Arch. Mech., **55**, 3, pp. 257–274, Warszawa 2003

On Bénard convection in a porous medium in the presence of throughflow and rotation in hydromagnetics

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The effect of throughflow on a layer of a rotating fluid heated from below in porous medium in the presence of a vertical magnetic field is considered. For the case of stationary convection, the rotation has always a stabilizing effect. The medium permeability has always a destabilizing effect whereas the magnetic field and the throughflow have always a stabilizing effects in the absence of rotation. But in the presence of rotation, the medium permeability is found to have a destabilizing effect whereas the magnetic field and the throughflow have a stabilizing effects under certain conditions. Graphs have been plotted by giving numerical values to the parameters, to depict the stability characteristics. The magnetic field and rotation introduce oscillatory modes in the system, which were nonexistent in their absence. The sufficient conditions for non-existence of the overstability are also obtained.

Key words: Thermal instability, throughflow, magnetic field, rotation, porous medium.

1. Introduction

The determination of the criterion for the onset of convection in a horizontal fluid layer heated uniformly from below is a classical problem associated with Lord Rayleigh and H. Bénard. The steady state conduction solution becomes unstable, and convection begins when the Rayleigh number R exceeds a certain critical value R_c. A comprehensive account of the onset of Bénard convection, under varying assumptions of hydromagnetics, has been given by Chandrasekhar [1]. In the classical problem, there is no flow of fluid across the horizontal boundaries. A slightly modified problem when a layer of fluid subjected to an adverse vertical temperature gradient with an imposed constant vertical motion downward/upward through the layer, called throughflow, produced by injection at one boundary and removal of fluid at the other boundary, is studied by Shvartsblat [2, 3, 4] and his results were summarized by

GERSHUNI and ZHUKHOVITSKII [5]. The throughflow is measured by a Péclet number P_e.

Shvartsblat pointed out that the problem is of interest because of the importance of possibility in controlling the convective instability by adjustment of the transverse throughflow and importance in control of convection by the adjustment of transverse throughflow and also due to its relevance in meteorology. He also found, for the case of conducting rigid permeable boundaries, that R_c was independent of the sign of P_e, and increased markedly with P_e increasing, i.e. the throughflow is stabilizing and is independent of the direction of the flow. GERSHUNI and ZHUKHOVITSKII ([5], p. 236) wrote that the stabilizing effect may be explained as follows. With increasing injection velocity, a temperature boundary layer forms at one of the boundaries. This decreases the effective thickness of the stratified layer of fluid which (at sufficiently large P_e) is of order $d_{\text{eff}} \sim d/P_e$, where d is the layer depth. On the other hand, the characteristic temperature difference across the layer remains fixed. The critical Rayleigh number defined in terms of d is thus of the order of $R_c \sim (d/d_{\text{eff}})^3$, so that it increases with the Péclet number according to $R_c \sim P_e^3$.

The effect of throughflow is in general quite complex. Not only is the basic temperature profile altered, but also in the perturbation equations certain contributions arise from the convection of both the temperature and velocity, and there is an interaction between all is these contributions. The meteorologists KRISHNAMURTI [6, 7, 8] and SOMERVILLE and GAL-CHEN [9] have discussed the effects of small amounts of throughflow, but their main interest in it was the measure of a vertical asymmetry and associated stability of hexagonal cells. NIELD [10] has studied the effect of vertical throughflow on the onset of convection in a fluid layer by considering the boundaries which are either rigid or free and either insulating or conducting. The effect of magnetic field on the stability of thermal flow is of interest to geophysics, particularly in the study of earth's core, when earth's mantle, which consists of conducting fluid, behaves like a porous medium that can become convectively unstable as a result of differential diffusion. Another application of the results of flow through a porous medium in the presence of a magnetic field is the study of stability of the convective geothermal flow.

The effect of vertical throughflow in a porous medium has not been extensively discussed so far, in spite of its natural occurrence in many geothermal and deep-sea hydrodynamic problems. The flow through porous media is of considerable interest for petroleum engineers and for specialists in geophysical fluid dynamics as stated in a book by CHIN [11]. A great number of applications in geophysics may be found in a book by PHILLIPS [12]. When the fluid slowly percolates through the pores of a rock, the gross effect is represented by Darcy's law. As a result of this macroscopic law, the usual viscous term in the equations of fluid motion is replaced by the resistance term $\left[-\dfrac{\mu}{k_1}\mathbf{q}\right]$, where μ is the viscosity of the fluid, k_1 is the medium permeability and $\mathbf{q}$ is the Darcian (filter) velocity of the fluid. HOMSY and SHERWOOD [13] and NIELD [14] have also studied the convective instability in porous medium with throughflow.

The purpose of the present study is to discuss the effect of throughflow (so-called mass-discharge) on thermal instability of the fluid in a porous medium in the presence of rotation in hydromagnetics by using the linearized stability theory and the normal mode analysis method. Earlier SPARROW [15] presented an experimental study of the heat transfer and temperature field in an enclosure in the presence of rotation and coolant throughflow. The *in-situ* processing of energy resources such as coal, oil shale, or geothermal energy, often involves the non-isothermal flow of fluids through porous medium. This throughflow is an integrated feature of in-situ processing, and it is of interest to assess its effect on the stability limits. Many operations and processes involving the thermal flow of rotating fluid through porous medium with throughflow commonly occur in geophysics, packed-bed processing, *in-situ* coal gasification and other problems.

2. Formulation of the problem and perturbation equations

Here we consider an infinite, horizontal, incompressible fluid layer of thickness d, with the uniform and prescribed vertical velocity w_0 at the horizontal boundaries, heated from below, so that the temperatures and densities at the bottom surface $z = 0$ are T_0 and ρ_0, and at the upper surface $z = d$ they are T_d and ρ_d, respectively, and that a uniform temperature gradient $\beta\,(= |dT/dz|)$ is maintained. Here w_0, the imposed vertical velocity is the magnitude of the throughflow. The gravity field $\mathbf{g} = (0,\ 0,\ -g)$, a uniform vertical magnetic field $\mathbf{H} = (0,\ 0,\ H)$ and a uniform vertical rotation $\mathbf{\Omega} = (0, 0, \Omega)$ act on the system. This fluid layer is flowing through an isotropic and homogeneous porous medium of porosity ε and medium permeability k_1.

Let ρ, p, T and $\mathbf{q} = (u, v, w)$ denote the fluid density, pressure, temperature and filter velocity, respectively. Then the momentum balance, mass balance and energy balance equations of fluid flowing through porous medium, following the Boussinesq approximation, are given by

$$\text{(2.1)}\qquad \frac{\rho_0}{\varepsilon}\left[\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{\varepsilon}(\mathbf{q}.\nabla)\,\mathbf{q}\right] = -\nabla p + (\rho_0 + \delta\rho)\,\mathbf{g} - \frac{\mu}{k_1}\mathbf{q} + \frac{\mu_e}{4\pi}(\nabla\times\mathbf{H})\times\mathbf{H} + \frac{2\rho_0}{\varepsilon}(\mathbf{q}\times\mathbf{\Omega}),$$

$$\text{(2.2)}\qquad \nabla.\mathbf{q} = 0,$$

(2.3) $$E\frac{\partial T}{\partial t} + (\mathbf{q}.\nabla)\,T = \kappa\nabla^2 T.$$

Here $E = \varepsilon + (1-\varepsilon)\left(\dfrac{\rho_s c_s}{\rho_0 c_v}\right)$ is a constant, while ρ_s, c_s and ρ_0, c_v stand for the density and heat capacity of the solid (porous matrix) material and the fluid, respectively.

The Maxwell equations yield

(2.4) $$\varepsilon\frac{d\mathbf{H}}{\mathrm{dt}} = (\mathbf{H}.\nabla)\,\mathbf{q} + \varepsilon\eta\nabla^2\mathbf{H},$$

(2.5) $$\nabla.\,\mathbf{H} = 0$$

where $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \dfrac{1}{\varepsilon}\mathbf{q}.\nabla$ stands for the convective derivative.

The equation of state is

(2.6) $$\rho = \rho_0\left[1 - \alpha\,(T - T_0)\right],$$

where the subscript zero refers to values at the reference level $z = 0$. In writing Eq. (2.1), use has been made of the Boussinesq approximation, which states that the density variations are ignored in all the terms in the equation of motion except the external force term.

The basic solution is

(2.7) $$\mathbf{q} = (0,\ 0,\ w_0)\,, \qquad T = -\beta z + T_0, \qquad \rho = \rho_0\,(1 + \alpha\beta z)\,,$$

where w_0 is the magnitude of the throughflow.

Here we use the linearized stability theory and the normal mode analysis method. Assume small perturbations around the basic solution, and let $\delta\rho$, δp, θ, $\mathbf{h}(h_x, h_y, h_z)$ and $\mathbf{q}' = (u, v, w)$, denote, respectively, the perturbations in fluid density ρ, pressure p, temperature T, magnetic field $\mathbf{H}$ $(0,\ 0,\ H)$ and velocity $\mathbf{q} = (0,\ 0,\ w_0)$. The change in density $\delta\rho$, caused mainly by the perturbation of the temperature θ, is given by

(2.8) $$\delta\rho = -\alpha\rho_0\theta.$$

Then the linearized perturbation equations of the fluid reduce to:

(2.9) $$\frac{1}{\varepsilon}\left[\frac{\partial u}{\partial t} + \frac{1}{\varepsilon}w_0\frac{\partial u}{\partial z}\right] = -\frac{1}{\rho_0}\frac{\partial}{\partial x}\delta p - \frac{\nu}{k_1}u + \frac{\mu_e H}{4\pi\rho_0}\left(\frac{\partial h_x}{\partial z} - \frac{\partial h_z}{\partial x}\right) + \frac{2}{\varepsilon}v\Omega,$$

$$\frac{1}{\varepsilon}\left[\frac{\partial v}{\partial t}+\frac{1}{\varepsilon}w_0\frac{\partial v}{\partial z}\right]=-\frac{1}{\rho_0}\frac{\partial}{\partial y}\delta p-\frac{\nu}{k_1}v+\frac{\mu_e H}{4\pi\rho_0}\left(\frac{\partial h_y}{\partial z}-\frac{\partial h_z}{\partial y}\right)-\frac{2}{\varepsilon}u\Omega, \tag{2.10}$$

$$\frac{1}{\varepsilon}\left[\frac{\partial w}{\partial t}+\frac{1}{\varepsilon}w_0\frac{\partial w}{\partial z}\right]=-\frac{1}{\rho_0}\frac{\partial}{\partial z}\delta p+g\alpha\theta-\frac{\nu}{k_1}w, \tag{2.11}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{2.12}$$

$$E\frac{\partial \theta}{\partial t}+w_0\frac{\partial \theta}{\partial z}=\beta w+\kappa\nabla^2\theta, \tag{2.13}$$

$$\frac{\partial h_x}{\partial x}+\frac{\partial h_y}{\partial y}+\frac{\partial h_z}{\partial z}=0, \tag{2.14}$$

$$\varepsilon\frac{\partial h_x}{\partial t}=H\frac{\partial u}{\partial z}+\varepsilon\eta\nabla^2 h_x, \tag{2.15}$$

$$\varepsilon\frac{\partial h_y}{\partial t}=H\frac{\partial v}{\partial z}+\varepsilon\eta\nabla^2 h_y, \tag{2.16}$$

$$\varepsilon\frac{\partial h_z}{\partial t}=H\frac{\partial w}{\partial z}+\varepsilon\eta\nabla^2 h_z. \tag{2.17}$$

Applying the operator $-\dfrac{\partial}{\partial x}$ to Eq. (2.9), and $-\dfrac{\partial}{\partial y}$ to Eq. (2.10), using (2.12) and adding, we get

$$\frac{1}{\varepsilon}\left[\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z}\right)+\frac{1}{\varepsilon}w_0\frac{\partial^2 w}{\partial z^2}\right] =\frac{1}{\rho_0}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\delta p-\frac{\nu}{k_1}\frac{\partial w}{\partial z}+\frac{\mu_e H}{4\pi\rho_0}\nabla^2 h_z-\frac{2\Omega}{\varepsilon}\zeta, \tag{2.18}$$

where $\zeta=\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}$ is z-component of vorticity.

Now applying $\frac{\partial}{\partial z}$ to Eq. (2.18) and $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ to Eq. (2.11) and adding, we get

$$\left[\frac{1}{\varepsilon}\frac{\partial}{\partial t}+\frac{w_0}{\varepsilon^2}\frac{\partial}{\partial z}+\frac{\nu}{k_1}\right]\nabla^2 w = g\alpha\left(\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}\right)\theta+\frac{\mu_e H}{4\pi\rho_0}\frac{\partial}{\partial z}\nabla^2 h_z-\frac{2\Omega}{\varepsilon}\frac{\partial\zeta}{\partial z}. \tag{2.19}$$

Applying $-\frac{\partial}{\partial y}$ to Eq. (2.9) and $\frac{\partial}{\partial x}$ to Eq. (2.10) using (2.12) and adding, we get

$$\left[\frac{1}{\varepsilon}\frac{\partial}{\partial t}+\frac{w_0}{\varepsilon^2}\frac{\partial}{\partial z}+\frac{\nu}{k_1}\right]\zeta=\frac{2\Omega}{\varepsilon}\frac{\partial w}{\partial z}+\frac{\mu_e H}{4\pi\rho_0}\frac{\partial\xi}{\partial z}, \tag{2.20}$$

where $\xi=\frac{\partial h_y}{\partial x}-\frac{\partial h_x}{\partial y}$ stand for the z-component of the current density.

Now applying $-\frac{\partial}{\partial y}$ to Eq. (2.15) and $\frac{\partial}{\partial x}$ to Eq. (2.16) and adding, we get

$$\left[\varepsilon\frac{\partial}{\partial t}-\varepsilon\eta\nabla^2\right]\xi=H\frac{\partial\zeta}{\partial z}. \tag{2.21}$$

3. The dispersion relation

Analyzing the disturbances appearing in two-dimensional waves, and considering the disturbances characterized by a particular wave number, we assume that the perturbation quantities are of the form

$$[w,\ h_z,\ \theta,\ \zeta,\ \xi] = [W(z),\ K(z),\ \Theta(z),\ \mathrm{Z}(z),\ \mathrm{X}(z)]\exp\left(ik_x x+ik_y y+nt\right), \tag{3.1}$$

where k_x, k_y are the wave numbers along the x- and y- directions, respectively, $k=\sqrt{(k_x^2+k_y^2)}$ is the resultant wave number and n is the growth rate which is, in general, a complex constant.

Expressing the coordinates x, y, z in the new unit of length d and letting $a=kd$, $\sigma=\frac{nd^2}{\nu}$, $p_1=\frac{\nu}{\kappa}$, $p_2=\frac{\nu}{\eta}$, $P_\ell=\frac{k_1}{d^2}$, $P_e'=\frac{w_0 d}{\kappa}$ and $D=\frac{d}{dz}$, Eqs. (2.19), (2.13), (2.17), (2.20) and (2.21), using (3.1) become

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_\ell}+\frac{P_e'}{\varepsilon^2 p_1}D\right]\left(D^2-a^2\right)W = -\frac{g\alpha a^2 d^2}{\nu}\Theta+\frac{\mu_e H d}{4\pi\rho_0\nu}\left(D^2-a^2\right)DK-\frac{2\Omega d^3}{\varepsilon\nu}DZ, \tag{3.2}$$

$$\left(D^2-a^2-Ep_1\sigma\right)\Theta-P_e'D\Theta=-\left(\frac{\beta d^2}{\kappa}\right)W, \tag{3.3}$$

$$\left(D^2-a^2-p_2\sigma\right)K=-\left(\frac{Hd}{\varepsilon\eta}\right)DW, \tag{3.4}$$

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_\ell}+\frac{P_e'}{\varepsilon^2 p_1}D\right]Z=\left(\frac{\mu_e H d}{4\pi\rho_0\nu}\right)DX+\left(\frac{2\Omega d}{\nu\varepsilon}\right)DW, \tag{3.5}$$

$$\left(D^2-a^2-p_2\sigma\right)X=-\left(\frac{Hd}{\varepsilon\eta}\right)DZ. \tag{3.6}$$

Eliminating Θ, K, X and Z between Eqs. (3.2)–(3.6), we get

$$\begin{aligned}&\Bigg[\left(D^2-a^2-Ep_1\sigma-P_e'D\right)\left(D^2-a^2-p_2\sigma\right)\left(\frac{\sigma}{\varepsilon}+\frac{1}{P_\ell}+\frac{P_e'}{\varepsilon^2 p_1}D\right)\\ &\quad\left\{\left(\frac{\sigma}{\varepsilon}+\frac{1}{P_\ell}+\frac{P_e'}{\varepsilon^2 p_1}D\right)\left(D^2-a^2-p_2\sigma\right)+QD^2\right\}\Bigg]\left(D^2-a^2\right)W\\ &\quad+\left(D^2-a^2-Ep_1\sigma-P_e'D\right)\left(D^2-a^2\right)\\ &\quad\left\{\left(\frac{\sigma}{\varepsilon}+\frac{1}{P_\ell}+\frac{P_e'}{\varepsilon^2 p_1}D\right)\left(D^2-a^2-p_2\sigma\right)+QD^2\right\}QD^2W\\ &\quad+T_A\left(D^2-a^2-Ep_1\sigma-P_e'D\right)\left(D^2-a^2-p_2\sigma\right)^2D^2W\\ &=\left(D^2-a^2-p_2\sigma\right)\left\{\left(\frac{\sigma}{\varepsilon}+\frac{1}{P_\ell}+\frac{P_e'}{\varepsilon^2 p_1}D\right)\left(D^2-a^2-p_2\sigma\right)+QD^2\right\}Ra^2W,\end{aligned} \tag{3.7}$$

where $R = \dfrac{g\alpha\beta d^4}{\nu\kappa}$ is the Rayleigh number, $Q = \dfrac{\mu_e H^2 d^2}{4\pi\rho_0\nu\varepsilon\eta}$ is the Chandrasekhar number, $T_A = \left(\dfrac{2\Omega d^2}{\varepsilon\nu}\right)^2$ is the modified Taylor number, $p_1 = \dfrac{\nu}{\kappa}$ is the Prandtl number and $P'_e = \dfrac{w_0 d}{\kappa}$ is the Péclet number accounting for the throughflow effect.

Consider the case when both boundaries are free as well as perfect conductors of heat, while the adjoining medium is perfectly conducting. The case of two free boundaries is slightly artificial but it enables us to find analytical solutions and to make some qualitative conclusions. The appropriate boundary conditions, with respect to which Equations (3.2)–(3.6) must be solved, are

$$W = D^2W = 0, \quad \Theta = 0, \quad DZ = 0, \tag{3.8}$$

at $z = 0$ and $z = 1$, $K = 0$ on the perfectly conducting boundaries, and h_x, h_y, h_z are continuous.

Using the above boundary conditions, it can be shown that all the even-order derivatives of W must vanish for $z = 0$ and 1 and hence the proper solution of W characterizing the lowest mode is

$$W = W_0 \sin \pi z, \tag{3.9}$$

where W_0 is a constant.

Substituting the proper solution $W = W_0 \sin \pi z$ in the resultant equation, we obtain the dispersion relation

$$\begin{aligned} R_1 = {} & \left(\frac{1+x}{x}\right)(1 + x + iEp_1\sigma_1 + P_e \cot \pi z) \\ & \times \left[\left(\frac{i\sigma_1}{\varepsilon} + \frac{1}{P} + \frac{P_e}{\varepsilon^2 p_1}\cot \pi z\right) + \frac{Q_1}{(1 + x + ip_2\sigma_1)}\right] \\ & + T_1 \left[\frac{(1 + x + iEp_1\sigma_1 + P_e \cot \pi z)(1 + x + ip_2\sigma_1)}{x\left\{\left(\frac{i\sigma_1}{\varepsilon} + \frac{1}{P} + \frac{P_e}{\varepsilon^2 p_1}\cot \pi z\right)(1 + x + ip_2\sigma_1) + Q_1\right\}}\right], \end{aligned} \tag{3.10}$$

where $R_1 = \dfrac{R}{\pi^4}$, $Q_1 = \dfrac{Q}{\pi^2}$, $T_1 = \dfrac{T_A}{\pi^4}$, $x = \dfrac{a^2}{\pi^2}$, $P = \pi^2 P_\ell$, $P_e = \dfrac{P'_e}{\pi}$ and $i\sigma_1 = \dfrac{\sigma}{\pi^2}$.

Equation (3.10) is the required dispersion relation including the effects of throughflow, magnetic field, rotation and medium permeability on the thermal instability of fluid in a porous medium.

4. The stationary convection

When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma = 0$. Putting $\sigma = 0$, the dispersion relation (3.10) reduces to

$$(4.1) \qquad R_1 = \left(\frac{1+x}{x}\right)(1+x+P_e\cot\pi z) \times \left(\left(\frac{1}{P}+\frac{P_e}{\varepsilon^2 p_1}\cot\pi z\right)+\frac{Q_1}{1+x}+\frac{T_1}{\left\{\left(\frac{1}{P}+\frac{P_e}{\varepsilon^2 p_1}\cot\pi z\right)(1+x)+Q_1\right\}}\right)$$

which expresses the modified Rayleigh number R_1 as a function of the dimensionless wave number x and the parameters P_e, Q_1, T_1 and p_1. The meaning of this relation (4.1) is that for all Rayleigh numbers less than that given by (4.1), disturbances in the wave number x will be stable; these disturbances will become marginally stable when the Rayleigh number equals the value given by (4.1); and when the Rayleigh number exceeds the value given by (4.1), the same disturbances will be unstable.

In order to investigate the effects of rotation, medium permeability, magnetic field and throughflow, we examine the natures of $\dfrac{dR_1}{dT_1}$, $\dfrac{dR_1}{dP}$, $\dfrac{dR_1}{dQ_1}$ and $\dfrac{dR_1}{dP_e}$ analytically. Equation (4.1) yields

$$(4.2) \qquad \frac{dR_1}{dT_1} = \frac{(1+x)(1+x+P_e\cot\pi z)}{x\left\{\left(\frac{1}{P}+\frac{P_e}{\varepsilon^2 p_1}\cot\pi z\right)(1+x)+Q_1\right\}}.$$

This shows that rotation has always a stabilizing effect on the thermal instability of a rotating fluid in a porous medium in the presence of throughflow.

Also Eq. (4.1) yields

$$(4.3) \qquad \frac{dR_1}{dP} = -\left(\frac{1+x}{x}\right)\frac{(1+x+P_e\cot\pi z)}{P^2}\left[1-\frac{T_1(1+x)}{\{X_1(1+x)+Q_1\}^2}\right],$$

$$(4.4) \qquad \frac{dR_1}{dQ_1} = \frac{(1+x+P_e\cot\pi z)}{x}\left[1-\frac{T_1(1+x)}{\{X_1(1+x)+Q_1\}^2}\right],$$

$$\frac{dR_1}{dP_e} = \left(\frac{1+x}{x}\right)\cot\pi z\left[\left(\frac{1}{P}+\frac{2P_e}{\varepsilon^2 p_1}\cot\pi z\right) + \frac{(1+x)}{\varepsilon^2 p_1}\left\{1-\frac{T_1(1+x)}{\{X_1(1+x)+Q_1\}^2}\right\}\right] + \frac{Q_1}{(1+x)} + \frac{T_1}{\{X_1(1+x)+Q_1\}^2}\{(1+x)+Q_1 P\}, \tag{4.5}$$

where

$$\left(\frac{P_e\cot\pi z}{\varepsilon^2 p_1}+\frac{1}{P}\right) = X_1.$$

Thus for stationary convection, the medium permeability has always a destabilizing effect, whereas the magnetic field and the throughflow have always a stabilizing effects on the thermal instability of fluid in a porous medium in the absence of rotation. But in the presence of rotation, the medium permeability is found to have a destabilizing effect whereas the magnetic field and the throughflow have a stabilizing effect if

$$T_1 < \frac{\left\{(1+x)\left(\varepsilon^2 p_1 + PP_e\cot\pi z\right)+Q_1\varepsilon^2 p_1 P\right\}^2}{\varepsilon^4 p_1^2 P^2(1+x)}. \tag{4.6}$$

If

$$T_1 > \frac{\left\{(1+x)\left(\varepsilon^2 p_1 + PP_e\cot\pi z\right)+Q_1\varepsilon^2 p_1 P\right\}^2}{\varepsilon^4 p_1^2 P^2(1+x)}, \tag{4.7}$$

then the medium permeability has always a stabilizing effect and the magnetic field has always a destabilizing effect, whereas the throughflow has a stabilizing or destabilizing effect on the system.

The dispersion relation (4.1) is analyzed numerically. In Fig. 1, R_1 is plotted against the wave number x for $p_1 = 7$, $P_e = 4$, $Q_1 = 2$, $P = 4$, $\varepsilon = 0.7$, $z = 0.25$ and $T_1 = 10, 20, 30, 40$. It is clear that the rotation has always a stabilizing effect as the Rayleigh number increases with the increase in the rotation parameter. In Fig. 2, R_1 is plotted against the wave number x for $p_1 = 7$, $P_e = 4$, $Q_1 = 2$, $T_1 = 0$, $\varepsilon = 0.7$, $z = 0.25$ and $P = 1, 2, 3, 4$. It is clear that the medium permeability has a destabilizing effect in the absence of rotation whereas in the presence of rotation parameter ($T_1 = 20$) (Fig. 3), medium permeability has a stabilizing effect for small wave numbers and destabilizing effect for higher wave numbers. This is because, in their simultaneous presence, there is a competition between the stabilizing role of rotation and destabilizing role of the medium permeability, and the rotation parameter succeeds in stabilizing a certain wave number range.

In Fig. 4, R_1 is plotted against the wave number x for $p_1 = 7$, $P = 4$, $\varepsilon = 0.7$, $z = 0.25$, $P_e = 4$, $T_1 = 0$ and $Q_1 = 1, 2, 3, 4$. It is observed that the magnetic field has always a stabilizing effect in the absence of rotation, whereas in the presence of rotation parameter ($T_1 = 20$) (Fig. 5), magnetic field has a destabilizing effect for small wave numbers and a stabilizing effect for higher wave numbers. This is because, in their simultaneous presence of medium permeability, rotation and magnetic field, there is a competition between the stabilizing role of rotation and magnetic field and a destabilizing role of medium permeability, and each parameter succeeds in stabilizing a certain wave number range. In Fig. 6, R_1 is plotted against the wave number x for $p_1 = 7$, $P = 4$, $Q_1 = 10$, $T_1 = 0$, $\varepsilon = 0.7$, $z = 0.25$ and $P_e = 4, 5, 6, 7$. It is observed that the throughflow has a stabilizing effect as the Rayleigh number increases with the increase in the throughflow parameter in the absence of rotation, whereas in the presence of rotation parameter ($T_1 = 100$) (Fig. 7), throughflow has a stabilizing effect or destabilizing effect on the system.

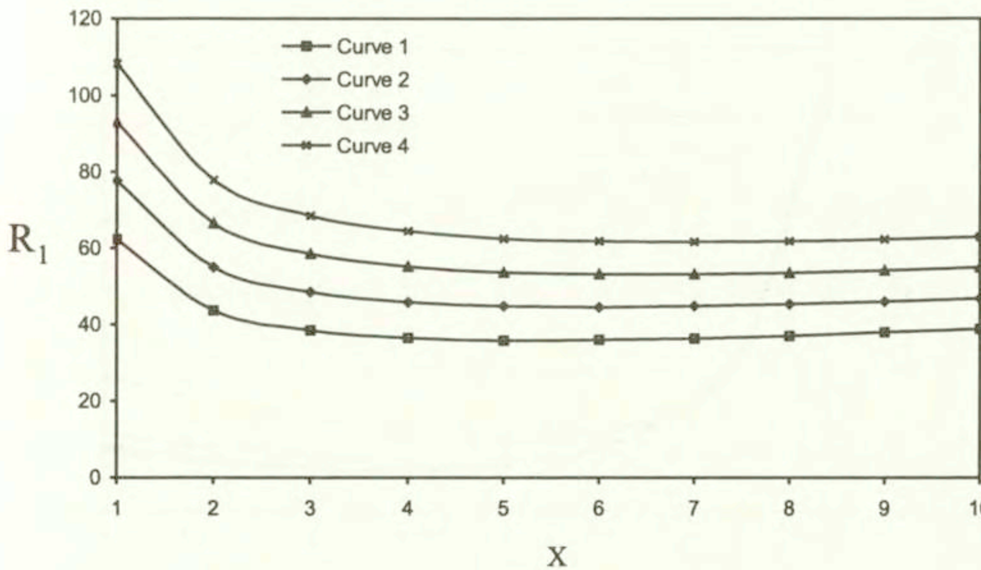

FIG. 1. The variation of Rayleigh number (R_1) with wave number (x) for $p_1 = 7$, $Q_1 = 5$, $P = 4$, $P_e = 4$, $\varepsilon = 0.7$, $z = 0.25$; $T_1 = 10$ for curve 1, $T_1 = 20$ for curve 2, $T_1 = 30$ for curve 3 and $T_1 = 40$ for curve 4.

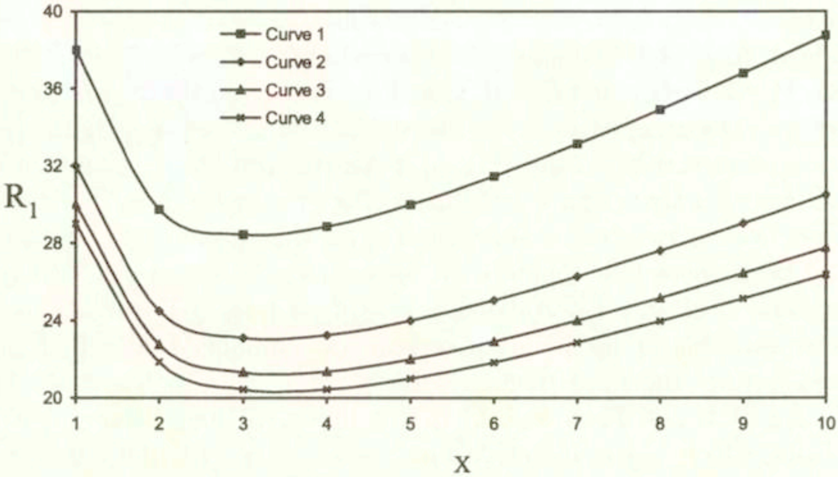

FIG. 2. The variation of Rayleigh number (R_1) with wave number (x) for $p_1 = 7$, $Q_1 = 2$, $T_1 = 0$, $P_e = 4$, $\varepsilon = 0.7$ $z = 0.25$; $P = 1$ for curve 1, $P = 2$ for curve 2, $P = 3$ for curve 3 and $P = 4$ for curve 4.

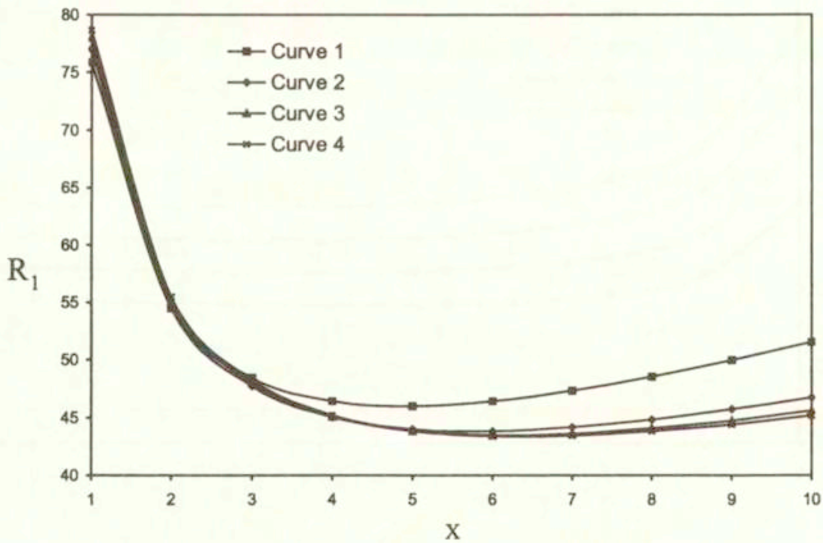

FIG. 3. The variation of Rayleigh number (R_1) with wave number (x) for $p_1 = 7$, $Q_1 = 2$, $T_1 = 20$, $P_e = 4$, $\varepsilon = 0.7$, $z = 0.25$; $P = 1$ for curve 1, $P = 2$ for curve 2, $P = 3$ for curve 3 and $P = 4$ for curve 4.

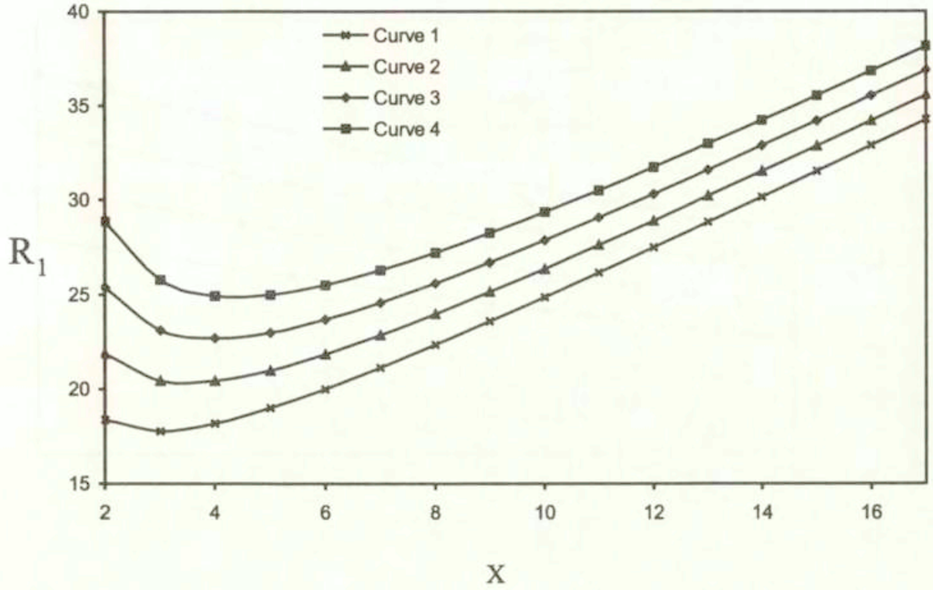

FIG. 4. The variation of Rayleigh number (R_1) with wave number (x) for $p_1 = 7$ $P_e = 4$, $T_1 = 0$, $P = 4$, $\varepsilon = 0.7$, $z = 0.25$; $Q_1 = 1$ for curve 1, $Q_1 = 2$ for curve 2, $Q_1 = 3$ for curve 3 and $Q_1 = 4$ for curve 4.

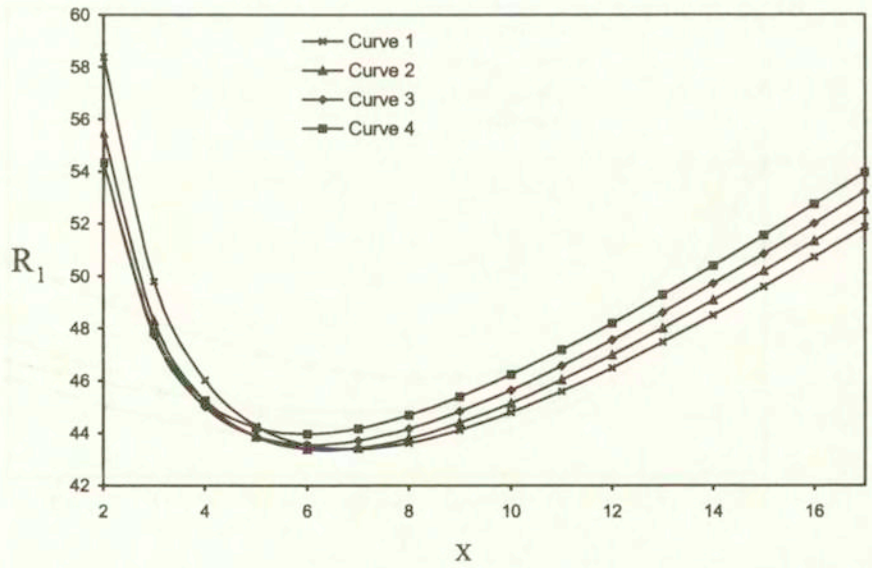

FIG. 5. The variation of Rayleigh number (R_1) with wave number (x) for $p_1 = 7$, $P_e = 4$, $T_1 = 20$, $P = 4$, $\varepsilon = 0.7$, $z = 0.25$; $Q_1 = 1$ for curve 1, $Q_1 = 2$ for curve 2, $Q_1 = 3$ for curve 3 and $Q_1 = 4$ for curve 4.

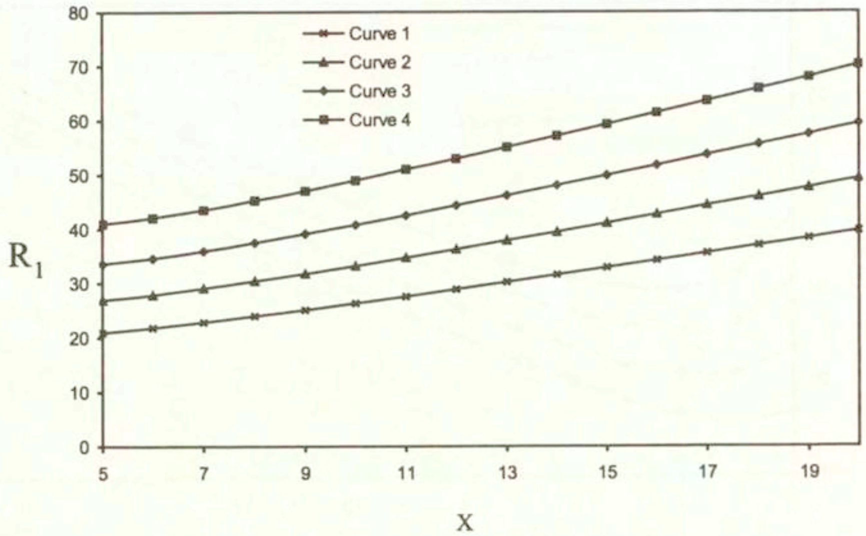

FIG. 6. The variation of Rayleigh number (R_1) with wave number (x) for $p_1 = 7$, $Q_1 = 2$, $T_1 = 0$, P = 4, $\varepsilon = 0.7$, z = 0.25; $P_e = 4$ for curve 1, $P_e = 5$ for curve 2, $P_e = 6$ for curve 3 and $P_e = 7$ for curve 4.

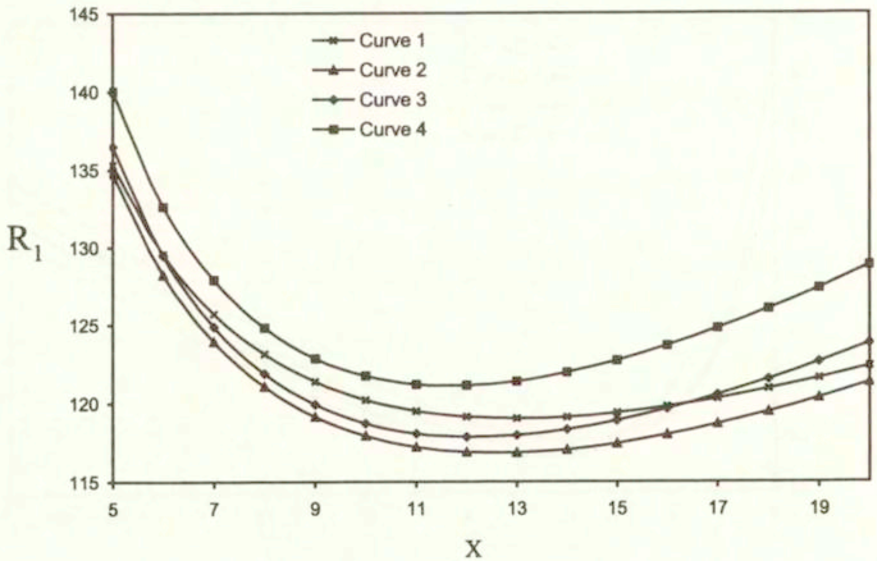

FIG. 7. The variation of Rayleigh number (R_1) with wave number (x) for $p_1 = 7$, $Q_1 = 2$, $T_1 = 100$, $P = 4$, $\varepsilon = 0.7$, z = 0.25; $P_e = 4$ for curve 1, $P_e = 5$ for curve 2, $P_e = 6$ for curve 3 and $P_e = 7$ for curve 4.

5. The case of oscillatory modes

Here we examine the possibility of oscillatory modes, if any, in the stability problem due to the presence of throughflow, magnetic field and medium permeability. Multiplying (3.2) by W^*, the complex conjugate of W, and using Eqs. (3.3) – (3.7) together with the boundary conditions (3.8), we obtain

$$(5.1)\quad \left(\frac{\sigma}{\varepsilon}+\frac{1}{P_\ell}\right)\left\langle |DW|^2+a^2|W|^2\right\rangle-\frac{g\alpha a^2\kappa}{\nu\beta}$$
$$\left[\left\langle |D\Theta|^2+a^2|\Theta|^2\right\rangle+Ep_1\sigma^*\left\langle|\Theta|^2\right\rangle-P_e'\langle D\Theta\,\Theta^*\rangle\right]$$
$$+d^2\left[\left(\frac{\sigma^*}{\varepsilon}+\frac{1}{P_\ell}\right)\left\langle|Z|^2\right\rangle-\frac{P_e'}{\varepsilon^2p_1}\left[\langle DZ^*\,\mathrm{Z}\rangle\right]\right]$$
$$+\frac{\mu_e\eta\varepsilon}{4\pi\rho_0\nu}\left[\left\langle\left(\left|D^2K\right|^2+2a^2|DK|+a^4|K|^2\right)\right\rangle+p_2\sigma^*\left\langle\left(|DK|^2+a^2|DK|\right)\right\rangle\right]$$
$$-\frac{P_e'}{\varepsilon^2p_1}\left[\langle D^2W^*\,\mathrm{DW}\rangle+a^2\langle DW^*\,\mathrm{W}\rangle\right]=0,$$

where $\langle ....\rangle=\int\limits_0^1(........)\,dz$.

Putting $\sigma=i\sigma_i$, where σ_i is real and equating the imaginary parts of Eq. (4.6), we obtain

$$(5.2)\qquad \sigma_i\begin{bmatrix}\dfrac{1}{\varepsilon}\left\langle|DW|^2+a^2|W|^2\right\rangle+\dfrac{g\alpha a^2\kappa}{\nu\beta}Ep_1\left\langle|\Theta|^2\right\rangle\\ -\dfrac{d^2}{\varepsilon}\left\langle|Z|^2\right\rangle-\dfrac{\mu_e\eta\varepsilon}{4\pi\rho_0\nu}p_2\left\langle|DK|^2+a^2|K|^2\right\rangle\end{bmatrix}=0.$$

It is clear from (5.2) that σ_i may be zero or non-zero, meaning that the modes may be non-oscillatory or oscillatory. But in the absence of magnetic field and rotation, (5.2) reduces to

$$(5.3)\qquad \sigma_i\left[\frac{1}{\varepsilon}\left\langle|DW|^2+a^2|W|^2\right\rangle+\frac{g\alpha a^2\kappa}{\nu\beta}Ep_1\left\langle|\Theta|^2\right\rangle\right]=0.$$

Here the quantity inside the brackets is positive definite. Hence

$$(5.4)\qquad \sigma_i=0.$$

This shows that whenever $\sigma_r = 0$ implies that $\sigma_i = 0$, then the stationary (cellular) pattern of flow prevails in the onset of instability. In other words, the principle of exchange of stabilities is valid for the fluid heated from below in porous medium with throughflow in the absence of magnetic field and rotation. The oscillatory modes are introduced due to the presence of magnetic field and rotation, which were non-existent in their absence.

6. The case of overstability

The present section is devoted to find the possibility as to whether instability may occur as overstability. Since we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it suffices to find the conditions for which (3.10) will admit solutions with σ_i real.

Equating real and imaginary parts of (3.10) and eliminating R_1 between them, we obtain

$$A_3c_1^3 + A_2c_1^2 + A_1c_1 + A_0 = 0, \tag{6.1}$$

where we have put $c_1 = \sigma_1^2$, $\left(\dfrac{P_e \cot \pi z}{\varepsilon^2 p_1} + \dfrac{1}{P}\right) = X_1$, $(b + P_e \cot \pi z) = X_2$, $b = 1 + x$ and

$$\begin{aligned} A_0 = {} & \left[\frac{1}{\varepsilon}\left(\frac{\varepsilon E p_1}{P^3} - T_1\right) + \left(\frac{P_e \cot \pi z}{\varepsilon^2 p_1}\right) E p_1 \left\{\left(\frac{P_e \cot \pi z}{\varepsilon^2 p_1}\right)^2 + \frac{2X_1}{P}\right\}\right] b^5 \\ & + \left[E p_1 Q_1 X_1 \{2X_1 + 1\} + T_1 \left\{\frac{P_e \cot \pi z}{\varepsilon}(E - \varepsilon) + \frac{E p_1}{P}\right\}\right] b^4 \\ & + \left[\frac{1}{\varepsilon} X_1^2 X_2 \left(b^2 - \varepsilon Q_1 p_2\right)\right] b^3 \\ & + \left[\frac{2Q_1}{\varepsilon} X_1 X_2 \left(b^2 - \varepsilon Q_1 p_2\right) + Q_1^3 (E p_1 - p_2) + p_2 Q_1 T_1 X_2\right] b^2 \\ & + \left[\frac{1}{\varepsilon} Q_1^2 P_e \cot \pi z \left(b^2 - \varepsilon Q_1 p_2\right)\right] b, \end{aligned} \tag{6.2}$$

$$A_3 = \frac{p_2^4}{\varepsilon^3}\left[b^2 + \left\{\frac{P_e \cot \pi z}{\varepsilon}(E + \varepsilon) + \frac{\varepsilon E p_1}{P}\right\} b\right]. \tag{6.3}$$

Since σ_1 is real for overstability, the three values of $c_1 (= \sigma_1^2)$ are positive. So the product of roots of (6.1) is positive, but this is impossible if $A_0 > 0$ and

$A_3 > 0$ $\left(\text{since the product of the roots of Eq. (6.1) is } -\dfrac{A_0}{A_3}\right)$. $A_0 > 0$ and $A_3 > 0$ are, therefore, sufficient conditions for the nonexistence of overstability.

It is clear from Eq. (6.2) and (6.3) that A_0 and A_3 are always positive if

$$Ep_1 > p_2, \quad \frac{\varepsilon E p_1}{P^3} > T_1 \quad \text{and} \quad b > \sqrt{Q_1 \varepsilon p_2}, \tag{6.4}$$

which implies that

$$\frac{E\nu}{\kappa} > \frac{\nu}{\eta}, \quad \kappa < E\left(\frac{\nu\varepsilon}{k_1}\right)^3\left(\frac{d^2}{4\Omega^2\pi^2}\right) \quad \text{and} \quad (1+x) > \left(\frac{\mu_e}{4\pi\rho_0}\right)^{1/2}\frac{Hd}{\eta\pi}, \tag{6.5}$$

i. e.

$$E\eta > \kappa, \quad \kappa < E\left(\frac{\nu\varepsilon}{k_1}\right)^3\left(\frac{d^2}{4\Omega^2\pi^2}\right) \text{ and } \left(1+\frac{k^2d^2}{\pi^2}\right) > \left(\frac{\mu_e}{4\pi\rho_0}\right)^{1/2}\frac{Hd}{\eta\pi}. \tag{6.6}$$

Thus, $\quad E\eta > \kappa, \quad \kappa < E\left(\dfrac{\nu\varepsilon}{k_1}\right)^3\left(\dfrac{d^2}{4\Omega^2\pi^2}\right)$ and $\left(1+\dfrac{k^2d^2}{\pi^2}\right) > \left(\dfrac{\mu_e}{4\pi\rho_0}\right)^{1/2}\dfrac{Hd}{\eta\pi}$

are the sufficient conditions for the nonexistence of overstability, the violation of which does not necessarily imply the occurrence of overstability.

Acknowledgement

The financial assistance to Dr. Sunil in the form of Research and Development Project [No. 25(0129)/02/EMR-II] of the Council of Scientific and Industrial Research (CSIR), New Delhi, is gratefully acknowledged.

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Received December 5, 2002; revised version April 7, 2003.