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# On the optimal design of viscoplastic bars under combined torsion with bending

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*Dedicated to Professor Piotr Perzyna*
*on the occasion of his* $70^{th}$ *birthday*

THE GOVERNING EQUATION for viscoplastic simultaneous torsion and bending of a prismatic bar is derived. Then a certain particular closed-form solution of this equation is found; it corresponds to an elliptic cross-section with the ratio of semi-axes depending on the bending-to-torsion ratio. This solution proves to be optimal if the optimization constraint is imposed on initiation (nucleation) and growth of material damage and if the material properties conform to the Huber–Mises–Hencky failure hypothesis.

## 1. Introduction

THERE EXISTS a great variety of formulations of optimal design problems of viscoplastic structures. Viscoplasticity – giving the most adequate description of mechanical properties of many materials, particularly under dynamic loadings – brings together the difficulties of the theory of plasticity and of creep mechanics: effective time factor and necessity of separation of plastically–active and plastically-passive processes, not always described by unique and experimentally verified criteria for various viscoplastic materials. Further problems pertain to quasi–static or dynamic loadings, a great variety of constitutive equations, various approaches to damage, their evolution and final rupture. A comprehensive treatment of various aspects of viscoplasticity has been given by Perzyna and his collaborators as a result of his over forty years research, initiated by the most often quoted fundamental papers [27], [29] and his early monograph [30].

Particularly much attention to optimal design of viscoplastic structures, both under quasi-static and dynamic loadings, has been paid by CEGIELSKI (partly with ŻYCZKOWSKI). In most cases the minimal volume served as the design objective, whereas the optimization constraints were divided into global and local ones. In the first group the following quantities were classified: the total energy dissipated during the process, the norm of residual displacements (e.g.

the maximal residual deflection of a beam), the norm of displacements maximal in time under impact etc. Local constraints were imposed on the unit dissipated energy, maximal (in time) reduced stress at individual points of the body, the minimal or maximal dimension of the cross-section, and similar.

In particular, quasi-static loadings were investigated by CEGIELSKI [5], where optimal shapes of a cantilever beam were considered as an example. The author analyzed the dependence of optimal shapes on constitutive equations, on distribution of loading in space and in time, and on the type of constraints adopted. The remaining papers were devoted to dynamic loadings: CEGIELSKI and ŻYCZKOWSKI [8] determined optimal shapes of bars under axial impact, CEGIELSKI [4] considered optimal beams for various impulse shapes, CEGIELSKI and ŻYCZKOWSKI [9] found optimal thickness distribution in circular cylindrical shells under dynamic combined loadings, and ŻYCZKOWSKI and CEGIELSKI [47] optimized beams under transverse impact. CEGIELSKI and ŻYCZKOWSKI [10] discussed optimal bars under dynamic axial loading in the range of finite strains, CEGIELSKI [6] optimized non-prismatic circular bars under dynamic twisting loadings, and finally, CEGIELSKI [7] considered optimal beams under dynamic bending and axial forces.

We mention also several other papers devoted to optimization of viscoplastic structures, based mainly on sensitivity analysis. ARORA *et al.* [1] compared material derivative and control volume approach in the case of the geometrically non-linear viscoplasticity. ZHANG *et al.* [41] derived design sensitivity coefficients by the boundary element method. ARORA *et al.* [2] LEE *et al.* [19] used a Lagrangian description and discussed in detail various constitutive equations of viscoplasticity. JAO and ARORA [14] considered optimization of viscoplastic structures described by an endochronic model. LEU and MUKHERJEE [22, 23] discussed sensitivity in finite-strain viscoplasticity. KULKARNI and NOOR [17] considered two-dimensional viscoplastic structures under dynamic loadings. A detailed review of optimization of viscoplastic structures is given in the survey paper by ŻYCZKOWSKI [43].

The present paper deals with optimal design of viscoplastic bars under simultaneous quasi-static torsion with bending. In view of the neglected dynamic effects it is assumed that both the twisting and bending moments are constant along the axis of the bar, hence optimal design is reduced to optimization of the cross-sectional shape. First the governing equation for viscoplastic torsion with bending for bars of arbitrary cross-section will be derived. Then a particular closed-form solution will be found; it corresponds to an elliptic cross section and generalizes that found by OBERWEIS and ŻYCZKOWSKI [26] for perfectly plastic materials. Then, following the paper by ŻYCZKOWSKI [44] who proved that this elliptic section satisfies the Drucker-Shield necessary condition of optimal plastic design, we are going to analyze the attributes of optimality of this section in viscoplasticity. In particular we shall prove that according to some approaches to damage mechanics, the initiation (nucleation) of damage starts uniformly along the whole contour line, hence the shape obtained may be called the „shape of uniform viscoplastic strength".

From among the related papers we mention here those by PIECHNIK [35] who solved the problem of simultaneous bending with torsion of a circular bar subject to nonlinear creep, by MEGUID *et al.* [24] on viscoplastic combined tension with torsion, by LAU and LISTVINSKY [18] (bending with torsion of a circular cylinder under creep conditions), finally by RYSZ and ŻYCZKOWSKI [37], who optimized a thin-walled cross-section under bending with torsion for a given creep rupture time (the „shape of uniform creep strength").

The present paper is based on the following assumptions:

1. A straight prismatic bar is subject to twisting moment $M_t$ and bending moment $M_b$, changing slowly in time $t$ (quasi-static loading). At the beginning no relation is assumed between the moments, but the solution obtained will be valid only for proportional changes of these moments.
2. The supports of the bar allow for free warping of all cross-sections. A particular example of the supports will be described below.
3. The material is viscoplastic and isotropic; in general, elastic strains and plastic hardening are allowed for, but in the first part of the paper a restriction to rigid-perfectly viscoplastic materials will be introduced.
4. The material is incompressible both in elastic and viscoplastic range. It is governed by the Huber-Mises-Hencky (HMH) failure hypothesis.
5. The analysis is confined to small strains.

## 2. Governing equations for torsion with bending

Consider a prismatic bar of arbitrary solid bisymmetric cross-section with the axis $z$, subject to simultaneous torsion and bending in the principal plane $yz$. The twisting and the bending moments are assumed to be constant along the axis. For example, it will be assumed that the cross-section $z = 0$ is clamped (but allowing for free warping), and the cross-section $z = l$ is free (Fig.1). The signs of the moments shown in Fig.1 are assumed to be compatible with the paper [44]. Then the distribution of velocities in engineering notation is given by (Hill [13])

$$\begin{aligned} \dot{u} &= -\frac{1}{2}\dot{\varkappa}xy + \dot{\vartheta}yz, \\ \dot{v} &= -\frac{1}{4}\dot{\varkappa}\left(-x^2 + y^2 + 2z^2\right) - \dot{\vartheta}xz, \\ \dot{w} &= \dot{\varkappa}yz + \dot{\vartheta}w_0(x, y), \end{aligned} \tag{2.1}$$

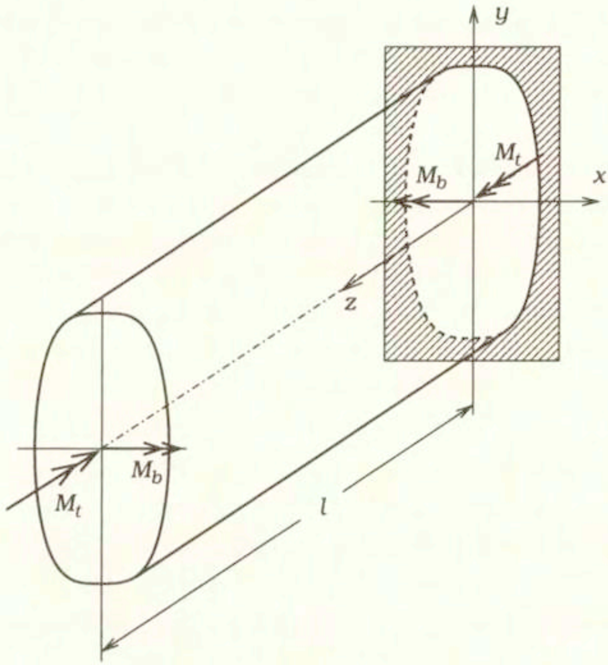


Fig. 1. Scheme of the bar.

for any constitutive equation under the assumption of incompressibility and small strains. In these equations dots denote derivatives with respect to the time t, and namely $\dot{\varkappa}$ the rate of curvature, $\dot{\vartheta}$ – the rate of unit angle of twist, and $w_0(x,y)$ – the warping function. The strain rates are as follows:

$$(2.2)\qquad \begin{gathered} \dot{\varepsilon}_x = \dot{\varepsilon}_y = -\frac{1}{2}\dot{\varkappa}y, \qquad \dot{\varepsilon}_z = \dot{\varkappa}y, \\ \dot{\gamma}_{xy} = 0, \qquad \dot{\gamma}_{zy} = \dot{\vartheta}\left(\frac{\partial w_0}{\partial y} - x\right), \qquad \dot{\gamma}_{zx} = \dot{\vartheta}\left(\frac{\partial w_0}{\partial x} + y\right). \end{gathered}$$

The equilibrium equations are in this case reduced to one equation

$$(2.3)\qquad \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0$$

with the relevant boundary condition

$$(2.4)\qquad \tau_{zx}\mathrm{d}y - \tau_{zy}\mathrm{d}x = 0.$$

The constitutive equations of viscoplasticity are assumed in the form proposed by Perzyna [27, 28, 30] under additional restriction to the Huber-Mises-Hencky hypothesis. In tensorial notation they have the form (with slightly changed notation)

$$\dot{e}_{ij} = \frac{1}{2G}\dot{s}_{ij} + \frac{1}{T_m}\left\langle \Phi\left(\frac{\sqrt{s_{kl}s_{kl}}}{\varkappa_h} - 1\right)\right\rangle \frac{s_{ij}}{\sqrt{s_{kl}s_{kl}}}, \tag{2.5}$$

where $s_{ij}$, $\dot{s}_{ij}$, and $\dot{e}_{ij}$ denote, in turn, the deviatoric stress, deviatoric stress rate, and the deviatoric strain rate components, $\Phi$ is the empirical overstress function, $\varkappa_h$ – the isotropic workhardening function, $T_m$ – the relaxation time, $G$ – Kirchhoff's modulus and $\langle\ \rangle$ denotes the Macauley bracket (ramp function). In our case (2.5) yields three scalar equations, hence together with (2.3) we have four equations for four unknowns $\sigma_z$, $\tau_{zx}$, $\tau_{zy}$, and $w_0$; all these functions depend on two spatial variables, $x$ and $y$, and on the time $t$.

## 3. A certain exact solution for rigid-visco-perfectly plastic materials

First we restrict our considerations to rigid-visco-perfectly plastic materials. Then the constitutive equations (2.5) are simplified to the form

$$\dot{e}_{ij} = \frac{1}{T_m}\left\langle \Phi\left(\sqrt{\frac{3}{2}}\frac{\sqrt{s_{kl}s_{kl}}}{\sigma_0} - 1\right)\right\rangle \frac{s_{ij}}{\sqrt{s_{kl}s_{kl}}}, \tag{3.1}$$

where $\sigma_0$ denotes the yield-point stress in uniaxial tension. Since the strain rate distribution is given here by (2.2), we are interested in an inverse form of (3.1), in order to calculate the stress components. To this aim we multiply each side of (3.1) by itself and after contraction obtain

$$\dot{e}_{ij}\dot{e}_{ij} = \frac{1}{T_m^2}\left[\left\langle \Phi\left(\sqrt{\frac{3}{2}}\frac{\sqrt{s_{kl}s_{kl}}}{\sigma_0} - 1\right)\right\rangle\right]^2. \tag{3.2}$$

Introducing the strain rate intensity $\dot{\varepsilon}_e$ and the stress intensity $\sigma_e$ by the formulae

$$\dot{\varepsilon}_e = \sqrt{\frac{2}{3}\dot{e}_{ij}\dot{e}_{ij}}, \qquad \sigma_e = \sqrt{\frac{3}{2}s_{ij}s_{ij}}, \tag{3.3}$$

we rewrite (3.2) in the form

$$\dot{\varepsilon}_e = \frac{1}{T_m}\sqrt{\frac{2}{3}}\left\langle \Phi\left(\frac{\sigma_e}{\sigma_0} - 1\right)\right\rangle. \tag{3.4}$$

The symbol $\dot{\varepsilon}_e$ should not be confused with the time derivative of the strain intensity $\varepsilon_e$; they are equal to each other just in a simple loading processes.

For plastically active processes (loading) we may invert (3.4) to the form

$$\sigma_e = \sigma_0 \left[ 1 + \Phi^{-1} \left( \sqrt{\frac{3}{2}} T_m \dot{\varepsilon}_e \right) \right], \tag{3.5}$$

where the symbol $\Phi^{-1}$ denotes the function inverse with respect to $\Phi$. Equation (3.1) expresses similarity of deviators, hence, taking (3.3) into account, we may write

$$s_{ij} = \frac{2}{3} \frac{\sigma_e}{\dot{\varepsilon}_e} \dot{e}_{ij} = \frac{2\sigma_0}{3\dot{\varepsilon}_e} \left[ 1 + \Phi^{-1} \left( \sqrt{\frac{3}{2}} T_m \dot{\varepsilon}_e \right) \right] \dot{\varepsilon}_{ij}. \tag{3.6}$$

Now we define the function $\Psi\left(\dot{\varepsilon}_e^2\right)$ as follows:

$$\frac{2\sigma_0}{3\dot{\varepsilon}_e} \left[ 1 + \Phi^{-1} \left( \sqrt{\frac{3}{2}} T_m \dot{\varepsilon}_e \right) \right] = \Psi\left(\dot{\varepsilon}_e^2\right), \tag{3.7}$$

hence

$$s_{ij} = \Psi\left(\dot{\varepsilon}_e^2\right) \dot{e}_{ij}. \tag{3.8}$$

The argument $\dot{\varepsilon}_e^2$ is here more convenient than $\dot{\varepsilon}_e$. Returning to engineering notation and making use of (2.2) we obtain

$$\begin{aligned} \sigma_z &= \frac{3}{2} \Psi \dot{\varkappa} y, \\ \tau_{zx} &= \frac{1}{2} \Psi \dot{\vartheta} \left( \frac{\partial w_0}{\partial x} + y \right), \\ \tau_{zy} &= \frac{1}{2} \Psi \dot{\vartheta} \left( \frac{\partial w_0}{\partial y} - x \right). \end{aligned} \tag{3.9}$$

The argument $\dot{\varepsilon}_e^2$ of the function $\Psi$ equals

$$\varepsilon_e^2 = \dot{\varkappa}^2 y^2 + \frac{1}{3} \dot{\vartheta}^2 \left[ \left( \frac{\partial w_0}{\partial x} + y \right)^2 + \left( \frac{\partial w_0}{\partial y} - x \right)^2 \right]. \tag{3.10}$$

Substituting (3.9) and (3.10) into the equilibrium equation (2.3) we obtain the equation for the warping function (final governing equation of the problem)

$$\begin{aligned} \Psi' \Bigg\{ & \left[ \left( \frac{\partial w_o}{\partial y} - x \right) \left( \frac{\partial^2 w_o}{\partial x \partial y} - 1 \right) + \left( \frac{\partial w_o}{\partial x} + y \right) \frac{\partial^2 w_o}{\partial x^2} \right] \left( \frac{\partial w_o}{\partial x} + y \right) \dot{\vartheta}^2 \\ & + \left[ \left( \frac{\partial w_o}{\partial y} - x \right) \frac{\partial^2 w_o}{\partial y^2} + \left( \frac{\partial w_o}{\partial x} + y \right) \left( \frac{\partial^2 w_o}{\partial x \partial y} + 1 \right) \right] \left( \frac{\partial w_o}{\partial y} - x \right) \dot{\vartheta}^2 \\ & + 3y \left( \frac{\partial w_o}{\partial y} - x \right) \dot{\varkappa}^2 \Bigg\} + \Psi \left( \frac{\partial^2 w_o}{\partial x^2} + \frac{\partial^2 w_o}{\partial y^2} \right) = 0, \end{aligned} \tag{3.11}$$

where $\Psi'$ denotes the derivative of $\Psi$ with respect to its argument $\dot{\varepsilon}_e^2$. In linear elasticity we have obviously $\Psi' = 0$, in (3.11) just the last term remains, and the warping $w_0$ is a harmonic function (HUBER [13]).

The nonlinear second-order Eq. (3.11) is rather complicated and its solutions depend, in general, on the shape of the function $\Psi = \Psi(\dot{\varepsilon}_e^2)$, but it may be easily checked that the harmonic function, well-known in elasticity,

$$w_o = Cxy \tag{3.12}$$

may satisfy (3.11) for any $\Psi$. Indeed, substituting (3.12) into (3.11) we find that the last term vanishes, and all the remaining terms may be divided by $\Psi' xy$. Then we obtain the following algebraic equation:

$$2C(C+1)\dot{\vartheta}^2 + 3\dot{\varkappa}^2 = 0, \tag{3.13}$$

and $C$ may be evaluated for a given ratio $\dot{\vartheta}^2/\dot{\varkappa}^2$. In this case the shearing stresses are equal

$$\begin{aligned} \tau_{zx} &= \frac{1}{2}\Psi\dot{\vartheta}(C+1)y, \\ \tau_{zy} &= \frac{1}{2}^{\Psi}\dot{\vartheta}(C-1)x, \end{aligned} \tag{3.14}$$

so their effective values depend on the function $\Psi$. Nevertheless, substituting (3.14) into the boundary condition (2.4) we realize that the final shape of the cross-section does not depend on $\Psi$, namely

$$(C+1)y \, \mathrm{d}y - (C-1)x \, \mathrm{d}x = 0. \tag{3.15}$$

This equation determines an ellipse, since from (3.13) it is seen that $-1 < C < 0$. If we write equation of this ellipse in the classical form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{3.16}$$

we find

$$C = -\frac{b^2 - a^{\ 2}}{b^2 + a^2} \tag{3.17}$$

and (3.13) results in

$$\frac{4}{3}\frac{\dot{\vartheta}^2}{\dot{\varkappa}^2} = \frac{(a^2+b^2)^2}{a^2(b^2-a^2)}. \tag{3.18}$$

Equation (3.18) restricts our considerations to simple loading/unloading processes (the strain rates remain proportional to each other at any point of the body). It coincides with Eq. (13) of the paper by OBERWEIS and ŻYCZKOWSKI [26], where a certain exact solution for combined bending with torsion of a perfectly plastic bar was found. Indeed, if (3.12) is valid for any function $\Psi$, then it must also be valid for perfect plasticity. Later, ŻYCZKOWSKI [44] proved that the ellipse (3.16) combined with (3.18) determines the optimal shape of the cross-section and in [45] he applied this relation to determine optimal shapes of the beams under the variable bending moment $M_b = M_b(z)$.

For the given ratio $\dot{\vartheta}/\dot{\varkappa}$ we can find from (3.18) the relevant ratio $a/b$, it means the shape of the elliptical cross-section. Equation (3.18) is biquadratic with respect to $a/b$ and we obtain

$$\frac{a^2}{b^2} = \frac{1}{1+\dfrac{4}{3}\dfrac{\dot{\vartheta}^2}{\dot{\varkappa}^2}}\left(\frac{2}{3}\frac{\dot{\vartheta}^2}{\dot{\varkappa}^2} - 1 \pm \sqrt{\frac{4}{9}\frac{\dot{\vartheta}^4}{\dot{\varkappa}^4} - \frac{8}{3}\frac{\dot{\vartheta}^2}{\dot{\varkappa}^2}}\right). \tag{3.19}$$

The radical in (3.19) must be non-negative, hence $\dot{\vartheta}^2/\dot{\varkappa}^2 \geqslant 6$. For $\dot{\vartheta}^2/\dot{\varkappa}^2 = 6$ we obtain one solution $a/b = 1/\sqrt{3}$, whereas for $\dot{\vartheta}^2/\dot{\varkappa}^2 > 6$ formula (3.19) determines two elliptic sections for which the solution (3.12) holds.

Substituting (3.12) and (3.17) into (3.10) we may write

$$\dot{\varepsilon}_e^2 = \frac{b^4}{b^2 - a^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\dot{\varkappa}^2, \tag{3.20}$$

or making use of (3.18),

$$\dot{\varepsilon}_e^2 = \frac{4}{3}\frac{a^2 b^4}{(a^2+b^2)^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\dot{\vartheta}^2. \tag{3.21}$$

## 4. Evaluation of external loadings

The bar under consideration is subject to twisting moment $M_t$ and bending moment $M_b$. They are expressed in terms of stresses as follows:

$$M_t = \iint_A (\tau_{zx}\, y - \tau_{zy}\, x)\,\mathrm{d}A, \tag{4.1}$$

$$M_b = 2\iint_{A/2} \sigma_z\, y\mathrm{d}A, \tag{4.2}$$

where $A$ is the cross-sectional area. Substituting (3.14) we may write

$$M_t = \frac{1}{a^2+b^2}\,\dot{\vartheta}\iint_A \Psi\left(\dot{\varepsilon}_e^2\right)\left(a^2y^2 + b^2x^2\right)\mathrm{d}x\ \mathrm{d}y. \tag{4.3}$$

It is convenient to introduce the variables $\rho$ and $\varphi$ by the substitution

$$x = a\ \rho\cos\varphi, \quad y = b\rho\sin\varphi, \quad \mathrm{d}A = ab\ \rho\ \mathrm{d}\rho\ \mathrm{d}\varphi, \tag{4.4}$$

then

$$M_t = \frac{2\pi a^3 b^3}{a^2+b^2}\ \dot{\vartheta} \int\limits_o^1 \Psi\left(\dot{\varepsilon}_e^2\right)\rho^3 \mathrm{d}\rho. \tag{4.5}$$

We call the (non–orthogonal) system of coordinates $\rho$, $\varphi$ the "polar–elliptic coordinates" since the name "elliptic coordinates" is ascribed to some other, orthogonal system (Morse and Feshbach [25]). Stress and strain components are retained in Cartesian system. The normal stresses $\sigma_z$ are given by (3.9), hence

$$M_b = \frac{3}{2}\pi a b^3 \dot{\varkappa} \int\limits_o^1 \Psi\left(\dot{\varepsilon}_e^2\right)\rho^3 \mathrm{d}\rho. \tag{4.6}$$

Introduce the ratio of the moments squared, like in the papers by Bochenek *at al.* [3] and by Życzkowski [44]

$$\eta = \left(\frac{M_b}{M_t}\right)^2. \tag{4.7}$$

Substituting (4.5), (4.6) and (3.18), we obtain for any function $\Psi(\dot{\varepsilon}_e^2)$

$$\eta = \frac{3}{4}\ \frac{b^2-a^2}{a^2} \tag{4.8}$$

and hence the ratio of semi-axes of the ellipse may be expressed in terms of external loadings as follows:

$$\frac{b}{a} = \sqrt{1+\frac{4}{3}\ \eta}. \tag{4.9}$$

This result coincides with that obtained by Życzkowski for perfect plasticity [44]. In contradistinction to (3.19), it gives one and only one solution for any value of $\eta$.

As an example we consider the power law, being an extension of Norton's creep law to viscoplasticity (Perzyna [30]).

$$\dot{e}_{ij} = \frac{1}{T_m}\left\langle \left(\sqrt{\frac{3}{2}}\ \frac{\sqrt{s_{kl}s_{kl}}}{\sigma_o} - 1\right)^{\delta} \right\rangle \frac{s_{ij}}{\sqrt{s_{kl}s_{kl}}}. \tag{4.10}$$

Then (3.5) and (3.7) take the form

$$\sigma_e = \sigma_o \left[ 1 + \left( \sqrt{\frac{3}{2}} T_m \frac{b^2}{\sqrt{b^2 - a^2}} \, \rho |\dot{\varkappa}| \right)^{1/\delta} \right], \tag{4.11}$$

$$\Psi = \frac{2\sigma_o (b^2 - a^2)}{3b^2 \rho |\dot{\varkappa}|} \left[ 1 + \left( \sqrt{\frac{3}{2}} \, T_m \frac{b^2}{\sqrt{b^2 - a^2}} \rho |\dot{\varkappa}| \right)^{1/\delta} \right] \tag{4.12}$$

(or a similar form expressed in terms of $\dot{\vartheta}$), and the moments are equal to

$$M_b = \frac{\pi}{3} \, ab\sqrt{b^2 - a^2} \, \sigma_o \left[ 1 + \frac{3\delta}{3\delta + 1} \left( \sqrt{\frac{3}{2}} T_m \frac{b^2}{\sqrt{b^2 - a^2}} |\dot{\varkappa}| \right)^{1/\delta} \right] \operatorname{sign} \dot{\varkappa}, \tag{4.13}$$

$$M_b = \frac{2\pi}{3\sqrt{3}} \, a^2 b \, \sigma_o \left[ 1 + \frac{3\delta}{3\delta + 1} \left( \sqrt{2} \, T_m \frac{ab^2}{a^2 + b^2} |\dot{\vartheta}| \right)^{1/\delta} \right] \operatorname{sign} \dot{\vartheta}. \tag{4.14}$$

Of course, in view of (3.18) the square brackets are identical to each other and (4.8) holds, but it is more natural to present $M_b$ in terms of $\dot{\varkappa}$ and $M_t$ in terms of $\dot{\vartheta}$, and hence the notation used. For $\delta \to 0$ we obtain the formulae derived in [44] under the assumption of perfect plasticity. Inversion of (4.13) and (4.14) so as to determine the functions $\dot{\varkappa} = \dot{\varkappa}(M_b)$ and $\dot{\vartheta} = \dot{\vartheta}(M_t)$ does not present any difficulties.

## 5. Extension to additional elastic strains and plastic hardening

In Sec. 3 we derived an exact solution for the problem under consideration but under additional restrictions to perfect plasticity and ommision of elastic strains. The solution is explicit and analytical if the function $\Phi^{-1}$ in the formula for $\Psi$ (3.7) may be determined analytically. Now we return to the more general constitutive equation (2.5) and prove that also in this case a similar solution holds, but $\Psi$ must be evaluated numerically.

We make use of the polar–elliptic coordinates (4.4), and assume the warping function $w_o$ in the form (3.12) with $C$ determined by (3.17). Hence Eqs. (2.2) look now as follows:

$$\begin{aligned} &\dot{\varepsilon}_x = \dot{\varepsilon}_y = -\frac{1}{2} \, \dot{\varkappa}\rho b \sin\varphi, \qquad \dot{\varepsilon}_z = \dot{\varkappa}\rho b \sin\varphi, \\ &\dot{\gamma}_{xy} = 0, \quad \dot{\gamma}_{zx} = \dot{\vartheta} \frac{2a^2}{b^2 + a^2} \, \rho b \sin\vartheta, \quad \dot{\gamma}_{zy} = -\dot{\vartheta} \frac{2b^2}{b^2 + a^2} \, \rho a \cos\varphi. \end{aligned} \tag{5.1}$$

where $\dot{\varkappa}$ and $\dot{\varphi}$ are interrelated by (3.18). Now we assume the solution in the form (3.9) with substituted (3.12)

$$(5.2)\qquad \begin{aligned} \sigma_z &= \frac{3}{2}\Psi\dot{\varkappa}\rho b\sin\varphi,\\ \tau_{zx} &= \Psi\dot{\vartheta}\frac{a^2}{b^2+a^2}\rho b\sin\varphi,\\ \tau_{zy} &= -\Psi\dot{\vartheta}\frac{b^2}{b^2+a^2}\rho a\ \cos\varphi. \end{aligned}$$

Equations (5.2) are regarded as a hypothesis: it will be shown that they can satisfy all the governing equations, and the equation for $\Psi$, generalizing (3.7), will be derived.

In general, $\Psi = \Psi(\rho,\varphi,t)$, but the equilibrium Eq. (2.3) results in $\partial\Psi/\partial\varphi = 0$, hence we assume $\Psi = \Psi(\rho,t)$. First we discuss plastically passive processes. Neglecting the bracket in (2.5) we realize that all the three independent equations for $\dot{\varepsilon}_z, \dot{\gamma}_{zx}$, and $\dot{\gamma}_{zy}$, are satisfied if

$$(5.3)\qquad \dot{\varkappa} = \frac{1}{2G}\ \frac{\mathrm{d}}{\mathrm{d}t}(\Psi\dot{\varkappa}).$$

We used here the symbol of ordinary derivative, since the spatial variable $\rho$ is not present. Integrating (5.3) we find

$$(5.4)\qquad \varkappa = \frac{\Psi\dot{\varkappa}}{2G} + C.$$

In the elastic range preceding viscoplastic deformations the constant $C$ vanishes, and hence

$$(5.5)\qquad \Psi = \frac{2G\varkappa}{\dot{\varkappa}}.$$

Further we prove that the plastic hardening depends only on the coordinate $\rho$, and does not depend on $\varphi$. It will be sufficient to restrict our considerations to isotropic hardening: in simple loading/unloading processes enforced by (3.18) other types of hardening (kinematic, distortional, etc.) do not bring any effects. Isotropic hardening is usually expressed in terms of the Odqvist parameter $I_{\varepsilon p}$ (strain–hardening) or of the plastic work $W^p$ (work–hardening). Here we consider only strain–hardening in view of simpler final expressions, but principal conclusions remain valid for work-hardening without change. The Odqvist parameter is defined as the length of the trajectory in the plastic strain space:

$$(5.6)\qquad I_{\varepsilon p} = \int\limits_0^t \sqrt{\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij}}\ \mathrm{d}\bar{t} = \int\limits_0^t \sqrt{\dot{e}_{ij}\dot{e}_{ij} - \frac{1}{G}\dot{e}_{ij}\dot{s}_{ij} + \frac{1}{4G^2}\dot{s}_{ij}\dot{s}_{ij}}\ \ \mathrm{d}\bar{t}$$

where $\bar{t}$ is the variable of integration. Making use of (5.1), (5.2) and (3.18) we calculate the invariants appearing in (5.6):

$$\dot{e}_{ij}\,\dot{e}_{ij} = \frac{3}{2}\frac{b^4}{b^2-a^2}\rho^2\dot{\varkappa}^2, \tag{5.7}$$

$$\dot{e}_{ij}\,\dot{s}_{ij} = \frac{3}{2}\frac{b^4}{b^2-a^2}\rho^2\dot{\varkappa}\frac{\mathrm{d}}{\mathrm{d}t}(\Psi\dot{\varkappa}), \tag{5.8}$$

$$\dot{s}_{ij}\,\dot{s}_{ij} = \frac{3}{2}\frac{b^4}{b^2-a^2}\rho^2\left[\frac{\mathrm{d}}{\mathrm{d}t}(\Psi\dot{\varkappa})\right]^2, \tag{5.9}$$

and hence

$$I_{\varepsilon p} = \sqrt{\frac{3}{2}}\frac{b^2\rho}{\sqrt{b^2-a^2}}\int\limits_0^t \left|\dot{\varkappa} - \frac{1}{2G}\frac{\mathrm{d}}{\mathrm{d}t}(\Psi\dot{\varkappa})\right| \;\mathrm{d}\bar{t} \tag{5.10}$$

or after integration, for monotonically increasing $|\varkappa|$,

$$I_{\varepsilon p} = \sqrt{\frac{3}{2}}\frac{b^2\rho}{\sqrt{b^2-a^2}}\;\left|\varkappa - \frac{\Psi\dot{\varkappa}}{2G}\right|\;\Bigg|_0^t = I_{\varepsilon p}(\rho, t). \tag{5.11}$$

It is seen from (5.5) that in the elastic range the bracket vanishes, hence the integration starts in the viscoplastic range, and we may simply omit the limits of integration $0, t$. The most important conclusion is that $I_{\varepsilon p}$ depends on $\rho$ and $t$, but does not depend on $\varphi$.

In order to have notation similar to (3. 1) we present now the strain-hardening function in the form

$$\begin{aligned} \varkappa_h &= \sigma_o\sqrt{\frac{2}{3}}\left[1 + f_h\left(I_{\varepsilon p}\right)\right] \\ &= \sigma_o\sqrt{\frac{2}{3}}\left[1 + f_h\left(\sqrt{\frac{3}{2}}\frac{b^2\rho}{\sqrt{b^2-a^2}}\left|\varkappa - \frac{\Psi\dot{\varkappa}}{2G}\right|\,\right)\right] \end{aligned} \tag{5.12}$$

where $f_h$ is a function to be determined experimentally. In applications, $f_h$ is often assumed as a linear function. Now we return to (2. 5), substitute (5.1), (5.2), (5. 12) and the expression resulting from (5. 2) for the stress intensity

$$\sigma_e = \sqrt{\frac{3}{2}s_{ij}s_{ij}} = \frac{3}{2}\frac{b^2}{\sqrt{b^2-a^2}}\Psi\rho\,|\dot{\varkappa}| \tag{5.13}$$

and realise, that all three equations for the components $\dot{\varepsilon}_z$, $\dot{\gamma}_{zx}$, and $\dot{\gamma}_{zy}$ cancelled, in turn, by $\sin\varphi$, $\sin\varphi$ and $\cos\varphi$, may simultaneously be satisfied. It takes place if the function $\Psi = \Psi(\rho, t)$ satisfies the nonlinear ordinary differential equation

$$\dot{\varkappa} = \frac{1}{2G}\frac{\mathrm{d}}{\mathrm{d}t}(\Psi\dot{\varkappa}) + \sqrt{\frac{2}{3}}\frac{\sqrt{b^2-a^2}}{T_m b^2\rho}\left\langle \Phi\left\{\frac{\frac{3}{2}\frac{b^2}{\sqrt{b^2-a^2}}\Psi\rho\dot{\varkappa}}{\sigma_o[1+f_h(I_{\varepsilon p})]}-1\right\}\right\rangle. \tag{5.14}$$

The symbol of partial derivative was not introduced into (5.13) since the variable $\rho$ may be regarded as a parameter and the differentiation with respect to $\rho$ does not appear.

The elastic-viscoplastic interface is determined by vanishing of the bracket in (5.14). We obtain the equation

$$\frac{3}{2}\frac{b^2}{\sqrt{b^2-a^2}}\Psi\rho\dot{\varkappa} = \sigma_o\left[1+f_h\left(\sqrt{\frac{3}{2}}\frac{b^2\rho}{\sqrt{b^2-a^2}}\left|\varkappa-\frac{\Psi\dot{\varkappa}}{2G}\right|\right)\right]. \tag{5.15}$$

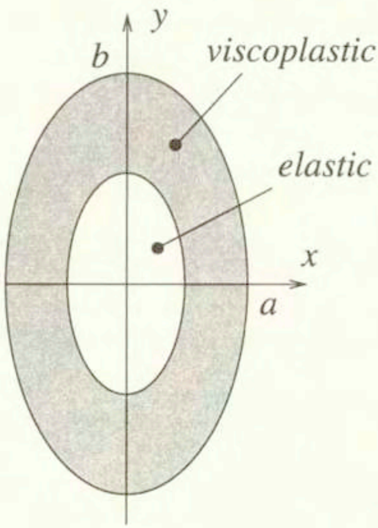


FIG. 2. The elastic and viscoplastic zones in the cross-section.

Considering the first plastification we substitute $\Psi$ determined for the elastic range (5.5), hence the elastic-viscoplastic interface is an ellipse, geometrically similar to the contour, and is described by the coordinate $\rho = \rho_b$

$$\rho_b = \frac{\sqrt{b^2-a^2}}{3b^2}\frac{\sigma_o}{G\varkappa} \tag{5.16}$$

(Fig.2). Of course, this ellipse decreases with increasing $\varkappa$.

## 6. Attributes of optimality of the solution obtained

In the introduction we pointed out the variety of optimization problems in viscoplasticity. Now we are going to prove that the solution obtained, namely the elliptic shape (3.16) satisfying the conditions (3.19) or (4.9), is indeed an optimal solution if we impose constraints on damage evolution and restrict the class of materials under consideration.

Strength of a viscoplastic structure is usually determined by reaching by the upper bound of the damage parameter a certain prescribed value, regarded as critical. There exist many various approaches to the concepts of damage and to relevant evolution equations. For our purposes any of them may be employed, provided the material of the structure (of the bar under torsion with bending) is governed exclusively by the HMH failure hypothesis. Indeed, from (5.13) it is seen that the stress intensity $\sigma_e = \sigma_{HMH}$ is constant along the contour $\rho = 1$ at any time $t$, hence the elliptic shape is the "shape of uniform viscoplastic strength". Though the shapes of uniform strength are not always optimal (for example, if the geometry changes are taken into account, ŚWISTERSKI *at al.* [40]), but in the case under consideration nothing like that takes place.

First we consider Kachanov's approach who introduced the damage parameter $D$ connected with formation of microcracks. He described the damage evolution under uniaxial tension by the equation

$$\dot{D} = \overline{C}\left(\frac{\sigma}{1-D}\right)^{\mu}, \tag{6.1}$$

where $\overline{C}$ and $\mu$ are temperature-dependent material constants. Several generalizations of (6.1) for multiaxial states were introduced. HAYHURST [12] proposed to replace $\sigma$ by the following stress invariant $\sigma_{red}$ (reduced stress)

$$\sigma_{red} = \alpha\sigma_{\mathrm{I}} + \beta J_{1\sigma} + \gamma\sigma_e \tag{6.2}$$

with $\alpha + \beta + \gamma = 1$; in this equation $\sigma_{\mathrm{I}}$ denotes the maximal principal stress (Galileo's hypothesis), and $J_{1\sigma} = \sigma_{kk}$ – the first stress invariant. Some particular cases of (6.2) were considered eariler by SDOBYRIEV [38] with $\beta = 0$, $\alpha = \gamma = \dfrac{1}{2}$, and by RABOTNOV [36] with $\beta = 0$. $\alpha + \gamma = 1$. It is seen that (6.1) with substituted (6.2) gives constant damage rate at the boundary of the elliptic cross-section under consideration if $\alpha = \beta = 0$, $\gamma = 1$. This takes place for example, for aluminium alloys Al-Mg-Si [12], and then the ellipse may be regarded as optimal.

In a series of papers Perzyna considered the damage as nucleation and growth of microvoids (porosity). In the papers [31, 32] he proposed to describe nucleation of the porosity parameter $\xi$ by the evolution equation

$$(\dot{\xi})_{nucl} = \frac{1}{T_m} h^*(\xi,\ \vartheta) \left[ \exp \frac{m^*(\vartheta)|I_n - \tau_n(\xi,\ \vartheta,\ I_{\varepsilon\rho})|}{k\vartheta} - 1 \right], \tag{6.3}$$

where $\vartheta$ denotes temperature (not to be confused with the notation introduced in Sec. 2 for the unit angle of twist), $k$ – the Boltzmann constant, $h^*(\xi,\vartheta)$ is a material function describing the microvoid interaction, $m^*(\vartheta)$ is a temperature-dependent coefficient, $\tau_n$ is the threshold stress for microvoid nucleation, and $I_n$ denotes the following stress invariant, similar to (6.2),

$$I_n = a_1 J_{1\sigma} + a_2 \sigma_e + a_3 (J_{3d})^{1/3}, \tag{6.4}$$

$a_i$ $(i = 1, 2, 3)$ are the material constants, and $J_{3d}$ is the third invariant of the stress deviator. Further, in a paper with DRABIK [34], the following evolution equation for the growth mechanism of $\xi$ was postulated:

$$(\dot{\xi})_{grow} = \frac{1}{T_m} \frac{g^*(\xi,\ \vartheta)}{x_o} \left[ I_g - \tau_{eq}(\xi,\ \vartheta,\ I_{\varepsilon p}) \right], \tag{6.5}$$

where $T_m x_o$ denotes dynamic viscosity of the material, $g^*(\xi,\vartheta)$ – the function describing the microvoid interaction, $I_g$ – the stress invariant like (6.4) with some other coefficients $b_i$, and $\tau_{eq}$, the void growth threshold stress. Finally, the evolution equation for $\xi$ is determined by the sum of (6.3) and (6.5). The details are presented by PERZYNA [33] in his contribution to the Handbook of Materials Behaviour Models [21]. It is seen that in the case $a_1 = a_2 = 0$ the microvoid nucleation is governed by the HMH hypothesis, and then the elliptic contour is optimal with respect to nucleation. Moreover, if $b_1 = b_2 = 0$, then the optimality pertains also to the microvoids growth. Unfortunately, not too many values of material constants $a_i$ and $b_i$ are available. Paper [33] quotes the relevant values for the AISI 4340 steel. The constant $b_3 = 0$ and $b_1$ is smaller than $b_2$, but $b_1$ is different from zero and for that steel the optimality of (3.16) does not take place.

In [46] ŻYCZKOWSKI proposed to express the damage evolution equations in terms of the unit dissipated power $\overline{\Psi}$, namely

$$\dot{D} = \frac{1}{C_d} \sqrt{\frac{\overline{\Psi}}{1 - D}}, \tag{6.6}$$

where $C_d$ is called the damage modulus. In the case of Perzyna's Eq. (2.5), for viscoplastic materials we obtain

$$\overline{\Psi} = s_{ij}\ \dot{e}^p_{ij} = \frac{1}{T_m} \left\langle \Phi \left( \frac{\sqrt{s_{kl} s_{kl}}}{\varkappa_h} - 1 \right) \right\rangle \sqrt{s_{ij} s_{ij}} \tag{6.7}$$

and hence

$$\dot{D} = \frac{1}{C_d}\sqrt{\frac{1}{(1-D)T_m}\left\langle \Phi\left(\frac{\sqrt{s_{kl}s_{kl}}}{\varkappa_h} - 1\right)\right\rangle \sqrt{s_{ij}s_{ij}}}. \tag{6.8}$$

It is seen that in this case $\dot{D}$ depends only on the second deviatoric stress invariant and hence the shape (3.16) is optimal. The hypothesis (6.6) was relatively well confirmed for various materials subject to nonlinear creep in uniaxial tension, but a similar confirmation for viscoplastic materials is lacking, and undoubtedly it may take place for a certain restricted class of materials only.

Extensive reviews of damage evolution equations are given by LEMAITRE [20], KRAJCINOVIC [16], SKRZYPEK and GANCZARSKI [39]. Many of them are expressed just in terms of the stress intensity $\sigma_e$ and then the shape (3.16) is optimal for viscoplastic bars under simultaneous torsion with bending if the strength expressed by damage evolution is assumed as the optimization constraint.

## 7. Conclusions

1. The paper gives a simple closed-form solution to the complicated nonlinear governing equation of viscoplastic bars under torsion with bending (3.11). It corresponds to an elliptic cross-section with the ratio of semi-axes depending on the ratio of twist to curvature, (3.19), or on the ratio of bending to twisting moment (4.9). This solution may be regarded as a bench mark to verify numerical methods applied for other shapes of the cross-section.

2. In his habilitation thesis GAJEWSKI [11] considered the dependence of optimal shapes on the constitutive equations adopted. In general, the optimal shapes depend on them, but Gajewski separated many cases of structural elements, loadings and constraints in which the final shape does not depend on constitutive equations. The present paper shows probably the first case of optimal design under combined loadings in which the solution also does not depend on constitutive equations.

3. The solution obtained is shown to be optimal if the constraints are imposed on initiation and growth of damage, considered as responsible for strength of the bar. Optimality takes place if the constitutive equations and the damage evolution equations are based on the HMH failure hypothesis, it means on the second deviatoric stress invariant.

4. Extension to linear combination with other stress invariants, like (6.2) or (6.4), is not possible. However, ŻYCZKOWSKI [42] considered general, non-

linear functions of the basic invariants leading to the "ad hoc" elliptic yield condition, namely, in the case under consideration,

$$\sigma_z^2 + c\,(\tau_{zx}^2 + \tau_{zy}^2) = \sigma_o^2, \tag{7.1}$$

with arbitrary positive constant $c$. Then the chances of deriving similar formulae as in the present paper are quite realistic.

5. Optimization of the viscoplastic bars subjected to torsion with bending with the constraints imposed on stiffness is undoubtedly much more difficult. However, if we express that stiffness (or rather compliance) by the total dissipated power, then one may expect that (3.16) with (3.19) or (4.9) gives also in this case the optimal solution, since we proved in Sec. 6 that the unit dissipated power is constant along any line $\rho$ =const. The proof of such a statement should be based on the general Eq. (3.11) and hence it seems rather complicated.

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