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Classification of thin shell models deduced from the nonlinear three-dimensional elasticity. Part I : the shallow shells

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THE PURPOSE of this paper is to construct a classification of asymptotic shell models (inferred from the non linear three-dimensional elasticity) with respect to the applied forces and to the geometrical data. To do this, we use a constructive approach based on a dimensional analysis of the nonlinear three-dimensional equilibrium equations, which naturally gives rise to the appearance of dimensionless numbers characterizing the applied forces and the geometry of the shell. In order to limit our study to one-scale problems, these dimensionless numbers are expressed in terms of the relative thickness ε of the shell, which is considered as the perturbation parameter. This leads, on the one hand, to distinguish shallow shells from strongly curved shells which have a different asymptotic behaviour, and on the other hand, to fix the applied force level. For each of these two classes of shells, using the usual asymptotic method, we propose a complete classification of two-dimensional shell models based on decreasing force levels, from severe to low. In the first part of this paper, we present the classification for shallow shells. We obtain successively the nonlinear membrane model, another membrane model, Koiter's non linear shallow shell model, and the linear Novozhilov-Donnell one, respectively for severe, high, moderate and low forces.

1. Introduction

THE STUDY of thin shells is the subject of numerous works in mechanical structure area. The main goal of these works is to predict the behaviour of the shell, when it is subjected to a known level of applied loads. To this end, many authors have proposed two-dimensional shell models whose resolution is less difficult than the classical three-dimensional equations. These two-dimensional models may be obtained from the three-dimensional elasticity using essentially three approaches[1)]. The first one is a direct approach which consists in introducing a priori assumptions in the three-dimensional Eqs. [27][17]. The second one is a

[1)] Even if other approaches exist

direct or surfacic approach where the shell is modelled as a surface embedded in $\mathbb{R}^3$ [3]-[5][16][48][25]. Finally, the third possible approach is based on asymptotic techniques. Contrary to the two first approaches, the asymptotic approaches, based on mathematical techniques developed by J. L. LIONS [28] for problems containing a small parameter, lead to a rigorous justification of two-dimensional shell models.

In linear shell theory, H. S. RUTTEN [40] and A. L. GOLDENVEIZER [24] have developed some ideas concerning the application of the asymptotic expansion method to the shell theory. F. JOHN [26] has also proposed another approach which is based on the estimation of the stresses and of their derivatives in the interior of the domain.

However, the first rigorous results have been obtained by P. DESTUYNDER [4][5] within the framework of linear elasticity. In these works, the author uses an intrinsic variational approach which makes appear explicitly the curvature of the shell middle surface. The application of the asymptotic expansion method leads to the Novozhilov-Donnell model in the case of shallow shells and to the linear membrane model in the case of strongly curved shells, also called general shells by other authors.

Another approach using local coordinates has been developed in [11][21][22]. For "general shells", the asymptotic expansion of linear three-dimensional variational equations leads to the classical linear membrane or to the pure bending model [14][21], according to whether the middle surface of the shell admits or not inextensional displacements. The importance of such inextensional displacements in shell theory, which does not modify the metric of the middle surface, is known since V. V. NOVOZHILOV [17] and A. L. GOLDENVEIZER [9]. The study of inextensional displacements in linear theory has been systematized by P. DESTUYNDER [14], E. SANCHEZ-PALENCIA [19][20], G. GEYMONA *et al.* [8] and D. CHOÏ [2].

These two asymptotic approaches have been extended to non-linear shell theory. For shallow shells, the Koiter nonlinear shell model[2)] and the non-linear Marguerre-von Kármán one have been deduced from three-dimensional non linear Eqs. [7][19][10]. For general shells, the non linear membrane model has been obtained whatever the geometric rigidity of the middle surface is [13]. The nonlinear pure bending model has been deduced in the case of non-inhibited shells[3)] [11]. Let us cite also the works of W. Z. Chien [6] who tried to classify the geometrically two-dimensional nonlinear shell models by evaluating the respective order of magnitude of the membrane and bending stress tensor contribution in 2D general equations.

[2)] Which is also named Donnell-Mushtari-Vlasov model.

[3)] In the nonlinear sense.

However, on the one hand the these approaches generally use a priori scaling assumptions on displacements which are unknowns of the problem. On the other hand, the results obtained do not enable to deduce a general classification which specifies the domain of validity of the two-dimensional shell models. In particular, the following paradox still subsists: when the nonlinear Koiter shallow shell model and the linear Novozhilov-Donnell one are deduced, respectively from the nonlinear and the linear elasticity, they are obtained for the same level of forces and the same deflections. However, it is well known that these two models reflect qualitatively different types of behaviour.

The aim of this paper is to present *a constructive method of classification* of asymptotic shell models from the *nonlinear three-dimensional elasticity*, which specifies the domain of validity of the obtained models. To do this, the asymptotic models are deduced from the level of applied forces and from the geometric properties of the middle surface of the shell.

In this approach, we use the following classical assumptions which enable to simplify considerably the calculations :

- Shallow shells are assumed to be totally clamped on the lateral surface, to avoid boundary layers. The study of these boundary layers is not the subject of this paper[4)].
- The fields of applied forces and displacements are assumed not to vary rapidly, in order to be able to use only one scale to characterize the displacements, and only one scale to characterize the forces[5)]. This is equivalent, as in [26][27][18], to consider that the wave length of strain is of L_0 order.

However, contrary to most of works on asymptotic justification of shell models, the scale of the displacements[6)] is not a data of the problem, but *is deduced from the scale of applied forces*. It is in this sense that the expression "without any a priori assumption" can be used to characterize the approach presented in this paper. This approach has been already validated in nonlinear plate theory [34]–[35], in linear shell theory [15] and extended to nonlinear shell theory [6]. Let us notice as well that our approach can be applied to elastic-plastic plate and shells [36].

First, using geometrical and physical reference quantities, we write the three-dimensional nonlinear equilibrium equations in a dimensionless form. This naturally makes appear dimensionless numbers $\mathcal{F}$ and $\mathcal{G}$ characterizing the level of body and surface forces, and two shape factors ε and $\mathcal{C}$, which characterize the

[4)]Thus we have no number characterizing the distance from a current point to the lateral boundary, as in [26].

[5)]This assumption is not necessary and can be dropped. In this case, we have multi-scale problems which are much more complicated. It is not the subject of this paper.

[6)]Which characterizes the order of magnitude of the displacements and is a priori an unknown of the problem.

thickness and the curvature of the shell. Then, to obtain a one-scale problem, the dimensionless numbers $\mathcal{F}, \mathcal{G}$ and $\mathcal{C}$ are linked to ε. This leads to distinguish, as in [4][5], shallow shells (where $\mathcal{C} = \varepsilon^2$) from strongly curved shells (where $\mathcal{C} = \varepsilon$), which have different asymptotic behaviours. For a given force level[7)], we make the asymptotic expansion of equations with respect to the small parameter ε.

Finally, to make out the classification of asymptotic shell models, we study decreasing force levels, from severe to low. For each force level, the order of magnitude of displacements and the corresponding two-dimensional model are deduced from asymptotic expansion of three-dimensional nonlinear equations. To each two-dimensional model that we obtain, we associate a minimization problem. When we consider lower force levels, two cases are possible. If the leading term of the expansion of the displacement is equal to zero, then we make a new dimensional analysis of the displacement. If it is different from zero, we continue the expansion of the three-dimensional equations. This constitutes the original character of this approach. Indeed, the scalings on displacements are progressively deduced from the level of applied forces.

For shallow shells, the classification leads to four kinds of models : a nonlinear membrane model, another membrane model[8)], the non-linear Koiter shallow shell model and the linear Novozhilov-Donnell model, obtained respectively for severe, high, moderate and low force levels. These results can be summarized in the following table:

Shell model	In-plane surface forces	Normal surface forces
Nonlinear membrane model	ε	ε
Another membrane model	ε^2	ε^3
Koiter's non linear model	ε^3	ε^4
Linear Novozhilov-Donnell model	$\varepsilon^{n\geq 4}$	ε^{n+1}

In the case of strongly curved shells, the classification obtained is more complex. It depends not only on the force levels, but also on the inhibited or not inhibited character of the middle surface in the linear and nonlinear sense. The classification of asymptotic shell models for strongly curved shells will not be presented here. It will be the subject of the second part of this paper.

2. The three-dimensional problem

In what follows, we index by a star (*) all dimensional variables. On the other hand, within the framework of large displacements, the reference and the current

[7)] A force level corresponds to a relation between $\mathcal{F}$, $\mathcal{G}$ and ε^p, which is chosen as the reference scale.

[8)] Which has to our knowledge no equivalent in the literature.

configuration cannot be confused. So the reference configuration variables will be indexed by ($_0$).

Let ω_0^* be a connected surface embedded in $\mathbb{R}^3$, whose diameter is L_0, with a "smooth enough" boundary γ_0^*. We note N_0 the unit normal to ω_0^* and C_0^* its curvature operator. Let $i^*_{\omega_0^*}$ be the identity mapping on ω_0^*, $T\omega_0^*$ the tangent bundle of ω_0^* (the collection of all tangent spaces corresponding to all points p_0^* of ω_0^*), I_0^* the identity on $T\omega_0^*$, Π_0^* the orthogonal projection onto $T\omega_0^*$ and J_0^* the $\pi/2$ rotation around N_0.

Let us consider a shell of $2h_0$ thickness, whose middle surface is ω_0^*. The shell itself occupies the domain $\overline{\Omega}_0^*$ in its reference configuration where $\Omega_0^* = \omega_0^* \times]-h_0, h_0[$ is an open set of $\mathbb{R}^3$. We denote q_0^* the generic point of $\overline{\Omega}_0^*$ and $\Gamma_0^{*\pm} = \overline{\omega}_0 * \times \{\pm h_0\}$ the upper and lower faces of the shell.To simplify the problem without loss of generality, we assume that the shell is clamped on all its lateral surface $\Gamma_0^* = \gamma_0^* \times [-h_0, h_0]$.

We assume that the shell, subjected to applied body forces $f^* : \overline{\Omega}_0^* \to \mathbb{R}^3$ and to surface forces $g^{*\pm} : \Gamma_0^{*\pm}\mathbb{R} \to \mathbb{R}^3$, occupies the set $\overline{\Omega}^*$ in its deformed configuration. In what follows, we set $f^* = f_t^* + f_n^* N_0$ (respectively $g^{*\pm} = g_t^{*\pm} + g_n^{*\pm} N_0$), decomposition of f^* (respectively of $g^{*\pm}$) onto $T\omega_0^* \oplus \mathbb{R}N_0$. Moreover we consider only thin shells (such as $h_0 << L_0$ and $h_0 \|C_0^*\|_\infty << 1$), subjected to dead loads which are independent of the configuration.

The unknown of the problem is then the displacement $U^* : \overline{\Omega}_0^* \to \mathbb{R}^3$ (or the mapping $\phi^* : \overline{\Omega}_0^* \to \mathbb{R}^3$) such that if $q_0^* \in \overline{\Omega}_0^*$ denotes the initial position of a material point, its position in the deformed configuration is $\phi^*(q_0^*) = q_0^* + U^*(q_0^*)$.

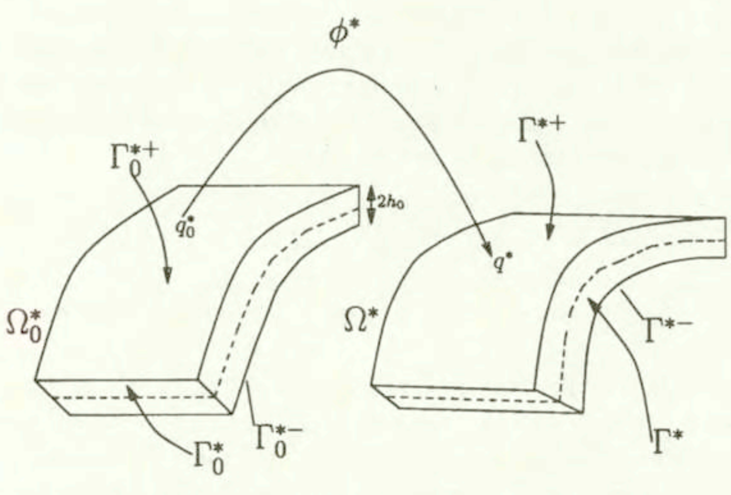

FIG. 1. Initial and final shell configuration.

In this paper we will use the following notations: $\dfrac{\partial}{\partial q_0^*}$ and Div^* denote respectively the gradient and the divergence in the three-dimensional space, $\dfrac{\partial}{\partial p_0^*}$, $\dfrac{\hat\partial}{\partial p_0^*}$ and div^* denote respectively the two-dimensional gradient, the covariant derivative and the two-dimensional divergence defined on ω_0^*. The overbar denotes the transposition operator with respect to the metric and $\mathrm{Tr}A$ the trace of the endomorphism A.

Within the framework of nonlinear elasticity, the displacement $U^*:\overline{\Omega_0^*}\to\mathbb{R}^3$ and the second Piola-Kirchhoff tensor Σ^* solve the nonlinear equilibrium equations

$$
\begin{aligned}
&\mathrm{Div}^*(\Sigma^*\overline{F}^*) = -\overline{f^*} && \text{in } \Omega_0^*,\\
&(F^*\Sigma^*)^{\pm}.N_0 = \pm g^{*\pm} && \text{on } \Gamma_0^{*\pm},\\
&U^* = 0 && \text{on } \Gamma_0^*,
\end{aligned}
\tag{2.1}
$$

where $F^* = \dfrac{\partial\phi^*(q_0^*)}{\partial q_0^*} = I_3 + \dfrac{\partial U^*}{\partial q_0^*}$ denotes the linear tangent map to the mapping function $q_0^*\to\phi^*(q_0^*) = q_0^* + U^*(q_0^*)$. Limiting our study to Saint-Venant Kirchhoff materials, the constitutive relation takes the following form :

$$\Sigma^* = \lambda\ \mathrm{Tr}(E^*)I_3 + 2\mu E^*$$

where $E^* = (\overline{F}^*F^* - I_3)/2$ denotes the nonlinear Green-Lagrange strain tensor, I_3 the identity of $\mathbb{R}^3$, λ and μ the Lamé constants of the material.

These equilibrium equations can be completed with the equation of continuity $\rho^*\mathrm{det}F^* = \rho_0^*$ where ρ_0^* and ρ^* denote respectively the voluminal mass of the material in the reference and in the deformed configuration. In what follows, we assume ρ^* to be bounded, which can be written

$$\mathrm{det}F^* = \det\left(\frac{\partial\phi^*(q_0^*)}{\partial q_0^*}\right) \geq a > 0 \quad \text{in} \quad \Omega_0^*, \tag{2.2}$$

where $a > 0$ is a constant independent of the geometry. This condition will be used later.

In the case of thin shells, for all material point q_0^* in $\overline{\Omega_0^*}$, we have the unique decomposition $q_0^* = p_0^* + z^*N_0$ where $p_0^*\in\overline{\omega_0^*}$ and $z^*\in[-h_0,h_0]$ denote respectively the orthogonal projection of q_0^* onto $\overline{\omega_0^*}$ and onto the normal N_0. It is then possible to decompose the displacement U^* onto $T\omega_0^*\oplus\mathbb{R}N_0$ as follows:

$$U^* = V^* + u^*N_0$$

where V^* is a field of tangent vectors and u^* a scalar field on ω_0^*.
In what follows, we set

$$G^* = \frac{\partial U^*}{\partial q_0^*} \qquad \text{and} \qquad \mathcal{H}^* = \Sigma^* \overline{F^*}.$$

Then, using the matrix notation, G^* can be decomposed onto $T\omega_0^* \oplus \mathbb{R}N_0$ as:

$$G^* = \begin{bmatrix} G_t^* & G_s^* \\ \overline{G}_s^{\prime *} & G_n^* \end{bmatrix} \tag{2.3}$$

with

$$G_t^* = \left(\frac{\hat{\partial} V^*}{\partial p_0^*} - u^* C_0^*\right)\kappa_0^{*-1}, \qquad G_s^* = \frac{\partial V^*}{\partial z^*},$$

$$G_s^{\prime *} = \kappa_0^{*-1}\left(C_0^* V^* + \overline{\frac{\partial u^*}{\partial p_0^*}}\right), \qquad G_n^* = \frac{\partial u^*}{\partial z^*},$$

and $\kappa_0^* = I_0 - z^* C_0^*$. As κ_0^* is invertible[9)], we have : $\kappa_0^{*-1} = I_0 + z^* C_0^* + z^{*2} C_0^{*2} + \dots$.
In the same way, E^* can be written :

$$E^* = \begin{bmatrix} E_t^* & E_s^* \\ \overline{E}_s^* & E_n^* \end{bmatrix} \tag{2.4}$$

where

$$2E_t^* = \overline{G}_t^* G_t^* + G_s^{\prime *} \overline{G}_s^{\prime *} + \overline{G}_t^* + G_t^*,$$

$$2E_s^* = \overline{G}_t^* G_s^* + G_n^* G_s^{\prime *} + G_s^* + G_s^{\prime *},$$

$$2E_n^* = \overline{G}_s^* G_s^* + G_n^{*2} + 2G_n^*.$$

Therefore, the second Piola-Kirchhoff tensor Σ^* takes the following form:

$$\Sigma^* = \begin{bmatrix} \Sigma_t^* & \Sigma_s^* \\ \overline{\Sigma}_s^* & \Sigma_n^* \end{bmatrix} \tag{2.5}$$

with

$$\Sigma_t^* = \lambda(\mathrm{Tr}\,(E_t^*) + E_n^*) + 2\mu E_t^*, \qquad \Sigma_s^* = 2\mu E_s^*,$$

$$\Sigma_n^* = \lambda \mathrm{Tr}\,(E_t^*) + (\lambda + 2\mu) E_n^*.$$

Finally, the matrix form of $\mathcal{H}^* = \Sigma^* \overline{F^*}$ becomes:

$$\mathcal{H}_t^* = \begin{bmatrix} \mathcal{H}_s^*, & \mathcal{H}_s^* \\ \overline{\mathcal{H}'}_s^* & \mathcal{H}_n^* \end{bmatrix} \tag{2.6}$$

---

9) Because $h_0 \|C_0^*\|_\infty << 1$ for a thin shell.

with

$$\mathcal{H}_t^* = \Sigma_t^* + \Sigma_t^* \overline{G}_t^* + \Sigma_s^* \overline{G}_s^*, \qquad \mathcal{H}_s^* = \Sigma_s^* + G_n^* \Sigma_s^* + \Sigma_t^* G_s'^*,$$
$$\mathcal{H}_s'^* = \Sigma_s^* + G_t^* \Sigma_s^* + \Sigma_n^* G_s^*, \qquad \mathcal{H}_n^* = \Sigma_n^* + G_n^* \Sigma_n^* + \overline{G'}_s^* \Sigma_s^*.$$

To finish, let us decompose the three-dimensional divergence onto $T\omega_0^* \oplus \mathbb{R}N_0$. Then the equilibrium Eqs. (2.1) can be written in $T\omega_0^* \oplus \mathbb{R}N_0$ as :

$$\begin{aligned}
&\mathrm{div}^* \left(\kappa_0^{*-1} \mathcal{H}_t^*\right) - \mathrm{div}^* \left(\kappa_0^{*-1}\right) \mathcal{H}_t^* - \overline{\mathcal{H}_s^*} \kappa_0^{*-1} C_0^* - \mathrm{Tr}(\kappa_0^{*-1} C_0^*) \overline{\mathcal{H}'}_s^* \\
&\qquad + \frac{\overline{\partial \mathcal{H}_s'^*}}{\partial z*} = -\overline{f}_t^*, \\
&\mathrm{div}^* \left(\kappa_0^{*-1} \mathcal{H}_s^*\right) - \mathrm{div}^* \left(\kappa_0^{*-1}\right) \mathcal{H}_s^* + \mathrm{Tr}(\mathcal{H}_t^* \kappa_0^{*-1} C_0^*) - \mathrm{Tr}(\kappa_0^{*-1} C_0^*) \overline{\mathcal{H}}_n^* \\
&\qquad + \frac{\partial \mathcal{H}_n^*}{\partial z*} = -f_n^*,
\end{aligned} \tag{2.7}$$

where we recall that div^* denotes the two-dimensional divergence on ω_0^*. The boundary conditions on the upper and lower faces $\Gamma_0^{*\pm}$ become:

$$\mathcal{H}_s'^{*\pm} = \pm g_t^{*\pm} \quad \text{and} \quad \mathcal{H}_n^{*\pm} = \pm g_n^{*\pm}. \tag{2.8}$$

The boundary conditions on the lateral surface Γ_0^* are given by:

$$V^* = 0 \quad \text{and} \quad u^* = 0. \tag{2.9}$$

3. Dimensional analysis of equations

3.1. The dimensionless numbers governing shell problems

Let us define the following dimensionless physical data and dimensionless unknowns of the problem :

$$\begin{aligned}
&p_0 = \frac{p_0^*}{L_0}, \quad z = \frac{z^*}{h_0}, \quad C_0 = \frac{C_0^*}{C_r}, \quad V = \frac{V^*}{V_r}, \quad u = \frac{u^*}{u_r}, \\
&f_n = \frac{f_t^*}{f_{tr}}, \quad g_n = \frac{f_n^*}{f_{nr}}, \quad g_n = \frac{g_t^*}{g_{tr}}, \quad g_t = \frac{g_n^*}{g_{nr}},
\end{aligned} \tag{3.1}$$

where the variables indexed by r are the reference ones. The new variables which appear without a star are dimensionless. In particular, $C_r = ||C_0^*||_\infty$ denotes the maximum curvature of the shell in its reference configuration.

To avoid any assumption on the order of magnitude of the normal and the tangential displacement components, the reference scales V_r and u_r are first assumed to be equal to L_0. Thus we allow a priori large displacements. If necessary, it will be always possible to define new reference scales for the displacements.

In what follows, to simplify the calculations, we set:

$$G = \varepsilon G^*, \quad E = \varepsilon^2 E^*, \quad \Sigma = \frac{\varepsilon^2}{\mu}\Sigma^* \quad \text{and} \quad \mathcal{H} = \frac{\varepsilon^3}{\mu}\mathcal{H}^*. \tag{3.2}$$

The so defined dimensionless variables naturally depend on ε. However, it is important to notice that this definition does not constitute any assumption on the order of magnitude of G^*, E^*, Σ^* or $\mathcal{H}^*$. It only enables to use dimensionless quantities which will be more practical for the asymptotic expansion of equations.

According to the previous notation, the dimensionless components of G are given by:

$$\begin{gathered} G_t = \left(\varepsilon \frac{\partial \hat{V}}{\partial_{p0}} - u\mathcal{C}C_0\right)\kappa_0^{-1}, \quad G_s = \frac{\partial V}{\partial z}, \\ G'_s = \kappa_0^{-1}\left(\mathcal{C}C_0 V + \varepsilon \frac{\overline{\partial u}}{\partial_{p0}}\right), \quad G_n = \frac{\partial u}{\partial z}, \end{gathered} \tag{3.3}$$

where

$$\kappa_0 = I_0 - \mathcal{C}zC_0 \qquad \text{and} \qquad \kappa_0^{-1} = I_0 + z\mathcal{C}C_0 + z^2\mathcal{C}^2C_0^3 + \ldots$$

This dimensional analysis naturally makes appear the dimensionless numbers $\varepsilon = h_0/L_0$ and $\mathcal{C} = h_0 C_r$ which characterize the geometry of the shell:

i) The first one, $\varepsilon = h_0/L_0$, ratio of the initial half-thickness of the shell to the diameter of the middle surface ω_0^* is a known parameter of the problem.

ii) The second one, $\mathcal{C} = h_0 C_r$, product of the half-thickness by the reference curvature of the shell, is as well a known parameter of the problem. For thin shells, ε and $\mathcal{C}$ are small parameters.

On the other hand, the dimensionless components of E are given by:

$$\begin{aligned} 2E_t &= \overline{G_t}G_t + G'_s\overline{G'_s} + \varepsilon(\overline{G_t} + G_t), \\ 2E_s &= \overline{G_t}G_s + G_nG'_s + \varepsilon(G'_s + G_s), \\ 2E_n &= \overline{G_s}G_s + G_n^2 + 2\varepsilon G_n, \end{aligned} \tag{3.4}$$

and the dimensionless components of Σ become:

$$\begin{gathered} \Sigma_t = \beta(Tr(E_t) + E_n)I_0 + 2E_t, \qquad \Sigma_s = 2E_s, \\ \Sigma_n = \beta Tr(E_t) + (\beta + 2)E_n, \end{gathered} \tag{3.5}$$

where $\beta = \dfrac{\lambda}{\mu}$. Finally, the dimensionless components of $\mathcal{H}$ are given by:

$$
(3.6)\qquad
\begin{gathered}
\mathcal{H}_t = \varepsilon\Sigma_t + \Sigma_t\overline{G_t} + \Sigma_s\overline{G_s}, \quad \mathcal{H}_s = \varepsilon\Sigma_s + G_n\Sigma_s + \Sigma_t G'_s,\\
\mathcal{H}'_s = \varepsilon\Sigma_s + G_t\Sigma_s + \Sigma_n G_s, \quad \mathcal{H}_n = \varepsilon\Sigma_n + G_n\Sigma_n + \overline{G'_s}\Sigma_s.
\end{gathered}
$$

Accordingly, the equilibrium Eq. (2.7) can be written in $\Omega_0 = \omega_0\times]-1,+1[$ in the dimensionless form:

$$
(3.7)\qquad
\begin{aligned}
&\varepsilon\left(\operatorname{div}\left(\kappa_0^{-1}\mathcal{H}_t\right) - \operatorname{div}\left(\kappa_0^{-1}\right)\mathcal{H}_t\right) - \mathcal{C}\left(\overline{\mathcal{H}}_s\kappa_0^{-1}C_0 + \operatorname{Tr}(\kappa_0^{-1}C_0)\overline{\mathcal{H}'}_s\right)\\
&\qquad\qquad + \frac{\overline{\mathcal{H}'_s}}{\partial z} = -\varepsilon^3\mathcal{F}\overline{f}_t,\\
&\varepsilon\left(\operatorname{div}\left(\kappa_0^{-1}\mathcal{H}_s\right) - \operatorname{div}\left(\kappa_0^{-1}\right)\mathcal{H}_s\right) + \mathcal{C}\left(\operatorname{Tr}(\mathcal{H}_t\kappa_0^{-1}C_0) - \operatorname{Tr}(\kappa_0^{-1}C_0)\mathcal{H}_n\right)\\
&\qquad\qquad + \frac{\partial\mathcal{H}_n}{\partial z} = -\varepsilon^3\mathcal{F}f_n.
\end{aligned}
$$

The dimensionless boundary conditions (2.8) on the upper and lower faces $\Gamma_0^{\pm} = \overline{\omega}_0 \times \{\pm 1\}$ are given by:

$$
(3.8)\qquad \mathcal{H}'^{\pm}_s = \pm\,\varepsilon^3\mathcal{G}g_t^{\pm} \qquad \text{and} \qquad \mathcal{H}_n^{\pm} = \pm\,\varepsilon^3\mathcal{G}g_n^{\pm}
$$

and the boundary conditions (2.9) on the lateral surface $\Gamma_0 = \gamma_0\times[-1,1]$ lead to:

$$
(3.9)\qquad V = 0 \qquad \text{and} \qquad u = 0.
$$

Thus the dimensional analysis of equations makes appear the other dimensionless numbers, already obtained in [33]-[35], which characterize the applied forces:

$$
(3.10)\qquad \mathcal{F}_t = \frac{h_0 f_{tr}}{\mu}, \quad \mathcal{F}_n = \frac{h_0 f_{nr}}{\mu}, \quad \mathcal{G}_t = \frac{g_{tr}}{\mu} \quad \text{and} \quad \mathcal{G}_n = \frac{g_{nr}}{\mu}.
$$

Indeed, the numbers $\mathcal{F}_t$ and $\mathcal{F}_n$ (respectively $\mathcal{G}_t$ and $\mathcal{G}_n$) represent the ratio of the resultant on the thickness of the body forces (respectively the ratio of the surface forces) to μ considered as a reference stress. These numbers only depend on known physical quantities and must be considered as known data of the problem.

3.2. Reduction to a one-scale problem

To obtain a one-scale problem, ε is chosen as the reference parameter. Therefore the other dimensionless numbers must be linked to ε.

On one hand, $\mathcal{C}$ must be linked to ε. This leads to distinguish shallow shells where $\mathcal{C} = \varepsilon^2$ (whose middle surface is close to a plate) from strongly curved shells where $\mathcal{C} = \varepsilon$. These two shell families have different asymptotic behaviours which have been already studied in the linear case [4][5]. In the first part of this paper, we will limit our study to shallow shells. On the other hand, the force ratios $\mathcal{F}_t, \mathcal{F}_n, \mathcal{G}_t$ and $\mathcal{G}_n$ must be linked to the powers of ε as well. This determines the level of applied forces. To make out a general classification for all the force levels, we should consider all the combinations $\mathcal{F}_t = \varepsilon^m$, $\mathcal{F}_n = \varepsilon^l$, $\mathcal{G}_t = \varepsilon^p$ and $\mathcal{G}_n = \varepsilon^q$, for m, l, p and q strictly positive integers. However, such a tiresome work is not necessary because the different two-dimensional shell models obtained are essentially determined by the first member of equilibrium equations. Hence, the study of all the combinations of force levels can be reduced to some particular ones. Let us define

$$\tau = \mathrm{Max}(\mathcal{F}_t, \mathcal{F}_n, \mathcal{G}_t, \mathcal{G}_n) \tag{3.11}$$

which will determine the corresponding two-dimensional model. If $\mathcal{T} = \varepsilon^p$, we will say that the force level is of ε^p order. The classification will be deduced with respect to decreasing values of $\mathcal{T}$, from $\mathcal{T} = \varepsilon$ (severe force level) to $\mathcal{T} = \varepsilon^n$, $n \geq 4$ (low force level).

- For severe applied forces we will consider the same level of normal and tangential forces $\mathcal{F}_t = \mathcal{G}_t = \mathcal{F}_n = \mathcal{G}_n = \varepsilon$.
- For high to low applied forces, we will consider a level of tangential forces more important than the level of normal forces : $\mathcal{F}_t = \mathcal{G}_t = \varepsilon^m$ and $\mathcal{F}_n = \mathcal{G}_n = \varepsilon^{m+1}$ for $m \geq 2$. This gap which naturally appears is due to the fact that the tangential and the normal direction do not play a symmetrical role for shallow shells. Once reduced to a one-scale problem, we make the asymptotic expansion of Eqs. (3.7)–(3.9) for decreasing force levels.

4. The nonlinear membrane model

4.1. Asymptotic expansion of 3D equilibrium equations

Let us consider in this section a shallow shell where $\mathcal{C} = \varepsilon^2$, subjected to a severe force level $\mathcal{G}_t = \mathcal{F}_t = \varepsilon$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon$. Thus problem (3.7)–(3.9) is now reduced to a one-scale problem with ε as the small parameter. The standard asymptotic expansion method leads to write the dimensionless solution (V, u) as a formal expansion with respect to ε:

$$V = V^0 + \varepsilon^1 V^1 + \varepsilon^2 V^2 + \ \dots \qquad \text{and} \qquad u = u^0 + \varepsilon^1 u^1 + \varepsilon^2 u^2 + \ \dots \tag{4.1}$$

The expansion (4.1) of (V,u) implies an expansion of G, E, Σ and $\mathcal{H}$:

$$(4.2)\qquad \begin{aligned} &G = G^0 + \varepsilon G^1 + \varepsilon^2 G^2 + \ \dots \qquad E = E^0 + \varepsilon E^1 + \varepsilon^2 E^2 + \ \dots \\ &\Sigma = \Sigma^0 + \varepsilon \Sigma^1 + \varepsilon^2 \Sigma^2 + \ \dots \quad \mathcal{H}^0 = \mathcal{H}^0 + \varepsilon \mathcal{H}^1 + \varepsilon^2 \mathcal{H}^2 + \ \dots \end{aligned}$$

To simplify the presentation, the expressions of G^0, G^1 ..., E^0, E^1 ..., Σ^0, Σ^1 ... and $\mathcal{H}^0$, $\mathcal{H}^1$ ... will be detailed later when necessary. We only recall that, for shallow shells where $\mathcal{C} = \varepsilon^2$, the dimensional components (3.3) of G become :

$$(4.3)\qquad \begin{aligned} &G_t = \varepsilon \left(\frac{\hat{\partial} V}{\partial p_0} - \varepsilon u C_0 \right) \kappa_0^{-1}, \qquad G_s = \frac{\partial v}{\partial z}, \\ &G_s' = \varepsilon \kappa_0^{-1} \left(\varepsilon C_0 V + \frac{\overline{\partial u}}{\partial p_0} \right), \qquad G_n = \frac{\partial u}{\partial z}, \end{aligned}$$

where $\kappa_0 = I_0 - \varepsilon^2 z C_0$ and $\kappa_0^{-1} = I_0 + \varepsilon^2 z C_0 + \varepsilon^4 (z C_0)^2 + \ \dots$. Then we have the following result:

RESULT 1.

For a severe force level $\mathcal{G}_t = \mathcal{F}_t = \varepsilon$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon$, the leading term (V^0, u^0) of the asymptotic expansion of (V,u) only depends on p_0 and solves the following non-linear membrane problem:

$$\operatorname{div} \left(n_t^0 \left(I_0 + \overline{\frac{\hat{\partial} V^0}{\partial p_0}} \right) \right) = -\overline{p_t} \qquad \text{in} \quad \omega_0,$$

$$\operatorname{div} \left(n_t^0 \overline{\frac{\partial u^0}{\partial p_0}} \right) = -p_n \qquad \text{in} \quad \omega_0,$$

$$(V^0, u^0) = (0,0), \quad \text{on} \quad \gamma_0$$

where $\quad n_t^0 = \dfrac{4\beta}{2+\beta} \operatorname{Tr}(\Delta_t^0)\ I_0 + 4\Delta_t^0,$

$$2\Delta_t^0 = \overline{\frac{\hat{\partial} V^0}{\partial p_0}} + \frac{\hat{\partial} V^0}{\partial p_0} + \overline{\frac{\hat{\partial} V^0}{\partial p_0}}\,\frac{\hat{\partial} V^0}{\partial p_0} + \overline{\frac{\partial u^0}{\partial p_0}}\,\frac{\partial u^0}{\partial p_0},$$

$$p_t = g_t^+ + g_t^- + \int_{-1}^{+1} f_t dz \quad \text{and} \quad p_n = g_n^+ + g_n^- + \int_{-1}^{+1} f_n dz.$$

P r o o f. The proof of this result is decomposed into 3 steps from i) to iii)

i) (V^0, u^0) depends only on p_0

We replace $\mathcal{H}$ by its expansion in the dimensionless equilibrium Eqs. (3.7) – (3.9) where the dimensionless numbers have been replaced by $\mathcal{C} = \varepsilon^2$, $\mathcal{G}_t = \mathcal{G}_n = \mathcal{F}_t = \mathcal{F}_n = \varepsilon$. We then obtain a chain of coupled problems $\mathcal{P}^0$, $\mathcal{P}^1$, $\mathcal{P}^2$ ..., corresponding respectively to the cancellation of the factor of ε^0, ε^1, ε^2 ...

Using (3.4) – (3.6) and (4.3), let us write explicitely the expressions of G^0, E^0, Σ^0 and $\mathcal{H}^0$. We have:

$$\tag{4.4} \begin{gathered} G_t^0 = 0, \quad G_s^0 = \frac{\partial V^0}{\partial z}, \quad G'^0_s = 0, \quad G_n^0 = \frac{\partial u^0}{\partial z}, \\ 2E_t^0 = 0, \quad 2E_s^0 = 0, \quad 2E_n^0 = \overline{G_s^0}\, G_s^0 + G_n^{0\,2}, \\ \Sigma_t^0 = \beta E_n^0\, I_0, \quad \Sigma_s^0 = 0, \quad \Sigma_n^0 = (\beta+2) E_n^0, \\ \mathcal{H}_t^0 = 0, \quad \mathcal{H}_s^0 = 0, \quad \mathcal{H}'^0_s = G_s^0 \Sigma_n^0, \quad \mathcal{H}_n^0 = G_n^0\, \Sigma_n^0. \end{gathered}$$

The cancellation of the factor of ε^0 in the expansion of dimensionless equations (3.7) – (3.9) leads to problem $\mathcal{P}^0$:

$$\frac{\partial \mathcal{H}'^0_s}{\partial z} = 0 \quad \text{and} \quad \frac{\partial \mathcal{H}_n^0}{\partial z} = 0 \quad \text{in} \quad \Omega_0,$$
$$\mathcal{H}'^{0\pm}_s = 0 \quad \text{and} \quad \mathcal{H}_n^{0\pm} = 0 \quad \text{on} \quad \Gamma_0^{\pm}.$$

So we have

$$\mathcal{H}'^0_s = 0 \quad \text{and} \quad \mathcal{H}_n^0 = 0 \qquad \text{in} \quad \Omega_0.$$

Using the expression (4.4) of $\mathcal{H}'^0_s$ and $\mathcal{H}_n^0$, we obtain:

$$\left(\left\| \frac{\partial V^0}{\partial z} \right\|^2 + \left(\frac{\partial u^0}{\partial z} \right)^2 \right) \frac{\partial V^0}{\partial z} = 0 \quad \text{and} \quad \left(\left\| \frac{\partial V^0}{\partial z} \right\|^2 + \left(\frac{\partial u^0}{\partial z} \right)^2 \right) \frac{\partial V^0}{\partial z} = 0 \quad \text{in} \quad \Omega_0$$

which leads to

$$\frac{\partial V^0}{\partial z} = 0 \quad \text{and} \quad \frac{\partial u^0}{\partial z} = 0 \quad \text{in} \quad \Omega_0$$

and implies that

$$\tag{4.5} V^0 = V^0(p_0) \quad \text{and} \quad u^0 = u^0(p_0).$$

Hence we get $\mathcal{H}^0 = \mathcal{H}^1 = \mathcal{H}^2 = 0$, and problems $\mathcal{P}^1$ and $\mathcal{P}^2$ are trivially satisfied.

ii) Expression of (V^1, u^1)
The cancellation of the factor of ε^3 in Eqs. (3.7) – (3.9) leads to the problem $\mathcal{P}^3$:

$$\frac{\partial \mathcal{H}_s'^3}{\partial z} = \frac{\partial \mathcal{H}_n^3}{\partial z} = 0 \quad \text{in} \quad \Omega_0,$$

$$\mathcal{H}_s'^{3\,\pm} = \mathcal{H}_n^{3\,\pm} = 0 \quad \text{on} \quad \Gamma_0^{\pm},$$

where the expression of $\mathcal{H}^3$ is obtained from (3.4) – (3.6) and (4.3) as follows:

$$\mathcal{H}_t^3 = \Sigma_t^2(\overline{G_t^1} + I_0) + \Sigma_s^2\overline{G_s^1}, \qquad \mathcal{H}_s^3 = \Sigma_s^2(G_n^1+1) + \Sigma_t^2 G_s'^1,$$

$$\mathcal{H}_s'^3 = (G_t^1 + I_0)\Sigma_s^2 + G_s^1\Sigma_t^2, \qquad \mathcal{H}_n^3 = (G_n^1+1)\Sigma_n^2 + \overline{G_s'^1}\Sigma_s^2,$$

$$\Sigma_t^2 = \beta(\mathrm{Tr}(E_t^2) + E_n^2)I_0 + 2E_t^2,$$

$$\Sigma_s^2 = 2E_s^2,$$

$$\Sigma_n^2 = \beta\mathrm{Tr}(E_t^2) + (\beta+2)E_n^2, \tag{4.6}$$

$$2E_t^2 = \overline{G_t^1}G_t^1 + \overline{G_t^1} + G_t^1 + G_s'^1\overline{G_s'^1},$$

$$2E_s^2 = \overline{G_t^1}G_s^1 + G_n^1 G_s'^1 + G_s'^1 + G_s^1,$$

$$2E_n^2 = \overline{G_s^1}G_s^1 + (G_n^1)^2 + 2G_n^1,$$

$$G_t^1 = \frac{\hat{\partial} V^0}{\partial p_0}, \quad G_s^1 = \frac{\partial V^1}{\partial z}, \quad G_s'^1 = \overline{\frac{\partial u^0}{\partial p_0}}, \quad G_n^1 = \frac{\partial u^1}{\partial z}.$$

Problem $\mathcal{P}^3$ then gives us :

$$\mathcal{H}_s'^3 = \mathcal{H}_n^3 = 0 \quad \text{in} \quad \Omega_0$$

Replacing $\mathcal{H}_s'^3$ et $\mathcal{H}_n^3$ by their expressions (4.6), we get:

$$(G_t^1 + I_0)\Sigma_s^2 + G_s^1\Sigma_n^2 = 0 \quad \text{and} \quad \overline{G_s'^1}\Sigma_s^2 + (G_n^1+1)\Sigma_n^2 = 0 \quad \text{in} \quad \Omega_0 \tag{4.7}$$

or equivalently, using matrix notations :

$$(I_3 + G^1)\begin{bmatrix} \Sigma_s^2 \\ \Sigma_n^2 \end{bmatrix} = 0 \quad \text{in} \quad \Omega_0. \tag{4.8}$$

Now, let us use of the equation of continuity (2.2)

$$\det F^* \geq a > 0$$

where a is independent of the geometry, hence independent of $\varepsilon = h_0/L_0$. As $G^0 = 0$, the expansion of G becomes

$$G = \varepsilon G^1 + \varepsilon^2 G^2 + \ \ldots$$

According to (3.2), the expansion of $F^* = I_3 + G^*$ reduces to

$$F^* = (I_3 + G^1) + \varepsilon G^2 + \cdots$$

and the equation of continuity leads to:

$$\det\left(I_3 + G^1\right) > 0$$

So $I_3 + G^1$ is invertible and Eq. (4.8) then implies that:

$$\Sigma_s^2 = 0 \quad \text{and} \quad \Sigma_n^2 = 0.$$

Hence, using the expressions (4.6) of Σ_s^2 and Σ_n^2, we get:

$$E_s^2 = 0 \qquad \text{and} \qquad E_n^2 = -\frac{\beta}{\beta+2}\mathrm{Tr}(E_t^2)$$

So we have :

$$\begin{gathered} E_t^2 = \Delta_t^0, \quad \Sigma_t^2 = \frac{2\beta}{\beta+2}\mathrm{Tr}(\Delta_t^0) I_0 + 2\Delta_t^0, \quad \mathcal{H}_t^3 = \Sigma_t^2\left(I_0 + \overline{\frac{\hat{\partial} V^0}{\partial p_0}}\right), \\ \mathcal{H}_s^3 = \Sigma_t^2 \overline{\frac{\partial u^0}{\partial p_0}}, \quad \mathcal{H}_s'^3 = \frac{\partial V^1}{\partial z}\Sigma_t^2 \quad \text{and} \quad \mathcal{H}_n^3 = 0, \end{gathered} \tag{4.9}$$

where $\quad 2\Delta_t^0 = \overline{\dfrac{\hat{\partial} V^0}{\partial p_0}} + \dfrac{\hat{\partial} V^0}{\partial p_0} + \overline{\dfrac{\hat{\partial} V^0}{\partial p_0}}\,\dfrac{\hat{\partial} V^0}{\partial p_0} + \overline{\dfrac{\partial u^0}{\partial p_0}}\,\dfrac{\partial u^0}{\partial p_0}.$

Now, let us define the mapping $\phi^1 = V^1 + (u^1 + z)N_0$ to simplify the expressions. Then Eq. (4.7) can be written with matricial notations :

$$\begin{gathered} 2E_s^2 = \left[I_0 + \overline{\frac{\hat{\partial} V^0}{\partial p_0}} \quad \overline{\frac{\partial u^0}{\partial p_0}} \right] \begin{bmatrix} \dfrac{\partial V^1}{\partial z} \\ \dfrac{\partial u^1}{\partial z} + 1 \end{bmatrix} = \Pi_0 \overline{(I_3 + G^1)}\, \frac{\partial \phi^1}{\partial z} = 0, \\ 2E_n^2 = \left|\left| \frac{\partial \phi^1}{\partial z} \right|\right|^2 - 1 = -\frac{2\beta}{\beta+2}\mathrm{Tr}(E_t^2), \end{gathered} \tag{4.10}$$

where Π_0 denotes the orthogonal projection onto $T\omega_0$.
In the first above equation $I_3 + G^1$ is invertible in $\mathbb{R}^3$, so $\Pi_0\overline{(I_3 + G^1)}$ is an operator of rank 2. On the other hand, we have :

$$\Pi_0\overline{(I_3 + G^1)} = \left[I_0 + \overline{\frac{\hat{\partial} V^0}{\partial p_0}} \quad \overline{\frac{\partial u^0}{\partial p_0}} \right].$$

So we deduce that $\Pi_0\overline{(I_3 + G^1)}$ is independent of z. Accordingly, the image of $\mathbb{R}^3$ by this operator is a plane independent of z. Then the relation $\Pi_0\overline{(I_3 + G^1)}\dfrac{\partial \phi^1}{\partial z} = 0$ implies that $\dfrac{\partial \phi^1}{\partial z} = \theta_0 N$, $\theta_0 \in \mathbb{R}$, where N denotes the normal to the plane image of $\mathbb{R}^3$ by the application $(I_3 + G^1)\Pi_0$.

As $2E_t^2 = \overline{\dfrac{\hat{\partial} V^0}{\partial p_0}} + \dfrac{\hat{\partial} V^0}{\partial p_0} + \overline{\dfrac{\hat{\partial} V^0}{\partial p_0}}\,\dfrac{\hat{\partial} V^0}{\partial p_0} + \overline{\dfrac{\partial u^0}{\partial p_0}}\,\dfrac{\partial u^0}{\partial p_0}$ is independent of z, the second Eq. (4.10) then implies that $\left\| \dfrac{\partial \phi^1}{\partial z} \right\| = \sqrt{1 - \dfrac{2\beta}{\beta + 2}\operatorname{Tr}(E_t^2)}$ is independent of z as well. So we have :

$$\phi^1 = U^1(p_0) + z\theta_0 N \tag{4.11}$$

with $\theta_0 = \sqrt{1 - \dfrac{2\beta}{\beta + 2}\operatorname{Tr}(E_t^2)}$,
where N is an unit vector such as $\Pi_0\overline{(I_3 + G^1)}N = 0$, and where U^1 depends only on p_0

iii) Nonlinear membrane equations
The cancellation of the factor of ε^4 in the expansion of Eqs. (3.7) – (3.9) leads to problem $\mathcal{P}^4$:

$$\operatorname{div}(\mathcal{H}_t^3) + \overline{\frac{\partial \mathcal{H}_s'^4}{\partial z}} = -\overline{f_t} \quad \text{in} \quad \Omega_0 \qquad \mathcal{H'}_s^{4\pm} = \pm g_t^{\pm} \quad \text{on} \quad \Gamma_0^{\pm}$$

$$\operatorname{div}(\mathcal{H}_s^3) + \frac{\partial \mathcal{H}_n^4}{\partial z} = -\overline{f_n} \quad \text{in} \quad \Omega_0 \qquad \mathcal{H}_n^{4\pm} = \pm g_t^{\pm} \quad \text{on} \quad \Gamma_0^{\pm}$$

Using (4.9), an integration from -1 to 1 with respect to z of the above equations leads to the equilibrium equations of Result 1.
The boundary conditions on γ_0 are obtained by assuming that the leading term (V^0, u^0) of the expansion of (V, u) satisfies the clamped condition on γ_0.

□

4.2. Comparison with existing models

In the literature, the two-dimensional shell models are generally obtained in a weak formulation. So the nonlinear membrane model obtained in Result 1 in a local formulation must be written in a weak formulation to be compared to other existing ones. To do this, let us define the space of admissible displacements :

$$V(\omega_0) = \Big\{ U = (V, u) \ : \ \omega_0 \longrightarrow \mathbb{R}^3, \text{ "smooth"}, \ (V, u) = (0, 0) \ \text{ on } \ \gamma_0 \Big\}$$

where V and u denote the tangential and normal components of the displacement U. Then, the two-dimensional membrane equations of Result 1 can be written in the following weak formulation:

RESULT 2.

For applied forces such as $\mathcal{G}_t = \mathcal{G}_n = \mathcal{F}_t = \mathcal{F}_n = \varepsilon$, the leading term (V^0, u^0) of the expansion of the displacement is solution of the weak problem :

Find $(V^0, u^0) \in V(\omega_0)$ such that:

$$\int\limits_{\omega_0} \operatorname{Tr}(n_t^0 \delta \Delta_t^0) d\omega_0 = \int\limits_{\omega_0} (\overline{p_t} \delta V^0 + p_n \delta u^0) d\omega_0 \qquad \forall (\delta V^0, \delta u^0) \in V(\omega_0) \tag{4.12}$$

with :

$$n_t^0 = \frac{4\beta}{2+\beta} \operatorname{Tr}(\Delta_t^0) I_0 + 4\Delta_t^0 \quad \text{and} \quad 2\Delta_t^0 = \overline{\frac{\hat{\partial} V^0}{\partial p_0}} + \frac{\hat{\partial} V^0}{\partial p_0} + \overline{\frac{\hat{\partial} V^0}{\partial p_0}} \, \frac{\hat{\partial} V^0}{\partial p_0} + \overline{\frac{\partial u^0}{\partial p_0}} \, \frac{\partial u_0}{\partial p_0}$$

where $\delta \Delta_t^0$ denotes the variation of Δ_t^0 due to the virtual displacements $(\delta V^0, \delta u^0)$ associated to (V^0, u^0).

The proof of this result is classical and uses the Stokes theorem. It will not be discussed here.

Thus, for a severe force level, we obtain a nonlinear membrane model whose weak formulation of Result 2 is different from the one obtained in [13] for "general shells"[10)]. However, it seems to be a generalization to shallow shells of the nonlinear membrane model obtained for plates in [21][34].

4.3. Back to physical variables

The return to physical variables in equations of Result 1 leads to define :

$$V^{*0} = V_r V^0 = L_0 V^0 \qquad \text{and} \qquad u^{*0} = u_r^0 u^0 = L_0 u^0$$

[10)] In the papers using a description of the shell with local coordinates, the "general shells" corresponds for us to strongly curved shells. In the second part of this paper, where the strongly curved shells are studied, we will see that we obtain the same nonlinear membrane model as in [13] for a severe force level.

We then have the following result:

RESULT 3.

For applied forces f^* and g^* such as $\mathcal{G}_t = \mathcal{G}_n = \mathcal{F}_t = \mathcal{F}_n = \varepsilon$, the displacement (V^{*0}, u^{*0}) depends only on p_0^* and verifies the following nonlinear membrane problem :

$$h_0 \operatorname{div}\left(n_t^{*0} \left(I_0 + \overline{\frac{\hat{\partial} V^{*0}}{\partial p_0^*}} \right) \right) = -\overline{p_t^*} \quad \text{in} \qquad \omega_0^*,$$

$$h_0 \operatorname{div}^* \left(n_t^{*0} \overline{\frac{\partial u^{*0}}{\partial p_0^*}} \right) = -p_n^* \quad \text{in} \qquad \omega_0^*,$$

$$V^{*0} = 0 \quad \text{and} \quad u^{*0} = 0 \quad \text{on,} \qquad \gamma_0^*$$

where:

$$n_t^{*0} = \frac{4\lambda\mu}{\lambda + 2\mu} \operatorname{Tr}(\Delta_t^{*0})\ I_0 + 4\mu \Delta_t^{*0},$$

$$2\Delta_t^{*0} = \overline{\frac{\hat{\partial} V^{*0}}{\partial p_0^*}} + \frac{\hat{\partial} V^{*0}}{\partial p_0^*} + \overline{\frac{\hat{\partial} V^{*0}}{\partial p_0^*}}\, \frac{\hat{\partial} V^{*0}}{\partial p_0^*} + \overline{\frac{\hat{\partial} V^{*0}}{\partial p_0^*}}\, \frac{\hat{\partial} V^{*0}}{\partial p_0^*},$$

$$p_t^* = g^{*+}_{\ t} + g^{*-}_{\ t} + \int_{-h_0}^{h_0} f_t^* dz^* \quad \text{and} \quad p_n^* = g^{*+}_{\ n} + g^{*-}_{\ n} + \int_{-h_0}^{h_0} f_n^* dz^*.$$

P r o o f. Let us define $n_t^{*0} = \mu n_t^0$, $p_t^* = g^{*+}_{\ t} + g^{*-}_{\ t} + \dfrac{\mathcal{G}_t}{\mathcal{F}_t} \displaystyle\int_{-h_0}^{h_0} f_t^* dz^*$ and $p_n^* = g^{*+}_{\ n} + g^{*-}_{\ n} + \dfrac{\mathcal{G}_n}{\mathcal{F}n} \displaystyle\int_{-h_0}^{h_0} f_n^* dz^*$. Going back to the physical variables in Result 1, we get :

$$L_0 \operatorname{div}\left(n_t^{*0} \left(I_0 + \overline{\frac{\hat{\partial} V^{*0}}{\partial p_0^*}} \right) \right) = -\frac{1}{\mathcal{G}_t} \overline{p_t^*} \quad \text{in} \qquad \omega_0^*,$$

$$L_0 \operatorname{div}^* \left(n_t^{*0} \overline{\frac{\partial u^{*0}}{\partial p_0^*}} \right) = -\frac{1}{\mathcal{G}_n} p_n^*, \quad \text{in} \qquad \omega_0^*,$$

$$V^{*0} = 0 \quad \text{and} \quad u^{*0} = 0 \quad \text{on,} \qquad \gamma_0^*.$$

□

According to the force level considered here, we obtain the equations of Result 3. In what follows, to save space, we won't give the dimensional equations associated to the other dimensionless models. However, it would be possible to obtain them in the same way.

5. Another membrane model

In this section, we assume that the shell is subjected to a high force level $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^2$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^3$. First we show that the displacement (V^0, u^0) which is a solution of the Result 1 is equal to zero. This proves that the reference scales of the displacement $V_r = u_r = L_0$ have been not properly chosen and we will make a new dimensional analysis of equilibrium equations with $V_r = u_r = h_0$. We then show that the new asymptotic expansion of equations leads to another membrane model.

5.1. Determination of the reference scales of the displacement

For a force level such as $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^2$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^3$, following the proof of Results 1 and 2, we obtain the same weak formulation without a right side, whose associated minimization problem is the following one :

Find $(V^0, u^0) \in V(\omega_0)$ *which minimizes on* $V(\omega_0)$ *the functional* $\mathcal{J} = \int\limits_{\omega_0} \alpha d\omega_0$

where $\alpha = \dfrac{2\beta}{\beta+2}\left[\mathrm{Tr}(\Delta_t)\right]^2 + 2\mathrm{Tr}({\Delta_t}^2)$,

and $2\Delta_t(V,u) = \overline{\dfrac{\hat{\partial} V}{\partial p_0}} + \dfrac{\hat{\partial} V}{\partial p_0} + \overline{\dfrac{\hat{\partial} V}{\partial p_0}}\,\dfrac{\hat{\partial} V}{\partial p_0} + \overline{\dfrac{\partial u}{\partial p_0}}\,\dfrac{\partial u}{\partial p_0}$.

It is easy to show that the displacement (V^0, u^0) which minimizes the functional $\mathcal{J}$ defined above is solution of Result 2.

As the density of energy α is positive and is equal to zero if and only if $\Delta_t = 0$, this minimization problem implies that $\Delta_t(V^0, u^0) = 0$. We now have to use the following lemma to prove that $(V^0, u^0) = (0,0)$:

LEMMA 1. In $V(\omega_0)$, the solution of the equation

$$\overline{\frac{\hat{\partial} V}{\partial p_0}} + \frac{\hat{\partial} V}{\partial p_0} + \overline{\frac{\hat{\partial} V}{\partial p_0}}\,\frac{\hat{\partial} V}{\partial p_0} + \overline{\frac{\partial u}{\partial p_0}}\,\frac{\partial u}{\partial p_0} = 0$$

is $(V^0, u^0) = (0,0)$ in ω_0

Proof. Let us explain $\int\limits_{\omega_0} \mathrm{Tr}\,(\Delta_t) d\omega_0$. We have :

$$\mathrm{Tr}(\Delta_t) = \mathrm{Tr}\left(\frac{\hat{\partial} V}{\partial p_0}\right) + \frac{1}{2}\mathrm{Tr}\left(\overline{\frac{\hat{\partial} V}{\partial p_0}}\,\frac{\hat{\partial} V}{\partial p_0}\right) + \frac{1}{2}\left\|\overline{\frac{\partial u}{\partial p_0}}\right\|^2$$

As $V = 0$ on γ_0, using the Stokes formula, we get:

$$\int_{\omega_0} \mathrm{Tr}\left(\frac{\hat{\partial} V}{\partial p_0}\right) d\omega_0 = 0.$$

So we have:

$$\int_{\omega_0} \mathrm{Tr}\,(\Delta_t) d\omega_0 = \frac{1}{2}\int_{\omega_0} \left(\mathrm{Tr}\left(\overline{\frac{\hat{\partial} V}{\partial p_0}}\,\frac{\hat{\partial} V}{\partial p_0}\right) + \left\|\overline{\frac{\partial u}{\partial p_0}}\right\|^2\right) d\omega_0.$$

Thus the equation $\Delta_t(V, u) = 0$ implies that $\int_{\omega_0} \mathrm{Tr}(\Delta_t) d\omega_0 = 0$ and we have:

$$\int_{\omega_0} \left(\mathrm{Tr}\left(\overline{\frac{\hat{\partial} V}{\partial p_0}}\,\frac{\hat{\partial} V}{\partial p_0}\right) + \left\|\overline{\frac{\partial u}{\partial p_0}}\right\|^2\right) d\omega_0 = 0.$$

As $\mathrm{Tr}\left(\overline{\frac{\hat{\partial} V}{\partial p_0}}\,\frac{\hat{\partial} V}{\partial p_0}\right) + \left\|\overline{\frac{\partial u}{\partial p_0}}\right\|^2$ is positive, we obtain:

$$\mathrm{Tr}\left(\overline{\frac{\hat{\partial} V}{\partial p_0}}\,\frac{\hat{\partial} V}{\partial p_0}\right) = 0 \quad \text{and} \quad \left\|\overline{\frac{\partial u}{\partial p_0}}\right\|^2 = 0 \quad \text{in} \quad \omega_0$$

which leads to :

$$\frac{\hat{\partial} V}{\partial p_0} = 0 \qquad \text{and} \qquad \overline{\frac{\partial u}{\partial p_0}} = 0 \qquad \text{in} \quad \omega_0.$$

Finally we have:

$$\frac{\partial \|V\|^2}{\partial p_0} = 2\overline{V}\,\frac{\hat{\partial} V}{\partial p_0} = 0 \qquad \text{and} \qquad \overline{\frac{\partial u}{\partial p_0}} = 0 \qquad \text{in} \quad \omega_0$$

which proves that $\|V\|$ and u are constant in ω_0. As V and u are zero on the boundary γ_0, they are equal to zero in all ω_0. This ends the proof of Lemma 1. □

Remarking that since $(V^0, u^0) = (0, 0)$, the expressions (4.11) of θ_0 and N reduce to $\theta_0 = 1$ and $N = N_0$. So we get:

(5.1) $$V^1 = V^1(p_0) \quad \text{and} \quad u^1 = u^1(p_0).$$

As we have proved that $(V^0, u^0) = (0,0)$, we get

$$V = \frac{V^*}{V_r} = \frac{V^*}{L_0} = \varepsilon V^1 + \varepsilon^2 V^2 + \cdots$$

$$u = \frac{u^*}{u_r} = \frac{u^*}{L_0} = \varepsilon u^1 + \varepsilon^2 u^2 + \cdots$$

which is equivalent to :

$$\widetilde{V} = \frac{V^*}{\varepsilon V_r} = \frac{V^*}{h_0} = V^1 + \varepsilon V^2 + \cdots = \widetilde{V^0} + \varepsilon\widetilde{V^1} + \varepsilon^2\widetilde{V^2} + \cdots$$

$$\widetilde{u} = \frac{u^*}{\varepsilon u_r} = \frac{u^*}{h_0} = u^1 + \varepsilon u^2 + \cdots = \widetilde{u^0} + \varepsilon\widetilde{u^1} + \varepsilon^2\widetilde{u^2} + \cdots$$

Accordingly we have proved that for the level forces considered here, the reference scales of the displacement $V_r = u_r = L_0$ have not been properly chosen. For the leading term of the expansion of the displacement to be of the order of one unit, the reference scales of the displacement must verify $(V_r, u_r) = (h_0, h_0)$. Therefore the dimensionless equilibrium equations must be written again with $V_r = h_0$ and $u_r = h_0$ as the new reference scales. The new dimensionless displacements will still be denoted $V = V^*/V_r$ and $u = u^*/u_r$.

5.2. The associated asymptotic model

With these new reference scales of the displacement $(V_r, u_r) = (h_0, h_0)$, which are directly deduced from the force level considered, we make a new dimensional analysis of equilibrium Eqs. (2.7) – (2.9). We obtain the same dimensionless Eqs. (3.7) – (3.9) where V and u must be changed into εV and εu in the expression (4.3) of G. Thus we have with these new reference scales of the displacement :

(5.2)
$$G_t = \varepsilon^2 \left(\frac{\hat{\partial} V}{\partial p_0} - u\varepsilon C_0 \right) \kappa^{-1}, \qquad G_s = \varepsilon \frac{\partial V}{\partial z},$$

$$G'_s = \varepsilon^2 \kappa^{-1} \left(\varepsilon C_0 V + \frac{\overline{\partial u}}{\partial p_0} \right), \qquad G_n = \varepsilon \frac{\partial u}{\partial z}.$$

The new expressions of E, Σ and H can be obtained from (3.4) – (3.6) and (5.2).

On the other hand, the asymptotic expansion of equations enables to write again the new dimensionless solution (V, u) of the new dimensionless problem as a formal expansion with respect to ε. This is equivalent to changing (V^i, u^i) into (V^{i-1}, u^{i-1}) for $i \geq 1$ in the previous results. In particular, relation (5.1) becomes:

$$V^0 = V^0(p_0) \quad \text{and} \quad u^0 = u^0(p_0)$$

Then, we have the following result:

RESULT 4.

For a force level such as $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^2$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^3$, the leading term (V^0, u^0) of the expansion of (V, u) depends only on p_0 and satisfies the membrane equations

$$
\begin{aligned}
&\operatorname{div}\left(n_t^0\right) = -\overline{p_t} && in \quad \omega_0,\\
&\operatorname{div}\left(n_t^0 \overline{\frac{\partial u^0}{\partial p_0}}\right) + \operatorname{Tr}\left(n_t^0 C_0\right) = -p_n - \operatorname{div}(M_t) && in \quad \omega_0,\\
&V^0 = 0 \qquad \text{and} \qquad u^0 = 0 && in \quad \gamma_0,
\end{aligned} \tag{5.3}
$$

where $\quad n_t^0 = \dfrac{4\beta}{2+\beta}\operatorname{Tr}\left(\Delta_t^0\right) I_0 + 4\Delta_t^0, \quad 2\Delta_t^0 = \overline{\dfrac{\hat{\partial} V^0}{\partial p_0}} + \dfrac{\hat{\partial} V^0}{\partial p_0}$

$$
p_t = g_t^+ + g_t^- + \int_{-1}^{1} f_t\, dz, \quad M_t = g_t^+ - g_t^- + \int_{-1}^{1} z\, f_t\, dz \quad \text{and} \quad p_n = g_n^+ + g_n^- + \int_{-1}^{1} f_n\, dz.
$$

P r o o f.

i) Expression of (V^1, u^1)

According to the new dimensional analysis (5.2), we have:

$$
\begin{aligned}
&G_t^2 = \frac{\hat{\partial} V^0}{\partial p_0}, \quad G_s^2 = \frac{\partial V^1}{\partial_z}, \quad G_s'^2 = \overline{\frac{\partial u^0}{\partial p_0}}, \quad G_n^2 = \frac{\partial u^1}{\partial z},\\
&2E_t^3 = \overline{\frac{\hat{\partial} V^0}{\partial p_0}} + \frac{\hat{\partial} V^0}{\partial p_0}, \quad 2E_s^3 = \frac{\partial V^1}{\partial z} + \overline{\frac{\partial u^0}{\partial p_0}}, \quad 2E_n^3 = 2\frac{\partial u^1}{\partial z},\\
&\Sigma_t^3 = \beta(\operatorname{Tr}(E_t^3) + E_n^3)I_0 + 2E_t^3, \quad \Sigma_s^3 = 2E_s^3,\\
&\Sigma_n^3 = \beta\operatorname{Tr}(E_t^3) + (\beta+2)E_n^3,\\
&\mathcal{H}_t^4 = \Sigma_t^3, \quad \mathcal{H}_s^4 = \Sigma_s^3 \quad \mathcal{H'}_s^4 = \Sigma_s^3 \quad \mathcal{H}_n^4 = \Sigma_n^3
\end{aligned} \tag{5.4}
$$

The cancellation of the factor of ε^4 in the new expansion of Eqs. (3.7)–(3.9) leads to problem $\mathcal{P}^4$:

$$
\frac{\partial \mathcal{H}_s'^4}{\partial z} = 0 \quad \text{and} \quad \frac{\partial \mathcal{H}_n^4}{\partial z} = 0 \quad \text{in} \quad \Omega_0, \quad \mathcal{H}_s'^{4\,\pm} = 0 \quad \text{and} \quad \mathcal{H}_n^{4\,\pm} = 0 \quad \text{on} \quad \Gamma_0^{\pm}
$$

which implies that

$$\mathcal{H}'^4_s = 0 \quad \text{and} \quad \mathcal{H}^4_n = 0 \qquad \text{in} \quad \Omega_0$$

Using (5.5), we get:

$$E^3_s = 0 \quad \text{and} \quad 2E^3_n = -\frac{\beta}{\beta+2}\mathrm{Tr}(E^3_t) \tag{5.5}$$

which can be written in terms of displacements :

$$\frac{\partial V^1}{\partial z} = -\overline{\frac{\partial u^0}{\partial p_0}} \quad \text{and} \quad \frac{\partial u^1}{\partial z} = -\frac{\beta}{\beta+2}\mathrm{Tr}(\Delta^0_t) \quad \text{in} \quad \Omega_0$$

and leads to

$$\begin{cases} u^1 = \underline{u}^1 - z\dfrac{\beta}{\beta+2}\mathrm{Tr}\,(\Delta^0_t) \\ \\ V^1 = \underline{V}^1 - z\overline{\dfrac{\partial u^0}{\partial p_0}} \end{cases} \tag{5.6}$$

where the fields of tangent vectors $\underline{V}^1$ and of scalars $\underline{u}^1$ depend only on p_0.

On the other hand, we have according to (5.4):

$$E^3_t = \Delta^0_t, \quad \Sigma^3_t = \mathcal{H}^4_t = \frac{1}{2}n^0_t \quad \text{and} \quad \mathcal{H}^4_s = \mathcal{H}'^4_s = 0 \tag{5.7}$$

where the expressions of n^0_t and Δ^0_t are those of Result 4.

ii) First membrane equation

The first equation of problem $\mathcal{P}^5$ then reduces to :

$$\begin{aligned} &\mathrm{div}\left(\mathcal{H}^4_t\right) + \overline{\frac{\mathcal{H}'^5_s}{\partial z}} = -\overline{f_t} \quad \text{in} \quad \Omega_0 \qquad \mathcal{H}'^{5\pm}_s = \pm g^\pm_t \quad \text{on} \quad \Gamma^\pm_0, \\ &\frac{\mathcal{H}^5_n}{\partial z} = 0 \qquad \text{in} \quad \Omega_0 \qquad \mathcal{H}^{5\,\pm}_n = 0 \qquad \text{on} \quad \Gamma^\pm_0. \end{aligned} \tag{5.8}$$

Using (5.7), an integration upon the thickness (from -1 to 1 with respect to z) of the first above equation leads to the first equation of Result 4:

$$\mathrm{div}(n^0_t) = -\overline{p_t}$$

where $p_t = g^+_t + g^-_t + \int_{-1}^{1} f_t\,dz.$

On the other hand, an integration of Eq. (5.8) with respect to z enables to calculate $\mathcal{H}'^5_s$ and $\mathcal{H}^5_n$. We obtain :

$$\mathcal{H}'^5_s = \frac{1}{2}\left(zp_t + g_t^+ - g_t^- + \int_z^1 f_t\,dz - \int_{-1}^z f_t\,dz \right),$$

$$\mathcal{H}^5_n = 0.$$

Moreover Eqs. (3.4) - (3.6) and (5.2), lead to :

$$\mathcal{H}^5_s = \Sigma^4_s + \Sigma^3_t \overline{\frac{\partial u^0}{\partial p_0}} \qquad \text{and} \qquad \mathcal{H}'^5_s = \Sigma^4_s$$

which implies that:

$$\mathcal{H}^5_s = \frac{1}{2}\left(zp_t + g_t^+ - g_t^- + \int_z^1 f_t\,dz - \int_{-1}^z f_t\,dz \right) + \Sigma^3_t \overline{\frac{\partial u^0}{\partial p_0}}. \tag{5.9}$$

iii) Second membrane equation

The second equation of problem $\mathcal{P}^6$ reduces to :

$$\operatorname{div}(\mathcal{H}^5_s) + \operatorname{Tr}\left(\mathcal{H}^4_t C_0\right) + \frac{\partial \mathcal{H}^6_n}{\partial z} = -f_n \quad \text{in} \quad \Omega_0,$$

$$\mathcal{H}^{6\,\pm}_n = \pm g^\pm_n \quad \text{on} \quad \Gamma^\pm_0.$$

Using (5. 7) and (5. 9), an integration upon the thickness leads to the second equation of Result 4:

$$\operatorname{div}\left(n^0_t \overline{\frac{\partial u^0}{\partial p_0}} \right) + \operatorname{Tr}(n^0_t C_0) = -p_n - \operatorname{div}(M_t) \qquad \text{in} \quad \omega_0$$

with $p_n = g^+_n + g^+_n + \int_{-1}^1 f_n dz$ and $M_t = g^+_t - g^+_t + \int_{-1}^1 z\, f_t\,dz$. Let us notice that we have used the relation :

$$\int_{-1}^1 \left(\int_z^1 f_t\,dz - \int_{-1}^z f_t\,dz \right) dz = 2\int_{-1}^1 z\, f_t\,dz.$$

The expansion of the clamped condition $U = 0$ on Γ_0 leads at the first order to :

$$(V^0, u^0) = (0,0) \qquad \text{on} \quad \gamma_0$$

which concludes the proof of Result 4.

□

5.3. A few comments

The membrane model obtained in Result 4 for a high force level has to our knowledge no equivalent in the literature. If we set $C_0 = 0$, it is different from the FÖPPL plate model [20] given also in [46]. There is no nonlinear term coupling the deflection u^0 to the tangential displacement in the expression of the membrane strain, as in the von Kármán model.

On the contrary, the model obtained here is linear in the sense explained later. It can be split into two linear problems verified by V^0 and u^0, as the simplified version of the FÖPPL model given by H. M. BERGER[11)] [1].

Indeed the first equation of Result 4:

$$\begin{cases} \operatorname{div}\left(n_t^0\right) = -\overline{p_t} & \text{in} \quad \omega_0, \\ V^0 = 0 \quad \text{on} \quad \gamma_0, \end{cases}$$

where $n_t^0 = \dfrac{4\beta}{2+\beta}\operatorname{Tr}(\Delta_t^0)\ I_0 + 4\Delta_t^0$ and $2\Delta_t^0 = \overline{\dfrac{\hat{\partial} V^0}{\partial p_0}} + \dfrac{\hat{\partial} V^0}{\partial p_0}$, is a linear equation which depends only on the tangential displacement V^0. To prove that this problem has unique solution V^0 in $[H_0^1(\omega_0)]^2$, let us write the above equations in the following weak formulation:

Find $V^0 \in [H_0^1(\omega_0)]^2$ such as

$$\int\limits_{\omega_0} \operatorname{Tr}\Big(n_t^0 \delta\Delta_t^0\Big) d\omega_0 = \int\limits_{\omega_0} \overline{p_t}\delta V^0 d\omega_0 \qquad \forall \delta V^0 \in [H_0^1(\omega_0)]^2 \tag{5.10}$$

where the expressions of n_t^0 and Δ_t^0 are those of Result 4.

It is possible to prove (see [4][5]) that the mapping

$$V \in [H_0^1(\omega_0)]^2 \to \left\{ \int\limits_{\omega_0} \operatorname{Tr}\Big((\Delta_t^0)^2\Big) \right\}^{1/2}$$

[11)] H. M. BERGER [1] used the Föppl-von Kármán theory to formulate the strain energy density of a deformed plate. He then made the simplifying (but irrational) assumption of ignoring the term containing the second invariant of strain (relative to the first invariant of strain). This leads to a set of two uncoupled problems. For further comments see also [25].

is a norm on $[H_0^1(\omega_0)]^2$ equivalent to the usual one, provided that the Christoffel symbols of the middle surface are small enough. Therefore the first term of the weak formulation is elliptic in $[H_0^1(\omega_0)]^2$. Then if the forces are smooth enough ($L^2(\omega_0)$), this problem has an unique solution in $[H_0^1(\omega_0)]^2$.

Once V^0 and n_t^0 determined, the second equation of Result 4 becomes a linear second order equation with respect to the normal displacement u^0.

Therefore, in the case of shallow shells and for a high level of surface forces, this asymptotic approach enables to construct a membrane model which is in fact linear, but which cannot be deduced from the linear three-dimensional elasticity.

6. The Koiter's nonlinear shallow shell model

6.1. New reference scales of the displacements

In this section, we consider a shallow shell subjected to a moderate force level such as $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^3$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^4$. Then, for this force level, following the proof of Result 4, it is possible to prove that the tangential displacement V^0 is solution of $(5.3)_1$ without a right side. As this equation has an unique solution provided that V^0 is smooth enough, we have $V^0 = 0$. Let us notice that the second equation of Result 4 is then trivially satisfied and the normal displacement u^0 is undetermined.

Therefore, as $V^0 = 0$ for the force level considered here, the reference scale of the tangential displacement $V_r = h_0$ is still not properly chosen. We have to consider $V_r = \varepsilon h_0$ for V^0 to be different from zero. Thus the dimensionless equilibrium equations must be written again with $V_r = \varepsilon h_0$ and $u_r = h_0$ as reference scales. The dimensionless components of the displacements will still be denoted V and u. As previously, the new dimensionless equations so obtained from (2.7)-(2.9) are the same as (3.7)-(3.9) where V must be changed into εV in the expression (5.2) of G. Thus we have :

$$(6.1)\qquad \begin{aligned} G_t &= \varepsilon^3 \left(\frac{\hat{\partial} V}{\partial p_0} - u C_0 \right) \kappa^{-1}, & G_s &= \varepsilon^2 \frac{\partial V}{\partial z}, \\ G'_s &= \varepsilon^2 \kappa^{-1} \left(\varepsilon^2 C_0 V + \overline{\frac{\partial u}{\partial p_0}} \right), & G_n &= \varepsilon \frac{\partial u}{\partial z}. \end{aligned}$$

The new expressions of E, Σ and H are then deduced from (3.4) – (3.6) and (6.1).

6.2. The asymptotic model

We write again the new tangential displacement V of the new dimensionless problem obtained with $V_r = \varepsilon h_0$ as a formal expansion with respect to ε. This is equivalent to change V^i into V^{i-1} for $i \geq 1$ in the results of the previous section. Thus relation (5.6) becomes :

$$u^0 = \zeta_n^0(p_0), \quad V^0 = \zeta_t^0(p_0) - z\overline{\frac{\partial \zeta_n^0}{\partial p_0}} \quad \text{and} \quad u^1 = u^1(p_0) \tag{6.2}$$

and implies that (V^0, u^0) is a Kirchhoff-Love displacement. Then, the new asymptotic expansion of equilibrium equations leads to the following result:

RESULT 5.

For given applied forces such as $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^3$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^4$, the leading term (V^0, u^0) of the expansion of (V, u) is a Kirchhoff-Love displacement which satisfies:

i) $u^0 = \zeta_n^0(p_0)$ and $V^0 = \zeta_t^0(p_0) - z\overline{\dfrac{\partial \zeta_n^0}{\partial p_0}}$.

ii) $\zeta^0 = (\zeta_t^0, \zeta_n^0)$ is a solution of the following equations :

$$\begin{aligned}
&\operatorname{div}\left(n_t^0\right) = -\overline{p_t} && \text{in} \quad \omega_0,\\
&\operatorname{div}(\overline{\operatorname{div} m_t^0}) + \operatorname{div}\left(n_t^0 \overline{\frac{\partial \zeta_n^0}{\partial p_0}}\right) + \operatorname{Tr}\left(n_t^0 C_0\right) = -p_n - \operatorname{div}(M_t) && \text{in} \quad \omega_0,\\
&\zeta_n^0 = \frac{\partial \zeta_n^0}{\partial \nu_0} = 0 \qquad \text{and} \qquad \zeta_t^0 = 0 && \text{on} \quad \gamma_0,
\end{aligned}$$

where ν_0 denotes the unit external normal to γ_0 and where

$$n_t^0 = \frac{4\beta}{2+\beta}\operatorname{Tr}\left(\Delta_t^0\right) I_0 + 4\Delta_t^0, \qquad 2\Delta_t^0 = \overline{\frac{\hat{\partial}\zeta_t^0}{\partial p_0}} + \frac{\hat{\partial}\zeta_t^0}{\partial_0} + \overline{\frac{\partial \zeta_n^0}{\partial p_0}}\,\frac{\partial \zeta_n^0}{\partial p_0} - 2\zeta_n^0 C_0,$$

$$m_t^0 = \frac{4\beta}{3(2+\beta)}\operatorname{Tr}\left(K_t^0\right) I_0 + \frac{4}{3}K_t^0, \qquad K_t^0 = -\frac{\hat{\partial}}{\partial p_0}\,\frac{\partial \zeta_n^0}{\partial p_0},$$

$$p_t = g_t^+ + g_t^- + \int_{-1}^{1} f_t\, dz, \qquad M_t = g_t^+ - g_t^- + \int_{-1}^{1} z\, f_t\, dz$$

$$\text{and} \quad p_n = g_n^+ + g_n^- + \int_{-1}^{+1} f_n\, dz.$$

P r o o f. The proof of this result is similar to the previous ones. Let us just give the main steps.

i) Problem $\mathcal{P}_5$ leads to the following expression of u^1 and V^1 which will be used later :

$$\begin{cases} u^1 = u^1(p_0) \\ V^1 = \underline{V}^1 - z\overline{\dfrac{\partial u^1}{\partial p_0}} \end{cases} \tag{6.3}$$

where $\underline{V}^1$ and $\underline{u}^2$ only depend on p_0.

ii) On the other hand, using (3.4) – (3.6), (6.1) and (6.2), we get:

$$E_t^4 = \Delta_t^0 + zK_t^0, \qquad \mathcal{H}_t^5 = \Sigma_t^4 = \frac{1}{2}(n_t^0 + 3zm_t^0) \tag{6.4}$$

where the expressions of n_t^0, m_t^0 Δ_t^0 and K_t^0 are those of Result 5.

iii) After a few calculations, the first equation of problem $\mathcal{P}_6$ leads to

$$\mathcal{H}_s^6 = \frac{1}{2}\left(zp_t + g_t^+ - g_t^- + \int_z^1 f_t dz \; - \int_{-1}^z f_t dz + \frac{3(1-z^2)}{2}\overline{div(m_t^0)} + \Sigma_t^0 \overline{\frac{\partial \zeta_n^0}{\partial p_0}} \right).$$

Finally, the integration upon the thickness of the first equation of problem $\mathcal{P}_6$ and of the second equation of problem $\mathcal{P}_7$ leads to the equations of the Koiter's nonlinear shallow shell model of the Result 5.

6.3. Comments

Accordingly, the nonlinear Koiter's shallow shell model has been rigorously justified by asymptotic expansion, without any a priori assumption. On the contrary, the order of magnitude of the displacements has been directly deduced from the force level considered. We so justify the scaling assumptions on the displacements generally made in the literature [11][19].

Let us notice that the existence of a unique solution of Koiter's shallow shell model has been proved in [2] when the applied forces are weak enough.

We recall that there exists two other shallow shell models, the Marguerre-von Kármán and the Marguerre one, which are very close to the Koiter's one. These two models, which only differ by the boundary conditions on the lateral surface, have been obtained by asymptotic expansion in the case of a particular description of the middle surface in local coordinates [10][7].

At last, let us notice that Koiter's shallow shell model is a generalization to shallow shells of the usual nonlinear plate model whose justification by asymptotic expansions can be found in [9][21][34].

7. The linear Novozhilov-Donnell model

The Novozhilov-Donnell model is generally obtained from the linear three-dimensional elasticity by making a priori assumptions (Kirchhoff-Love assumptions) and neglecting the terms of second order with respect to the curvature [17]. Later, many authors have tried to justify rigorously this model by asymptotic expansion of the linear three-dimensional equations. One finds a justification in [11] for a particular parametrization of the middle surface in local coordinates, in [4][5] by using an intrinsic variational formulation and in [15][17] from the local equilibrium equations. In this section, we propose to justify rigorously the linear Novozhilov-Donnell model from asymptotic expansion of the *nonlinear* equilibrium equations. This will enable us to determine precisely its domain of validity.

7.1. Order of magnitude of displacements

For a low force level such as $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^4$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^5$, we obtain the same equations as in Result 5 without a right side. The associated minimization problem implies that $\Delta_t^0 = K_t^0 = 0$ and then that $(\zeta_t^0, \zeta_n^0) = (0,0)$ (see [6]).

Since we have proved that $(V^0, u^0) = (0,0)$, the reference scales of the displacements $V_r = h_0\varepsilon$ and $u_r = h_0$ don't correspond to the low force level considered here. We have to make a new dimensional analysis of equilibrium equations (2.7) – (2.9) with $V_r = \varepsilon^2 h_0$ and $u_r = \varepsilon h_0$ as the new reference scales. We then obtain the same dimensionless Eqs. (3.7) – (3.9) where V and u must be changed into εV and εu in the previous expression (6.1) of the components of G. Hence we have now:

$$G_t = \varepsilon^4 \left(\frac{\hat{\partial} V}{\partial p_0} - uC_0 \right) \kappa^{-1} \qquad G_s = \varepsilon^3 \frac{\partial V}{\partial z},$$

(7.1)

$$G_s' = \varepsilon^3 \kappa^{-1} \left(\varepsilon^2 C_0 V + \overline{\frac{\partial u}{\partial p_0}} \right) \qquad G_n = \varepsilon^2 \frac{\partial u}{\partial z},$$

and the new expressions of E, Σ and H will be calculated using (3.4) – (3.6).

7.2. The associated linear asymptotic model

The asymptotic expansion method enables to write again the new dimensionless solution (V, u) corresponding to $V_r = \varepsilon^2 h_0$ and $u_r = \varepsilon h_0$ as a formal expansion with respect to ε. This is equivalent to the change (V^i, u^i) into (V^{i-1}, u^{i-1}) for $i \geq 1$ in the results of the previous section. In particular (6.3) becomes

$$u^0 = \zeta_n^0(p_0), \quad V^0 = \zeta_t^0(p_0) - z\overline{\frac{\partial \zeta_n^0}{\partial p_0}} \tag{7.2}$$

which proves that (V^0, u^0) is still a Kirchhoff-Love displacement. On the other hand, according to (3.4) – (3.6) and (7.1), the first non-zero terms of the expansion of G, E, Σ and $\mathcal{H}$ are now given by:

$$\begin{aligned}
&\mathcal{H}_t^6 = \Sigma_t^5, \quad \mathcal{H}_s^6 = \Sigma_s^5, \quad \mathcal{H'}_s^6 = \Sigma_s^5, \quad \mathcal{H}_n^6 = \Sigma_n^5,\\
&\Sigma_t^5 = \beta(\mathrm{Tr}(E_t^5) + E_n^5)I_0 + 2E_t^5, \quad \Sigma_s^5 = 2E_s^5,\\
&\Sigma_n^5 = \beta\mathrm{Tr}(E_t^5) + (\beta+2)E_n^5\\
&2E_t^5 = \overline{\frac{\hat{\partial} V^0}{\partial p_0}} + \frac{\hat{\partial} V^0}{\partial p_0} - 2u^0 C_0,\\
&2E_s^5 = \frac{\partial V^1}{\partial z} + \overline{\frac{\partial u^1}{\partial p_0}},\\
&2E_n^5 = 2\frac{\partial v^2}{\partial z},\\
&G_t^4 = \frac{\hat{\partial} V^1}{\partial p_0} - u^1 C_0, \quad G_s^4 = \frac{\partial V^1}{\partial z},\\
&G_s'^4 = \overline{\frac{\partial u^1}{\partial p_0}}, \quad G_n^4 = \frac{\partial u^2}{\partial z}.
\end{aligned} \tag{7.3}$$

We then have the following result:

RESULT 6.

For a low force level such as $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^4$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^5$, the leading term (V^0, u^0) of the expansion of (V, u) is a Kirchhoff-Love displacement which verifies :

i) $u^0 = \zeta_n^0(p_0)$ and $V^0 = \zeta_t^0(p_0) - z\overline{\dfrac{\partial \zeta_n^0}{\partial p_0}}$.

ii) $\zeta^0 = (\zeta_t^0, \zeta_n^0)$ is solution of the dimensionless Novozhilov-Donnell model:

$$\operatorname{div}\left(n_t^0\right) = -\overline{p_t} \qquad \text{in} \quad \omega_0,$$

$$\operatorname{div}(\overline{\operatorname{div} m_t^0}) + \operatorname{Tr}(n_t^0 C_0) = -p_n - \operatorname{div}(M_t) \quad \text{in} \quad \omega_0,$$

$$\zeta_n^0 = \frac{\partial \zeta_n^0}{\partial \nu_0} = 0 \qquad \text{and} \qquad \zeta_t^0 = 0 \qquad \text{on} \quad \gamma_0,$$

where ν_0 denotes the unit external normal along γ_0 and where

$$n_t^0 = \frac{4\beta}{2+\beta}\operatorname{Tr}(\Delta_t^0)\ I_0 + 4\Delta_t^0, \qquad \Delta_t^0 = \frac{1}{2}\left(\overline{\frac{\hat{\partial}\zeta_t^0}{\partial p_0}} + \frac{\hat{\partial}\zeta_t^0}{\partial p_0} - 2\zeta_n^0 C_0\right),$$

$$m_t^0 = \frac{4\beta}{3(2+\beta)}\operatorname{Tr}(K_t^0) I_0 + \frac{4}{3}K_t^0, \qquad K_t^0 = -\frac{\hat{\partial}}{\partial p_0}\overline{\frac{\partial \zeta_n^0}{\partial p_0}},$$

$$p_t = g_t^+ + g_t^- + \int_{-1}^{1} f_t\, dz, \quad M_t = g_t^+ - g_t^- + \int_{-1}^{1} z\, f_t\, dz$$

and $p_n = g_n^+ + g_n^- + \int_{-1}^{1} f_n\, dz.$

P r o o f. The proof of this result is similar to the previous ones and is left to the reader.

Contrary to the existing justifications of the Novozhilov-Donnell model, the approach explained here enables us to deduce it directly from the *nonlinear three-dimensional elasticity*. This result is fundamental because it specifies its domain of validity.

Indeed, the linear Novozhilov-Donnell model is proved to be valid for weaker force levels as the nonlinear Koiter's shallow shell one. These forces lead to deflections of εh_0 order and not of h_0 order, as we could think according to the existing justifications of the Novozhilov-Donnell model from the linear elasticity [5][15].

On the other hand, let us notice that the so obtained Novozhilov-Donnell model is an extension of the linear Kirchhoff-Love plate model. Indeed, if the curvature operator C_0 takes the value zero, we find again the classical linear

Kirchhoff-Love model, which has been already justified by asymptotic expansion from the linear three-dimensional elasticity in [8][4][33]. One finds also in [12][35] a justification from the nonlinear three-dimensional elasticity. At least, we recall that the existence and the unicity of the solution of the Novozhilov-Donnell model has been proved in [4] for shallow shells.

7.3. Domain of validity of the Novozhilov-Donnell model

We have proved that the linear Novozhilov-Donnell model is valid for a low force level $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^4$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^5$ which leads to deflections of εh_0 order. In fact, the linear Novozhilov-Donnell is valid for lower force levels $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^p$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^{p+1}$, $p \geq 5$. These force levels lead to deflections of $\varepsilon^{p-3}h_0$ order.

Indeed, for a force level $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^p$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^{p+1}$, $p \geq 5$, we obtain the Novozhilov-Donnell model of Result 6 without a right side, whose unique solution is $(V^0, u^0) = (0,0)$. Therefore according to the same argument as previously, the reference scales (V_r, u_r) of the displacement must be chosen equal to $(h_0\varepsilon^{p-2}, h_0\varepsilon^{p-3})$. A new dimensional analysis of the equations with $V_r = h_0\varepsilon^{p-2}$, $u_r = h_0\varepsilon^{p-3}$ and a new asymptotic expansion lead again to the Novozhilov-Donnell model of Result 6.

It is important to notice that for sufficiently weak force levels (of ε^p order with $p \geq 4$), the problem becomes linear with respect to the displacements and the asymptotic model that we obtain is the Novozhilov-Donnell one. This result means that the linear Novozhilov-Donnell model can be used for sufficiently weak force levels of $\varepsilon^{p\geq 4}$ order, where the dimensionless numbers $\mathcal{G}_t, \mathcal{F}_t, \mathcal{G}_n, \mathcal{F}_n$ are known quantities of the problem.

8. Conclusion

The method of classification of asymptotic shell models developed in this paper is constructive. It leads to a classification from the level of applied forces without any a priori assumption[12)]. On the contrary, the order of magnitude of the displacements (characterized by the reference scales V_r and u_r) and the corresponding two-dimensional model are directly deduced from the force levels. These force levels are characterized by the dimensionless numbers $\mathcal{F}_t$, $\mathcal{F}_n$, $\mathcal{G}_t$,$\mathcal{G}_n$ which are known data of the problem.

In this paper, we have studied only a combination of $(\mathcal{F}_t, \mathcal{F}_n, \mathcal{G}_t, \mathcal{G}_n)$ for each value of $\tau = \mathrm{Max}(\mathcal{F}_t, \mathcal{F}_n, \mathcal{G}_t, \mathcal{G}_n)$. However, the study of the other combinations is not fundamental; it would lead to the same two-dimensional models with a

12)In the sense defined in the Introduction.

different right side. The following table resumes the so obtained classification with respect to τ:

τ	(V_r, u_r)	Shell model	$\Delta_t^0,\ K_t^0$
ε	(L_0, L_0)	*non linear membrane model* $\operatorname{div}\left(n_t^0\left(I_0+\overline{\hat\partial V^0/\partial p_0}\right)\right)=-p_t$ $\operatorname{div}\left(n_t^0\,\overline{\partial u^0/\partial p^0}\right)=-p_n$ $V^0_{\vert\gamma_0}=u^0{}_{\vert\gamma_0}=0$	$2\Delta_t^0=\overline{\frac{\hat\partial V^0}{\partial p_0}}+\frac{\hat\partial V^0}{\partial p_0}+$ $\overline{\frac{\hat\partial V^0}{\partial p_0}}\,\frac{\hat\partial V^0}{\partial p_0}+\overline{\frac{\partial u^0}{\partial p_0}}\,\frac{\partial u^0}{\partial p_0}$
ε^2	(h_0, h_0)	*another membrane model* $\operatorname{div}(n_t^0)=-p_t$ $\operatorname{div}\left(n_t^0\,\overline{\partial u^0/\partial p^0}\right)+\operatorname{Tr}(n_t^0 C_0)$ $=-p_n-\operatorname{div}(M_t)$ $V^0_{\vert\gamma_0}=u^0{}_{\vert\gamma_0}=0$	$2\Delta_t^0=\overline{\frac{\hat\partial V^0}{\partial p_0}}+\frac{\hat\partial V^0}{\partial p_0}$
ε^3	$(\varepsilon h_0, h_0)$	*non linear Koiter's shallow shell model* $\operatorname{div}(n_t^0)=-p_t$ $\operatorname{div}(\overline{\operatorname{div} m_t^0})+\operatorname{div}\left(n_t^0\,\overline{\partial\zeta_n^0/\partial p^0}\right)$ $+\operatorname{Tr}(n_t^0 C_0)=-p_n-\operatorname{div}(M_t)$ $\zeta_t^0\vert_{\gamma_0}=\zeta_n^0\vert_{\gamma_0}=\frac{\partial\zeta_n^0}{\partial\nu_0}\vert_{\gamma_0}=0$	$2\Delta_t^0=\overline{\frac{\hat\partial\zeta_t^0}{\partial p_0}}+\frac{\hat\partial\zeta_t^0}{\partial p_0}+$ $\overline{\frac{\partial\zeta_n^0}{\partial p_0}}\,\frac{\partial\zeta_n^0}{\partial p_0}-2\zeta_n^0 C_0$ $K_t^0=-\frac{\hat\partial}{\partial p_0}\,\overline{\frac{\partial\zeta_n^0}{\partial p_0}}$
$\varepsilon^{p\geq 4}$	$h_0(\varepsilon^{p-2},\varepsilon^{p-3})$	*linear Novozhilov-Donnell model* $\operatorname{div}(n_t^0)=-p_t$ $\operatorname{div}(\overline{\operatorname{div} m_t^0})+\operatorname{Tr}(n_t^0 C_0)$ $=-p_n-\operatorname{div}(M_t)$ $\zeta_t^0\vert_{\gamma_0}=\zeta_n^0\vert_{\gamma_0}=\frac{\partial\zeta_n^0}{\partial\nu_0}\vert_{\gamma_0}=0$	$2\Delta_t^0=\overline{\frac{\hat\partial\zeta_t^0}{\partial p_0}}+\frac{\hat\partial\zeta_t^0}{\partial p_0}$ $-2\zeta_n^0\,C_0$ $K_t^0=-\frac{\hat\partial}{\partial p_0}\,\overline{\frac{\partial\zeta_n^0}{\partial p_0}}$

where $n_t^0=\dfrac{4\beta}{\beta+2}\operatorname{Tr}(\Delta_t^0)I_0+4\Delta_t^0$ and $m_t^0=\dfrac{4\beta}{3(\beta+2)}\operatorname{Tr}(K_t^0)I_0+\dfrac{4}{3}K_t^0.$

On the other hand, the classification deduced from the three-dimensional nonlinear elasticity enables us to specify the domain of validity of the obtained two-dimensional shell models, thanks to the dimensionless numbers naturally introduced.

In particular we have proved that the usual linear Novozhilov-Donnell model is valid for applied force levels weaker than the ones for which the nonlinear Koiter's shallow shell model is obtained.

These forces lead to deflections of εh_0 order and not of h_0 order. This result is important and underlines the pathology of the results obtained from asymptotic expansion of linear three-dimensional equilibrium equations, which are already an "expansion at the first order" of nonlinear equilibrium equations. Indeed, when the linear Novozhilov-Donnell model is deduced from the linear three-dimensional [32], it seems to have the same domain of validity as the nonlinear Koiter's shallow shell model (deflections of h_0 order).

Finally, let us notice the constructive character of this approach. Indeed another membrane model, which has to our knowledge no equivalent in the literature, has been put in a prominent position for high force levels. This model cannot be obtained from the linear elasticity.

In the second part of this paper, we will study the strongly curved shells which have a different asymptotic behaviour. In this case, the classification is more complex : it depends not only on the force levels, but also on the existence of inextensional displacements which keep invariant the metric of the middle surface of the shell.

Appendix A. Intrinsic formalism of surface theory

We recall here the principal notations of the intrinsic formalism of surface theory used in this paper. It is inspired from the works of J.M. Souriau [45], R. Valid [47][48][25], J. Breuneval [3][4][5] and P. Destuynder [4][5].

Parametrized surface

Let U be an open set of $\mathbb{R}^2$ and

$$\begin{array}{rlll} f & : & U & \to \mathbb{R}^3 \\ & & x=(u,v) & \mapsto p=f(x) \end{array}$$

an embedding in $\mathbb{R}^3$ (see Fig. 2). Then $\omega = f(U)$ is called a surface embedded in $\mathbb{R}^3$ and U the open set of reference of the system of local coordinates (f,U). We assume here that f is smooth enough $(C^2(U))$.

Local basis of ω

The independent vectors $a_1 = \dfrac{\partial f}{\partial u}$ and $a_2 = \dfrac{\partial f}{\partial v}$ span a vectorial space called tangent space at $p = f(x)$ to ω and denoted $T_p\omega$. We denote (a_1, a_2) the natural or the local basis of $T_p\omega$ and S the matrix defined by $S = (a_1, a_2)$.

Finally we define $N = \dfrac{a_1 \wedge a_2}{\| a_1 \wedge a_2 \|}$ the unit normal at p to ω. Therefore, each vector W of $\mathbb{R}^3$ can be split into

$$W = \Pi W + N\overline{N}W = V + uN$$

where $V = \Pi W$ and u denote respectively the orthogonal projection of W onto $T_p\omega$ and the normal N, the overbar the operator of transposition.

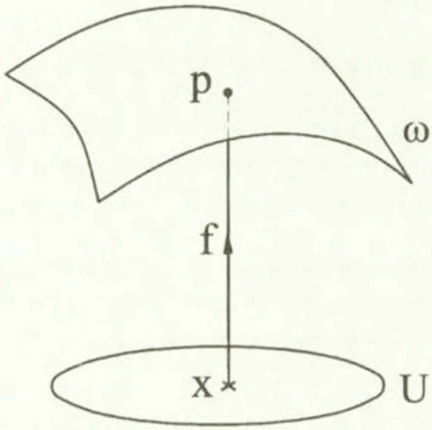

FIG. 2. Parametrization of the surface ω.

First fundamental form.

At each point p of ω, the scalar product of $\mathbb{R}^3$ implies a scalar product on $T_p\omega$:

$$\overline{dp}V = \overline{dx}(\overline{S}S)Y$$

where $dp = Sdx$ and $V = SY$ denote two tangent vectors of $T_p\omega$. We can so define, when p varies on ω, a field of covariant tensors $g \in T_p^*\omega \otimes T_p^*\omega$ where $T_p^*\omega$ denotes the dual space of $T_p\omega$.

DEFINITION 1. *The field of quadratic forms associated to g is called the first fundamental form of the surface ω. In the local or natural basis, it is represented by the matrix:* $G = \overline{\dfrac{\partial p}{\partial x}} \dfrac{\partial p}{\partial x} = \overline{S}S$.

Covariant derivative of a field of tangent vectors

Let $p \mapsto dp = Sdx$ and $p \mapsto V = SY$ be two fields of tangent vectors at p to ω. The derivative dV of the vector field V in the direction dp is not generally tangent to ω.

DEFINITION 2. *We define on ω a derivation ∇ for which the derivative of a tangent vector field is tangent :*

$$\begin{aligned} \nabla \ : \ & T\omega \times T\omega \ \to T_p\omega, \\ & (dp, V) \quad \mapsto \quad \nabla_{dp} V \stackrel{\text{def}}{=} \Pi dV, \end{aligned}$$

$\nabla_{dp}V$ is the covariant derivative of the tangent vector field V in the direction dp, denoted also $\hat{d}V$. In the local basis, we have:

$$\nabla_{dp} V = S[dY + \Gamma(dx, Y)].$$

Γ is the Christoffel operator whose components in the local basis are the Christoffel symbols $\Gamma^{\delta}_{\alpha\beta}$ and Π the orthogonal projection onto $T_p\omega$.

Second fundamental form

The normal part of the derivative dV in the direction dp of the tangent vector field $p \mapsto V$ can be represented in the local basis by the bilinear symmetric form F such as $\overline{N}dV = \overline{dx}FY$. We have

$$F = -\overline{\frac{\partial N}{\partial x}}\,\frac{\partial p}{\partial x} = -\overline{\frac{\partial p}{\partial x}}\,\frac{\partial N}{\partial x}$$

where $\dfrac{\partial N}{\partial x}$: $\mathbb{R}^2 \to \mathbb{R}^3$ denotes the linear tangent mapping to the field $x \mapsto N$.

DEFINITION 3. *The quadratic form associated to F is called the second fundamental form of the surface ω.*

Curvature operator

The linear tangent mapping $\dfrac{\partial N}{\partial p}$: $dp \mapsto dN = \dfrac{\partial N}{\partial p}dp$ of the field $p \mapsto N(p)$ defines an endomorphism of the tangent plane $T_p\omega$. Indeed, as $\overline{N}N = 1$, we have $\overline{N}dN = 0$ which implies that $dN \in T_p\omega$.

DEFINITION 4. *The endomorphism $C = -\dfrac{\partial N}{\partial p}$ is called curvature operator of the surface ω. It is symmetric with respect to the scalar product.*

Let us notice that in the local basis associated to the system of local coordinates (f, U), the operator C is represented by the matrix $G^{-1}F$. Indeed, we have:

$$F = \overline{\frac{\partial p}{\partial x}}\,C\,\frac{\partial p}{\partial x} = \overline{S}CS \qquad \text{and} \qquad S^{-1}CS = (\overline{S}S)^{-1}\overline{S}CS = G^{-1}F.$$

Derivative of a field of tangent vectors

Let $p \mapsto dp$ and $p \mapsto V$ be two tangent vector fields. Then the derivative dV of the tangent vector field V in the direction dp can be written in the intrinsic form :

$$dV = \hat{d}V + (\overline{dp}CV)N$$

where C denotes the curvature operator and $\hat{d}V = \frac{\hat{\partial} V}{\partial p} dp$ – the covariant derivative of V in the direction dp. We have also :

$$\frac{\partial V}{\partial p} = \frac{\hat{\partial} V}{\partial p} + N\overline{V}C$$

Derivative of a vector field of $\mathbb{R}^3$ defined on a surface

Let $p \mapsto W = V + uN$ be a vector field defined in ω which takes its values in $\mathbb{R}^3$ and $p \mapsto dp$ a field of tangent vectors. We then define the derivative dW of the vector field $p \mapsto W$ in the direction dp as : $dW = dV + duN + udN$. The associated tangent linear mapping

$$\begin{aligned} \frac{\partial W}{\partial p} \ :\ T_p\omega &\to \mathbb{R}^3, \\ dp &\mapsto \frac{\partial W}{\partial p} dp = dW, \end{aligned}$$

can be written:

$$\frac{\partial W}{\partial p} = \left[\frac{\hat{\partial} V}{\partial p} - uC\right] + N\left[\overline{V}C + \frac{\partial u}{\partial p}\right],$$

Classical two-dimensional divergence

The divergence of a tangent vector field V defined on a surface ω is given by:

$$\operatorname{div}(V) = \operatorname{Tr}\left(\frac{\hat{\partial} V}{\partial p}\right)$$

where Tr denotes the trace operator and $\frac{\hat{\partial}}{\partial p}$ the covariant derivative on ω.

The divergence of a field of endomorphisms A_t of the tangent plane $T_p\omega$ can be defined as follows

$$\operatorname{div}(A_t)V = \operatorname{div}(A_t V) - \operatorname{Tr}\left(A_t \frac{\hat{\partial} V}{\partial p}\right)$$

for all tangent vector field V defined on ω.

Particular divergence div_{t3}

It is possible to generalize the classical two-dimensional divergence of a field of endomorphisms of $T_p\omega$ to a field of operators $A_{t_3} : \omega \mapsto \mathcal{L}(\mathbb{R}^3, T_p\omega)$, denoted div_{t_3} as follows (see [47][49]) :

$$\mathrm{div}_{t_3}(A_{t_3})W = \mathrm{div}(A_{t_3}W) - \mathrm{Tr}\Big(A_t \frac{\hat{\partial} V}{\partial p}\Big)$$

for all vector field W: $\omega \mapsto \mathbb{R}^3$of ω.

The divergence div_{t_3} enables to write equations in a more compact form and to simplify the calculations. However, it can be linked to the classical two-dimensional divergence as follows:

LEMMA 2. Let A_{t3} be a field of operator defined on ω which takes its values in $\mathcal{L}(\mathbb{R}^3, T_p\omega)$. Then the field A_{t3} can be split as follows: $A_{t3} = A_t + A_s\overline{N}$ where $A_t = A_{t3}\Pi$ is a field of endomorphisms of $T_p\omega$ and $A_s = A_{t3}N$ a field of tangent vectors to ω. Moreover, it can be proved that:

$$\mathrm{div}_{t3}(A_{t3}) = \mathrm{div}\,(A_t) - \overline{A_s}C + (\mathrm{div}\,(A_s) + \mathrm{Tr}\,(A_tC))\,\overline{N}.$$

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