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Multicriterial optimization of composite element reinforced by two families of fibres

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Abstract. The paper deals with a composite element in which the matrix is reinforced with two families of parallel continuous fibres inclined to the x_1 axis at the angles ϑ_1 and ϑ_2. The stress and strain states were determined in an element subjected to normal and tangential loads. The problem of two-criteria optimization is considered. Minimum strain energy and minimum cost of composite element were chosen as criteria. The strain energy is determined with respect to the system of principal axes of stress. Three independent variables: the angle directing the first family of fibres, the angle between two families and volume fraction of fibres are selected as the design variables. Examining particular load cases in composites made with epoxy resin reinforced with carbon fibres elements and in high performance fibre reinforced cementitious composite elements, optimum solutions have been determined in the sense of assumed criteria.

Key words: multicriterial optimization, composite element with two non-orthogonal families of fibres, minimum strain energy.

1. Introduction

The optimization of fibre reinforcement was considered since many years for advanced composites. Many of the papers deal with the composite materials reinforced by one or orthogonal families of fibres. The fibre direction was often selected as a design variable and the optimum fibre direction was determined from various mechanical criteria. Many papers deal with the problem of finding the optimal orientation of orthotropic axes for an elastic body in order to maximize or minimize the stiffness of the body (the elastic energy is assumed as a meaningful measure of the global stiffness) Banichuk [1], Pedersen [2–4], Sacchi Landriani and Rovati [5]. One of the earlier results on optimal orientation of material symmetry axes can be found in the work [1], where necessary conditions for optimal distribution of material properties in orthotropic bodies subjected to plane state of stress are given. The same optimality conditions for an orthotropic material was independently obtained by Pedersen [2,3] and Sacchi, Rovati [5]. They showed, that for orthotropic materials the principal strain and stress directions are aligned when the criterion for optimal orientation is satisfied. In most cases it was obtained alignment between principal stress directions, principal strain directions and principal material directions too. However, optimal orientations exist for which the principal axes of material differ from those of the principal strains. Pedersen [2,3] performed a systematic study of the optimal solutions in the plane stress problem for the case of a material that had low shear stiffness and for a material that had high shear stiffness. The more general problem to maximize or minimize stiffness was studied by Pedersen and Bendsoe [6], Pedersen and Cheng [7]. The optimal orientation of fibres was determined by means of the above-mentioned material parameter – shear stiffness, which appears in the expression for the strain energy density, and of another, corresponding parameter which describes the strain energy density in terms of stress.

The description of a composite element presented by Marks [8–10] is a continuous description taking into account the physical properties of the matrix and fibres and taking into account the assumption of strain compatibility between fibres and matrix. A similar model of a fibre composite body was presented by Świtka [11]. The aforementioned paper concerned elastic plates made of fibrous composite and loaded in bending. The expression for the tensor of internal forces in the central plane of the plate has a form similar to the expression for the mean stress in the composite element, [8]. This expression differs in the number of fibre families and in the term for determination of density of r-th family of fibres.

The papers by Marks [9,10] differ basically from those that concern problems of optimization of the orientation of fibres in orthotropic bodies, presented in the papers specified above. The papers [9,10] deal with the optimization of a composite element made of matrix reinforced with two non-orthogonal families of continuous fibres. The minimum strain energy is chosen as the optimization criterion similar to those in other papers on orthotropic materials. The necessary conditions for the minimum of strain energy described in the system of principal stress directions are determined on the basis of the Kuhn-Tucker theorem. From analytical solution of the optimization problem for two fibre families, three solutions have been found, in which a global minimum of the strain energy can be searched for. The solutions are: two family of fibres are aligned and placed along the direction of principal stress corresponding to its greater absolute value, two fibre families placed along the principal directions and two non-orthogonal fibre families satisfying the certain set of equations. The optimal solution depends on the material constants of the matrix and fibres, on the magnitude of the load, and on the volumetric

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content of fibres. In the cases of two kinds of composites of basically different properties of fibres and matrix identified are the ranges of principal stresses ratio at which the global minimum of the strain energy corresponds either to the first, to the second or to the third solution [12].

The present work is a development of the aforementioned papers. Two-criteria optimization of a disc element formed of the matrix reinforced with continuous, non-orthogonal fibres belonging to two families is considered. Minimum strain energy and minimum cost of composite element were assumed as the criteria of optimization. The fibre directions of two families and volume fraction of fibres are selected as the design variables. The necessary conditions of minimum strain energy were determined from Kuhn-Tucker theorem. The strain energy is determined with respect to the system of principal axes of stress.

The obtained solutions are applied to design of elements of building structures. The optimum directions of the fibre families and volume fraction of the fibre reinforcement are determined in the composite of epoxy matrix reinforced with carbon fibres and in the high performance cementitious composite reinforced by steel fibres. These structural elements are subjected to various normal and tangential loads.

2. Basic assumptions and constitutive equations

Let us consider the composite element in the shape of a disc, in which the matrix is reinforced by two families of parallel fibres inclined at angles ϑ_1 and ϑ_2 to x_1 axis, see Fig. 1. Every family is constituted of continuous, thin fibres placed in parallel relatively to each other in the plane parallel to the middle plane of the disc element. The fibres of a given family have a common constant direction and are densely distributed. The composite element is in the plane state of stress which can be described by three stress components of a generalized plane stress state $\sigma_{\alpha\beta}$ $\alpha,\beta = 1,2$. The stress components $\sigma_{\alpha\beta}$ correspond to certain mean values over the thickness of the plate. Following assumptions were taken regarding the materials: the matrix is isotropic, matrix as well as thin fibres are linear elastic and homogeneous, strain compatibility is ensured between the fibres and the matrix. Taking advantage of generalized stress state and the assumptions concerning the materials, physical relationships were obtained defining stress components in the composite in the following form [8]

$$\begin{aligned}\sigma_{\alpha\beta} = {} & \frac{E^{(m)}}{1+\nu}\left(\varepsilon_{\alpha\beta} + \frac{\nu}{1-\nu}\delta_{\alpha\beta}\varepsilon_{\delta\delta}\right)\frac{h_m}{h} \\ & + E^{(s)}\frac{h_a}{h}\varepsilon_{\gamma\delta}a_\gamma a_\delta a_\alpha a_\beta \\ & + E^{(s)}\frac{h_b}{h}\varepsilon_{\gamma\delta}b_\gamma b_\delta b_\alpha b_\beta\end{aligned} \tag{1}$$

where $E^{(m)}$ – Young's modulus of matrix, $E^{(s)}$ – Young's modulus of fibres, ν – Poisson's ratio of the matrix, h_m - thickness of matrix, h_a and h_b - thicknesses of fibre layers having direction vectors **a** and **b** ($h = h_m + h_a + h_b$). Physical relations (1) have the form of relationships describing homogeneous and anisotropic material. Taking into account the form of vectors tangent to the first and the second fibre family, the stress components in the composite can be presented as follows:

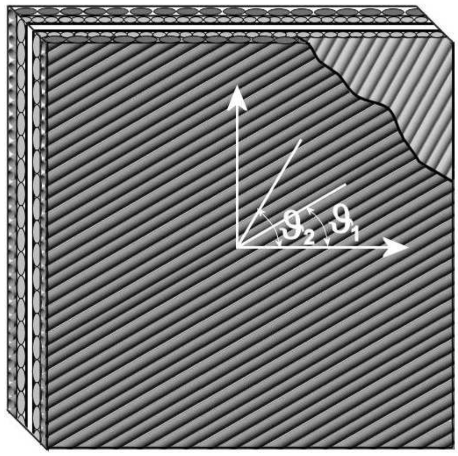

Fig. 1. Composite element reinforced by two families of fibres

$$\begin{aligned}\sigma_{11} &= C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\left(2\varepsilon_{12}\right), \\ \sigma_{22} &= C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22} + C_{23}\left(2\varepsilon_{12}\right), \\ \sigma_{12} &= C_{13}\varepsilon_{11} + C_{22}\varepsilon_{23} + C_{33}\left(2\varepsilon_{12}\right),\end{aligned} \tag{2}$$

where:

$$\begin{aligned}C_{11} = {} & \frac{E^{(m)}}{(1+\nu)(1-\nu)}\frac{h_m}{h} \\ & + E^{(s)}\left(\frac{h_a}{h}\cos^4\vartheta_1 + \frac{h_b}{h}\cos^4\vartheta_2\right),\end{aligned}$$

$$\begin{aligned}C_{12} = {} & \frac{\nu E^{(m)}}{(1+\nu)(1-\nu)}\frac{h_m}{h} \\ & + E^{(s)}\left(\frac{h_a}{h}\sin^2\vartheta_1\cos^2\vartheta_1 + \frac{h_b}{h}\sin^2\vartheta_2\cos^2\vartheta_2\right),\end{aligned}$$

$$C_{13} = E^{(s)}\left(\frac{h_a}{h}\sin\vartheta_1\cos^3\vartheta_1 + \frac{h_b}{h}\sin\vartheta_2\cos^3\vartheta_2\right),$$

$$\begin{aligned}C_{22} = {} & \frac{E^{(m)}}{(1+\nu)(1-\nu)}\frac{h_m}{h} \\ & + E^{(s)}\left(\frac{h_a}{h}\sin^4\vartheta_1 + \frac{h_b}{h}\sin^4\vartheta_2\right),\end{aligned}$$

$$C_{23} = E^{(s)}\left(\frac{h_a}{h}\sin^3\vartheta_1\cos\vartheta_1 + \frac{h_b}{h}\sin^3\vartheta_2\cos\vartheta_2\right),$$

$$\begin{aligned}C_{33} = {} & \frac{E^{(m)}}{2(1+\nu)}\frac{h_m}{h} \\ & + E^{(s)}\left(\frac{h_a}{h}\sin^2\vartheta_1\cos^2\vartheta_1 + \frac{h_b}{h}\sin^2\vartheta_2\cos^2\vartheta_2\right).\end{aligned}$$

From Eq. (2) the strain components are determined; they take the following form:

$$\begin{aligned}\varepsilon_{11} &= S_{11}\sigma_{11} + S_{12}\sigma_{22} + S_{13}\sigma_{12}, \\ \varepsilon_{22} &= S_{12}\sigma_{11} + S_{22}\sigma_{22} + S_{23}\sigma_{12}, \\ 2\varepsilon_{12} &= S_{13}\sigma_{11} + S_{23}\sigma_{22} + S_{33}\sigma_{12}.\end{aligned}$$

Assuming equal distribution of fibres in two directions, that is $\frac{h_a}{h} = \frac{h_b}{h} = \frac{1}{2}\eta$, where $\eta = \frac{h-h_m}{h}$ defines the volume fraction of fibres; denoting in addition the angle between fibre families by α, the components of matrix S_{ij} are given in the following form:

$$S_{11} = \frac{1}{D}\left\{\frac{\left(E^{(m)}\right)^2}{(1+\nu)^2(1-\nu)}(1-\eta)^2 + \frac{E^{(m)}E^{(s)}}{4(1+\nu)(1-\nu)} \right.$$
$$\times\ (1-\eta)\eta + \frac{3E^{(m)}E^{(s)}}{8(1+\nu)}(1-\eta)\eta$$
$$- \frac{E^{(m)}E^{(s)}}{8(1-\nu)}(1-\eta)\eta\ \cos 2(2\vartheta_1+\alpha)\cos 2\alpha$$
$$- \frac{E^{(m)}E^{(s)}}{2(1+\nu)}(1-\eta)\eta\ \cos(2\vartheta_1+\alpha)\cos\alpha$$
$$\left. + \frac{1}{8}\left(E^{(s)}\right)^2(\eta)^2\left[\cos\alpha - \cos(2\vartheta_1+\alpha)\right]^2\sin^2\alpha\right\},$$

$$S_{12} = -\frac{1}{D}\left\{\frac{\nu\left(E^{(m)}\right)^2}{(1+\nu)^2(1-\nu)}(1-\eta)^2 + \frac{E^{(m)}E^{(s)}}{8(1-\nu)}\right.$$
$$\times(1-\eta)\eta\left[1-\cos 2(2\vartheta_1+\alpha)\cos 2\alpha\right]$$
$$\left. - \frac{\left(E^{(s)}\right)^2}{16}\eta^2\left[\cos 2\alpha - \cos 2(2\vartheta_1+\alpha)\right]\sin^2\alpha\right\},$$

$$S_{13} = \frac{1}{D}\left\{-\frac{E^{(m)}E^{(s)}}{2(1+\nu)}(1-\eta)\eta\sin(2\vartheta_1+\alpha)\cos\alpha\right.$$
$$- \frac{E^{(m)}E^{(s)}}{4(1-\nu)}(1-\eta)\eta\sin 2(2\vartheta_1+\alpha)\cos 2\alpha$$
$$- \frac{\left(E^{(s)}\right)^2}{4}\eta^2\left[\cos\alpha - \cos(2\vartheta_1+\alpha)\right]$$
$$\left.\times\sin(2\vartheta_1+\alpha)\sin^2\alpha\right\},$$

$$S_{22} = \frac{1}{D}\left\{\frac{\left(E^{(m)}\right)^2}{(1+\nu)^2(1-\nu)}(1-\eta)^2 + \frac{E^{(m)}E^{(s)}}{4(1+\nu)(1-\nu)}\right.$$
$$\times(1-\eta)\eta + \frac{3E^{(m)}E^{(s)}}{8(1+\nu)}(1-\eta)\eta$$
$$- \frac{E^{(m)}E^{(s)}}{8(1-\nu)}(1-\eta)\eta\cos 2(2\vartheta_1+\alpha)\cos 2\alpha$$
$$+ \frac{E^{(m)}E^{(s)}}{2(1+\nu)}(1-\eta)\eta\cos(2\vartheta_1+\alpha)\cos\alpha$$
$$\left. + \frac{1}{8}\left(E^{(s)}\right)^2(\eta)^2\left[\cos\alpha + \cos(2\vartheta_1+\alpha)\right]^2\sin^2\alpha\right\}, \tag{3}$$

$$S_{23} = -\frac{1}{D}\left\{\frac{E^{(m)}E^{(s)}}{2(1+\nu)}(1-\eta)\eta\sin(2\vartheta_1+\alpha)\cos\alpha\right.$$
$$- \frac{E^{(m)}E^{(s)}}{4(1-\nu)}(1-\eta)\eta\sin 2(2\vartheta_1+\alpha)\cos 2\alpha$$
$$+ \frac{\left(E^{(s)}\right)^2}{4}\eta^2\left[\cos\alpha + \cos(2\vartheta_1+\alpha)\right]$$
$$\left.\times\sin(2\vartheta_1+\alpha)\sin^2\alpha\right\},$$

$$S_{33} = \frac{2}{D}\left\{\frac{\left(E^{(m)}\right)^2}{(1+\nu)(1-\nu)}(1-\eta)^2 + \frac{E^{(m)}E^{(s)}}{(1+\nu)(1-\nu)}\right.$$
$$\times(1-\eta)\eta - \frac{E^{(m)}E^{(s)}}{4(1-\nu)}(1-\eta)\eta$$
$$\times\left[1-\cos 2(2\vartheta_1+\alpha)\cos 2\alpha\right] + \frac{1}{4}\left(E^{(s)}\right)^{(2)}$$
$$\left.\times\ \eta^2\sin^2(2\vartheta_1+\alpha)\sin^2\alpha\right\}$$

and

$$D = \frac{E^{(m)}(1-\eta)}{(1+\nu)(1-\nu)}$$
$$\times\left[\frac{\left(E^{(m)}\right)^2(1-\eta)^2}{(1+\nu)} + \frac{E^{(m)}E^{(s)}\eta(1-\eta)}{(1+\nu)}\right.$$
$$\left. + \frac{\left(E^{(s)}\right)^2\eta^2}{4}\left(1-\cos^4\alpha - \nu\sin^4\alpha\right)\right].$$

The strain energy of composite element may be expressed by the relation from [8]

$$U = \frac{1}{2}\iiint_V \sigma_{ij}\varepsilon_{ij}dV = h\iint_\Omega W d\Omega, \tag{4}$$

where the integrand is

$$W = \frac{1}{2}\sigma_{\alpha\beta}\varepsilon_{\alpha\beta} = \frac{1}{2}\left\{S_{11}(\sigma_{11})^2 + 2S_{12}\sigma_{11}\sigma_{22} + S_{22}(\sigma_{22})^2\right.$$
$$\left. + 2S_{13}\sigma_{11}\sigma_{12} + 2S_{23}\sigma_{22}\sigma_{12} + S_{33}(\sigma_{12})^2\right\}.$$

In order to simplify the expression of function W, a new coordinate system $(y_1,\ y_2)$ is introduced, inclined at angle β to coordinate (x_1, x_2), which is coincident with principal axes of stress and denote the stress tensor components by σ_I, σ_{II} ($\sigma_I = \sigma_{y_1y_1}$, $\sigma_{II} = \sigma_{y_2y_2}$, $\sigma_{y_1y_2} = 0$). The both coordinate system are Cartesian and orthogonal. In the new coordinate system the strain energy W is:

$$W = \frac{1}{2}\left(S_{11}\sigma_I^2 + 2S_{12}\sigma_I\sigma_{II} + S_{22}\sigma_{II}^2\right),$$

where $S_{\gamma\delta}(\gamma,\beta = 1,2)$ are functions defined by Eqs. (3) depending on variables $(\vartheta_1+\beta), \alpha$. After substituting expressions S_{11}, S_{12}, S_{22} the strain energy in the system of principal axes of stress is expressed in the following form [6]:

$$W = \frac{1}{2D}\left\{-(\sigma_{II}-\sigma_I)^2\left[\left[\frac{E^{(m)}E^{(s)}}{8(1-\nu)}\eta(1-\eta)\cos 2\alpha\right.\right.\right.$$
$$\left. - \frac{\left(E^{(s)}\right)^2}{16}\eta^2\sin^2\alpha\right]\cos 2\left[(2\vartheta_1+\alpha)-2\beta\right]$$
$$\left. - \frac{\left(E^{(s)}\right)^2}{16}\eta^2\sin^2\alpha\right] + (\sigma_{II}+\sigma_I)^2\frac{\left(E^{(s)}\right)^2}{8}\eta^2\sin^2\alpha\cos^2\alpha$$
$$+ \left(\sigma_{II}^2 - \sigma_I^2\right)\left[\frac{E^{(m)}E^{(s)}}{2(1+\nu)}\eta(1-\eta) + \frac{\left(E^{(s)}\right)^2}{4}\eta^2\sin^2\alpha\right]$$
$$\times\cos\left[(2\vartheta_1+\alpha)-2\beta\right]\cos\alpha$$

$$+ \left(\sigma_I^2 - 2\nu\sigma_I\sigma_{II} + \sigma_{II}^2\right) \frac{\left(E^{(m)}\right)^2 (1-\eta)^2}{(1+\nu)^2 (1-\nu)}$$
$$+ \left(\sigma_I^2 + \sigma_{II}^2\right) \frac{3E^{(m)}E^{(s)}\eta (1-\eta)}{8(1+\nu)} \tag{5}$$
$$+ \left(\sigma_I^2 - (1+\nu)\sigma_I\sigma_{II} + \sigma_{II}^2\right) \frac{E^{(m)}E^{(s)}\eta (1-\eta)}{4(1+\nu)(1-\nu)} \Bigg\} .$$

3. Basic notions of multicriteria optimisation

The search of the composite materials with better and better properties, particularly the materials used in building structures, are a subject of many investigations, both theoretical and experimental. For such materials various requirements are formulated, often contradictory. In such situation the multicriteria optimization can help out in finding of a material with the required properties.

The basic notions in formulating a multicriteria optimization problem are: decision variables, constrains and optimization criteria, also known as objective functions.

The decision variables are usually expressed in the form of a vector $\boldsymbol{x}^T = (x_1, x_2, ..., x_n)$ in an n-dimensional space called the decision space. Every point in the space corresponds to a composite with n decision variables.

In optimization of materials, unconstrained extrema of the objective function are seldom looked for. A great number of constrains are usually imposed, defining the feasible region $\boldsymbol{\Omega}$. The feasible region $\boldsymbol{\Omega}$ is usually only a part of the n-dimensional space of the decision variables given by $\boldsymbol{\Omega} = \{\boldsymbol{x}\varepsilon R^n\colon\ \boldsymbol{h}(\boldsymbol{x}) = 0,\ \boldsymbol{g}(\boldsymbol{x}) \leq 0\}$.

The multicriteria optimization consists in the choice of the best solution from many possible variants on the basis of many criteria, i.e. on the selection of a vector $\boldsymbol{f}^T = (f_1, f_2, ..., f_k)$ corresponding to an objective function. A multicriteria optimization problem can be therefore treated as an optimization problem of an objective function vector. The objective functions space is k-dimensional. An objective region $\boldsymbol{f}(\boldsymbol{\Omega})$ is a part of the objective space.

The solution $\boldsymbol{x}^{id}$ which makes every objective function reach its extremum independently, is called the ideal solution of multicriteria optimization. In the case of the search for the minimum $\boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{x}^{id}$ is therefore the ideal solution of multicriteria problem if $\boldsymbol{x}^{id} \in \boldsymbol{\Omega}$ and $\boldsymbol{f}(\boldsymbol{x}^{id}) \leq \boldsymbol{f}(\boldsymbol{x})$ for every $\boldsymbol{x} \in \boldsymbol{\Omega}$. As the objective functions are usually in conflict, the ideal solution does not exist in most cases. This means that all criteria can simultaneously obtain their minimum values. Such criteria are referred to as cooperating criteria and the related solution is called the ideal solution.

The solution, in which none of the objective functions can be improved without simultaneous deterioration of at least one of the remaining objective functions, is called the non-dominated solution. Vector $\boldsymbol{x}^*$ is a non-dominated solution when no such $\boldsymbol{x} \in \boldsymbol{\Omega}$ exists that $f_j(\boldsymbol{x}) \leq f_j(\boldsymbol{x}^*)$ at $j \in J = 1, 2, ..., k$ and $f_j(\boldsymbol{x}) < f_j(\boldsymbol{x}^*)$ for at least one $j \in J$. The search for non-dominated solutions is called optimization in the Pareto-sense. The Pareto solution in general is not unique. Many $\boldsymbol{x}^*$ vectors usually exist in the $\boldsymbol{\Omega}$ space, to which corresponds the vector $\boldsymbol{f}^* = \boldsymbol{f}(\boldsymbol{x}^*)$ constituting the set of compromises.

In view of the great number of non-dominated solutions, it is necessary to select the best one on the basis of an additional criterion. Such a solution is called the preferred solution. The preferred solution $\boldsymbol{x}^{pr}$ is a non-dominated selected on the basis of an additional criterion. It corresponds to the values $\boldsymbol{f}(\boldsymbol{x}^{pr})$ contained within the objective region and is considered to be the best solution.

A solution of a multicriteria optimization problem includes, therefore, objective quantities, to which belong:

- the set of compromises,
- the ideal point,

and the quantities which depend on additional preferences:

- preferred solution, that is the vector of objective functions $\boldsymbol{f}^{pr}$ and the corresponding vector of decision variables $\boldsymbol{x}^{pr}$.

If there are no additional preferences, the preferred solution is assumed to be the point belonging to the set of compromises, situated nearest to the ideal point and the corresponding vector of the decision variables [13,14].

4. Multicriterial optimization of orientation and volume fraction of fibres

The optimization criteria are as follows:

- minimum strain energy of the composite element (4)
- minimum cost of the composite element. The cost is expressed in the following form [10]:

$$K(\eta) = (1-\eta)k_1 + \eta k_2, \tag{6}$$

where k_1 and k_2, are unit costs of matrix and fibres, respectively. Here $k_2 > k_1$.

There are three independent design variables: the angle of inclination ϑ_1 of one of fibre families to x_1 axis, the angle α between the two families and the volume of fibre reinforcement η. The constrains for the variables are:

$$0 \leqslant \vartheta_1 \leqslant \pi, \quad 0 \leqslant \alpha \leqslant \pi, \quad \underline{\eta} \leqslant \eta \leqslant \overline{\eta},$$

where $\underline{\eta}, \overline{\eta}$ are lower and upper limit fractions of fibres.

To solve the problem a substitute objective function is constructed:

$$F^* = W - \mu_1\vartheta_1 + \mu_2(\vartheta_1 - \pi) - \mu_3\alpha + \mu_4(\alpha - \pi)$$
$$+ \mu_5\left(\underline{\eta} - \eta\right) + \mu_6(\eta - \bar{\eta}) . \tag{7}$$

The necessary conditions for the minimum strain energy are derived from the Kuhn-Tucker theorem in the following form:
– the conditions of equality are:

$$\vartheta_1 \left(\frac{\partial W}{\partial \vartheta_1} - \mu_1 + \mu_2\right) = 0,$$
$$\alpha \left(\frac{\partial W}{\partial \alpha} - \mu_3 + \mu_4\right) = 0, \tag{8}$$
$$\eta \left(\frac{\partial W}{\partial \eta} - \mu_5 + \mu_6\right) = 0,$$

$$- \mu_1 \vartheta_1 = 0, \quad \mu_2 (\vartheta_1 - \pi) = 0, \quad -\mu_3 \alpha = 0,$$
$$\mu_4 (\alpha - \pi) = 0, \quad \mu_5 (\underline{\eta} - \eta) = 0, \quad \mu_6 (\eta - \overline{\eta}) = 0;$$

– the conditions of inequality are:

$$\frac{\partial W}{\partial \vartheta_1} - \mu_1 + \mu_2 \geqslant 0, \;\; \frac{\partial W}{\partial \alpha} - \mu_3 + \mu_4 \geqslant 0, \;\; \frac{\partial W}{\partial \eta} - \mu_5 + \mu_6 \geqslant 0,$$
$$\frac{\partial F^*}{\partial \mu_i} \leqslant 0, \;\; i = 1, ..., 6, \;\; \vartheta_1 \geqslant 0, \;\; \alpha \geqslant 0, \;\; \eta \geqslant 0, \;\; \mu_i \geqslant 0. \tag{9}$$

By virtue of the Kuhn-Tucker's theorem (8), (9), it clearly appears that the global minimum of the strain energy can occur at one of the solutions of the system of equations presented below:

1°

$$\begin{aligned} &\sin\left[(2\vartheta_1 + \alpha) - 2\beta\right] = 0 \\ &\sin\alpha = 0 \\ &\eta = \overline{\eta} \end{aligned} \tag{10}$$

2°

$$\sin\left[(2\vartheta_1 + \alpha) - 2\beta\right] = 0, \tag{11}$$

$$\begin{aligned}
&\frac{E^{(m)}}{(1-\nu^2)}(1-\eta) \\
&\times \left\{ \left[(\sigma_{II} - \sigma_I)^2 \left(\frac{E^{(m)} E^{(s)}}{2(1-\nu)} \eta(1-\eta) + \frac{\left(E^{(s)}\right)^2}{4} \eta^2 \right) \right.\right. \\
&\times \cos\alpha - \left(\sigma_{II}^2 - \sigma_I^2\right) \left(\frac{E^{(m)} E^{(s)}}{2(1+\nu)} \eta (1-\eta) \right. \\
&\left. + \frac{\left(E^{(s)}\right)^2}{4} \eta^2 \left(\sin^2\alpha - 2\cos^2\alpha\right) \right) \cos\left[(2\vartheta_1 + \alpha) - 2\beta\right] \\
&\left. + (\sigma_I + \sigma_{II})^2 \; \frac{\left(E^{(s)}\right)^2}{4} \eta^2 \cos\alpha \cos 2\alpha \right] \\
&\times \left(\frac{\left(E^{(m)}\right)^2}{(1+\nu)} (1-\eta)^2 + \frac{E^{(m)} E^{(s)}}{(1+\nu)} \eta (1-\eta) \right. \\
&\left. + \frac{\left(E^{(s)}\right)^2}{4} \eta^2 \left(1 - \cos^4\alpha - \nu \sin^4\alpha\right) \right) \\
&+ \frac{\left(E^{(s)}\right)^2}{2} \eta^2 \left[- (\sigma_{II} - \sigma_I)^2 \right. \\
&\times \left(\frac{E^{(m)} E^{(s)}}{4(1-\nu)} \eta (1-\eta) \cos 2\alpha - \frac{\left(E^{(s)}\right)^2}{4} \eta^2 \sin^2\alpha \right) \\
&+ \left(\sigma_{II}^2 - \sigma_I^2\right) \left(\frac{E^{(m)} E^{(s)}}{(1+\nu)} \eta (1-\eta) + \frac{\left(E^{(s)}\right)^2}{2} \eta^2 \sin^2\alpha \right) \\
&\times \cos\left[(2\vartheta_1 + \alpha) - 2\beta\right] \cos\alpha \\
&+ 2\left(\sigma_I^2 - 2\nu \sigma_I \sigma_{II} + \sigma_{II}^2\right) \frac{\left(E^{(m)}\right)^2}{(1+\nu)^2 (1-\nu)} (1-\eta)^2 \\
&+ \left(\sigma_I^2 - (1+\nu) \sigma_I \sigma_{II} + \sigma_{II}^2\right) \frac{E^{(m)} E^{(s)}}{2(1+\nu)(1-\nu)} \eta (1-\eta) \\
&+ \left(\sigma_I^2 + \sigma_{II}^2\right) \frac{3E^{(m)} E^{(s)}}{4(1+\nu)} \eta (1-\eta) + (\sigma_I + \sigma_{II})^2 \frac{\left(E^{(s)}\right)^2}{4} \\
&\left. \times \, \eta^2 \sin^2\alpha \cos^2\alpha \right] \left(\nu \sin^2\alpha - \cos^2\alpha\right) \cos\alpha \Big\} = 0 \\
&\eta = \overline{\eta}
\end{aligned}$$

3°

$$\begin{aligned} &\cos\alpha = 0, \\ &\cos\left[(2\vartheta_1 + \alpha) - 2\beta\right] = 0 \\ &\eta = \overline{\eta}. \end{aligned} \tag{12}$$

In order to obtain a solution of the two-criterial optimization problem, we should determine a compromise set, an ideal solution, and a preferred solution. For this purpose we should construct the normalized functions in the following form:

$$\begin{aligned} \Phi_1 (\vartheta_1, \alpha, \eta) &= \frac{W (\vartheta_1, \alpha, \eta)}{W_o} \\ \Phi_2 (\eta) &= \frac{K (\eta)}{K_o}, \end{aligned} \tag{13}$$

where $W(\vartheta_1, \alpha, \eta)$ is the function (5), and $K(\eta)$ the function of cost (6); W_o and K_o are the maximum values of these functions in the compromise set. Because $K_o = (1 - \bar{\eta})k_1 + \bar{\eta}k_2$, the function Φ_2 depends on the ratio of unit costs of fibres to that of matrix

$$\Phi_2 (\eta) = \frac{(1-\eta) + \eta \frac{k_2}{k_1}}{(1-\overline{\eta}) + \overline{\eta} \frac{k_2}{k_1}}. \tag{14}$$

In order to obtain an ideal solution we should search minimum of the strain energy $W(\vartheta_1, \alpha, \eta)$ and the function of cost $K(\eta)$. The strain energy attains minimum for upper limit fraction of fibres $\eta = \overline{\eta}$ and for the angles of fibre inclination $\vartheta_1 = \overset{o}{\vartheta}_1$ and $\alpha = \overset{o}{\alpha}$. The determined angles are obtained from the solutions of the system of the Eqs. (10) or (11) or (12). The solutions depend on type of composite, they depend therefore on material constants of matrix and fibres, on upper limit volume fraction of fibres $\overline{\eta}$ and on the imposed external loads p, q, τ. The function of cost of the composite element (6) is the linear function of the decision variable η and attains minimum at lower limit fraction of fibres $\underline{\eta}$. Therefore the ideal solution is characterized by following coordinates:

$$\begin{aligned} \Phi_1^{id} &= \frac{W\left(\overset{o}{\vartheta}_1, \, \overset{o}{\alpha}, \, \overline{\eta}\right)}{W_o}, \\ \Phi_2^{id} &= \frac{1 - \underline{\eta} + \underline{\eta} \frac{k_2}{k_1}}{k_o}, \end{aligned} \tag{15}$$

here $k_0 = 1 - \overline{\eta} + \overline{\eta} \frac{k_2}{k_1}$, and W_o is the minimal value of the strain energy W for $\eta = \underline{\eta}$ and for the angles $\vartheta_1 = \overset{*}{\vartheta}_1$, $\alpha = \overset{*}{\alpha}$, which are obtained from the solutions of the system of Eqs. (12) or (13) or (14).

The function of cost is a linear function of the decision variable η and the function of the strain energy is the function of three decision variables $\vartheta_1, \alpha, \eta$. Therefore to obtain the compromise set, we should search for different values of the $\eta \in \left[\underline{\eta}, \overline{\eta}\right]$ the values of the function Φ_2 (14) and minimum values of the function (13). The preferred solution may be obtained as the Euclidean metric function [13,14] expressing the distance between the set of compromises and the ideal point, and its minimum is determined:

$$T(\vartheta_1, \alpha, \eta) = \left\{ \left[\Phi_1(\vartheta_1, \alpha, \eta) - \Phi_1^{id}\right]^2 + \left[\Phi_2(\eta) - \Phi_2^{id}\right]^2 \right\}^{\frac{1}{2}}. \tag{16}$$

It is supposed that the composite element is subjected to two normal loads: p along the axis x_2 and $q = kp$ along the axis x_1 and to a tangent load $\tau = lp$ in the following type of composites:

- the composite of epoxy matrix reinforced by carbon fibres with the material constants ($E^{(m)} = 3.5$ GPa, $E^{(s)} = 220$ GPa, $\nu = 0.35$) with the lower and upper limit fractions $\underline{\eta} = 0.1$ and $\overline{\eta} = 0.6$, respectively. Three examples are considered with the following ratios of loads: $k = 0.5$ $l = 0.3$; $k = 0.5$ $l = 0$; $k = -0.5$ $l = 0.3$. In the function of cost, the ratio of the unit costs of the carbon fibres to epoxy matrix $k_2/k_1 = 7.4$ is accepted.
- the high performance cementitious composite reinforced by steel fibres with the material constants ($E^{(m)} = 40$ GPa, $E^{(s)} = 210$ GPa, $\nu = 0.23$) with the lower limit fraction $\underline{\eta} = 0.02$ and upper $\overline{\eta} = 0.08$. Two examples are considered with the following ratios of loads ($k = 0.5$ $l = 0.3$; $k = -0.5$ $l = 3$). In the function of cost, the ratio of the unit costs of the steel fibres to concrete B70 $k_2/k_1 = 100$ is accepted.

5. Minimum strain energy and minimum cost of a composite element made of epoxy matrix reinforced with two families of carbon fibres

In the first example ($k = 0.5$ $l = 0.3$) the function W reaches minimum when the decision variables satisfy the set of Eqs. (11). The solution of these equations has the following form $\vartheta_1 = 36.17°$, $\alpha = 57.46°$, $\eta = 0.6$. The obtained angles of fibre inclination satisfy the condition $2\vartheta_1 + \alpha = 2\beta + \pi$. The function $\Phi_2(\eta)$ attains its minimum for $\eta = 0.1$ and $k_o = 4.84$. The constant W_o is the value of the function W in point $\vartheta_1 = 44.18°$, $\alpha = 41.44°$, $\eta = 0.1$ and $W_o = 0.2485$. This is the maximum value of the function W in the compromise set. In the non-dimensional space of objective functions $\Phi_1\Phi_2$ (Fig. 2) the compromise set is introduced, the ideal point A is determined by the following coordinates:

$$\Phi_1^{id} = 0.2325, \quad \Phi_2^{id} = 0.3388.$$

As the preferred solution is admitted point D of compromise set that is closest to the ideal solution. Its coordinates are obtained from the Eq. (16):

$$\Phi_1 = 0.4569, \quad \Phi_2 = 0.5769.$$

In the space of the design variables this corresponds to

$$\vartheta_1 = 37.86°, \quad \alpha = 54.08°, \quad \eta = 0.28.$$

In the second example ($k = 0.5$ $l = 0$) the strain energy attains its minimum when the decision variables too satisfy the Eqs. (11). Then the solution of the equations is $\vartheta_1 = 55.1°$, $\alpha = 69.8°$, $\eta = 0.6$. The constant W_o is adopted as the value of the function W in the point $\vartheta_1 = 59.35°$, $\alpha = 61.30°$, $\eta = 0.1$ and $W_o = 0.2485$. The ideal solution is determined by the following coordinates:

$$\Phi_1^{id} = 0.2321, \quad \Phi_2^{id} = 0.3388.$$

The preferred solution is proposed to be the point which is closest to the ideal solution and is determined by the following coordinates

$$\Phi_1 = 0.4580, \quad \Phi_2 = 0.5757.$$

In the space of the design variables, this corresponds to:

$$\vartheta_1 = 56.03°, \quad \alpha = 67.94°, \quad \eta = 0.2791.$$

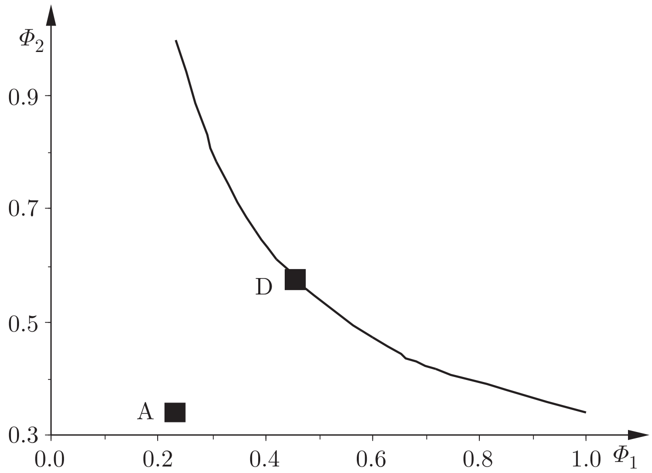

Fig. 2. Compromise set for a composite element of epoxy matrix reinforced with carbon fibres for $k = 0.5$ and $l = 0.3$

In the third example ($k = -0.5$ $l = 0.3$) the function W attains minimum when the decision variables satisfy the set of Eq. (12). The solution of these equations has the following form $\vartheta_1 = \beta + 90° = 79.10°$, $\alpha = 90°$, $\eta = 0.6$. Here for upper limit fraction of fibres two orthogonal fibre families coincide with the directions of principal stresses. The value of the function W in the point $\vartheta_1 = \beta + 90°$, $\alpha = 90°$, $\eta = 0.1$ is accepted as the constant $W_o = 0.3702$. The coordinates of the ideal solution are

$$\Phi_1^{id} = 0.2014, \quad \Phi_2^{id} = 0.3388.$$

The preferred point in the compromise set is determined by the coordinates:

$$\Phi_1 = 0.4196, \quad \Phi_2 = 0.5693.$$

In the space of the design variables this corresponds to

$$\vartheta_1 = 79.1°, \quad \alpha = 90°, \quad \eta = 0.2743.$$

In the considered examples the obtained compromise sets have similar shape as presented in Fig. 2. In the first two examples, we suppose that the composite element is subjected to the tensile load p along the axis x_2, to the tensile load $q = 0.5p$

applied along the axis x_1 and in the first example to the tangent load $\tau = 0.3p$. It may be concluded, that the preferred solutions correspond to the composite element with a volume fraction of fibres equal to 28% and with two families of fibres, which are inclined at following angles:

– in the first example the first family of fibres is inclined at an angle $37.86°$ to the axis x_1 and the angle between both families is $54.08°$,
– in the second example the first family of fibres is inclined at an angle $56.03°$ and the angle between both families is $67.94°$.

The obtained angles satisfy therefore the condition $2\vartheta_1 + \alpha = 2\beta + \pi$. In the third example it is supposed that the composite element is subjected to the tensile load p along the axis x_2, to the compressive load $q = -0.5p$ applied along the axis x_1 and to the tangent load $\tau = 0.3p$. In this example, the optimal solution is the composite element with two orthogonal fibre families coincide with the directions of principal stresses and with a volume fraction of fibres equal to 27%.

6. Minimum strain energy and minimum cost for a high performance cementitious composite element reinforced by steel fibres

In the first example ($k = 0.5$ $l = 0.3$) the function W reaches minimum when the decision variables satisfy the set of equations (10). This solution of these equations has the following form $\vartheta_1 = \beta + 90° = 64.9°$, $\alpha = 0°$, $\eta = 0.08$. Here for upper limit fraction of fibres the single family of fibre corresponds to the principal direction of stress. The function Φ_2 attains its minimum for $\eta = 0.02$ and $k_o = 8.92$. The function W for $\vartheta_1 = 64.9°$, $\alpha = 0°$, $\eta = 0.02$ is adopted as the constant W_o. In the non-dimensional space of objective function (Fig. 3) the compromise set is introduced, the ideal point A is determined from the condition (15) and has the following coordinates:

$$\Phi_1^{id} = 0.8373, \quad \Phi_2^{id} = 0.3341.$$

As the preferred solution the point D of compromise set, that is closest to the ideal solution, is admitted. Its coordinates are obtained from the Eq. (16):

$$\Phi_1 = 0.9868, \quad \Phi_2 = 0.3783,$$

which corresponds to the design variables

$$\vartheta_1 = 64.9°, \quad \alpha = 0°, \quad \eta = 0.02398.$$

In the second example ($k = -0.5$ $l = 3$) the strain energy attains its minimum when the decision variables satisfy the Eq. (12). Then the solution of the equations is $\vartheta_1 = \beta + 90° = 52.02°$, $\alpha = 90°$, $\eta = 0.08$. The constant W_o is the value of the function W in the point $\vartheta_1 = 52.02°$, $\alpha = 90°$, $\eta = 0.02$. The ideal solution is determined by the following coordinates:

$$\Phi_1^{id} = 0.8867, \quad \Phi_2^{id} = 0.3341.$$

In the considered example the preferred point in the compromise set is determined by the coordinates:

$$\Phi_1 = 0.9960, \quad \Phi_2 = 0.3549.$$

In the space of the design variables, this corresponds to:

$$\vartheta_1 = 52.02°, \quad \alpha = 90°, \quad \eta = 0.0219.$$

In the considered examples the obtained compromise sets have the similar shape as presented in Fig. 3. In the first example the preferred solution correspond to the composite element with one family of fibres inclined in the direction coinciding with the principal stress axis $\vartheta_1 = \beta + 90° = 64.9°$ and with the volume fraction of fibres equal to 2.4%. In the second example the preferred solution correspond to the composite element with two orthogonal fibre families coincide with the directions of principal stresses $\vartheta_1 = \beta + 90° = 52.02°$and with the volume fraction of fibres equal to 2.2%.

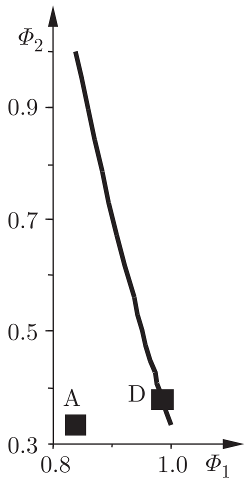

Fig. 3. Compromise set for a high performance cementitious composite element reinforced by steel fibres for $k = 0.5$ and $l = 0.3$

7. Concluding remarks

The problem of two-criteria optimization was considered. Minimum strain energy and minimum cost of composite element were assumed as criteria.

On the basis of the analytical solution (10), (11), (12) it was found, that the strain energy of the composite element attains minimum at the upper limit fraction of fibres and at one of the three following arrangements of fibres: two family of fibres are aligned and placed along the direction of principal stress corresponding to its greater absolute value, two fibre families placed along the principal directions, two non-orthogonal fibre families satisfying the set of Eqs. (11). The function of the cost attains minimum at the lower limit fraction of the fibres, so these criteria are therefore in a conflict.

Examining particular load cases in composite elements of epoxy resin reinforced by carbon fibres and in high performance fibre reinforced cementitious composite elements, the angles of inclination of fibres and the fraction of fibres have been determined in the sense of assumed criteria. The optimal angles of inclination of fibres depend on the material constants of matrix and fibres, on the magnitude of load and on the maximal fraction of fibres. In high performance fibre reinforced cementitious composite elements the optimal fraction of fibres is almost equal to the minimal fraction $\underline{\eta}$. Here the criterion of cost in which the ratio of the unit costs of the steel fibres to concrete matrix is accepted as the high value, has a great influence in determining the preferred solutions. At the low ratio k_2/k_1, the fraction of fibres in the preferred solution is larger.

The obtained solutions are local. They concern an internal differential element. Such formulation of the designing problem of an optimal distribution of non-orthogonal fibres and their volumetric contents (in terms of both criteria), can be applied to design composite structures with certain boundary conditions, which can be realized by numerical procedures based on the finite element method.

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