $\underline{\mathbf{E}}^{(0)} \equiv \ln(\underline{\mathbf{U}})$ (Hencky) strain or $\underline{\mathbf{E}}^{(1)} = \underline{\mathbf{U}} - \underline{\mathbf{I}}$ (Biot) strain, the components $E_{12}^{(*)}$ are nonlinear functions of the loading parameter $\gamma$. In order to characterize constitutive properties of the invesigated material, in material description, it is necessary to prepare charts of the corresponding components of work-conjugated pair of strain and stress tensors, hence in the case of component $E_{12}^{(2)}$ of the Green Lagrange strain there should be given component $\sigma_{12}^{(2)}$ of the II Piola–Kirchoff stress tensor.

There exists a gap between theoretical and experimental works discussing the simple shear test when large deformation are induced in the tested material. In theoretical works on constitutive relations the none or few experimental conditions of the SST proof for verification of theoretical concepts are discussed, while in experimental works the theoretical knowledge on the SST proof in finite deformations regime is rarely taken into account. The present work is directed at filling this gap. In the present work we will formulate an interpretation of the chart $\sigma_{12}^{(N)} \leftrightarrow \gamma$ for a simple shear test, and we will analyze applicability of SST proof to identification of material properties, simultaneously taking into account the theoretical and experimental conditions.

## 2. Homogeneous process of simple shear

Homogeneous deformation of simple shear is defined by the following formula

$$\underline{\mathbf{x}} = (X_1 + \gamma X_2)\,\underline{\mathbf{e}}_1 + X_2\,\underline{\mathbf{e}}_2 + X_3\,\underline{\mathbf{e}}_3 \tag{2.1}$$
$$\Leftrightarrow\ x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3,$$

where $\underline{\mathbf{x}}$ and $\underline{\mathbf{X}}$ denote position vector of the material point in actual configuration $\mathrm{B}_t$ and in reference (initial) configuration $\mathrm{B}_o$ – respectively, $\gamma$ denotes the deformation loading parameter. The $\{\underline{\mathbf{e}}_\alpha\}$, $\{\underline{\mathbf{E}}_i\}$ denote fixed (laboratory) orthonormal coordinate frames defined in actual and reference configurations (see Fig. 2). In the present paper the vectors $\underline{\mathbf{e}}_i$, $\underline{\mathbf{E}}_i$ have the same physical directions ($\underline{\mathbf{e}}_i = \underline{\mathbf{E}}_i$), and we on purpose distinguish them by large and small letters in order to make it easier for the reader to immediately distinguish the material (defined on reference configuration) and spatial (defined on actual configuration) geometrical objects – strain and stress measures.

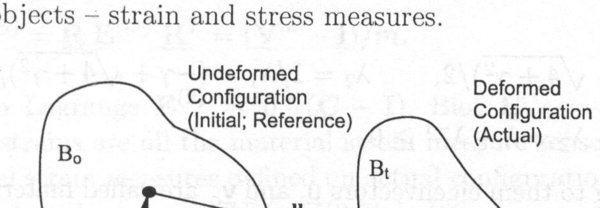



FIG. 2. Basic concepts used for description of finite deformations of materials.

Tensor of deformation gradient $\underline{\mathbf{F}} \equiv \partial x_i/\partial X_j\, \underline{\mathbf{e}}_i \otimes \underline{\mathbf{E}}_j$ in the case of simple shear has the form

$$\underline{\mathbf{F}} = \underline{\mathbf{e}}_1 \otimes \underline{\mathbf{E}}_1 + \underline{\mathbf{e}}_2 \otimes \underline{\mathbf{E}}_2 + \underline{\mathbf{e}}_3 \otimes \underline{\mathbf{E}}_3 + \gamma\, \underline{\mathbf{e}}_1 \otimes \underline{\mathbf{E}}_2. \tag{2.2}$$

The theorem on polar decomposition of the second order tensors ensures the existence of unique representation of tensor $\underline{\mathbf{F}}$ in the form $\underline{\mathbf{F}} = \underline{\mathbf{R}}\,\underline{\mathbf{U}} = \underline{\mathbf{V}}\,\underline{\mathbf{R}}$, where $\underline{\mathbf{R}} = 2A(\underline{\mathbf{e}}_1 \otimes \underline{\mathbf{E}}_1 + \underline{\mathbf{e}}_2 \otimes \underline{\mathbf{E}}_2) + \gamma\, A(\underline{\mathbf{e}}_1 \otimes \underline{\mathbf{E}}_2 - \underline{\mathbf{E}}_2 \otimes \underline{\mathbf{e}}_1)$ is called the material rotation tensor ($\underline{\mathbf{R}}^T \underline{\mathbf{R}} = \underline{\mathbf{I}}$); $A = (4+\gamma^2)^{-1/2}$. Symmetric positive-definite tensors $\underline{\mathbf{U}} = (\underline{\mathbf{F}}^T \underline{\mathbf{F}})^{1/2}$ and $\underline{\mathbf{V}} = (\underline{\mathbf{F}}\,\underline{\mathbf{F}}^T))^{1/2}$, called right and left stretch tensors, have the following forms in the case of simple shear deformation, when expressed in fixed coordinate frames

$$\underline{\mathbf{U}} = \begin{bmatrix} 2A & \gamma A & 0 \\ \gamma A & (2+\gamma^2)A & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{\mathbf{E}}_i \otimes \underline{\mathbf{E}}_j,$$

$$\underline{\mathbf{V}} = \begin{bmatrix} (2+\gamma^2)A & \gamma A & 0 \\ \gamma A & 2A & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j. \tag{2.3}$$

The spectral decomposition theorem ensures existence of the following representations of tensors $\underline{\mathbf{U}}$ and $\underline{\mathbf{V}}$

$$\underline{\mathbf{U}} = \sum \lambda_i \underline{\mathbf{u}}_i \otimes \underline{\mathbf{u}}_i, \qquad \underline{\mathbf{V}} = \sum \lambda_\alpha \underline{\mathbf{v}}_\alpha \otimes \underline{\mathbf{v}}_\alpha. \tag{2.4}$$

The scalars $\lambda_i = \lambda_\alpha > 0$ called principal stretches, eigenvalues of the characteristic equation $\det(\underline{\mathbf{F}}^T \underline{\mathbf{F}} - \lambda^2 \underline{\mathbf{I}}) = 0$, have the same values for tensors $\underline{\mathbf{U}}$ and $\underline{\mathbf{V}}$

$$\lambda_1 = (\gamma + \sqrt{4+\gamma^2})/2, \qquad \lambda_2 = 1/\lambda_1 = (-\gamma + \sqrt{4+\gamma^2})/2, \qquad \lambda_3 = 1,$$
$$\gamma = \lambda_1 - \lambda_2 = \lambda - \lambda^{-1} \geq 0. \tag{2.5}$$

Corresponding to them eigenvectors $\underline{\mathbf{u}}_i$ and $\underline{\mathbf{v}}_\alpha$ are called material and spatial principal directions of stretch tensors $\underline{\mathbf{U}}$ and $\underline{\mathbf{V}}$ ($\underline{\mathbf{v}}_I = \underline{\mathbf{R}}\,\underline{\mathbf{u}}_I$, $I = 1,2,3$). During simple shear deformation they constantly rotate, and their instantaneous location is described by the following formula

$$\tan(2\theta_L) = -2/\gamma, \qquad \pi/4 \leq \theta_L \leq \pi/2;$$
$$\tan(2\theta_E) = 2/\gamma, \qquad 0 \leq \theta_E \leq \pi/4, \tag{2.6}$$

where $\theta_L$ denotes instantaneous angle between $\underline{\mathbf{u}}_i$ and $\underline{\mathbf{E}}_i$, while $\theta_E$ denotes instantaneous angle between $\underline{\mathbf{v}}_\alpha$ and $\underline{\mathbf{e}}_\alpha$, $\theta_R$ denotes instantaneous angle between the respective Lagrangian and Eulerian principal axes (see Fig. 3).

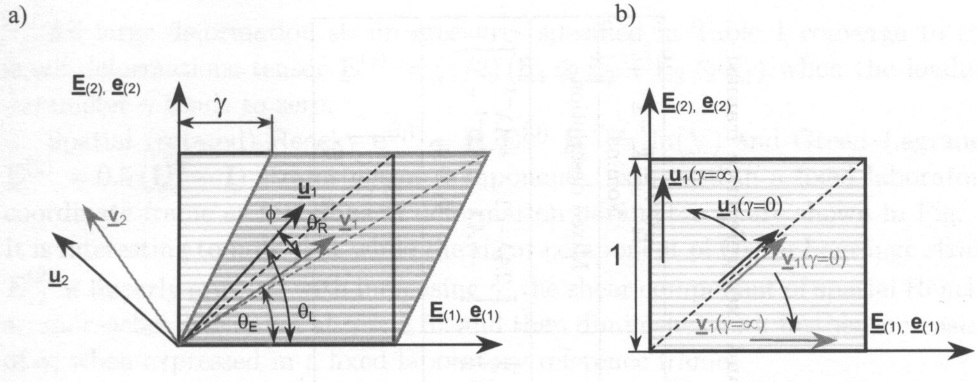



FIG. 3. Homogeneous simple shear. Location of Lagrangian $\underline{\mathbf{u}}_1$ and Eulerian $\underline{\mathbf{v}}_1$ eigenvectors with increasing deformation parameter $\gamma$. On the right, initial ($\gamma = 0$) and final ($\gamma = \infty$) positions of vectors $\underline{\mathbf{u}}_i$, $\underline{\mathbf{v}}_\alpha$ are shown.

In Fig. 3b, the initial ($\gamma = 0$) and final position ($\gamma = \infty$) of material $\underline{u}_1$ and spatial $\underline{v}_1$ principal axes is shown. We have also indicated the direction of their motion with the growth of deformation ($\gamma\nearrow$). Simple shear deformation is an isochoric deformation as $J = \det(\underline{\mathbf{F}}) = dv/dV = \rho_0/\rho = 1$.

## 2.1. *Strain measures at large deformations*

Hill introduced a family of material strain measures defined on the reference configuration by the formula $(2.7)_1$

$$
\begin{aligned}
&\underline{\mathbf{E}}^{(m)} = (\underline{\mathbf{U}}^m - \underline{\mathbf{I}})/m, \qquad m \neq 0, \qquad \underline{\mathbf{E}}^{(0)} = \ln(\underline{\mathbf{U}}), \qquad m = 0,\\
&\underline{\mathbf{e}}^{(m)} = \underline{\mathbf{R}}\,\underline{\mathbf{E}}^{(m)}\,\underline{\mathbf{R}}^T = (\underline{\mathbf{V}}^m - \underline{\mathbf{I}})/m.
\end{aligned} \tag{2.7}
$$

The Green–Lagrange $\underline{\mathbf{E}}^{(2)} = 0.5\,(\underline{\mathbf{C}} - \underline{\mathbf{I}})$, Biot $\underline{\mathbf{E}}^{(1)} = (\underline{\mathbf{U}} - \underline{\mathbf{I}})$, Hencky $\underline{\mathbf{E}}^{(0)} = \ln(\underline{\mathbf{U}})$ strains are all the material strain measure tensors. Corresponding to them spatial strain measures defined on actual configuration (generalized Hill measures) are based on tensor $\underline{\mathbf{V}}$ (see $(2.7)_2$). It is convenient for purposes of the present paper to introduce "nominal" strain tensor $\underline{\mathbf{E}}^{(N)} = (\underline{\mathbf{F}} - \underline{\mathbf{I}})$ work-conjugate in the sense of Hill (see $(2.8)_5$) with the tensor of nominal stress $\underline{\sigma}^{(N)}$ in order to be able to characterize the material energetically. In a general case, the tensor $\underline{\mathbf{E}}^{(N)}$ does not satisfy the requirements posed on the strain tensor appropriate for describing large deformations namely invariance with respect to rigid rotation. Similarly, the small strains tensor $\underline{\mathbf{E}}^{(s)} \equiv 0.5(\underline{\mathbf{F}} + \underline{\mathbf{F}}^T - 2\,\underline{\mathbf{I}})$ is also not invariant with respect to body rotation. Formula for various strain measures components expressed in fixed and principal axes (rotating) reference frames valid for a simple shear test are listed in Table 1.

**Table 1.** **Various strain measures components expressed in principal axes (rotating) and fixed reference frames – simple shear deformation.**

| Strain measure | $\underline{\mathbf{e}}^{(0)} = \ln(\underline{\mathbf{V}})$ Rotated Hencky | $\underline{\mathbf{E}}^{(1)} = (\underline{\mathbf{U}} - \underline{\mathbf{I}})$ Biot | $\underline{\mathbf{E}}^{(2)} = 0.5\,(\underline{\mathbf{U}}^2 - \underline{\mathbf{I}})$ Green Lagrange | $\underline{\mathbf{E}}^{(N)} = \underline{\mathbf{F}} - \underline{\mathbf{I}}$ Nominal |
|---|---|---|---|---|
| Type of description | Spatial description (Euler) | Material description (Lagrangian) | | Mixed description |
| Principal axes coordinate frame | $\begin{bmatrix} \ln(\lambda) & 0 \\ 0 & \ln(\lambda^{-1}) \end{bmatrix}$ $\underline{\mathbf{v}}_i \otimes \underline{\mathbf{v}}_j$ | $\begin{bmatrix} (\lambda - 1) & 0 \\ 0 & (\lambda^{-1} - 1) \end{bmatrix}$ $\underline{\mathbf{u}}_i \otimes \underline{\mathbf{u}}_j$ | $\frac{1}{2}\begin{bmatrix} (\lambda^2 - 1) & 0 \\ 0 & (\lambda^{-2} - 1) \end{bmatrix}$ $\underline{\mathbf{u}}_i \otimes \underline{\mathbf{u}}_j$ | $\begin{bmatrix} 2A\lambda - 1 & \gamma A\lambda^{-1} \\ -\gamma A\lambda & 2A\lambda^{-1} - 1 \end{bmatrix}$ $\underline{\mathbf{u}}_i \otimes \underline{\mathbf{u}}_j$ |
| Fixed axes coordinate frame | $\frac{\ln(\lambda)}{\sqrt{4+\gamma^2}}\begin{bmatrix} \gamma & 2 \\ 2 & -\gamma \end{bmatrix}$ $\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j$ | $\begin{bmatrix} 2A - 1 & \gamma A \\ \gamma A & (2+\gamma^2)A - 1 \end{bmatrix}$ $\underline{\mathbf{E}}_i \otimes \underline{\mathbf{E}}_j$ | $\frac{1}{2}\begin{bmatrix} 0 & \gamma \\ \gamma & \gamma^2 \end{bmatrix}$ $\underline{\mathbf{E}}_i \otimes \underline{\mathbf{E}}_j$ | $\begin{bmatrix} 0 & \gamma \\ 0 & 0 \end{bmatrix}$ $\underline{\mathbf{e}}_i \otimes \underline{\mathbf{E}}_j$ |

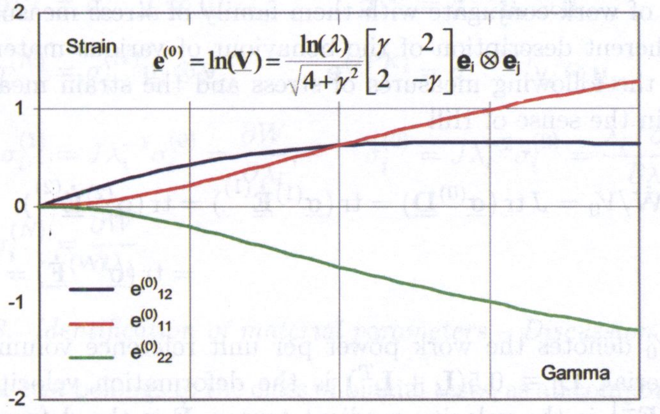
All components containing index three (3) are in in the case of SST identically equal to zero.

$A = (4 + \gamma^2)^{-1/2} = (\lambda + \lambda^{-1})^{-1}$.

All large deformation strain measures specified in Table 1 converge to the small deformations tensor $\underline{\mathbf{E}}^{(s)} = (\gamma/2)\,(\underline{\mathbf{E}}_1 \otimes \underline{\mathbf{E}}_2 + \underline{\mathbf{E}}_2 \otimes \underline{\mathbf{E}}_1)$ when the loading parameter $\gamma$ tends to zero.

Spatial (rotated) Hencky $\underline{\mathbf{e}}^{(0)} = \underline{\mathbf{R}}\ \underline{\mathbf{E}}^{(0)}\ \underline{\mathbf{R}}^T = \ln(\underline{\mathbf{V}})$ and Green–Lagrange $\underline{\mathbf{E}}^{(2)} = 0.5\,(\underline{\mathbf{U}}^2 - \underline{\mathbf{I}})$ strain tensors components, expressed in a fixed laboratory coordinate frame as functions of deformation parameter $\gamma$, are shown in Fig. 4. It is interesting to note that while the shear component of Green Lagrange strain $E^{(2)}_{12}$ is linearly growing with increasing $\gamma$, the shear component of spatial Hencky strain reaches maximum at $\gamma = 3.02$ and then diminishes with further increasing of $\gamma$, when expressed in a fixed laboratory reference frame.

a)




b)

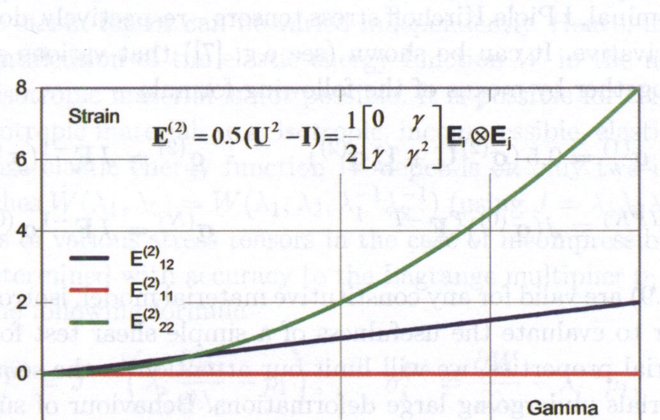



FIG. 4. Spatial (rotated) Hencky $\underline{\mathbf{e}}^{(0)} = \underline{\mathbf{R}}\ \underline{\mathbf{E}}^{(0)}\ \underline{\mathbf{R}}^T = \ln(\underline{\mathbf{V}})$ and Green Lagrange $\underline{\mathbf{E}}^{(2)} = 0.5\,(\underline{\mathbf{U}}^2 - \underline{\mathbf{I}})$ strain tensors components, expressed in a fixed laboratory coordinate frame, as functions of the deformation parameter $\gamma$.

We would also like to note that while the "volumetric" part of Hencky strain tensor gives a null value $e_v^{(0)} = 0$ in accordance with the physical situation of incompressibility in the case of simple shear deformation, the "volumetric" part of the Green Lagrange strain is not zero $E_v^{(2)} \neq 0$. Hence, in the case of this last measure the description of "volumetric" and shear effects are coupled. Independent the theoretical description of material behaviour of these two types of effects is difficult by means of the Green Lagrange strain measure.

### 2.2. *Stress measures at large deformations*

Together with the family of strain measures $(2.7)_1$, HILL [5] introduced also the concept of work-conjugate with them family of stress measures, in order to enable a coherent description of the behaviour of various materials. It can be shown that the following measures of stress and the strain measures are work conjugated in the sense of Hill

$$\dot{\mathbf{W}}/V_0 = J\,\mathrm{tr}\,(\underline{\boldsymbol{\sigma}}^{(0)}\underline{\mathbf{D}}) = \mathrm{tr}\,(\underline{\boldsymbol{\sigma}}^{(1)}\underline{\dot{\mathbf{E}}}^{(1)}) = \mathrm{tr}\,(\underline{\boldsymbol{\sigma}}^{(2)}\underline{\dot{\mathbf{E}}}^{(2)}) = \mathrm{tr}\,(\underline{\boldsymbol{\sigma}}^{(N)}\underline{\dot{\mathbf{F}}}) = \mathrm{tr}\,(\underline{\boldsymbol{\sigma}}^{(IPK)}\underline{\dot{\mathbf{F}}}^{T}), \tag{2.8}$$

where $\dot{\mathbf{W}}/V_0$ denotes the work power per unit reference volume of the undeformed material, $\underline{\mathbf{D}} \equiv 0.5(\underline{\mathbf{L}} + \underline{\mathbf{L}}^T)$ is the deformation velocity tensor, $\underline{\mathbf{L}} \equiv \partial\underline{\mathbf{v}}/\partial\underline{\mathbf{x}} = \underline{\dot{\mathbf{F}}}\underline{\mathbf{F}}^{-1}$ is the velocity gradient tensor, $\underline{\dot{\mathbf{F}}}$ is the deformation gradient velocity tensor, $\underline{\boldsymbol{\sigma}}^{(0)}$, $\underline{\boldsymbol{\sigma}}^{(1)}$, $\underline{\boldsymbol{\sigma}}^{(2)}$, $\underline{\boldsymbol{\sigma}}^{(N)}$, $\underline{\boldsymbol{\sigma}}^{(IPK)}$ denote the Cauchy, Biot, II Piola Kirchoff, Nominal, I Piola Kirchoff stress tensors – respectively, dot denotes usual material derivative. It can be shown (see e.g. [7]) that various stress measures are linked together by means of the following formula

$$\begin{aligned} &\underline{\boldsymbol{\sigma}}^{(1)} = 0.5\,(\underline{\boldsymbol{\sigma}}^{(2)}\underline{\mathbf{U}} + \underline{\mathbf{U}}\,\underline{\boldsymbol{\sigma}}^{(2)}), \qquad &&\underline{\boldsymbol{\sigma}}^{(2)} = J\,\underline{\mathbf{F}}^{-1}(\underline{\boldsymbol{\sigma}}^{(0)})^T\,\underline{\mathbf{F}}^{-T}, \\ &\underline{\boldsymbol{\sigma}}^{(IPK)} \equiv J(\underline{\boldsymbol{\sigma}}^{(0)})^T\underline{\mathbf{F}}^{-T}, &&\underline{\boldsymbol{\sigma}}^{(N)} = J\,\underline{\mathbf{F}}^{-1}\underline{\boldsymbol{\sigma}}^{(0)}. \end{aligned} \tag{2.9}$$

Formulae (2.9) are valid for any constitutive material model, isotropic or anisotropic. In order to evaluate the usefulness of a simple shear test for identification of the material properties, we will limit our attention in the sequel to isotropic elastic materials undergoing large deformations. Behaviour of such materials is completely known if we know the form of elastic energy function $W = W(\lambda_i)$, which must be an isotropic function of principal stretches $\lambda_i$. It can be shown (see [7]) that then the Cauchy stress tensor $\underline{\boldsymbol{\sigma}}^{(0)}$ must be collinear with tensor $\underline{\mathbf{V}}$ and takes the form $(2.10)_1$. Comparing the increments of elastic energy once expressed by elastic energy function $W$, and next by the work of external forces

$(2.10)_2$, the following formula can be obtained for principal values of the Cauchy stress tensor expressed in terms of derivatives of function $W$

$$(2.10)\quad \left\{ \underline{\sigma}^{(0)} = \frac{\partial W}{\partial \underline{e}^{(0)}} = \sigma_i^{(0)} \underline{v}_i \otimes \underline{v}_i; \qquad i = 1,2,3, \right.$$

$$dW = \left(\frac{\partial W}{\partial \lambda_i}\right) d\lambda_i = J\,\mathrm{tr}(\underline{\sigma}^{(0)}\underline{\mathbf{D}}) \Big\} \Leftrightarrow \Big\{ \sigma_i^{(0)} = J^{-1}\lambda_i \frac{\partial W}{\partial \lambda_i} \Big\}.$$

Corresponding formula for other stress measures are obtained with the aid of relations (2.9)

$$\underline{\sigma}^{(1)} = \sigma_i^{(1)} \underline{u}_i \otimes \underline{u}_i, \qquad \underline{\sigma}^{(2)} = \sigma_i^{(2)} \underline{u}_i \otimes \underline{u}_i,$$

$$\underline{\sigma}^{(N)} = \sigma_i^{(N)} \underline{u}_i \otimes \underline{v}_j, \qquad \underline{\sigma}^{(IPK)} = \sigma_i^{(IPK)} \underline{v}_i \otimes \underline{u}_j,$$

$$(2.11)\quad \sigma_i^{(1)} = J\lambda_i^{-1}\sigma_i^{(0)} = \frac{\partial W}{\partial \lambda_i}, \qquad \sigma_i^{(2)} = J\lambda_i^{-2}\sigma_i^{(0)} = \frac{\lambda_i^{-1}\partial W}{\partial \lambda_i},$$

$$\sigma_i^{(N)} = \frac{\partial W}{\partial \lambda_i}.$$

### *2.3. Identification of material parameters – Discussion*

Simple shear test belongs to the class of biaxial tests, as all components $\sigma_{3j}^{(*)} \equiv 0$, $j = 1,2,3$ are identically equal to zero (plane state of stress). In this case only two principal values of stress tensor remain non-zero and only two principal stretches of the strain tensor can be varied independently. Hence, in SST proof a complete identification of the elastic energy function $W$ in the most general case of elastic isotropic material is not possible. It is possible for certain special subclasses of isotropic materials, e.g. isotropic, incompressible, elastic materials, since in this case elastic energy function $W$ depends on only two independent principal stretches $W(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_1^{-1}\lambda_2^{-1})$ (using $J = \lambda_1\lambda_2\lambda_3 = 1$). The principal values of various stress tensors in the case of incompressible, isotropic material are determined with accuracy to the Lagrange multiplier $p_1$ and can be expressed by the following formula

$$(2.12)\quad \sigma_i^{(0)} = J^{-1}\left(\lambda_i \frac{\partial W}{\partial \lambda_i} - p_1\right), \qquad \sigma_i^{(1)} = \frac{\partial W}{\partial \lambda_i} - \lambda_i^{-1} p_1,$$

$$\sigma_i^{(2)} = \lambda_i^{-1}\frac{\partial W}{\partial \lambda_i} - \lambda_i^{-2} p_1, \qquad \sigma_i^{(N)} = \frac{\partial W}{\partial \lambda_i} - \lambda_i^{-1} p_1.$$

It results from the fact that application of any hydrostatic pressure $p$ does not influence the strain state of the incompressible material.

It is worth to note here that the SST proof can be also useful for investigation of the behaviour of compressible materials ($J \neq 1$) when the volumetric behaviour of the material can be separated from the shear behaviour (such a situation takes place for metals in a broad range of pressures). Then SST can serve for determination of "shear part" of the elastic energy function $W$ in full analogy with the case of incompressible materials, while the "volumetric part" must be identified with the aid of additional volumetric test.

The Cauchy stress tensor $\underline{\boldsymbol{\sigma}}^{(0)}$ has a simple physical interpretation of its components $\sigma_{ij}^{(0)} \equiv df_{ij}^t/da_i$, where $df_{ij}^t$ denotes the force component in direction $\underline{\mathbf{e}}_j$ operating on the cross-section area with direction vector $\underline{\mathbf{e}}_i$ (see also Fig. 2). The strain measure work conjugated with the Cauchy stress is the spatial Hencky strain tensor $\underline{\mathbf{e}}^{(0)}$. This strain measure in a "natural" way uncouples description of the volumetric and shear deformation effects, what suggests that using this pair for theoretical description of incompressible materials behavior is particularly advantageous. This can be easily illustrated by analysis of components of the strain measure tensors $\underline{\mathbf{e}}^{(0)}$ and $\underline{\mathbf{E}}^{(2)}$ in the case of investigated here simple shear deformation. This deformation is isochoric, as $\det(\underline{\mathbf{F}}) = 1$. In accordance with physical conditions, the Hencky strain tensor $\underline{\mathbf{e}}^{(0)}$ immediately reveals no volumetric changes $\operatorname{tr}(\underline{\mathbf{e}}^{(0)}) = 0$, but trace of the Green Lagrange strain tensor is non-zero $\operatorname{tr}(\underline{\mathbf{E}}^{(2)}) \neq 0$. This suggests that in the case of this later strain measure description of the shear and volumetric effects is mutually coupled (cf. Table 1). Direct experimental determination of components of the Cauchy stress tensor is in experimental practice difficult and thus expensive, as it requires measurement of actual cross-section areas $da_i$. If the deformation gradient $\underline{\mathbf{F}}$ is known with good approximation, it is more convenient and economically justified to measure experimentally the variation of forces $df_{ij}^t(\gamma)$ and measuring only the initial cross-sections $dA_i = \text{const}$, on which these forces operate. Such experimental data allow to determine the components of nominal stress tensor $\sigma_{ji}^{(N)} = \sigma_{ij}^{(IPK)} \equiv df_{ij}^t/dA_i$. Inverted formula $(2.9)_4$ enables the determination of the components of Cauchy stress tensor from the components of nominal stress tensor

$$
\begin{aligned}
&\sigma_{22}^{(0)} = \sigma_{22}^{(N)}, \qquad \sigma_{12}^{(0)} = \sigma_{21}^{(0)} = \sigma_{12}^{(N)} + \gamma\,\sigma_{22}^{(N)},\\
&\sigma_{11}^{(0)} = \sigma_{22}^{(0)} + \gamma\,\sigma_{12}^{(0)}, \qquad \sigma_{3i}^{(0)} \equiv 0,\\
&\sigma_{12}^{(N)} = df_{21}^t/dA_2, \qquad \sigma_{22}^{(N)} = df_{22}^t/dA_2.
\end{aligned}
\tag{2.13}
$$

In the above formula, for determination of component $\sigma_{11}^{(0)}$, the so-called universal relation was used, which is valid for each material remaining in the class of isotropic hyperelastic materials (see e.g. [7]). Please note that even for reconstruction of relation of the Cauchy shear stress, spatial Hencky shear strain $\sigma_{12}^{(0)} \leftrightarrow e_{12}^{(0)}$, the relation $\sigma_{12}^{(N)} \leftrightarrow \gamma/2$ is not sufficient, cf. $(2.13)_2$. In order to reconstruct the Cauchy stress tensor components from the nominal stress tensor components, experimental determination of at least two components of the nominal stress tensor namely $\sigma_{12}^{(N)}$ and $\sigma_{22}^{(N)}$ is required.

We are now in a position to deliver the interpretation of $\sigma_{12}^{(N)} \leftrightarrow \gamma$ experimental curve, specified the most often as a result of SST proof within the context of large deformations. Firstly, using the formula $(2.8)_5$, it can be stated that such a relation determines the capacity of elastic energy storage of the investigated material subjected to a SST proof, in the class of hyperelastic materials, regardless of the specific constitutive relation that the material obeys. For any hyperelastic material submitted to a SST proof $\dot{F}_{12} = \dot{\gamma}$ is the only non-zero component of the deformation gradient velocity tensor and $dW/V_0 = \sigma_{12}^{(N)} d\gamma$. Secondly, using the formula $(2.9)_2$ and $(2.9)_5$ ($\underline{\boldsymbol{\sigma}}^{(2)} = \underline{\boldsymbol{\sigma}}^{(N)} \underline{\mathbf{F}}^{-T}$) and special form of the deformation gradient $\underline{\mathbf{F}}$, it can be shown that for any material submitted to a SST proof $\underline{\boldsymbol{\sigma}}_{21}^{(2)} = \underline{\boldsymbol{\sigma}}_{12}^{(N)}$. Hence, the relation $\sigma_{12}^{(N)} \leftrightarrow \gamma/2$ is equivalent to the relation $\boldsymbol{\sigma}_{21}^{(2)} \leftrightarrow E_{21}^{(2)}$ in a large deformations regime (see also Table 2). Hence, this relation characterizes not only energetic but also constitutive behaviour of the material in this sense that it is the characteristic of the corresponding stress and strain components of a work conjugate pair of stress and strain tensors. However, this relation constitutes only a part of information, which can be obtained from the SST proof and not to be easily interpreted, as the second Piola Kirchoff stress tensor does not have any simple physical interpretation. We shall underline again that this information is insufficient to reconstruct from it a convenient connection $\sigma_{12}^{(0)} \leftrightarrow e_{12}^{(0)}$ which has clear physical interpretation. In order to obtain convenient constitutive information about the material behaviour available theoretically from the SST proof, the shearing and normal forces $df_{21}^t(\gamma)$, $df_{22}^t(\gamma)$ must be simultaneously measured in the simple shear test.

We will now illustrate how important it is to determine experimentally two components of nominal stress in the SST proof. The stress response of two "known" – in the sense that their constitutive relations does not have to be identified – incompressible ($J = \lambda_1\lambda_2\lambda_3 = 1$) isotropic elastic materials submitted to simple shear deformation are shown in Table 2.

The first material model with elastic energy function $\hat{W}_1 = 0.5\, G\, [(\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2 - 3]$ is used for description of the behaviour of rubberlike materials (in the sequel we will refer to this model as a "rubberlike" material). The constitutive law of this material model, when expressed in terms of work-conjugate pair $\underline{\boldsymbol{\sigma}}^{(0)} \leftrightarrow \underline{\mathbf{e}}^{(0)}$ (Cauchy stress$\leftrightarrow$spatial Hencky stain), is nonlinear. It is interesting to note that the same law, when expressed in terms of not work-conjugate pair $\underline{\boldsymbol{\sigma}}^{(0)} \leftrightarrow \underline{\mathbf{e}}^{(2)}$ (Cauchy stress$\leftrightarrow$Green Lagrange strain), gives a linear relation $\underline{\boldsymbol{\sigma}}^{(0)} \equiv 2G\underline{\mathbf{e}}^{(2)}$. The second material model with elastic energy function $\hat{W}_2 = G\, [\ln(\lambda_1)^2 + \ln(\lambda_2)^2 + \ln(\lambda_3)^2]$ is often used for description of elastic shear

**Table 2.** Various stress measures components for "rubberlike" and "metallike" material constitutive models of incompressible, hyperelastic, isotropic materials resulting from energy functions $\hat{W}_1$, $\hat{W}_2$ expressed in principal axes (rotating) and fixed reference frames – simple shear deformation.

| Stress measure | $\underline{\sigma}^{(0)} = \partial W/\partial \underline{e}^{(0)}$ Cauchy | $\underline{\sigma}^{(1)} \equiv 0.5(\underline{\sigma}^{(2)}\underline{U}+\underline{U}\,\underline{\sigma}^{(2)})$ Biot | $\underline{\sigma}^{(2)} \equiv J\,\underline{F}^{-1}\underline{\sigma}^{(0)T}\underline{F}^{-T}$ II Piola Kirchoff | $\underline{\sigma}^{(N)} \equiv J\,\underline{F}^{-1}\underline{\sigma}^{(0)}$ Nominal |
|---|---|---|---|---|
| Type of description | Spatial description (Eulerian) | Material description (Lagrangian) | | Mixed description |
| Material model | Rubberlike (NeoHookean) material $\hat{W}_1 = 0.5\,G\,[(\lambda_1)^2+(\lambda_2)^2+(\lambda_3)^2-3]$ $(\underline{\sigma}^{(0)} \equiv 2G\underline{e}^{(2)})$ | | | |
| Principal axes coordinate frame | $G\begin{bmatrix}\lambda^2-1 & 0\\ 0 & \lambda^{-2}-1\end{bmatrix}$ $\underline{\mathbf{v}}_i \otimes \underline{\mathbf{v}}_j$ | $G(\lambda-\lambda^{-1})\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ $\underline{\mathbf{u}}_i \otimes \underline{\mathbf{u}}_j$ | $G\begin{bmatrix}1-\lambda^{-2} & 0\\ 0 & 1-\lambda^2\end{bmatrix}$ $\underline{\mathbf{v}}_i \otimes \underline{\mathbf{u}}_j$ | $G(\lambda-\lambda^{-1})\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ $\underline{\mathbf{u}}_i \otimes \underline{\mathbf{v}}_j$ |
| Fixed axes coordinate frame | $G\begin{bmatrix}\gamma^2 & \gamma\\ \gamma & 0\end{bmatrix}$ $\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j$ | $\dfrac{G}{\sqrt{4+\gamma^2}}\begin{bmatrix}-\gamma^2 & 2\gamma\\ 2\gamma & \gamma^2\end{bmatrix}$ $\underline{\mathbf{E}}_i \otimes \underline{\mathbf{E}}_j$ | $G\begin{bmatrix}-\gamma^2 & \gamma\\ \gamma & 0\end{bmatrix}$ $\underline{\mathbf{E}}_i \otimes \underline{\mathbf{E}}_j$ | $G\begin{bmatrix}0 & \gamma\\ \gamma & 0\end{bmatrix}$ $\underline{\mathbf{E}}_i \otimes \underline{\mathbf{e}}_j$ |
| Material model | Metallike material $\hat{W}_2 = G\,[\ln(\lambda_1)^2+\ln(\lambda_2)^2+\ln(\lambda_3)^2]$, $(\underline{\sigma}^{(0)} \equiv 2G\underline{e}^{(0)})$ | | | |
| Principal axes coordinate frame | $2G\ln(\lambda)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ $\underline{\mathbf{v}}_i \otimes \underline{\mathbf{v}}_j$ | $2G\ln(\lambda)\begin{bmatrix}\lambda^{-1} & 0\\ 0 & -\lambda\end{bmatrix}$ $\underline{\mathbf{u}}_i \otimes \underline{\mathbf{u}}_j$ | $2G\ln(\lambda)\begin{bmatrix}\lambda^{-2} & 0\\ 0 & -\lambda^2\end{bmatrix}$ $\underline{\mathbf{u}}_i \otimes \underline{\mathbf{v}}_j$ | $2G\ln(\lambda)\begin{bmatrix}\lambda^{-1} & 0\\ 0 & -\lambda\end{bmatrix}$ $\underline{u}_i \otimes \underline{v}_j$ |
| Fixed axes coordinate frame | $2G\ln(\lambda)/\sqrt{4+\gamma^2}\cdot$ $\cdot\begin{bmatrix}\gamma & 2\\ 2 & -\gamma\end{bmatrix}\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}_j$ | $2G\ln(\lambda)\begin{bmatrix}-\gamma & 1\\ 1 & 0\end{bmatrix}$ $\underline{\mathbf{E}}_i \otimes \underline{\mathbf{E}}_j$ | $2G\ln(\lambda)/\sqrt{4+\gamma^2}\cdot$ $\cdot\begin{bmatrix}-\gamma(3+\gamma^2) & 2+\gamma^2\\ 2+\gamma^2 & -\gamma\end{bmatrix}$ $\underline{\mathbf{E}}_i \otimes \underline{\mathbf{E}}_j$ | $2G\ln(\lambda)/\sqrt{4+\gamma^2}$ $\cdot\begin{bmatrix}-\gamma & 2+\gamma^2\\ 2 & -\gamma\end{bmatrix}$ $\underline{\mathbf{E}}_i \otimes \underline{\mathbf{e}}_j$ |
| Work conjugate stress, strain pairs | $\underline{\sigma}^{(0)} \leftrightarrow \underline{e}^{(0)}$ | $\underline{\sigma}^{(1)} \leftrightarrow \underline{E}^{(1)}$ | $\underline{\sigma}^{(2)} \leftrightarrow \underline{E}^{(2)}$ | $\underline{\sigma}^{(N)} \leftrightarrow \underline{E}^{(N)} = \underline{F} - \underline{I}$ |

All components $\sigma_{3j}^{(*)} \equiv 0$, $j = 1, 2, 3$.

behaviour of metals. In the sequel we will refer to this model as "metallike" material. The name "metallike" is used because real metallic materials never undergo large elastic shear strains, instead they start to flow plastically. The constitutive law of "metallike" material model, when expressed in terms of the work-conjugate pair $\underline{\boldsymbol{\sigma}}^{(0)} \leftrightarrow \underline{\mathbf{e}}^{(0)}$, is linear $\underline{\boldsymbol{\sigma}}^{(0)} \equiv 2G\underline{\mathbf{e}}^{(0)}$ in Eulerian description. It is nonlinear when expressed in terms of $\underline{\boldsymbol{\sigma}}^{(2)} \leftrightarrow \underline{\mathbf{E}}^{(2)}$ conjugate pair in the Lagrangian description.

Various stress measures components for "rubberlike" and "metallike" material constitutive models of incompressible, hyperelastic, isotropic materials resulting

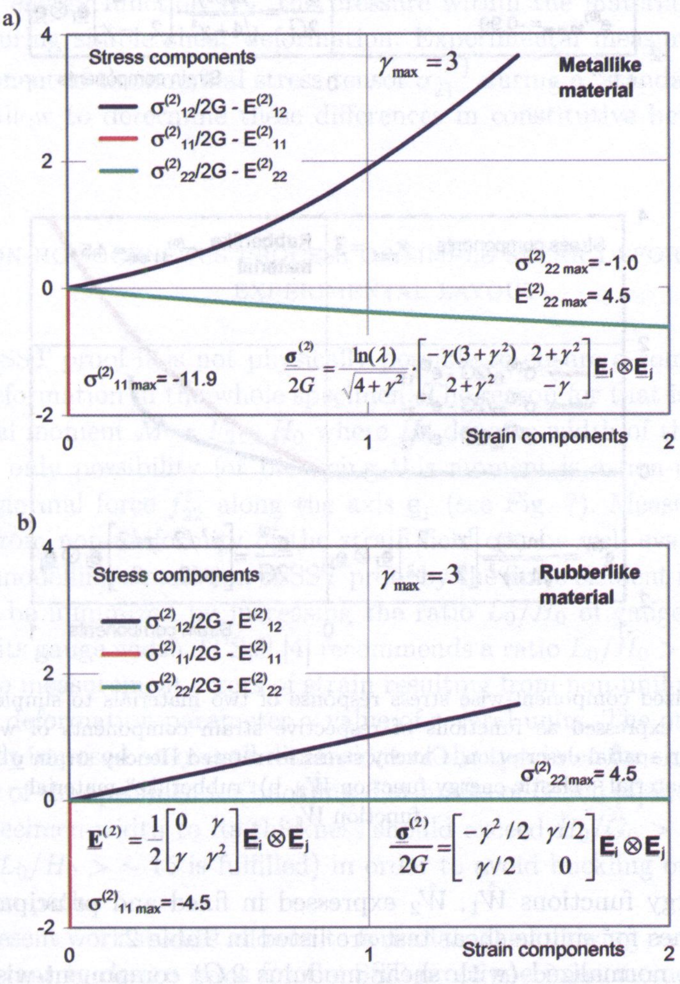



FIG. 5. Normalized component-wise stress response of two materials submitted to simple shear deformation loading expressed as functions of the respective strain components of work-conjugate strain measure in material description, II Piola–Kirchoff stress ↔ Green Lagrange strain $\underline{\boldsymbol{\sigma}}^{(2)}/2G \leftrightarrow \underline{\mathbf{E}}^{(2)}$, a) "metallike" material – elastic energy function $\hat{W}_2$, b) "rubberlike" material – elastic energy function $\hat{W}_1$.

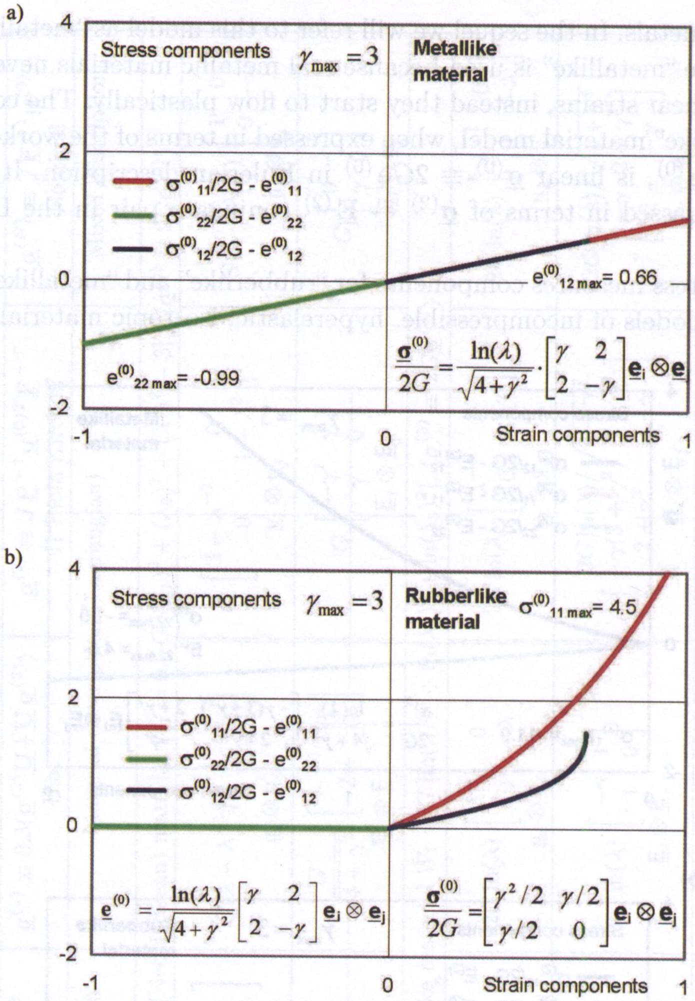



FIG. 6. Normalized component-wise stress response of two materials to simple shear deformation loading expressed as functions of respective strain components of work-conjugate strain measure in spatial description, Cauchy stress ↔ rotated Hencky strain $\underline{\sigma}^{(0)}/2G \leftrightarrow \underline{\mathbf{e}}^{(0)}$, a) "metallike" material – elastic energy function $\hat{W}_2$, b) "rubberlike" material – elastic energy function $\hat{W}_1$.

from the energy functions $\hat{W}_1$, $\hat{W}_2$ expressed in fixed and principal (rotating) reference frames for simple shear test are listed in Table 2.

In Fig. 5a normalized (with shear modulus $2G$) component-wise stress response of two materials to simple shear deformation loading is presented as a set of functions of the respective work-conjugate strain measure in material description (II Piola Kirchoff stress and Green Lagrange strain; $\underline{\sigma}^{(2)}/2G \leftrightarrow \underline{\mathbf{E}}^{(2)}$). In Fig. 5a response is shown of "metallike" material characterized by elastic energy function $\hat{W}_2$ and in Figure 5b response is shown of "rubberlike" material characterized by elastic energy function $\hat{W}_1$. In Fig. 6a and 6b analogous curves are presented in spatial description (Cauchy stress and rotated Hencky strain $\underline{\sigma}^{(0)}/2G \leftrightarrow \underline{\mathbf{e}}^{(0)}$).

In the case of a rubberlike material described by the elastic energy function $\hat{W}_1$, considerable pressure builds up $-p \equiv \mathrm{tr}\,(\underline{\sigma}^{(0)})/3 = (\sigma_1^{(0)} + \sigma_2^{(0)} + 0)/3 = G\gamma^2/3$ during simple shear deformation (see Table 2). No volumetric strain $e_v \equiv \mathrm{tr}\,(\underline{\mathbf{e}}^{(0)})/3 = 0$ accompanies this pressure, as from the definition the material is incompressible (see Table 1). In the case of "metallike" material described by the elastic energy function $\hat{W}_2$, the pressure within the material remains zero ($p = 0$) during simple shear deformation. Experimental measurement of only one component of the nominal stress tensor $\sigma_{21}^{(N)}$ during a "standard" SST proof does not allow to determine these differences in constitutive behaviour of the materials.

## 3. Non-homogeneous process of simple shear deformation – experimental layout

In the SST proof it is not physically possible to ensure a completely homogeneous deformation in the whole specimen. The reason for that is generation of a rotational moment $M = F_{21} \cdot H_0$ where $H_0$ denotes width of the deformation path. The only possibility for balancing this moment is a non-uniform distribution of normal force $f_{22}^t$ along the axis $\underline{\mathbf{e}}_1$ (see Fig. 7). Measurement errors resulting from non-uniformity of the strain field can be well evaluated by performing a modelling simulation of SST proof by the finite element method. These errors can be minimized by increasing the ratio $L_0/H_0$ of gauge length of the sample to its gauge width. G'Sell [4] recommends a ratio $L_0/H_0 > \sim 15$, in order to limit the measurement errors of strain resulting from non-uniformity to a few percent at deformation parameter $\gamma$ value of several units. The other important requirement imposed on overall dimensions of the specimen for simple shear test is stability of the specimen for buckling. Estimates of G'Sell [4] indicate that the ratio of specimen width to its thickness should exceed $H_0/G_0 > \sim 3$ (when the condition $L_0/H_0 > \sim 15$ is fulfilled) in order to avoid buckling of the specimen deformation path.

The present work studies allowed to formulate the following recommendation: it is important to design grips for the SST proof specimen in such a way as to prevent the slip of the specimen not only in the shear direction („12"), but also in the normal direction („12"), since in the case of some materials there may develop considerable normal stress in the course of simple shear deformation (see Table 2, "rubberlike" material).

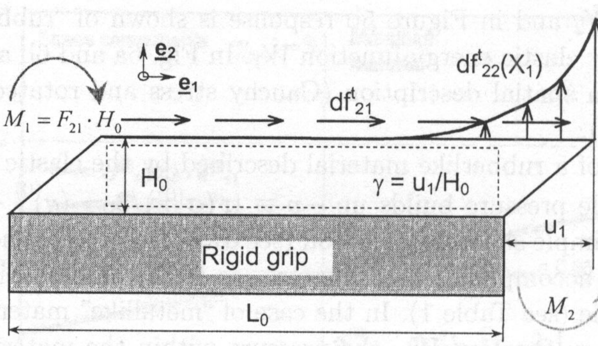



FIG. 7. Schematic drawing of non-homogeneous distribution of normal force $df^t_{22}(X_1)$ in simple shear test ensuring mechanical balance. .

## 4. Concluding remarks

In "standard" at present Simple Shear Test (SST) proof there are measured and recorded shear force as function of deformation parameter gamma $df_{21}(\gamma)$ and initial cross-section on which shear force operates $dA_2$. These data enable determination of relationship $\sigma_{12}^{(N)} \leftrightarrow \gamma$, i.e. variation of component („12") of the nominal stress tensor (non-symmetric) versus the deformation parameter $\gamma$. This relation constitutes at present a standard result of the SST proof. Its interpretation in the large deformations regime brings about certain difficulties as to what information it reveals. The theoretical analysis performed herein indicated that:

• the area under the curve $\sigma_{12}^{(N)} \leftrightarrow \gamma/2$ determines, in the case of hyperelastic materials, the elastic energy stored within the tested material as a result of simple shear deformation. Hence this relation characterizes the material energetically $(dW/V_0 = \sigma_{12}^{(N)}\, d\gamma)$;

• the relation $\sigma_{12}^{(N)} \leftrightarrow \gamma/2$ in the case of SST proof is equivalent to the relation $\sigma_{21}^{(2)} \leftrightarrow E_{21}^{(2)}$ (component („12") of the II Piola Kirchoff stress tensor versus component („12") of the Green Lagrange strain tensor) within large deformations context. This interpretation is valid only, when with a good approximation, homogeneous deformation takes place in the gauge section of the specimen. Hence, the relation $(\sigma_{12}^{(N)} \leftrightarrow \gamma/2) \Leftrightarrow (\sigma_{21}^{(2)} \leftrightarrow E_{21}^{(2)})$ carries also information about the constitutive behaviour of the material but in an inconvenient form (II Piola Kirchoff stress does not have any clear physical interpretation) and only a partial one – possible differences in the behaviour of tested material resulting from volumetric effects can not be evaluated. An attempt to transform the relationship $\sigma_{12}^{(N)} \leftrightarrow \gamma/2$ to the convenient relation $\sigma_{12}^{(0)} \leftrightarrow e_{12}^{(0)}$ – component („12") of the Cauchy stress versus component („12") of the rotated Hencky strain, having a clear physical interpretation is impossible due to lack of knowledge of component $\sigma_{22}^{(N)}$ of the nominal stress tensor, cf. $(2.13)_2$. This component can be evaluated if normal force $df_{22}^t(t)$ would be recorded, cf. $(2.13)_6$, which is not the case in the standard SST proof. Due to that reason, further technological development of the SST proof experimental technique is strongly recommended in such a way as to measure in a standard manner two forces: shear – $df_{21}^t$ and normal – $df_{22}^t$. Only this kind of measurements will allow to present this proof results in a physically clear and convenient form. Namely, the relations between components of the Cauchy stress versus respective components of the logarithmic strain.

The simple shear test allows to determine the material behaviour in response to deformation involving rotation of the principal axes of strain. There arises a natural question: to what other proof results should the SST proof results be compared. It seems that the best suited for that purpose is the pure shear test (PST), since in both of these tests, at the same stage of deformation expressed by the same value of deformation parameter $\gamma$, the (corresponding) principal strains have the same values, and the only difference between these two schemes of deformation is that in the first case principal axes rotate and in the second case they are fixed during the whole deformation process. Comparison of results of these two tests will enable us to evaluate how different evolution of microstructure influences the behaviour of the tested material.

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