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Analytical expressions of effective constants for a piezoelectric composite reinforced with square cross-section fibers

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THE PURPOSE of this paper is to present analytical expressions of the effective elastic, piezoelectric and dielectric constants of reinforced piezoelectric composite materials with unidirectional fibers periodically distributed in a square matrix, as obtained by means of the "double asymptotic homogenization" method. The cross-section of the fibers is square. Each periodic cell of the medium is a binary piezoelectric composite wherein both phases are homogeneous piezoelectric materials with transversely isotropic properties. Comparison between the derived theoretical predictions of characteristic parameters and the existing experimental results shows a rather good agreement. The results obtained in the present paper were verified by means of the universal relations of Schulgasser. Numerical computation of the effective properties can be realized without difficulties.

Key words: Piezocomposite, effective properties, asymptotic homogenization, fiber-reinforced, composite material.

1. Introduction

The piezocomposite materials have been used for hydrophone applications and transducers for medical imaging. The determination of the overall properties of piezocomposite, according to the physical and geometric characteristics of their components, is very important for applications.

Different techniques have been reported to estimate the effective electro-elastic properties of piezoelectric laminate composites. For example, in [1] the effective coefficients of a bi-laminate medium of 4mm symmetry using the hypotheses of equivalent homogeneity were obtained. In [2], the effective behavior of bi-laminate media of hexagonal symmetry using the theory of uniform fields in heterogeneous media by means of appropriate boundary conditions was determined. An effective medium model of layered composites has been investigated starting from a physical reasoning given in [3]. In the case of very fine composite structures, quasistatic and iso-strain (constant strain) approximations were used to derive the effective properties of composites (see, [4, 5]) which include the effective elasticity, permittivity, piezoelectricity, and density; some physical parameters useful for different applications were derived. Basing on the asymptotic homogenization method, analytical formulae for the overall properties of layered piezoelectric composites can be found in [6, 7].

Unidirectional fibrous composite with square cross-sections of fibers has been investigated only by means of numerical solution of the local problems. For instance, the finite element method was applied in [8] for determine the global properties of a piezocomposite. The Ritz method was used in [6] to investigate a rectangular cross-section in a square matrix for thermopiezoelectric composites.

In the present paper, using "double homogenization", (e.g., the homogenization is applied two times in different directions, according to the geometric configuration of the composite), analytical expressions of the effective coefficients are obtained, in a composite with square fibers distributed periodically into square cells. Somehow, the present work is related to a recent work by ANDRIANOV *et al.* [9, 10], where an asymptotic approach and Padé approximants are proposed for evaluating effective elastic and heat conductivity properties respectively, of two-component periodic composites with fibrous inclusions. In order to apply the so-called "double homogenization", the problem is divided into two homogenization stages: 1) the composite structure is homogenized, that is, the effective coefficients for a unidirectional structure in the direction x_2 are obtained; 2) afterwards, the effective coefficients are calculated for the composite 2–2 in the other direction x_1. In this case, due to the symmetry of the composite, the double homogenization can be easily realized. The main theoretical aspects used in this work can be found in [7]. These results for square fibers are compared with the expressions obtained in [7] for the laminate composites 2–2, and with some theoretical and experimental results reported in the literature, [11]. The universal relations given in [12] are satisfied for the set of coefficients calculated in the present work.

2. Theoretical procedure

Piezoelectric materials are characterized by the following different material coefficients: C (elastic), e (piezoelectric) and ε (dielectric), which are the fourth, third and second order tensors, respectively. When these materials are heterogeneous and periodic, the material coefficients are X-periodic functions. Here X denotes the periodic cell. Applying the method of asymptotic homogenization, the material coefficients are transformed into new physical coefficients $\overline{C}$ (elastic), $\overline{e}$ (piezoelectric) and $\overline{\varepsilon}$ (dielectric), which represent the homogeneous properties or effective coefficients. To obtain these coefficients it is necessary to solve a set of local problems, which are represented by a system of partial differential equations. In case of circular transverse section of fibers, an analytical solution for this system is obtained making use of the potential methods of the complex variable and properties of the Weierstrass elliptic functions, [13–17]. On the other hand, analytical closed forms of the effective coefficients of piezoelectric composites with square transverse sections of the fibers using asymptotic homogenization have not been reported yet. Therefore the purpose of the present work is to present an alternative form for the computation of analytical expressions for such composites.

Let us suppose that we have a composite material with unidirectional square fibers periodically distributed, where each periodic cell is a binary homogeneous piezoelectric medium with square symmetry in welded contact at the interface. Both the matrix and the fibers are assumed to be composed of homogeneous piezoelectric materials with 6 mm hexagonal symmetry.

2.1. Effective coefficients for the first homogenization in the direction x_2

In [7], the effective coefficients were obtained for piezoelectric laminated composites. Let us suppose that we have a laminate material formed by a piezoelectric phase (phase 1) of elastic, piezoelectric, dielectric and density constants $C_{ij}^{(1)}$, $e_{ij}^{(1)}$, $\varepsilon_{ij}^{(1)}$, $\rho^{(1)}$ respectively; and a phase of piezoelectric polymer (phase 2) of elastic, piezoelectric, dielectric and density parameters denoted by: $C_{ij}^{(2)}$, $e_{ij}^{(2)}$, $\varepsilon_{ij}^{(2)}$, $\rho^{(2)}$.

The effective coefficients for a piezoelectric laminate in the direction x_2 are exactly expressed as follows:

Effective elastic constants

$$C_{11}^{*} = -\frac{\Gamma_L^2\alpha_{12} + \Gamma_L(-C_{11}^{(1)}\beta_{12} - \alpha_{12} + C_{11}^{(2)}\beta_{12}) - C_{11}^{(2)}\beta_{12}}{\beta_{12}},$$

$$(2.1)\qquad C_{12}^{*} = -\frac{(C_{12}^{(1)} - C_{12}^{(2)})C_{22}^{(1)}(\Gamma_L - 1) - C_{12}^{(1)}\beta_{12}}{\beta_{12}},$$

$$C_{13}^{*} = -\frac{\Gamma_L^2\alpha_2^{(2)} + \Gamma_L(-C_{13}^{(1)}\beta_{12} - \alpha_{22} + C_{13}^{(2)}\beta_{12}) - C_{13}^{(2)}\beta_{12}}{\beta_{12}},$$

(2.1) [cont.]

$$C^*_{22} = -\frac{C^{(1)}_{22}C^{(2)}_{22}}{\beta_{12}},$$

$$C^*_{23} = -\frac{(C^{(1)}_{23} - C^{(2)}_{23})C^{(1)}_{22}(\Gamma_L - 1) - C^{(1)}_{23}\beta_{12}}{\beta_{12}},$$

$$C^*_{33} = -\frac{\Gamma_L^2\alpha_{32} + \Gamma_L(-C^{(1)}_{33}\beta_{12} - \alpha_{32} + C^{(2)}_{33}\beta_{12}) - C^{(2)}_{33}\beta_{12}}{\beta_{12}},$$

$$\begin{aligned}C^*_{44} = {} & \frac{-2.\varepsilon^{(2)}_{22}\Gamma_L\left(e^{(1)}_{24}e^{(2)}_{24}\right)^2\beta_{32}\beta_{42} + (2.\varepsilon^{(2)}_{22}\Gamma_L\beta_{32}\beta_{42} + \beta_{32}(\beta_{42})^2)\left(e^{(2)}_{24}\right)^4}{\beta_{32}\varepsilon^{(1)}_{22}\varepsilon^{(2)}_{22}\beta_{42}} \\ & + \frac{\varepsilon^{(2)}_{22}\Gamma_L\left(e^{(1)}_{24}\right)^2\beta_{32}\beta_{42} - \left(C^{(1)}_{44}\varepsilon^{(2)}_{22}\right)^2\varepsilon^{(1)}_{22} - C^{(1)}_{44}\varepsilon^{(1)}_{22}\left(\varepsilon^{(2)}_{22}\right)^2\beta_{32}}{\beta_{32}\varepsilon^{(1)}_{22}\varepsilon^{(2)}_{22}\beta_{42}} \\ & + \frac{(-\beta_{32}(\beta_{42})^2 - \varepsilon^{(2)}_{22}\Gamma_L\beta_{32}\beta_{42})\left(e^{(2)}_{24}\right)^2 C^{(1)}_{44}C^{(2)}_{44}\left(\varepsilon^{(1)}_{22}\right)^2\varepsilon^{(2)}_{22}}{\beta_{32}\varepsilon^{(1)}_{22}\varepsilon^{(2)}_{22}\beta_{42}} \\ & + \frac{\Gamma_L^2\left(\varepsilon^{(2)}_{22}\right)^2\beta_{32}\alpha_{42} + C^{(1)}_{22}\left(\varepsilon^{(1)}_{22}\right)^2\varepsilon^{(2)}_{22}\beta_{32} - \left(C^{(1)}_{44}\right)^2\beta_{42}\varepsilon^{(2)}_{22}\varepsilon^{(1)}_{22}}{\beta_{32}\varepsilon^{(1)}_{22}\varepsilon^{(2)}_{22}\beta_{42}},\end{aligned}$$

$$C^*_{55} = C^{(1)}_{55}\Gamma_L + C^{(2)}_{55} - C^{(2)}_{55}\Gamma_L,$$

$$C^*_{66} = \frac{1}{2}\frac{(C^{(1)}_{22} - C^{(1)}_{12})(C^{(2)}_{22} - C^{(2)}_{12})}{\beta_{22} - \beta_{12}}.$$

Effective piezoelectric constants

(2.2)

$$e^*_{24} = \frac{(-e^{(1)}_{24} + e^{(2)}_{24})\varepsilon^{(2)}_{22}\Gamma_L + e^{(2)}_{24}\beta_{42}}{\beta_{42}},$$

$$e^*_{32} = -\frac{(e^{(1)}_{32} - e^{(2)}_{32})C^{(1)}_{22}\Gamma_L + (-e^{(1)}_{32} + e^{(2)}_{32})C^{(1)}_{22} - e^{(1)}_{32}\beta_{12}}{\beta_{12}},$$

$$e^*_{15} = e^{(1)}_{15}\Gamma_L + e^{(2)}_{15} - e^{(2)}_{15}\Gamma_L,$$

$$e^*_{33} = -\frac{\Gamma_L e^{(2)}_{33}\beta_{12} - \Gamma_L\alpha_{52} - \Gamma_L e^{(1)}_{33}\beta_{12} + \Gamma_L^2\alpha_{52} - e^{(2)}_{33}\beta_{12}}{\beta_{12}},$$

$$e^*_{31} = -\frac{-e^{(2)}_{31}\beta_{12} + \Gamma_L^2\alpha_{72} - \Gamma_L\alpha_{72} - \Gamma_L e^{(x_1)}_{31}\beta_{12} + \Gamma_L e^{(2)}_{31}\beta_{12}}{\beta_{12}}.$$

Effective dielectric constants

$$
\begin{aligned}
&\varepsilon^*_{11} = \varepsilon^{(1)}_{11}\Gamma_L + \varepsilon^{(2)}_{11} - \varepsilon^{(2)}_{11}\Gamma_L,\\
\text{(2.3)}\qquad &\varepsilon^*_{33} = \frac{-\Gamma_L\varepsilon^{(2)}_{33}\beta_{12} + \Gamma_L\varepsilon^{(1)}_{33}\beta_{12} - \Gamma_L^2\alpha_{72} + \varepsilon^{(2)}_{33}\beta_{12} + \Gamma_L\alpha_{72}}{\beta_{12}},\\
&\varepsilon^*_{22} = -\frac{\varepsilon^{(1)}_{22}\varepsilon^{(2)}_{22}}{\beta_{42}}.
\end{aligned}
$$

Effective density constant

$$
\rho^* = \rho^{(1)}\Gamma_L + \rho^{(2)} - \rho^{(2)}\Gamma_L. \tag{2.4}
$$

where

$$
\begin{aligned}
\beta_{12} &= -\Gamma_L C^{(2)}_{22} - C^{(1)}_{22} + C^{(1)}_{22}\Gamma_L,\\
\beta_{22} &= -\Gamma_L C^{(2)}_{12} - C^{(1)}_{12} + C^{(1)}_{12}\Gamma_L,\\
\beta_{32} &= -\Gamma_L C^{(1)}_{44} + C^{(2)}_{44} - C^{(2)}_{44}\Gamma_L,\\
\beta_{42} &= -\Gamma_L \varepsilon^{(2)}_{22} - \varepsilon^{(1)}_{22} + \varepsilon^{(1)}_{22}\Gamma_L,\\
\alpha_{12} &= (C^{(1)}_{12} - C^{(2)}_{12})(C^{(1)}_{12} - C^{(2)}_{12}),\\
\alpha_{22} &= (C^{(1)}_{12} - C^{(2)}_{12})(C^{(1)}_{23} - C^{(2)}_{23}),\\
\alpha_{32} &= (C^{(1)}_{23} - C^{(2)}_{23})(C^{(1)}_{23} - C^{(2)}_{23}),\\
\alpha_{42} &= (e^{(1)}_{24} - e^{(2)}_{24})(e^{(1)}_{24} - e^{(2)}_{24}),\\
\alpha_{52} &= (C^{(1)}_{23} - C^{(2)}_{23})(e^{(1)}_{32} - e^{(2)}_{32}),\\
\alpha_{62} &= (C^{(1)}_{12} - C^{(2)}_{12})(e^{(1)}_{32} - e^{(2)}_{32}),\\
\alpha_{72} &= -(e^{(1)}_{32} - e^{(2)}_{32})(e^{(1)}_{32} - e^{(2)}_{32}).
\end{aligned}
$$

The expressions C^*_{ij}, e^*_{ij}, ε^*_{ij} and ρ^* represent the effective coefficients in the Structure I (Fig. 1) and we denote by Γ_L the volume fraction of piezoelectric material in this structure.

2.2. Effective coefficients for the second homogenization in the direction x_1

Now, the laminated composite is made of Phase 1 whose properties are the averaged coefficients in the direction x_2 calculated previously, namely: C^*_{ij}, e^*_{ij}, ε^*_{ij}, ρ^* and a phase of piezoelectric polymer (Phase 2) of elastic, piezoelectric, dielectric and density properties denoted by: $C^{(2)}_{ij}$, $e^{(2)}_{ij}$, $\varepsilon^{(2)}_{ij}$, $\rho^{(2)}$.

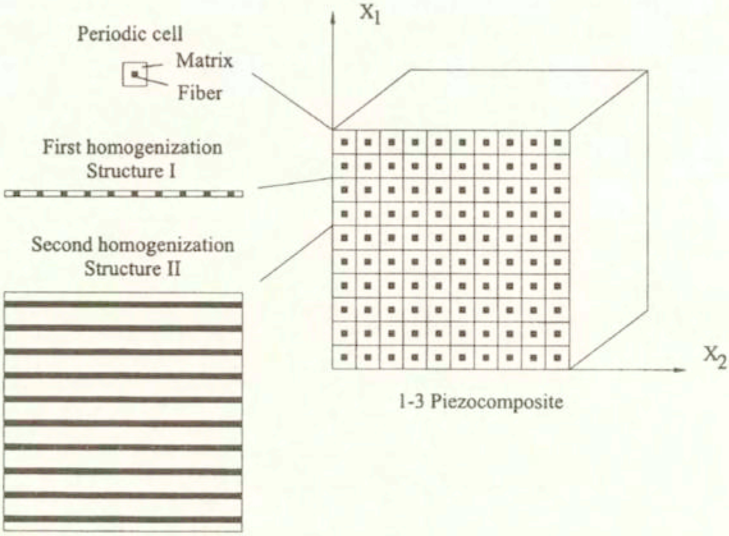

FIG. 1. Schematic diagram of a 1–3 piezoelectric composite, illustrating the periodic cell and two homogenization stages.

The effective coefficients of the composite in the direction x_1 are exactly given in the following form:

Effective elastic constants

(2.5)
$$
\begin{aligned}
\overline{C}_{11} &= -\frac{C^*_{11}C^{(2)}_{11}}{\beta_{11}},\\
\overline{C}_{12} &= -\frac{(C^*_{12}-C^{(2)}_{12})C^*_{11}(\Gamma_L-1)-C^*_{12}\beta_{11}}{\beta_{11}},\\
\overline{C}_{13} &= -\frac{(C^*_{13}-C^{(2)}_{13})C^*_{11}(\Gamma_L-1)-C^*_{13}\beta_{11}}{\beta_{11}},\\
\overline{C}_{22} &= -\frac{\Gamma_L^2\alpha_{11}+\Gamma_L(-C^*_{22}\beta_{11}-\alpha_{11}+C^{(2)}_{22}\beta_{11})-C^{(2)}_{22}\beta_{11}}{\beta_{11}},\\
\overline{C}_{23} &= -\frac{\Gamma_L^2\alpha_{21}+\Gamma_L(-C^*_{23}\beta_{11}-\alpha_{21}+C^{(2)}_{23}\beta_{11})-C^{(2)}_{23}\beta_{11}}{\beta_{11}},\\
\overline{C}_{33} &= -\frac{\Gamma_L^2\alpha_{31}+\Gamma_L(-C^*_{33}\beta_{11}-\alpha_{31}+C^{(2)}_{33}\beta_{11})-C^{(2)}_{33}\beta_{11}}{\beta_{11}},\\
\overline{C}_{44} &= C^*_{44}\Gamma_L+C^{(2)}_{44}-C^{(2)}_{44}\Gamma_L,
\end{aligned}
$$

(2.5) [cont.]

$$\overline{C}_{55} = \frac{-2.\varepsilon_{11}^{(2)}\Gamma_L\left(e_{15}^{*}e_{15}^{(2)}\right)^2\beta_{31}\beta_{41} + (2.\varepsilon_{11}^{(2)}\Gamma_L\beta_{31}\beta_{41} + \beta_{31}\beta_{41}^2)\left(e_{15}^{(2)}\right)^4}{\beta_{31}\varepsilon_{11}^{*}\varepsilon_{11}^{(2)}\beta_{41}}$$
$$+ \frac{\varepsilon_{11}^{(2)}\Gamma_L\left(e_{15}^{*}\right)^2\beta_{31}\beta_{41} - \left(C_{55}^{*}\varepsilon_{11}^{(2)}\right)^2\varepsilon_{11}^{*} - C_{55}^{*}\varepsilon_{11}^{*}\left(\varepsilon_{11}^{(2)}\right)^2\beta_{31}}{\beta_{31}\varepsilon_{11}^{*}\varepsilon_{11}^{(2)}\beta_{41}}$$
$$+ \frac{(-\beta_{31}\beta_{41}^2 - \varepsilon_{11}^{(2)}\Gamma_L\beta_{31}\beta_{41})\left(e_{15}^{(2)}\right)^2 C_{55}^{*}C_{55}^{(2)}\left(\varepsilon_{11}^{*}\right)^2\varepsilon_{11}^{(2)}}{\beta_{31}\varepsilon_{11}^{*}\varepsilon_{11}^{(2)}\beta_{41}}$$
$$+ \frac{\Gamma_L^2\left(\varepsilon_{11}^{(2)}\right)^2\beta_{31}\alpha_{41} + C_{55}^{*}\left(\varepsilon_{11}^{*}\right)^2\varepsilon_{11}^{(2)}\beta_{31} - \left(C_{55}^{*}\right)^2\beta_{41}\varepsilon_{11}^{(2)}\varepsilon_{11}^{*}}{\beta_{31}\varepsilon_{11}^{*}\varepsilon_{11}^{(2)}\beta_{41}},$$

$$\overline{C}_{66} = \frac{1}{2}\frac{(C_{11}^{*} - C_{12}^{*})(C_{11}^{(2)} - C_{12}^{(2)})}{\beta_{21} - \beta_{11}},$$

Effective piezoelectric constants

$$\overline{e}_{15} = \frac{(-e_{15}^{*} + e_{15}^{(2)})\varepsilon_{11}^{2}\Gamma_L + e_{15}^{(2)}\beta_{41}}{\beta_{41}},$$

$$\overline{e}_{31} = -\frac{(e_{31}^{*} - e_{31}^{(2)})C_{11}^{*}\Gamma_L + (-e_{31}^{*} + e_{31}^{(2)})C_{11}^{*} - e_{31}^{*}\beta_{11}}{\beta_{11}},$$

(2.6)
$$\overline{e}_{24} = e_{24}^{*}\Gamma_L + e_{24}^{(2)} - e_{24}^{(2)}\Gamma_L,$$

$$\overline{e}_{33} = -\frac{\Gamma_L e_{33}^{(2)}\beta_{11} - \Gamma_L\alpha_{51} - \Gamma_L e_{33}^{*}\beta_{11} + \Gamma_L^2\alpha_{51} - e_{33}^{(2)}\beta_{11}}{\beta_{11}},$$

$$\overline{e}_{32} = -\frac{-e_{32}^{(2)}\beta_{11} + \Gamma_L^2\alpha_{71} - \Gamma_L\alpha_{71} - \Gamma_L e_{32}^{*}\beta_{11} + \Gamma_L e_{32}^{(2)}\beta_{11}}{\beta_{11}}.$$

Effective dielectric constants

$$\overline{\varepsilon}_{22} = \varepsilon_{22}^{*}\Gamma_L + \varepsilon_{22}^{(2)} - \varepsilon_{22}^{(2)}\Gamma_L,$$

(2.7)
$$\overline{\varepsilon}_{33} = \frac{-\Gamma_L\varepsilon_{33}^{(2)}\beta_{11} + \Gamma_L\varepsilon_{33}^{*}\beta_{11} - \Gamma_L^2\alpha_{71} + \varepsilon_{33}^{(2)}\beta_{11} + \Gamma_L\alpha_{71}}{\beta_{11}},$$

$$\overline{\varepsilon}_{11} = -\frac{\varepsilon_{11}^{*}\varepsilon_{11}^{(2)}}{\beta_{41}}.$$

Effective density constant

(2.8)
$$\overline{\rho} = \rho^{*}\Gamma_L + \rho^{(2)} - \rho^{(2)}\Gamma_L,$$

where

$$\begin{aligned}
\beta_{11} &= -\Gamma_L C_{11}^{(2)} - C_{11}^{*} + C_{11}^{*}\Gamma_L,\\
\beta_{21} &= -\Gamma_L C_{12}^{(2)} - C_{12}^{*} + C_{12}^{*}\Gamma_L,\\
\beta_{31} &= -\Gamma_L C_{55}^{*} + C_{55}^{(2)} - C_{55}^{(2)}\Gamma_L,\\
\beta_{41} &= -\Gamma_L \varepsilon_{11}^{(2)} - \varepsilon_{11}^{*} + \varepsilon_{11}^{*}\Gamma_L,\\
\alpha_{11} &= (C_{12}^{*} - C_{12}^{(2)})(C_{12}^{*} - C_{12}^{(2)}),\\
\alpha_{21} &= (C_{12}^{*} - C_{12}^{(2)})(C_{13}^{*} - C_{13}^{(2)}),\\
\alpha_{31} &= (C_{13}^{*} - C_{13}^{(2)})(C_{13}^{*} - C_{13}^{(2)}),\\
\alpha_{41} &= (e_{15}^{*} - e_{15}^{(2)})(e_{15}^{*} - e_{15}^{(2)}),\\
\alpha_{51} &= (C_{13}^{*} - C_{13}^{(2)})(e_{31}^{*} - e_{31}^{(2)}),\\
\alpha_{61} &= (C_{12}^{*} - C_{12}^{(2)})(e_{31}^{*} - e_{31}^{(2)}),\\
\alpha_{71} &= -(e_{31}^{*} - e_{31}^{(2)})(e_{31}^{*} - e_{31}^{(2)}).
\end{aligned}$$

The expressions $\overline{C}_{ij}$, $\overline{e}_{ij}$, $\overline{\varepsilon}_{ij}$ and $\overline{\rho}$ denote the effective coefficients in the composite Structure II. These expressions represent the effective coefficients of the composite for square reinforcement fibers. The volume fraction of fibers is expressed according to $\Gamma_F = (\Gamma_L)^2$.

3. Applications to transducers. Results

One of the important applications of the piezoelectric composite materials appears in transducers used for medical imaging applications. The desired properties are a high electromechanical coupling coefficient K_t (0.6 to 0.7) and a low acoustic impedance Z (< 7.5 MRayls). Now, the case in which two different homogeneous phases are involved in the composite is studied. The effective properties of the composite can now be computed from Eqs. (2.5)–(2.8). A set of important physical parameters for pulse-echo transducer applications can be calculated. For instance, let us mention the electromechanical piezoelectric coupling coefficients K_t and K_p, the specific acoustic impedance Z, the longitudinal wave speed V_l and the hydrostatic charge coefficient d_h. They are given by the following formulae (see, [18]):

(3.1)
$$\overline{K}_t = \sqrt{1 - \frac{\overline{C}_{33}}{\overline{C}_{33}^{D}}},$$

$$\overline{K}_p = \sqrt{\frac{2}{1-\overline{\sigma}}}\overline{K}_{31},$$

(3.1) [cont.]

$$\overline{Z} = \overline{\rho}\overline{V}_3^D,$$

$$\overline{V}_l = \sqrt{\frac{\overline{C}_{33}^D}{\overline{\rho}}},$$

$$\overline{d}_h = \overline{d}_{31} + \overline{d}_{32} + \overline{d}_{33},$$

where

$$\overline{C}_{33}^D = \overline{C}_{33} + \overline{e}_{33}^2(\overline{\varepsilon}_{33})^{-1},$$

$$\overline{K}_{31} = \frac{\overline{d}_{31}}{\sqrt{\overline{\varepsilon}_{33}^T \overline{S}_{11}}},$$

(3.2)

$$\overline{\sigma} = -\frac{\overline{S}_{12}}{\overline{S}_{11}},$$

$$\overline{S}_{ij} = (-1)^{i+j}\frac{\triangle_{ij}}{\triangle},$$

$$\overline{d}_{mi} = \overline{e}_{mj}\overline{S}_{ji},$$

$$\overline{\varepsilon}_{mn}^T = \overline{d}_{mp}\overline{e}_{np} + \overline{\varepsilon}_{mn}$$

The superscripts D and T at a given symbol in (3.1) and (3.2) mean that the relevant quantity is measured at constant electric displacement D or at constant stress T; $\overline{d}_{ij}$ is the piezoelectric coefficient of the composite, $\overline{S}_{ij}$ are components of the effective compliance tensor $\overline{S}$, $\overline{\sigma}$ is the Poisson's ratio, $\triangle$ is determinant of the $\overline{C}_{ij}$ matrix and $\triangle_{ij}$ is the minor obtained by excluding the i-th row and j-th column. Some properties of the composite are presented as functions of the volume fraction of piezoelectric phase and also their implications for the design of pulse-echo ultrasonic transducers are shown. Material parameters of the piezoelectric and polymer phases used in the calculations are shown in Table 1.

A composite of $PZT-5A$ rods embedded in a passive polymer Araldite is now considered. The material values appearing in Table 1 were taken from [7, 11]. In Fig. 2, the electromechanical piezoelectric coupling coefficients $\overline{K}_t$ and $\overline{K}_p$ are plotted as functions of the piezoelectric volume fraction. The dotted line corresponds to the laminated composite 2–2 and the solid line to the fibrous composite 1–3. We observe that the value of $\overline{K}_t$ for the fibrous composite is greater than that for the laminated composite. Also we can appreciate that for the same values of piezoelectric volume fraction, the 1–3 composite has a smaller value of $\overline{K}_p$. Therefore, the composite 1–3 has better physical properties than the composite 2–2 for applications in transducers used for medical imaging applications.

Table 1. Material parameters

	PZT 5A	ARALDITE	TLZ-5	VDF/TrFE copolymer
$C_{11}^E\ (10^{10}\mathrm{N/m^2})$	12.10	0.546	12.6	0.85
$C_{12}^E\ (10^{10}\mathrm{N/m^2})$	7.54	0.294	7.95	0.36
$C_{13}^E\ (10^{10}\mathrm{N/m^2})$	7.52	0.294	8.41	0.36
$C_{33}^E\ (10^{10}\mathrm{N/m^2})$	11.10	0.546	10.9	0.99
$e_{33}\ (\mathrm{C/m^2})$	15.8	–	24.8	−0.29
$e_{31}\ (\mathrm{C/m^2})$	−5.4	–	−6.5	0.008
$\varepsilon_{33}^S/\varepsilon_0$	916	7.0	1813	6.0
$\rho\,(10^3\mathrm{Kg/m^3})$	7.75	1.17	7.898	1.88

$\varepsilon_0 = 8.85\ 10^{-12}\ \mathrm{C^2/Nm^2}$ (permittivity of free space)

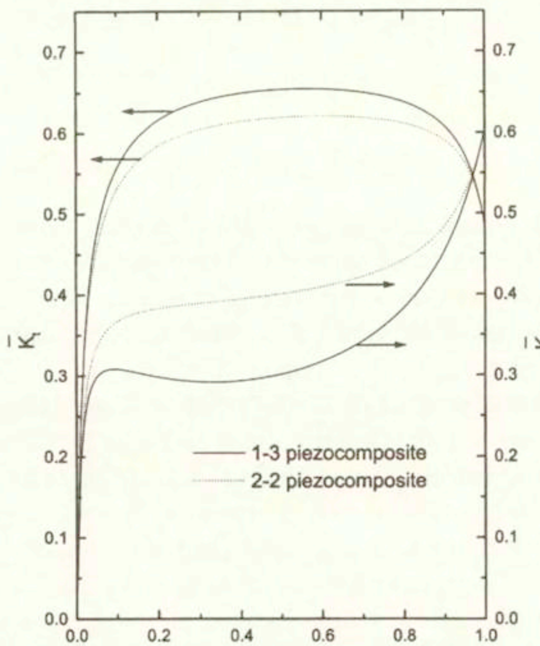

FIG. 2. Plot of electromechanical coupling coefficients $\overline{K}_t$ and $\overline{K}_p$ versus volume fraction of piezoelectric. The laminate composite 2–2 (continuous line) and the fibrous composite 1–3 (dotted line).

The analytical expressions that were derived using the "double homogenization" method can equally be used for a passive or piezoelectric matrix. The second example is for a 1–3 composite of $TLZ-5$ piezoelectric rods embedded in a piezoelectric copolymer $VDR/TrFE$. The properties are displayed in Table 1. In Fig. 3(a–f), the parameters: short-circuit stiffness constant (E-constant) $\overline{C}_{33}$, open-circuit stiffness constant (D-constant) $\overline{C}_{33}^{D}$ and dielectric constant $\overline{\varepsilon}_{33}^{T}/\varepsilon_0$, specific acoustic impedance $\overline{Z}$, electromechanical coupling constant $\overline{K}_t$, longitudinal wave speed $\overline{V}_l$ are plotted against volume fraction of the fibers, respectively. The continuous curve is the result of computation using the model proposed by the authors. The dotted curve for the *approximate theory* and the bullet points of the experimental results were reported in [11]. The agreement is seen to be quite good. The theoretical values show essentially the same trend as the experimental data [11] as well as the *approximate theory* used in [11] which was introduced by Smith and Auld (see, [4–5]). In this sense, six simplifying approximations to extract the essential physics are introduced en [4] and [5]. In connection with that, the main assumptions can be summarized as follows. First, the authors assume that the strain and electric field are independent of x and y throughout the individual phases. This is clearly not true in detail, as finite element calculations reveal. The expectation is that this approximation captures the physical behavior in an average sense. Second, they add the usual simplifications made in analyzing the thickness mode oscillations in a large, thin, electrode plate (symmetry in the $x-y$ plane, $E_1 = E_2 = 0$, etc). The third approximation embodies the picture that the ceramic and polymer move together in a uniform thickness oscillation. Thus the vertical strains (in the z direction) are the same in both phases. This is clearly not always true as the laser probe measurements of the displacements of oscillating composite plates reveal. Fourth, they describe the electric fields in two phases. Since the faces of the composite plates are equi-potential, they take the electrofields to be the same in both phases. Fifth approximation concerns the lateral interaction between the phases. They assume that the lateral stresses are equal in both phases and that the lateral strain in ceramics is compensated by a complementary strain in the polymer, so that the composite as a whole is laterally clamped. Sixth approximation deals with the dependent coordinates. Since the lateral periodicity is sufficiently fine, the authors obtain the effective total stress and electric displacement by averaging over the contributions of the constituent phases (rule of phases for both x_3-components of strains and electric displacement).

Note that the results of the "simple" physical analysis of Auld–Smith [4–5] agree remarkably well with the "rational" homogenization method, while the Auld–Smith results concern a thin plate with square inclusions whereas the "double homogenization" technique is applied to a body which is infinite in di-

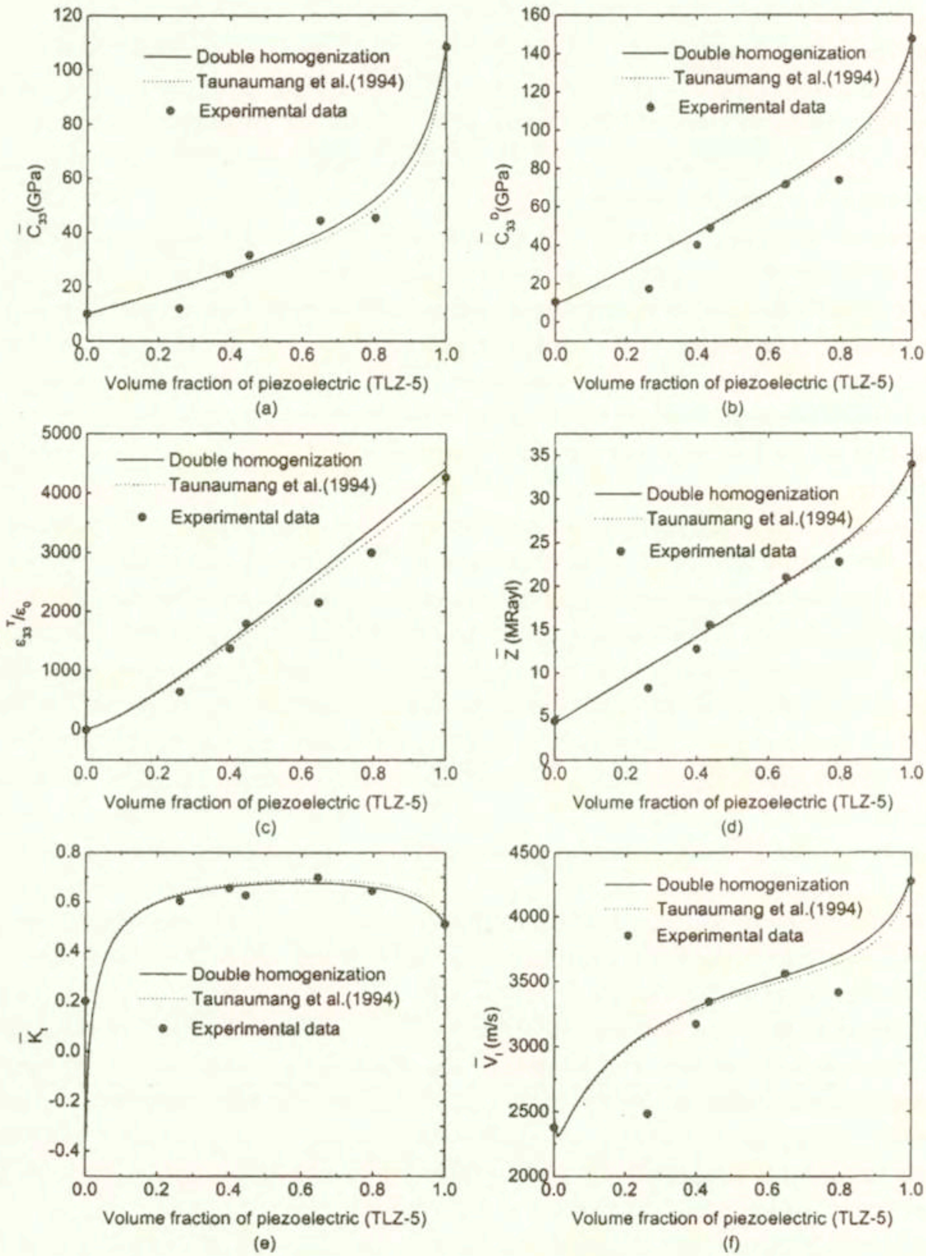

FIG. 3. Comparison between the "double asymptotic homogenization" with the theoretical and experimental results: (a) short-circuit stiffness constant (E-constant) $\overline{C}_{33}$, (b) open-circuit stiffness constant (D-constant) $\overline{C}_{33}^{D}$, (c) dielectric constant $\overline{\varepsilon}_{33}^{T}/\varepsilon_0$, (d) acoustic impedance $\overline{Z}$, (e) electromechanical coupling constant $\overline{K}_t$, (f) stiffened longitudinal velocity $\overline{V}_l$.

rection x_3. This may correspond to different hypotheses concerning the states of stress and strains in the body as in the case in a plate and the cross-section of an infinite cylinder. This may explain the difference observed the last value $\overline{d}_h$. This hydrostatic charge coefficient $\overline{d}_h$ is plotted as a function of the fiber volume fraction in Fig. 4. The solid line for the values are obtained using the "double asymptotic homogenization" method. The dotted line for the calculated values and the experimental bullet points were taken from [11]. This proves that the agreement between the experimental data and the "double homogenization" prediction is very good.

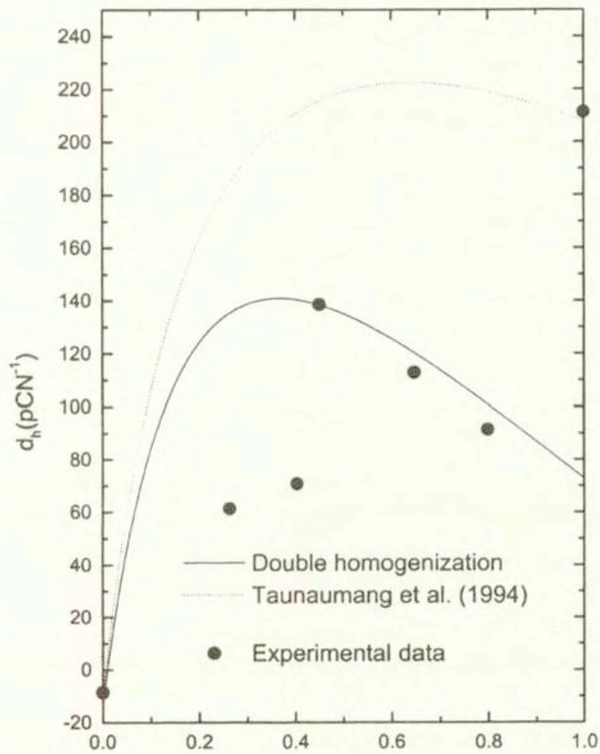

FIG. 4. The hydrostatic charge constant $\overline{d}_h$ versus fiber volume fraction.

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References

1. A. A. GREKOV, S. O. KRAMAROV and A. A. KUPRIENKO, *Effective properties of a transversely isotropic piezocomposite with cylindrical inclusions*, Ferroelectrics, **76**, 43–48, 1987.

2. Y. BENVENISTE and G. J. DVORAK, *Uniform fields and universal relations in piezoelectric composites*, J. Mech. Phys. Solids **40**, 1295–1312, 1992.

3. F. CRACIUN, L. SORBA, E. MOLINARI and M. PAPPALARDO, *A coupled-mode theory for periodic piezoelectric composites*, IEEE Trans. Ultrason. Ferroelectr. and Freq. Control, **36**, 50–56, 1989.

4. W. A. SMITH and B. A. AULD, *Modeling 1–3 composite piezoelectrics: thickness-mode oscillations*, IEEE Trans. Ultrason. Ferroelectr. and Freq. Control, **38**, 40–47, 1991.

5. W. A. SMITH, *Modeling 1–3 composite piezoelectrics: hydrostatic response*, IEEE Trans. Ultrason. Ferroelectr. and Freq. Control, **40**, 41–49, 1993.

6. A. GALKA, J. J. TELEGA and R. WOJNAR, *Some computational aspects of homogenization of thermopiezoelectric composites*, Comput. Assisted Mech. Engng. Sci. **3**, 133–154, 1996.

7. J. B. CASTILLERO, J. A. OTERO, R. R. RAMOS and A. BOURGEAT, *Asymptotic homogenization of laminated piezocomposite materials*, Int. J. Solids Struct., **35**, 527–541, 1998.

8. J. PASTOR, *Homogenization of linear piezoelectric media*, Mech. Res. Commun., **24**, 145–150, 1997.

9. IGOR V. ANDRIANOV, V. V. DANISHEVSKIĬ and DIETER WEICHERT, *Asymptotic determination of effective elastic properties of composite materials with fibrous square-shaped inclusions*, Eur. J. Mech. A/Solids, **21**, 1019–1036, 2002.

10. I. V. ANDRIANOV, G. A. STARUSHENKO, V. V. DANISHEVSKIĬ and S. TOKARZEWSKI, *Homogenization procedure and Padé approximants for effective heat conductivity of composite materials with cylindrical inclusions having square cross section*, Proc. R. Soc. Lond. A, **455**, 3401–3413, 1999.

11. H. TAUNAUMANG, I. L. GUY and H. L. W. CHAN, *Electromechanical properties of 1–3 piezoelectric ceramic/piezoelectric polymer composites*, J. Appl. Phys. **76**, 484–489, 1994.

12. K. SCHULGASSER, *Relationships between the effective properties of transversely isotropic piezoelectric composites*, J. Mech. Phys. Solids **40**, 473–479, 1992.

13. R. GUINOVART-DÍAZ, J. BRAVO-CASTILLERO, R. RODRÍGUEZ-RAMOS and F. J. SABINA, *Closed-form expressions for the effective coefficients of fibre-reinforced composite with transversely isotropic constituents. I. Elastic and hexagonal symmetry*, J. Mech. Phys. Solids, **49**, 1445–1462, 2001.

14. F. J. Sabina, R. Rodríguez-Ramos, J. Bravo-Castillero and R. Guinovart-Díaz, *Closed-form expressions for the effective coefficients of fibre-reinforced composite with transversely isotropic constituents. I. Piezoelectric and hexagonal symmetry*, J. Mech. Phys. Solids, **49**, 1463–1479, 2001.

15. R. Rodríguez-Ramos, F. J. Sabina, R. Guinovart-Díaz and J. Bravo-Castillero, *Closed-form expressions for the effective coefficients of a fiber-reinforced composite with transversely isotropic constituents. I. Elastic and square symmetry*, Mech. Mat., **33**, 223–235, 2001.

16. J. Bravo-Castillero, R. Guinovart-Díaz, F. J. Sabina and R. Rodríguez-Ramos, *Closed-form expressions for the effective coefficients of a fiber-reinforced composite with transversely isotropic constituents. II. Piezoelectric and square symmetry*, Mech. Mat., **33**, 237–248, 2001.

17. R. Guinovart-Díaz, J. Bravo-Castillero, R. Rodríguez-Ramos and F. J. Sabina, *Overall properties of piezocomposite materials 1-3*, Mat. Lett., **48**, 93–98, 2001.

18. D. A. Berlincourt, D. R. Curran and H. Jaffe, *Piezoelectric and piezomagnetic materials and their function in transducers*, Phys. of Acoust., **1A**, W. P. Mason [Ed.] Academic Press, New York 1964.

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