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Fluctuating flow of a third order fluid past an infinite plate with variable suction

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The two-dimensional flow problem of a third order incompressible fluid past an infinite porous plate is discussed when the suction velocity normal to the plate, as well as the the external flow velocity, varies periodically with time. The governing partial differential equation is of third order and nonlinear. Analytic solution is obtained using the series method. Expressions for the velocity and the skin friction have been obtained in a dimensionless form. The results of viscous and second order fluids can be recovered as special cases of this problem. Finally, several graphs are plotted and discussed.

1. Introduction

The oscillating flows play an important role in many engineering applications. The study of such flows was first initiated by Lighthill [1] who studied the effects of free stream oscillations on the boundary layer flows of viscous, incompressible fluid past an infinite plate. Thereafter Stuart [2] extended it to study a two-dimensional flow past an infinite, porous plate with constant suction when the free stream oscillates in time about a constant mean. After the appearance of Lighthill's [1] classic paper on the response of skin friction in laminar flow due to fluctuations in the free stream, considerable interest has been developed in the subject of boundary layers which have a regular fluctuating flow superimposed on the mean boundary flow. A large number of papers dealing with this subject have appeared, cf. for example Watson [3], Messiha [4], Kelly [5] and Lal [6]. The idea has been also extended to magnetohydrodynamic flows, Suryaprakasaro [7], and the elastico-viscous flows, Kaloni [8], Soundalgekar and Puri [9] and Puri [10]. The boundary layer suction is a very effective method for prevention of the separation. The effects of different arrangements and configurations of the suction holes and slits on the undesired phenomenon of separation have been studied extensively by various scholars, and have been compiled by Lachman [11]. In technological fields, the boundary layer phenomenon in non-Newtonian fluids has recently become a fascinating problem, under a wide range of geometrical, dynamical and rheological conditions.

Some experiments by Barnes *et al.* [12] confirmed that an increase in the flow rate is possible and that the phenomenon appears to be governed by the shear-dependent viscosity. In fact, in [13] Walters and Townsend show that the mean flow rate is unaffected by second-order viscoelasticity. Although the second-order model is able to predict the normal stress differences which are characteristic of non-Newtonian liquids, it is not shear thinning or thickening, the shear viscosity is constant. Third-order model exhibits shear-dependent viscosity, for a simple-shearing motion ($u' = (\gamma y', 0, 0)$), where γ is the rate of strain. The relation between the shearing stress and the rate of strain is given by $S_{xy} = \mu\left(1 \mp T_s^2\gamma^2\right)\gamma$, where T_s is the shear relaxation time (its reciprocal is the characteristic rate of strain at which the apparent shear viscosity noticeably decreases or increases), and μ is the lower limiting viscosity. Experiments made by Bruce [14] has shown that there are materials that exhibit: (1) strong normal stresses but are weakly shear thinning or thickening (class 1 a, b); (2) roughly equal normal and shear effects (class 2 a, b); (3) weak normal stresses, but they are strongly shear thinning or thickening (class 3 a, b).

Since many years there has been much interest in the effect of a variable suction velocity on the flow field. Regarding the elasto-viscous (Walters liquid B$'$) model, Soundalgekar and Puri [9] obtained the perturbation solution for the fluctuating flow of the elasto-viscous fluids past an infinite plate with variable suction.

As far as the authors are aware, no attempt has been made to examine the effect of the variable suction velocity on the flow fields of third-order fluids past an infinite plate. In the present work such an attempt has been considered. Literature survey revealed no previous attempts on studying this problem, even in the constant suction velocity case. The external flow velocity in the present paper is taken as $U_0'\left[1+\in e^{iw't'}\right]$ and the suction velocity is assumed to be of the form $v_0'\left[1+\in Ae^{iw't'}\right]$, where v_0' is a non-zero constant mean suction velocity, $\in$ is small and A is a positive constant such that $\in A \leq 1$. By neglecting higher powers of $\in$, approximate solutions are obtained for the velocity field in the boundary layer.

2. The constitutive model

The incompressible, homogeneous fluid of third order is a simple fluid of the differential type whose Cauchy stress tensor has the representation [15]

$$\mathbf{T} = -p\mathbf{I}+\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2+\beta_1\mathbf{A}_3+\beta_2\left(\mathbf{A}_1\mathbf{A}_2+\mathbf{A}_2\mathbf{A}_1\right)+\beta_3\left(\mathrm{tr}\mathbf{A}_1^2\right)\mathbf{A}_1, \tag{2.1}$$

where $-p\mathbf{I}$ is the indeterminate part of the stress due to the constraint of incompressibility, $\mu, \alpha_1, \alpha_2, \beta_1, \beta_2$ and β_3 are material constants, and the tensors A_n, $n = 1, 2, 3$ are defined through [16]

$$\begin{aligned}
&\mathbf{A}_1 = (\mathrm{grad}\mathbf{V}) + (\mathrm{grad}\mathbf{V})^T, \\
(2.2)\qquad &\mathbf{A}_n = \left(\frac{\partial}{\partial t'} + \mathbf{V}.\nabla\right)\mathbf{A}_{n-1} + \mathbf{A}_{n-1}(\mathrm{grad}\mathbf{V}) + (\mathrm{grad}\mathbf{V})^T\mathbf{A}_{n-1}, \; n > 1,
\end{aligned}$$

where $\mathbf{V}$ is the velocity and t' is the time.

JOSEPH [17] proved that the rest state of fluids of grade n, $n \neq 1$, any is unstable in the spectral sense of linearized theory when the ratio of the coefficients of $\mathbf{A}_n$ and $\mathbf{A}_{n-1}$ in the constitutive equation is negative. Hence, if $\alpha_1 < 0$ then the above model exhibits unacceptable stability characteristics. On the other hand, Eq. (2.1) must be consistent with thermodynamics principles. The thermodynamic of fluid model by Eq. (2.1) has been the object of a detailed study by FOSDICK and RAJAGOPAL [18]. They have shown that the Eq. (2.1) to be compatible with thermodynamics, and the free energy to be minimum when the fluid is at rest, the material constants should satisfy the relations

$$\begin{aligned}
&\mu \geq 0, \qquad \alpha_1 \geq 0, \qquad \beta_1 = \beta_2 = 0, \\
(2.3)\qquad &\beta_3 \geq 0, \qquad -\sqrt{24\mu\beta_3} \leq \alpha_1 + \alpha_2 \leq \sqrt{24\mu\beta_3}.
\end{aligned}$$

It is easy to see that the ratio of the coefficients of A_2 and A_3 in the form of $\mathbf{T}$, i.e. the "ratio" $\dfrac{\alpha_1}{0}$, does not satisfy neither the hypothesis of JOSEPH [17] nor the hypothesis of RENARDY [19], who assumed the coefficients α_{n-1} ($n \geq 5$ and here 3) of $\mathbf{A}_n$ is non-zero for instability. We also point out that the retarded motion approximation does not lead the models. Thus subject was clearly explained by DUNN and RAJAGOPAL [20]. Therefore, the model of Eq. (2.1) reduces to:

$$(2.4)\qquad \mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3\left(\mathrm{tr}\mathbf{A}_1^2\right)\mathbf{A}_1.$$

The equation of motion, in the absence of body forces, is

$$(2.5)\qquad \rho'\frac{d\mathbf{V}}{dt'} = \mathrm{div}\mathbf{T},$$

where ρ' is the density of the fluid in the dimensional form and $\dfrac{d}{dt'}$ is the material derivative. The fluid is incompressible, thus only isochoric (i.e. volume preserving) flows are possible, i.e. the flow satisfies the constraints

$$(2.6)\qquad \mathrm{div}\mathbf{V} = 0.$$

We consider a two-dimensional incompressible fluid flow along an infinite plane porous wall. The flow is independent of the distance parallel to the wall and the suction velocity normal to the wall is directed towards it and varies periodically with time about a non-zero constant mean value v_0'. The x'-axis is taken along the wall, y'-axis normal to the wall. Dash denotes dimensional quantities. Thus for the problem under consideration, we seek a velocity field of the form

$$\mathbf{V} = \left[u'\left(y', t'\right), v', 0\right], \tag{2.7}$$

where $v' < 0$ is the suction velocity.

From Eqs. (2.6) and (2.7)

$$\frac{\partial v'}{\partial y'} = 0. \tag{2.8}$$

It is evident from Eq. (2.8) that v' is a function of time only. Hence we consider v' in the form [4]

$$v' = -v_0'(1 + \in Ae^{i\omega' t'}). \tag{2.9}$$

The negative sign in Eq. (2.9) indicates that the suction velocity normal to the wall is directed towards the wall. In view of Eqs. (2.4), (2.7) and (2.9), Eq. (2.5) takes the form

$$\frac{\partial u'}{\partial t'} - v_0'(1 + \in Ae^{i\omega' t'})\frac{\partial u'}{\partial y'} = -\frac{1}{\rho'}\frac{\partial P'}{\partial x'} + \nu\frac{\partial^2 u'}{\partial y'^2} + \frac{\alpha_1}{\rho'}\left[\frac{\partial^3 u'}{\partial y'^2 \partial t'} - v_0'\left(1 + \in Ae^{i\omega' t'}\right)\frac{\partial^3 u'}{\partial y'^3}\right] + \frac{6\beta_3}{\rho'}\left(\frac{\partial u'}{\partial y'}\right)^2 \frac{\partial^2 u'}{\partial y'^2}, \tag{2.10}$$

$$\frac{\partial v'}{\partial t'} = -\frac{1}{\rho'}\frac{\partial P'}{\partial y'}, \tag{2.11}$$

where

$$\nu = \frac{\mu}{\rho'},$$

$$P' = p' - (2\alpha_1 + \alpha_2)\left(\frac{\partial u'}{\partial y'}\right)^2.$$

From Eqs. (2.9) and (2.11), it is clear that $\frac{\partial P'}{\partial y'}$ is small in the boundary layer and can be neglected [9]. Hence the pressure is taken to be constant along any normal and is given by its value outside the boundary layer. If $U'(t')$ is the stream velocity parallel to the wall just outside the boundary layer, then

$$-\frac{1}{\rho'}\frac{\partial P'}{\partial x'} = \frac{dU'}{dt'}$$

and the Eq. (2.10) takes the form

$$\frac{\partial u'}{\partial t'} - v_0'\left(1+\in Ae^{i\omega' t'}\right)\frac{\partial u'}{\partial y'} = \frac{dU'}{dt'} + \nu\frac{\partial^2 u'}{\partial y'^2} + \frac{\alpha_1}{\rho'}\left[\frac{\partial^3 u'}{\partial y'^2 \partial t'} - v_0'\left(1+\in Ae^{i\omega' t'}\right)\frac{\partial^3 u'}{\partial y'^3}\right] + \frac{6\beta_3}{\rho'}\left(\frac{\partial u'}{\partial y'}\right)^2\frac{\partial^2 u'}{\partial y'^2}. \tag{2.12}$$

The boundary conditions are

$$u' = 0 \quad \text{at} \quad y' = 0 \quad \text{and} \quad u' = U'(t') \quad \text{as} \quad y' \longrightarrow \infty. \tag{2.13}$$

We introduce dimensionless quantities defined by

$$\begin{gathered} y = \frac{y' v_0'}{\nu}, \quad t = \frac{v_0'^2 t'}{4\nu}, \quad \omega = \frac{4\nu\omega'}{v_0'^2}, \\ \alpha = \frac{\alpha_1 v_0'^2}{\rho\nu^2}, \quad u = \frac{u'}{U_0'}, \quad U = \frac{U'}{U_0'}, \quad \in_1 = \frac{6\beta_3}{\rho'\nu^3}U_0'^2 v_0'^2, \end{gathered} \tag{2.14}$$

where U_0' is the reference velocity and ω' is the frequency. Equation (2.12) takes the dimensionless form

$$\frac{1}{4}\frac{\partial u}{\partial t} - (1+\in Ae^{i\omega t})\frac{\partial u}{\partial y} = \frac{1}{4}\frac{dU}{dt} + \frac{\partial^2 u}{\partial y^2} + \alpha\left[\frac{1}{4}\frac{\partial^3 u}{\partial y^2 \partial t} - (1+\in Ae^{i\omega t})\frac{\partial^3 u}{\partial y^3}\right] + \in_1\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}, \tag{2.15}$$

subject to the conditions

$$u = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad u \longrightarrow U \quad \text{as} \quad y \longrightarrow \infty, \tag{2.16}$$

where

$$U = 1+\in e^{i\omega t}. \tag{2.17}$$

3. Perturbation solution

We note that the resulting equation of motion (2.15) is of the third order. Moreover, this equation is nonlinear as compared to the cases of the second order, elastic-viscous [9] and Newtonian flow [4] equations. As a result, it seems to be impossible to obtain the general solution in a closed form for arbitrary values of all parameters appearing in the nonlinear equation. Even in the case of constant suction and elastic-viscous fluid [8], all analytic solutions obtained so far are based on the assumptions that one or more of the parameters are zero or small. Therefore, we seek the solution of the problem as a power series expansion in the small parameters $\in_1$. Accordingly, we assumed that the velocity component u can be expanded in powers of $\in_1$ as follows:

$$u(y, \in_1) = u_0(y) + \in_1 u_1(y) + \dots \tag{3.1}$$

Substituting Eq. (3.1) into Eq. (2.15) and the boundary conditions (2.16), and then collecting terms of the same powers of $\in_1$, one obtains the following systems of partial differential equations along with appropriate boundary conditions.

System of order zero

$$\frac{1}{4}\frac{\partial u_0}{\partial t} - (1+ \in Ae^{i\omega t})\frac{\partial u_0}{\partial y} = \frac{i\omega}{4} \in e^{i\omega t} + \frac{\partial^2 u_0}{\partial y^2} + \alpha\left[\frac{1}{4}\frac{\partial^3 u_0}{\partial y^2 \partial t} - \left(1+ \in Ae^{i\omega t}\right)\frac{\partial^3 u_0}{\partial y^3}\right], \tag{3.2}$$

$$u_0 = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad u_0 \longrightarrow 1+ \in e^{i\omega t} \quad \text{as} \quad y \longrightarrow \infty. \tag{3.3}$$

System of order one

$$\frac{1}{4}\frac{\partial u_1}{\partial t} - (1+ \in Ae^{i\omega t})\frac{\partial u_1}{\partial y} = \frac{\partial^2 u_1}{\partial y^2} + \alpha\left[\frac{1}{4}\frac{\partial^3 u_1}{\partial y^2 \partial t} - \left(1+ \in Ae^{i\omega t}\right)\frac{\partial^3 u_1}{\partial y^3}\right] + \left(\frac{\partial u_0}{\partial y}\right)^2 \frac{\partial^2 u_0}{\partial y^2}, \tag{3.4}$$

$$u_1 = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad u_1 \longrightarrow 0 \quad \text{as} \quad y \longrightarrow \infty. \tag{3.5}$$

Zeroth-order solution

We note that the zeroth order mathematical problem is same as that of SOUNDALGEKAR and PURI [9] except that $(-k)$ is replaced by α in Eq. (3.2). Thus, in order to avoid repetition, the details of calculations are omitted and the solution is directly given by

$$u_0(y,t) = 1 - e^{-y} - \alpha y e^{-y} + \in e^{i\omega t}\begin{bmatrix} 1 - Se^{-hy} - (1-S)e^{-y} + Lye^{-hy} \\ -\alpha\begin{pmatrix} (1-S)\,e^{-hy} \\ -(1-y)\,e^{-y}\end{pmatrix}\end{bmatrix}, \tag{3.6}$$

where

$$h = \left[\frac{\sqrt{1+i\omega}+1}{2}\right], \tag{3.7}$$

$$L = \frac{h^2\left(h+\dfrac{i\omega}{4}\right)\left(1-\dfrac{4iA}{\omega}\right)}{\sqrt{1+i\omega}}, \tag{3.8}$$

$$S = 1 - \frac{4iA}{\omega}. \tag{3.9}$$

First-order solution

Now, let

$$u_1(y,t) = f_1(y) + \in e^{i\omega t} f_2(y). \tag{3.10}$$

Substituting Eqs. (3.6) and (3.10) in Eq. (3.4) and boundary conditions (3.5), comparing nonharmonic and harmonic terms and neglecting coefficients of $\in^2$, we get

$$\alpha\frac{d^3 f_1}{dy^3} - \frac{d^2 f_1}{dy^2} - \frac{df_1}{dy} = e^{-3y}(1+\alpha y), \tag{3.11}$$

$$\alpha\frac{d^3 f_2}{dy^3} - \left(1+\frac{i\omega\alpha}{4}\right)\frac{d^2 f_2}{dy^2} - \frac{df_2}{dy} + \frac{i\omega}{4}f_2 = A\frac{df_1}{dy} + B_1 - \alpha\left[A\frac{d^3 f_1}{dy^3} + B_2\right], \tag{3.12}$$

$$\text{at} \quad y = 0 \quad \text{and} \quad y \longrightarrow \infty : f_1 = f_2 = 0, \tag{3.13}$$

where

$$B_1 = e^{-3y}\left[\frac{A}{2} - 3(1-S)\right] + \frac{A}{6}e^{-y} + (h^2S - 2hS)e^{-(h+2)y}, \tag{3.14}$$

$$\begin{aligned} B_2 = {} & e^{-y}\left[\frac{A}{12}(9+2y) + 2 - \frac{A}{6}\right] \\ & - e^{-3y}\left[\frac{A}{4}(9+y^2-2y) - \frac{A}{6}(y-1) - \frac{9A}{2} - 1 + 3y + S - 2Sy\right] \\ & - e^{-(h+2)y}\left[2h(1-S) + 4hL - 2L - h^2 - h^2Ly\right]. \end{aligned} \tag{3.15}$$

There have been several investigations devoted to study the existence and uniqueness of the solutions to the equations governing the flows of fluids of differential type [21–23]. These equations are usually higher order partial differential equations than the Navier–Stokes equations. Hence the issue of whether the "no-slip" boundary condition is sufficient to have a well-posed problem is very important. This question can not be answered by any generality for fluids of differential type of complexity n, for arbitrary n. However, if attention is confined to fluids of grade 2 or grade 3, one can provide some definite answers, while some partial answers are also possible for fluids of grade n [24].

Before proceeding with the solution of Eqs. (3.11) and (3.12), it would be interesting to remark here that although in the classical viscous case ($\alpha = 0$) we encounter differential equations of order two [2, 4], the presence of the material parameter of the second order fluid increases the order to three. It would therefore seem that the additional boundary condition must be imposed in order to get a unique solution. In order to overcome such a difficulty, several authors have studied an acceptable additional condition. FOSDICK and BERSTEIN [25] have studied the flow in the annular region between two porous rotating cylinders. They assumed one of the constants in the solutions to be zero. However, there is no apparent reason for such a choice. FRATER [26] has studied the asymptotic suction flow. Since only two of the coefficients in the solution can be found by the no-slip condition, he imposes an extra condition that the solution tends to the Newtonian value as the coefficient of the higher derivative in the equation approaches zero. However, the perturbation expansion may give correct results under certain conditions [27]. Thus following [8, 28], we overcome the difficulty in the present study using perturbation expansion for small material parameter α and assume the solution in the form as follows [28]:

$$\begin{aligned} f_1 &= f_{01} + \alpha f_{11} + O(\alpha^2), \\ f_2 &= f_{02} + \alpha f_{12} + O(\alpha^2), \end{aligned} \tag{3.16}$$

which is valid for small values of α only. Putting Eq. (3.16) in Eqs. (3.11) and (3.12) and equating the coefficient of α we obtain

$$\frac{d^2 f_{01}}{dy^2} + \frac{df_{01}}{dy} = -e^{-3y}, \tag{3.17}$$

$$\frac{d^2 f_{11}}{dy^2} + \frac{df_{11}}{dy} = -\frac{df_{01}^3}{dy^3} + ye^{-3y}, \tag{3.18}$$

$$\frac{d^2 f_{02}}{dy^2} + \frac{df_{02}}{dy} - \frac{i\omega}{4} f_{02} = B_1, \tag{3.19}$$

$$\frac{d^2 f_{12}}{dy^2} + \frac{df_{12}}{dy} - \frac{i\omega}{4} f_{12} = -\frac{d^3 f_{02}}{dy^3} - \frac{i\omega}{4}\frac{df_{02}}{dy} - B_2, \tag{3.20}$$

$$\begin{aligned} f_{01} = f_{11} = f_{02} = f_{12} = 0 \ \text{at} \ y = 0, \\ f_{01} = f_{11} = f_{02} = f_{12} = 0 \ \text{as} \ y \longrightarrow \infty. \end{aligned} \tag{3.21}$$

Solving Eqs. (3.17) to (3.20) under the boundary conditions (3.21), we have, in view of Eq. (3.16),

$$f_1 = \frac{1}{12}\left[e^{-y}\left\{2 + \alpha\left(9 + 2y\right)\right\} - e^{-3y}\left\{\alpha\left(9 + y^2 - 2y\right) + 2\right\}\right], \tag{3.22}$$

$$\begin{aligned} f_2 = {} & M_1\left(e^{-3y} - e^{-hy}\right) + N_1\left(e^{-y} - e^{-hy}\right) + P_1\left(e^{-(h+2)y} - e^{-hy}\right) \\ & - \alpha\begin{pmatrix} M_2\left(e^{-3y} - e^{-hy}\right) + N_2\left(e^{-y} - e^{-hy}\right) + P_2\left(e^{-(h+2)y} - e^{-hy}\right) \\ + \dfrac{i}{3\omega}e^{-3y}\left(36y - 24Sy + 3Ay^2 - 2Ay\right) \end{pmatrix}, \end{aligned} \tag{3.23}$$

where

$$M_1 = \frac{2\left[A - 6\left(1 - S\right)\right]}{24 - i\omega}, \quad N_1 = \frac{2iA}{3\omega}, \quad P_1 = \frac{\left(h^2 - 2h\right)S}{h^2 + 3h + 2 - \dfrac{i\omega}{4}}, \tag{3.24}$$

$$M_2 = \frac{i}{\omega}\left(36M_1 + 9A + 8 - 8S\right), \quad N_2 = \frac{-4}{3\omega^2}\left(12N_1 - 13A - 24\right), \tag{3.25}$$

$$P_2 = \frac{2h(1 - S) + 4hL - 2L - h^2 - h^2 L}{h^2 + 3h - 2 - \dfrac{i\omega}{4}}. \tag{3.26}$$

In view of Eqs. (3.10), (3.22) and (3.23), we have

$$u_1 = \frac{1}{12}\Big[e^{-y}\left\{2+\alpha\left(9+2y\right)\right\} - e^{-3y}\left\{\alpha\left(9+y^2-2y\right)+2\right\}\Big] + \in e^{i\omega t}\begin{pmatrix} M_1\left(e^{-3y}-e^{-hy}\right)+N_1\left(e^{-y}-e^{-hy}\right) \\ +P_1\left(e^{-(h+2)y}-e^{-hy}\right) \\ -\alpha\begin{pmatrix} M_2\left(e^{-3y}-e^{-hy}\right)+N_2\left(e^{-y}-e^{-hy}\right) \\ +P_2\left(e^{-(h+2)y}-e^{-hy}\right) \\ +\frac{i}{3\omega}e^{-3y}\Big(36y-24Sy+3Ay^2-2Ay\Big)\end{pmatrix}\end{pmatrix}. \tag{3.27}$$

Now from Eqs. (3.6) and (3.27), the velocity field in the boundary layer is given by

$$u = 1-e^{-y}-\alpha y e^{-y} + \in e^{i\omega t}\begin{pmatrix} 1-Se^{-hy}-(1-S)e^{-y}- \\ \alpha\left\{(1-S)\,e^{-hy}-(1-y)\,e^{-y}\right\} \\ +Lye^{-hy}\end{pmatrix} \tag{3.28}$$
$$+ \in_1 \frac{1}{12}\Big[e^{-y}\left\{2+\alpha\left(9+2y\right)\right\} - e^{-3y}\left\{\alpha\left(9+y^2-2y\right)+2\right\}\Big]$$
$$+ \in_1 \in e^{i\omega t}\begin{pmatrix} M_1\left(e^{-3y}-e^{-hy}\right)+N_1\left(e^{-y}-e^{-hy}\right) \\ +P_1\left(e^{-(h+2)y}-e^{-hy}\right) \\ -\alpha\begin{pmatrix} M_2\left(e^{-3y}-e^{-hy}\right)+N_2\left(e^{-y}-e^{-hy}\right) \\ +P_2\left(e^{-(h+2)y}-e^{-hy}\right) \\ +\frac{i}{3\omega}e^{-3y}\left(36y-24Sy+3Ay^2-2Ay\right)\end{pmatrix}\end{pmatrix}.$$

The real, u_r, and the imaginary, u_i, parts of this expression, respectively, yield

$$u_r = 1-e^{-y}\big(1+\alpha y\big) + \frac{\epsilon_1}{12}\Big[e^{-y}\big\{2+\alpha(2y+9)\big\} - e^{-3y}\big\{\alpha\big(y^2-2y+9\big)+2\big\}\Big] + \epsilon\big\{M_r\cos(\omega t) - M_i\sin(\omega t)\big\}, \tag{3.29}$$

$$u_i = \epsilon\left(M_r\sin(\omega t) + M_i\cos(\omega t)\right), \tag{3.30}$$

where

$$M_r = m_{r_{10}} + \epsilon_1 m_{r_{11}}, \tag{3.31}$$

$$M_i = m_{i_{10}} + \epsilon_1 m_{i_{11}}. \tag{3.32}$$

The parameter functions $m_{r_{10}}, m_{i_{10}}, m_{r_{11}}$ and $m_{i_{11}}$ involved in u_r, u_i and M_r, M_i are explicitly computed, and are listed in the Appendix.

The other interesting aspect of the solution (3.28) is, however, the prediction of the shear stress near the wall. From Eq. (4) the expression for the shear stress is given by

$$P'_{x'y'} = \mu \frac{\partial u'}{\partial y'} + \frac{\alpha_1}{\rho'} \left[\frac{\partial^2 u'}{\partial y' \partial t'} - v'_0 \left(1 + \in A e^{i\omega' t'}\right) \frac{\partial^2 u'}{\partial y'^2} \right] + 2\beta_3 \left(\frac{\partial u'}{\partial y'} \right)^3, \tag{3.33}$$

which in virtue of Eq. (2.14) reduces to

$$P_{xy} = \frac{P'_{x'y'}}{U'_0 v'_0 \rho'} = \frac{\partial u}{\partial y} + \frac{\alpha}{4} \left[\frac{\partial^2 u}{\partial y \partial t} - 4\left(1 + \in A e^{i\omega t}\right) \frac{\partial^2 u}{\partial y^2} \right] + \frac{\in_1}{3} \left(\frac{\partial u}{\partial y} \right)^3. \tag{3.34}$$

where u is given by Eq. (3.28).

4. Discussions

In order to investigate the effects of the third order fluid on the velocity profile near the plate (both in case of constant and variable suction), we have plotted u_r against y in Figs. 1 to 4 for the different values of ϵ, ϵ_1, A, ω, α and $\omega t = \pi/2$. From Figs. 1 and 2 we observe that the velocity profile increases with fixed ω and large values of ϵ_1. Figure 3 is prepared to bring out the effects of the variable suction velocity on the separation of the fluid at the plate for large frequency. It is evident from this figure that velocity increases with an increase in ω, in A and ϵ_1, the third order fluid parameter. Further, for fixed ϵ_1, increase in ϵ, A and ω increases the velocity and then the two velocities coincide (see Fig. 4).

In Figs. 5 to 9 the fluctuating parts are plotted for different values of ϵ, ϵ_1, ω, α, A and for $\omega t = \pi/2$. For $A = 0$, it is noted that an increase in ϵ_1 with fixed ϵ and ω (Fig. 5) leads to a decrease in M_r, but with increase in ϵ_1 and for $\epsilon = 0.2$ and $\omega = 10$, M_r is almost the same. Figure 6 shows the effect of ϵ_1 in case of variable suction. In this case, it is noted that increase in ϵ_1 leads to a decrease in M_r first then the curves tend to coincide. Further, it is clear from Fig. 7 that for $\epsilon_1 = 0.7$ and increase in A and ω, results in a decrease in M_r, and ultimately the curves are almost the same. In case of non-Newtonian fluids at large ω and increase in ϵ_1 there is a fall of M_i (Fig. 8), which is not observed in Newtonian fluids. From Fig. 9, one can conclude that an increase in A and ω leads to an increase in M_i first; then there arises a decrease, then increase and finally it reaches zero level.

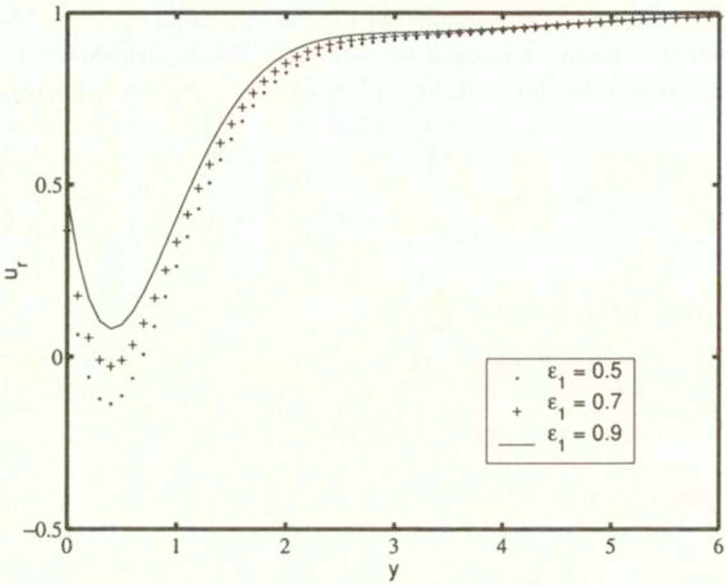

FIG. 1. Graphs for the parameter values $\alpha = 0.7, \epsilon = 0.5, \omega t = \pi/2, A = 0, \omega = 10$.

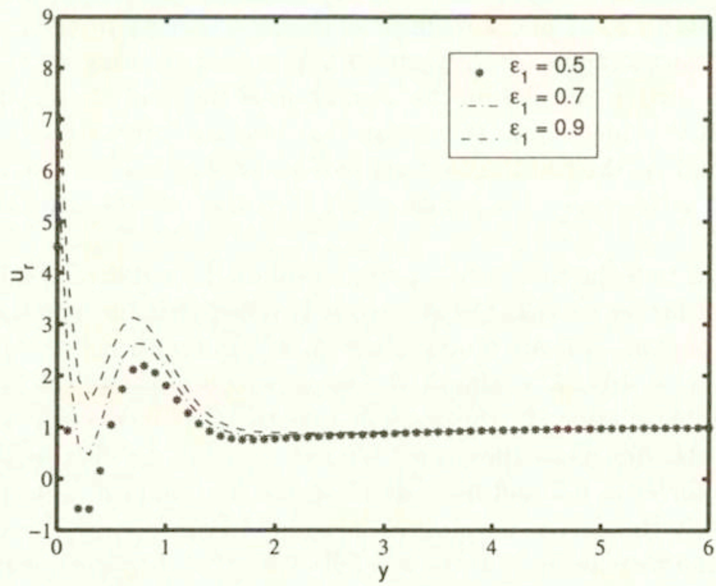

FIG. 2. Graphs for the parameter values $\alpha = 0.8, \epsilon = 0.5, \omega t = \pi/2, A = 0, \omega = 100$.

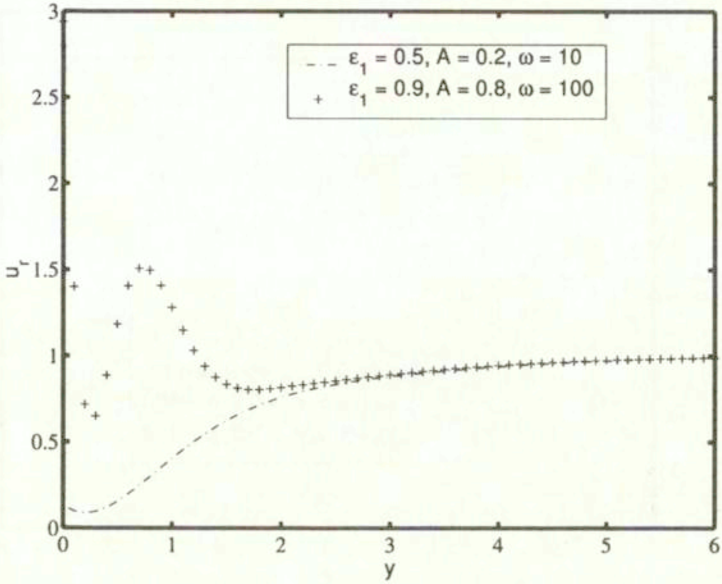

FIG. 3. Graphs for the parameter values $\epsilon = 0.2, \alpha = 0.8, \omega t = \pi/2$.

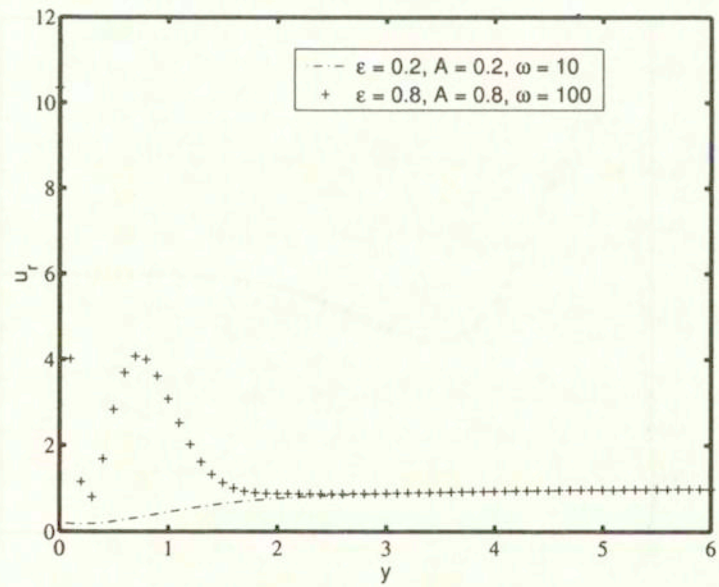

FIG. 4. Graphs for the parameter values $\epsilon_1 = 0.7, \alpha = 0.9, \omega t = \pi/2$.

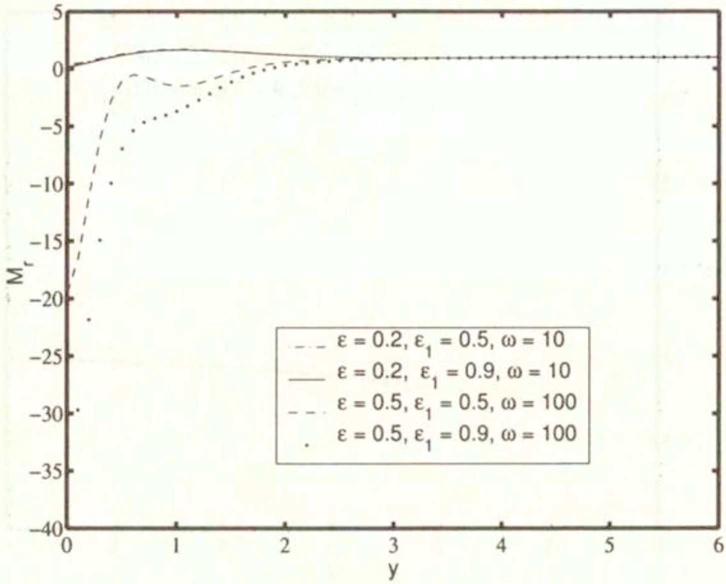

FIG. 5. Graphs for the parameter values $\alpha = 0.6, A = 0, \omega t = \pi/2$.

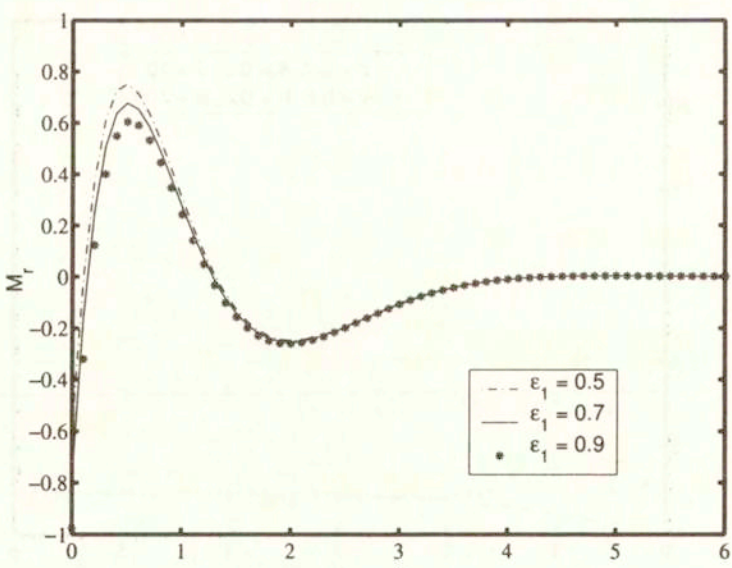

FIG. 6. Graphs for the parameter values $\alpha = 0.7, \omega t = \pi/2, \epsilon = 0.2, A = 0.4, \omega = 10$.

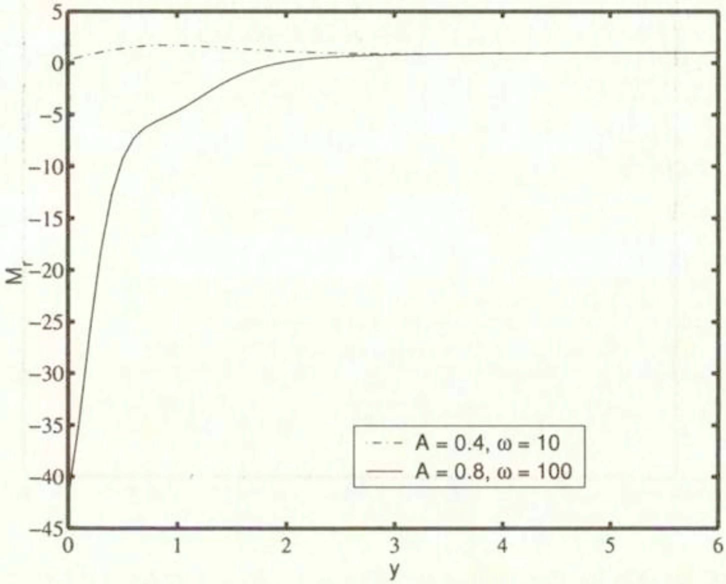

FIG. 7. Graphs for the parameter values $\alpha = 0.9, \omega t = \pi/2, \epsilon = 0.2, \epsilon_1 = 0.7$.

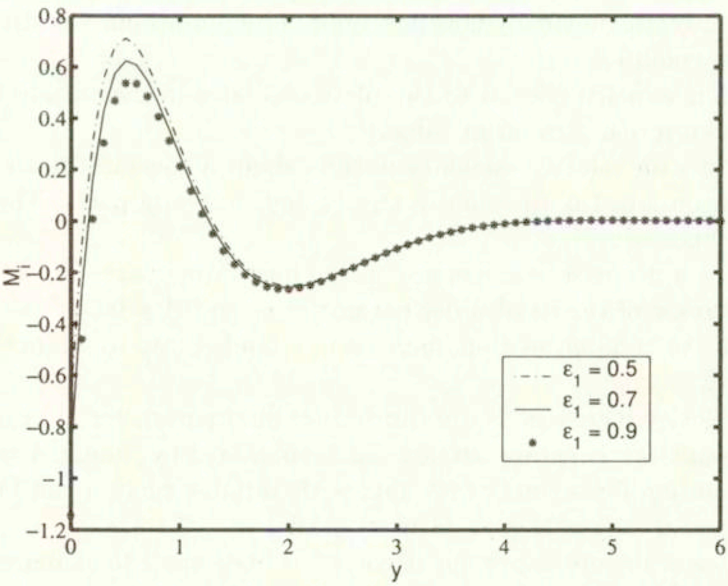

FIG. 8. Graphs for the parameter values $\alpha = 0.8, \omega t = \pi/2, \epsilon = 0.2, A = 0.4, \omega = 10$.

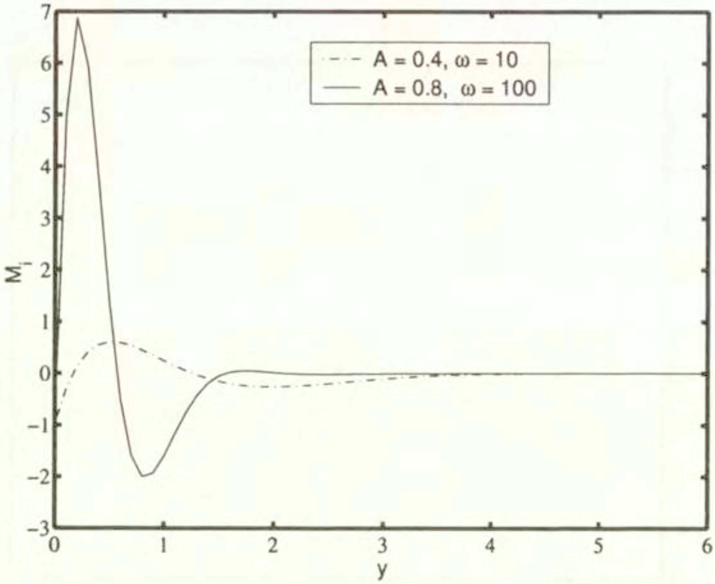

FIG. 9. Graphs for the parameter values $\alpha = 0.7, \omega t = \pi/2, \epsilon = 0.2, \epsilon_1 = 0.9$.

5. Conclusions

In this paper, the unsteady flow past an infinite porous plate is studied under the following conditions:
(i) the suction velocity normal to the plate oscillates in magnitude but not in direction about a non-zero mean value,
(ii) the free stream velocity oscillates in time about a constant mean value.
The solution obtained is the sum of steady and unsteady parts. The following results are obtained:

1. There is a decrease and increase in the fluctuating parts M_r and M_i with the increase of the third order parameter ϵ_1 and $A \neq 0$.
2. Increase of variable suction, increase in ϵ_1 and A lead to an increase in the velocity.
3. The velocity increases as the third order fluid parameter increases.
4. The results for constant suction can be obtained by taking $A = 0$.
5. The solution for second-order fluid with variable suction can be obtained as a special case of this problem by taking $\epsilon_1 = 0$.

As far as the authors are aware, no attempt has been made to examine the effect of variable suction velocity for second order fluids. However, a second order fluid exhibits normal stresses but is not shear thinning; the shear viscosity is constant. The third order approximation of a simple fluid exhibits shear-dependent viscosity. Keeping this fact in view, the problem considered for the third order fluid in this paper is more general.

Appendix

Equation (3.28) is a very complex algebraic equation. In order to split it into real and imaginary parts, for brevity, we define the following list of parameters:

$$m_{r_1} := \sqrt{\frac{1+\sqrt{1+\omega^2}}{2}}, \qquad m_{i_1} := \sqrt{\frac{-1+\sqrt{1+\omega^2}}{2}},$$

$$m_{r_2} := \frac{1}{2} + \frac{1}{2} m_{r_1}, \qquad m_{i_2} := \frac{1}{2} m_{i_1},$$

$$m_{r_3} := \frac{m_{r_1}}{m_{r_1}^2 + m_{i_1}^2}, \qquad m_{i_3} := -\frac{m_{i_1}}{m_{r_1}^2 + m_{i_1}^2},$$

$$m_{r_4} := m_{r_3}\left(R_4 + 4BR_3\right) - m_{i_3}\left(R_3 - 4BR_4\right),$$

$$m_{i_4} := m_{r_3}\left(R_3 - 4BR_4\right) + m_{i_3}\left(R_4 + 4BR_3\right),$$

$$m_{r_5} := 96A/R_5, \qquad m_{i_5} := 2A\left(\omega - (24)^2/\omega\right)/R_5,$$

$$m_{r_6} := \left\{R_6\left(R_7 + 4BR_8\right) + R_9\left(R_8 - 4BR_7\right)\right\} / \left(R_6^2 + R_9^2\right),$$

$$m_{i_6} := \left\{R_6\left(R_8 - 4BR_7\right) - R_9\left(R_7 + 4BR_8\right)\right\} / \left(R_6^2 + R_9^2\right),$$

$$m_{r_7} := -\frac{1}{\omega}\left(36 m_{i_5} + 32B\right), \quad m_{i_7} := \frac{1}{\omega}\left(36 m_{r_5} + 9A\right),$$

$$m_{r_8} := \frac{4}{3\omega^2}\left(13A + 24\right), \qquad m_{i_8} := -\frac{32A}{3\omega^3},$$

$$m_{r_9} := \left\{\left(-8Bm_{i_2} + 4R_{10} - R_{11} - R_{12}\right)R_{16} + \left(8Bm_{r_2} + 4R_{13} - 2R_{14} - R_{15}\right)R_9\right\} \div \left(R_{16}^2 + R_9^2\right),$$

$$m_{i_9} := \left\{\left(8Bm_{r_2} + 4R_{13} - 2R_{14} - R_{15}\right)R_{16} - \left(-8Bm_{i_2} + 4R_{10} - R_{11} - R_{12}\right)R_9\right\} \div \left(R_{16}^2 + R_9^2\right),$$

$$m_{r_{10}} := 1 - e^{-m_{r_2}y}(\cos(m_{i_2}y) - 4B\sin(m_{i_2}y)) - \alpha\big(4Be^{-m_{r_2}y}\sin(m_{i_2}y) - (1-y)e^{-y}\big)$$
$$+ ye^{-m_{r_2}y}\left(m_{r_4}\cos(m_{i_2}y) + m_{i_4}\sin(m_{i_2}y)\right),$$
$$m_{i_{10}} := e^{-m_{r_2}y}(4B\cos(m_{i_2}y) + \sin(m_{i_2}y)) - 4Be^{-y} - 4B\alpha e^{-m_{r_2}y}\cos(m_{i_2}y)$$
$$+ ye^{-m_{r_2}y}\left(m_{i_4}\cos(m_{i_2}y) - m_{r_4}\sin(m_{i_2}y)\right),$$

$$m_{r_{11}} := (m_{r_5} - \alpha m_{r_7})\left(e^{-3y} - e^{-m_{r_2}y}\cos(m_{i_2}y)\right) - (m_{i_5} - \alpha m_{i_7})\,e^{-m_{r_2}y}\sin(m_{i_2}y)$$
$$- \alpha m_{r_8}\left(e^{-y} - e^{-m_{r_2}y}\cos(m_{i_2}y)\right) - (2B/3 + m_{i_8})\,e^{-m_{r_2}y}\sin(m_{i_2}y)$$
$$+ 32\alpha Ae^{-3y}/\omega^2 + (m_{r_6} - \alpha m_{r_9})\,e^{-2y},$$
$$m_{i_{11}} := (m_{r_5} - \alpha m_{r_7})\,e^{-m_{r_2}y}\sin(m_{i_2}y) + (m_{i_5} - \alpha m_{i_7})\left(e^{-3y} - e^{-m_{r_2}y}\cos(m_{i_2}y)\right)$$
$$- \alpha m_{r_8}e^{-m_{r_2}y}\sin(m_{i_2}y) + (2B/3 - \alpha m_{i_8})\left(e^{-y} - e^{-m_{r_2}y}\cos(m_{i_2}y)\right)$$
$$- \left(12y + 3Ay^2 - 2Ay\right)\alpha e^{-3y}/(3\omega) + (m_{i_6} - \alpha m_{i_9})\,e^{-2y},$$

where

$$\begin{aligned}
R_1 &:= m_{r_2}^2 - m_{i_2}^2, \quad R_2 := m_{i_2} + \omega/4, \quad R_3 := 2m_{r_2}^2 m_{i_2} + R_1R_2,\\
R_4 &:= m_{r_2}R_1 - 2m_{r_2}m_{i_2}R_2, \quad R_5 := (24)^2 + \omega^2, \quad R_6 := R_1 + 3m_{r_2} + 2,\\
R_7 &:= R_1 - 2m_{r_2}, \quad R_8 := 2m_{r_2}m_{i_2} - 2m_{i_2}, \quad R_9 := 2m_{r_2}m_{i_2} + 3m_{i_2} - \omega/4,\\
R_{10} &:= m_{r_2}m_{r_4} - m_{i_2}m_{i_4}, \quad R_{11} := 2m_{r_4} + R_1,\\
R_{12} &:= m_{r_4}R_1 - 2m_{r_2}m_{i_2}m_{i_4}, \quad R_{13} := m_{i_2}m_{r_4} + m_{r_2}m_{i_4},\\
R_{14} &:= m_{i_4} + m_{r_2}m_{i_2}, \quad R_{15} := m_{i_4}R_1 + 2m_{r_2}m_{i_2}m_{i_4}, \quad R_{16} := R_1 + 3m_{r_2} - 2,\\
B &:= A/\omega.
\end{aligned}$$

The parameter functions $h, L, S, M_1, M_2, N_1, N_2, P_1$ and P_2 of Eqs. (3.7)–(3.9) and (3.24)–(3.26) can now be expressed in terms of these m_rs and m_is as follows:

$$h = \frac{1}{2} + \frac{1}{2}m_{r_1} + i\frac{1}{2}m_{i_1} = m_{r_2} + im_{i_2}, \qquad L = m_{r_4} + im_{i_4}, \qquad S = 1 - i4B,$$
$$M_1 = m_{r_5} + im_{i_5}, \qquad M_2 = m_{r_7} + im_{i_7}, \qquad N_1 = i2B/3,$$
$$N_2 = m_{r_8} + im_{i_8}, \qquad P_1 = m_{r_6} + im_{i_6}, \qquad P_2 = m_{r_9} + im_{i_9}.$$

Substituting the values of these parameters, Eq. (3.28) can be split into real and imaginary parts (the calculation is very lengthy and tedious but straightforward), u_r and u_i, as given in Eqs. (3.29) and (3.30), with

$$M_r = m_{r_{10}} + \epsilon_1 m_{r_{11}}, \qquad M_i = m_{i_{10}} + \epsilon_1 m_{i_{11}}.$$

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