

Thermodynamical admissibility of Biot's model of poroelastic saturated materials

K. WILMAŃSKI

*Weierstrass Institute for Applied Analysis
and Stochastics, Berlin, Germany,
www.wias-berlin.de/private/wilmansk*

*Dedicated to Professor Piotr Perzyna
on the occasion of his 70th birthday*

THE PAPER is devoted to the analysis of nonlinear elastic two-component models of saturated porous media, which in a linear form, reproduce as closely as possible the classical Biot's model [1]. We present a full evaluation of the second law of thermodynamics for four classes of nonlinear models. It is proven that two of them in which there is no dependence on higher gradients cannot lead to the Biot's model. On the other hand, two other models in which a dependence on the gradient of porosity is introduced, yield linear constitutive relations but not the field equations appearing in the Biot's model. However, a recombination of partial stresses and momentum sources leads to Biot's equations. This analysis together with earlier publications on the subject exhausts the discussion of the question of thermodynamical admissibility of the Biot's model.

1. Introduction

Continuous models of porous and granular materials which account for the relative motion of components (diffusion) rely on the theory of two-component immiscible mixtures in which one of the components is a solid. The first linear model of this type has been developed by M. A. Biot in early 40ies (see: [1] for the full account of Biot's papers on this subject). This model had an enormous success among engineers and geophysicists who applied it in soil mechanics, mechanics of sediments on the sea bottom, propagation of waves in porous and granular materials, experimental testing of such materials etc. Many phenomena of practical importance such as tortuosity which did not appear in the original model of Biot, have been added *ad hoc* to the model mostly by reinterpretation of original contributions.

Unfortunately, the attempts to verify foundations of the Biot's model and to develop such models in the spirit of modern continuum thermodynamics have been vehemently criticized by admirers of the Biot's model, since such attempts were a blasphemy in the holy shrine of infallibility. It is, certainly, true that the

Biot's model provides an explanation of many important observations in porous materials such as the existence of the second sound called Biot's slow wave or the structure of surface waves (existence of additional modes of propagation). Simultaneously it is rather easy to observe that the Biot's field equations violate fundamental laws of thermodynamics: the second law of thermodynamics and the principle of material objectivity.

The classical Biot's model relies on two partial momentum balance equations

$$(1.1) \quad \begin{aligned} \rho_0^S \frac{\partial \mathbf{v}^S}{\partial t} + \rho_{12} \frac{\partial \mathbf{v}^F}{\partial t} &= \operatorname{div} \mathbf{T}^S + \pi (\mathbf{v}^F - \mathbf{v}^S), \\ \rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} + \rho_{12} \frac{\partial \mathbf{v}^S}{\partial t} &= \operatorname{div} \mathbf{T}^F - \pi (\mathbf{v}^F - \mathbf{v}^S), \end{aligned}$$

with the following constitutive relations for stresses (e.g. [2])

$$(1.2) \quad \begin{aligned} \mathbf{T}^S &= A\mathbf{e}\mathbf{1} + 2G\mathbf{e}^S + Q\epsilon\mathbf{1}, \\ \mathbf{T}^F &= \sigma\mathbf{1}, \quad \sigma \equiv -p^F = Q\epsilon + R\epsilon, \end{aligned}$$

where we use the following notations appearing also further in this work: ρ_0^S, ρ_0^F are constant partial mass densities of the solid component (skeleton) and of the fluid component, respectively, ρ_{12} is the acceleration coupling constant ("added mass"), $\mathbf{v}^S, \mathbf{v}^F$ are macroscopic velocities of components, A, G, Q, R, π are material parameters, \mathbf{e}^S describes small deformations of the skeleton (Almansi-Hamel deformation tensor), and ϵ describes volume changes of the fluid component¹⁾

In my earlier works on this subject (e.g. [3], [4]) I have presented a rather immediate proof that the Biot's model indeed violates the principle of material objectivity. This is related to the coupling through accelerations, i.e. to the existence of the added mass ρ_{12} .

The thermodynamical admissibility is a more subtle issue. It will be presented in this work and it concerns the coupling constant Q .

One should distinguish two cases. The first one appears when changes of porosity are described by an additional field equation. It may be a second order equation proposed in the model of GOODMAN and COWIN [5], a first order evolution equation proposed by BOWEN [6] or a balance equation proposed in

¹⁾ Within the frame of macroscopical description, all these fields must possess a purely macroscopical interpretation. In our case the volume changes of the skeleton are described by the third invariant of the deformation tensor of the skeleton. This reduces to $\operatorname{tr} \mathbf{e}^S$ for small deformations. For the fluid, the mass ϵ of volume changes are related to changes of partial (smeared-out) mass densities

$$\epsilon = \frac{\rho_0^F - \rho^F}{\rho_0^F}.$$

We return to other quantities describing volume changes further in this paper (e.g. Subsec. 2. 3).

my own papers (e.g. [7]). In these models the coupling between partial stresses cannot have the form proposed by Biot due to identities following from the second law of thermodynamics. The second case appears when changes of porosity are described by a constitutive law (see: [8]). This case has not been systematically investigated and it is the subject of this work.

We prove that a nonlinear poroelastic model without contributions of higher gradients cannot produce the Biot's model. We mean by higher gradients of constitutive variables a gradient of porosity n , a gradient of the deformation gradient of skeleton \mathbf{F}^S , or a gradient of one of the partial velocities. A model in which the gradient of the deformation gradient \mathbf{F}^S appears leads to the necessity of introducing couple stresses and it becomes a Cosserat-type continuum. On the other hand, the gradients of velocities introduce viscous effects which in turn destroy the hyperbolicity of field equations. This is not desired in the wave analysis. In addition, none of these effects appears in the case of Biot's model. Consequently it remains the gradient of porosity as a reasonable possibility. We shall prove that a correction of a dependence on the gradient of porosity is sufficient for thermodynamical admissibility of Biot's stress relations. However such a correction yields automatically additional contributions to momentum balance equations. They must contain a term with the gradient of porosity as well and this is not the case in the Biot's model.

These considerations seem to exhaust the available possibilities and yield the conclusion that only constitutive relations for stresses in the Biot's model can be constructed in a thermodynamically admissible way provided a classical multicomponent model is extended to higher gradients. Field equations must be different.

We complete the paper with a remark on acoustic waves. We show that linear models constructed in Subsection 2.3 yield the existence of bulk acoustic P1-, S-, and P2-waves as predicted by Biot's model but their speeds of propagation are different.

2. Thermodynamic construction of models

2.1. Fields and field equations

In order to appreciate couplings between partial stresses of a poroelastic saturated medium we consider the two following two-component prototypes of the "Biot's" model for large deformations.

For the first class of models labeled (I) we assume that the real fluid in pores is incompressible. This assumption means that we have two possibilities to control changes of the geometry of the medium. On the one hand we can macroscopically deform the skeleton, and we measure this deformation by the macroscopic

deformation gradient \mathbf{F}^S . Doing so we keep the porosity constant, i.e. the fraction n of the volume of the representative volume element (*REV*) corresponding to the fluid component does not change ("undrained conditions"). We can also deform pores by drainage and in this way change the porosity n . Simultaneously we keep unchanged the macroscopic geometry, i.e. $\mathbf{F}^S = \text{const}$. Under the incompressibility assumption the latter deformation is solely related to changes of mass densities. For the fluid component these changes are given by the relation

$$(2.1) \quad \rho_t^F = n\rho^{FR}, \quad \rho^{FR} = \text{const},$$

where ρ^{FR} is the so-called true mass density. The incompressibility assumption yields this quantity to be constant. The index t means that the mass density refers to the unit volume of the mixture in the current configuration.

In the second class of models labeled (C) we allow for arbitrary changes of mass densities ρ_t^F, ρ_t^S but the porosity is assumed to be given by a "constitutive" relation. It has been shown [9] that in *thermodynamical equilibria* such a relation must have the form $n = \bar{n} \left(\frac{\rho_t^F}{\rho_t^S} \right)$. In the simplest case which we consider in this work we assume this relation to have the following form

$$(2.2) \quad n = n_0 \frac{\rho_t^F}{\rho_0^F} \frac{\rho_0^S}{\rho_t^S},$$

where ρ_0^F, ρ_0^S, n_0 denote reference constant values of partial mass densities and of the porosity. The above relation is assumed to hold also in thermodynamical nonequilibrium.

Let us note that in general the porosity n may change as well without accompanying changes of macroscopic mass densities ρ_t^F, ρ_t^S . This happens when changes of n are compensated by changes of real mass densities ρ^{FR}, ρ^{SR} in such a way that their products $n\rho^{FR}, (1-n)\rho^{SR}$ remain constant. Such changes are not controllable on the macroscopic level. They must proceed spontaneously. Consequently they yield a **relaxation** of porosity characteristic for microstructural variables. Relaxation properties are always dissipative and do not appear in thermodynamical equilibria. These phenomena require a source term in an equation describing changes of porosity. Such a general model was discussed elsewhere (e.g. [7]).

We shall discuss further properties of the relation (2.2) when we linearize the model.

For technical reasons it is easier to work in a so-called Lagrangian description (e.g. [7],[8]). In this description we refer all quantities to a chosen reference configuration \mathcal{B}_0 of the skeleton for which $\mathbf{F}^S = \mathbf{1}$.

Fields which describe mechanical processes in such a system are as follows:

1. The reference partial mass density of the fluid component

$$(2.3) \quad \rho^F = \rho^F(\mathbf{X}, t) = \rho_t^F J^S \equiv n J^S \rho^{FR}, \quad J^S := \det \mathbf{F}^S, \quad \mathbf{X} \in \mathcal{B}_0.$$

2. The field of partial velocity of the skeleton $\dot{\mathbf{x}}^S(\mathbf{X}, t)$ on the macroscopic level of description.
3. The field of partial velocity of the fluid $\dot{\mathbf{x}}^F(\mathbf{X}, t)$ on the macroscopic level of description.
4. The macroscopic deformation gradient of the skeleton $\mathbf{F}^S(\mathbf{X}, t)$.

By means of the velocity fields one can define the macroscopic filter velocity $\mathbf{w}(\mathbf{X}, t)$, and its corresponding Lagrangian image $\dot{\mathbf{X}}^F(\mathbf{X}, t)$ appearing in balance equations

$$(2.4) \quad \mathbf{w} := \dot{\mathbf{x}}^F - \dot{\mathbf{x}}^S, \quad \dot{\mathbf{X}}^F := \mathbf{F}^{S-1} \mathbf{w}.$$

The reference partial mass density of the skeleton $\rho^S \equiv \rho_0^S$ does not appear among those fields because it is constant in time if we assume that there is no mass exchange between components. Its current value is given by the relation

$$(2.5) \quad \rho_t^S = \rho_0^S J^{S-1},$$

which satisfies identically the partial mass conservation law in Eulerian description.

Summing up we can write the following relations for the porosity in the two above classes of the models

$$(2.6) \quad \begin{aligned} \text{(I)-models} & : n = J^{S-1} \frac{\rho^F}{\rho^{FR}}, \quad \rho^{FR} = \text{const}, \\ \text{(C)-models} & : n = \frac{\rho^F}{\rho^{FR}}, \quad \rho_0^{FR} = \frac{\rho_0^F}{n_0} = \text{const}. \end{aligned}$$

The fields must fulfil the following balance equations in Lagrangian description

1. **Mass conservation of the fluid component**

$$(2.7) \quad \frac{\partial \rho^F}{\partial t} + \text{Div}(\rho^F \dot{\mathbf{X}}^F) = 0.$$

2. **Momentum balance for the skeleton**

$$(2.8) \quad \rho^S \frac{\partial \dot{\mathbf{x}}^S}{\partial t} = \text{Div} \mathbf{P}^S + \hat{\mathbf{p}}.$$

3. Momentum balance for the fluid

$$(2.9) \quad \rho^F \left(\frac{\partial \dot{\mathbf{x}}^F}{\partial t} + \dot{\mathbf{X}}^F \cdot \text{Grad } \dot{\mathbf{x}}^F \right) = \text{Div} \mathbf{P}^F - \hat{\mathbf{p}}.$$

In addition, the deformation gradient \mathbf{F}^S must fulfil **integrability conditions** yielding the existence of the field of motion of the skeleton. They consist of two parts.

4. Kinematic compatibility condition relates the time derivative and the gradient of velocity

$$(2.10) \quad \frac{\partial \mathbf{F}^S}{\partial t} = \text{Grad } \dot{\mathbf{x}}^S.$$

The second part – a geometrical compatibility condition is a symmetry relation

$$(2.11) \quad \text{Grad } \mathbf{F}^S = (\text{Grad } \mathbf{F}^S)^T,$$

or, in Cartesian coordinates

$$(2.12) \quad \frac{\partial F_{kK}^S}{\partial X^L} = \frac{\partial F_{kL}^S}{\partial X^K},$$

where the small index refers to Eulerian coordinates, and the capital index to Lagrangian coordinates.

Obviously, all operators Grad, Div, $\frac{\partial}{\partial X^K}$ refer to Lagrangian variables.

As we have mentioned conditions (2.10), (2.11) yield the existence of the field of motion of skeleton, say $\chi^S(\mathbf{X}, t)$, whose derivatives give the deformation gradient and the partial velocity, vis.

$$(2.13) \quad \mathbf{F}^S = \text{Grad } \chi^S, \quad \dot{\mathbf{x}}^S = \frac{\partial \chi^S}{\partial t}.$$

Momentum balance equations contain partial Piola-Kirchhoff stresses $\mathbf{P}^S, \mathbf{P}^F$ which are related to the usual Cauchy stresses by the following transformation rules

$$(2.14) \quad \mathbf{T}^S = J^{S-1} \mathbf{P}^S \mathbf{F}^{ST}, \quad \mathbf{T}^F = J^{S-1} \mathbf{P}^F \mathbf{F}^{ST}.$$

Momentum equations contain as well the source $\hat{\mathbf{p}}$ which is the diffusion force.

We have to perform a closure in order to obtain field equations from the above balance relations. For poroelastic materials we consider further two models following from two choices of constitutive variables. Namely we choose either

$$(2.15) \quad \mathcal{C}^{(1)} = \left\{ \rho^F, \mathbf{F}^S, \dot{\mathbf{X}}^F \right\},$$

or

$$(2.16) \quad \mathcal{C}^{(2)} = \left\{ \rho^F, \mathbf{F}^S, \dot{\mathbf{X}}^F, \text{Grad}n \right\}.$$

The porosity n does not appear among these variables because it is either given by the relation (2.2) or by the relation (2.3). Consequently it can be eliminated from the set of independent constitutive variables.

The following functions must be given in terms of constitutive relations:

$$(2.17) \quad \mathcal{F} = \left\{ \mathbf{P}^S, \mathbf{P}^F, \hat{\mathbf{p}}, \psi^S, \psi^F \right\},$$

where ψ^S, ψ^F denote partial Helmholtz free energies appearing further in the second law of thermodynamics.

For reasons of material objectivity we should choose not only the *relative* velocity as the variable but also one of the *objective measures* of deformation. We shall do so further in this note. However the exploitation of the second law of thermodynamics is easier if we impose the objectivity after the exploitation of the entropy inequality.

For any of the choices of constitutive variables, the constitutive relations are assumed to have the form of the relation

$$(2.18) \quad \mathcal{F} = \mathcal{F} \left(\mathcal{C}^{(\alpha)} \right), \quad \alpha = 1, 2,$$

which is sufficiently smooth for all operations which we perform in the sequel.

Let us make two methodological remarks. Our interest in comparison with the Biot's model as well as in the analysis of acoustic waves is limited to linear models. Consequently the above presented nonlinear models are an overkilling. We do so on purpose because the exploitation of the second law of thermodynamics for a model with linear constitutive laws cannot be made consistent with explicit nonlinear contributions to field equations. This yields serious flaws of the thermodynamical analysis known in all nonlinear field theories.

Secondly we should point out that the restriction to incompressible real fluids in the class (I) of models **does not lead to any constraints**. This may be simply interpreted as a change of variables: changes of the partial mass density of the fluid are replaced by corresponding changes of the porosity. There is no reaction force on such a "constraint". This is different from the cases which

were considered previously (e.g. [10], [9]). In those cases the model possesses an additional equation for porosity which, in turn, is considered to be a real microstructural variable with spontaneous relaxation properties. In such models the incompressibility assumption yields the existence of the reaction pressure, and it requires a special structure of constitutive relations. We shall not discuss this problem in this work.

Let us mention in passing that the above change of variables may lead to some mathematical problems due to the restriction $0 < n < 1$. As we are primarily interested in a linear model changes of porosity with respect to its initial value n_0 are small. The choice of a real value of n_0 , say between 0.1 and 0.6, guarantees that this restriction is indeed fulfilled. In addition thermodynamical restrictions are derived by means of the partial mass density for which such a restriction does not appear.

2.2. Thermodynamic admissibility

2.2.1. Second law of thermodynamics. We present here solely the second law of thermodynamics for two-component systems for which the temperature is constant. If this is the case it may be formulated as follows (e.g. [11]). For any solution of field equations (i.e. for any *thermodynamical process*) the following inequality

$$(2.19) \quad \rho^S \frac{\partial \psi^S}{\partial t} + \rho^F \left(\frac{\partial \psi^F}{\partial t} + \dot{\mathbf{X}}^F \cdot \text{Grad} \psi^F \right) - \mathbf{P}^S \cdot \frac{\partial \mathbf{F}^S}{\partial t} - \mathbf{P}^F \cdot \text{Grad} \dot{\mathbf{x}}^F - \hat{\mathbf{p}} \cdot \mathbf{w} \leq 0,$$

must be satisfied identically.

The main technical problem with the exploitation of this inequality is the limitation to solutions of field equations. This can be eliminated by means of Lagrange multipliers introduced to thermodynamics by I-SHIH LIU ([7]). Namely it can be shown that the following inequality

$$(2.20) \quad \rho^S \frac{\partial \psi^S}{\partial t} + \rho^F \left(\frac{\partial \psi^F}{\partial t} + \dot{\mathbf{X}}^F \cdot \text{Grad} \psi^F \right) - \mathbf{P}^S \cdot \frac{\partial \mathbf{F}^S}{\partial t} - \mathbf{P}^F \cdot \text{Grad} \dot{\mathbf{x}}^F - \hat{\mathbf{p}} \cdot \mathbf{w} - \Lambda^n \left\{ \frac{\partial \rho^F}{\partial t} + \text{Div} \left(\rho^F \dot{\mathbf{X}}^F \right) \right\} - \Lambda^F \cdot \left\{ \frac{\partial \mathbf{F}^S}{\partial t} - \text{Grad} \dot{\mathbf{x}}^S \right\}$$

$$(2.20) \quad \left. \begin{array}{l} \text{[cont.]} \\ - \Lambda^{vS} \cdot \left\{ \rho^S \frac{\partial \dot{\mathbf{x}}^S}{\partial t} - \text{Div } \mathbf{P}^S - \hat{\mathbf{p}} \right\} \\ - \Lambda^{vF} \cdot \left\{ \rho^F \left(\frac{\partial \dot{\mathbf{x}}^F}{\partial t} + \dot{\mathbf{X}}^F \cdot \text{Grad } \dot{\mathbf{x}}^F \right) - \text{Div } \mathbf{P}^F + \hat{\mathbf{p}} \right\} \leq 0, \end{array} \right\}$$

must hold for any fields and not only for solutions of field equations. The multipliers $\Lambda^n, \Lambda^F, \Lambda^{vS}, \Lambda^{vF}$ are functions of constitutive variables $\mathcal{C}^{(\alpha)}$ for α equal either to 1 or to 2. We proceed to discuss the consequences of the above condition for these two different models.

2.2.2. $\mathcal{C}^{(1)}$ -models. We consider the model with constitutive variables given by the relation (2.15). It is not necessary to distinguish between (I)-models and (C)-models because the only difference appears in the final results due to the substitution of either (2.6)₁ or (2.6)₂.

It is easy to check that the chain rule of differentiation in the inequality (2.21) yields the linearity of this inequality with respect to the following derivatives

$$(2.21) \quad \left\{ \frac{\partial \rho^F}{\partial t}, \frac{\partial \mathbf{F}^S}{\partial t}, \frac{\partial \dot{\mathbf{x}}^S}{\partial t}, \frac{\partial \dot{\mathbf{x}}^F}{\partial t} \right\},$$

as well as

$$(2.22) \quad \{ \text{Grad } \rho^F, \text{Grad } \mathbf{F}^S, \text{Grad } \dot{\mathbf{x}}^S, \text{Grad } \dot{\mathbf{x}}^F \}.$$

Consequently, as the inequality must hold for all fields, the coefficients of these derivatives have to vanish. We obtain from the contributions of time derivatives (2.21) the following relations:

$$(2.23) \quad \begin{aligned} \Lambda^n &= \rho^S \frac{\partial \psi^S}{\partial \rho^F} + \rho^F \frac{\partial \psi^F}{\partial \rho^F}, \\ \mathbf{P}^S + \Lambda^F &= \rho^S \frac{\partial \psi^S}{\partial \mathbf{F}^S} + \rho^F \frac{\partial \psi^F}{\partial \mathbf{F}^S} \\ &\quad - \rho^S \left(\mathbf{F}^{S-T} \frac{\partial \psi^S}{\partial \dot{\mathbf{X}}^F} \right) \otimes \dot{\mathbf{X}}^F - \rho^F \left(\mathbf{F}^{S-T} \frac{\partial \psi^F}{\partial \dot{\mathbf{X}}^F} \right) \otimes \dot{\mathbf{X}}^F, \\ \rho^S \Lambda^{vS} &= - \rho^F \Lambda^{vF} = - \rho^S \frac{\partial \psi^S}{\partial \dot{\mathbf{X}}^F} - \rho^F \frac{\partial \psi^F}{\partial \dot{\mathbf{X}}^F}. \end{aligned}$$

On the other hand, the coefficients of spatial derivatives (2.22) lead to the following identities:

$$\left(\rho^F \frac{\partial \psi^F}{\partial \rho^F} - \Lambda^n \right) \dot{\mathbf{X}}^F + \frac{\partial \mathbf{P}^{ST}}{\partial \rho^F} \Lambda^{vS} + \frac{\partial \mathbf{P}^{FT}}{\partial \rho^F} \Lambda^{vF} = 0,$$

$$\text{sym}^{23} \left(\frac{\partial \psi^F}{\partial \mathbf{F}^S} + \Lambda^n \mathbf{F}^{S-T} \right) \otimes \dot{\mathbf{X}}^F = 0,$$

$$(2.24) \quad \Lambda^F = \mathbf{P}^F = -\rho^F \Lambda^n \mathbf{F}^{S-T}.$$

It remains the residual inequality

$$(2.25) \quad \hat{\mathbf{p}} \cdot \mathbf{w} \geq 0,$$

which defines the *dissipation density* of the system.

For technical reasons we make the following simplifying assumption:

$$(2.26) \quad \frac{\partial \psi^S}{\partial \dot{\mathbf{X}}^F} = \frac{\partial \psi^F}{\partial \dot{\mathbf{X}}^F} = 0.$$

One could proceed also without this simplification but we aim at the construction of a linear model for which such contributions would be neglected anyway.

Combination of relations (2.23)₁ and (2.24)₁ yields now

$$(2.27) \quad \Lambda^n = \rho^F \frac{\partial \psi^F}{\partial \rho^F}, \quad \frac{\partial \psi^S}{\partial \rho^F} = 0.$$

The second part of this relation has the most important bearing on the structure of interactions described by the model. Namely the partial free energy of the skeleton does not react on changes of the porosity. We see in a moment what is the reaction of partial stresses on this property. Such a conclusion would be impossible if we performed the exploitation of the second law for a linear model.

Relation (2.24)₂ leads after easy calculations to the relation

$$(2.28) \quad \frac{\partial \psi^F}{\partial \mathbf{F}^S} = -\Lambda^n \mathbf{F}^{S-T}.$$

Hence relations (2.23)₂ and (2.24)₂ yield the following relations for partial Piola-Kirchhoff stresses:

$$\mathbf{P}^S = \rho^S \frac{\partial \psi^S}{\partial \mathbf{F}^S}, \quad \mathbf{P}^F = -\rho^{F2} \frac{\partial \psi^F}{\partial \rho^F} \mathbf{F}^{S-T},$$

or, after the transformation to partial Cauchy stresses, described by (2.14)

$$(2.29) \quad \mathbf{T}^S = \rho_t^S \frac{\partial \psi^S}{\partial \mathbf{F}^S} \mathbf{F}^{ST}, \quad \mathbf{T}^F = -p^F \mathbf{1}, \quad p^F := \rho_t^{F2} \frac{\partial \psi^F}{\partial \rho_t^F}.$$

These are classical thermodynamical relations for elastic materials and ideal fluids, respectively. The most important property of these relations is the fact that identities (2.27)₂ and (2.28) yield

$$(2.30) \quad \mathbf{T}^S = \mathbf{T}^S(\mathbf{F}^S), \quad p^F = p^F(\rho_t^F).$$

The latter requires an assumption on isotropy and it shall be proven in the next subsection.

Consequently one **cannot obtain Biot's model** by linearization of the above model. Couplings between partial stresses which are appearing in the original Biot's model with the material constant Q would have to violate the second law of thermodynamics. This property seems to be common for all multicomponent models of poroelastic materials which do not contain higher gradients among constitutive variables (see also remarks in [4]). It is also a known property of miscible mixtures of fluids. In the latter case the lack of constitutive dependence on gradients of partial mass densities leads to a model called **simple mixtures** in which partial stresses are not coupled in a constitutive way [12].

Obviously the model contains coupling due to the relative motion of components and described by the source $\hat{\mathbf{p}}$. For this reason, solutions of the boundary – initial value problems and consequently local values of partial stresses follow from the **coupled** field equations.

We should also mention that the similarity of the relation for p^F for incompressible real fluids ((I)-models) to the relation for compressible fluids ((C)-models) is misleading. This relation has an entirely different physical interpretation. For compressible fluids the relation (2.29)₂ yields the following **linear form** of the constitutive relation for the partial pressure

$$(2.31) \quad p^F = p_0^F + \kappa (\rho^F - \rho_0^F),$$

where p_0^F is the reference pressure and ρ_0^F – the corresponding reference partial mass density. In such a case the compressibility coefficient κ describes elastic properties of the fluid. Simultaneously its square root specifies the speed of the longitudinal wave in the fluid. This is not the case for the model with the incompressibility assumption. Since the real fluid in this case is incompressible, it has no elastic properties. Consequently the relation for p^F describes its dependence on changes of porosity which are due to microscopical morphological changes of the skeleton such as a redistribution of grains. Hence the acoustic properties related to such a constitutive relation for the pressure cannot be extracted from microscopic properties of the real fluid component.

Incidentally such a model supports the views advocated by W. G. GRAY (e.g. see: [13]) that macroscopic constitutive relations of components in the macroscopic model cannot be directly related to constitutive properties of real components, and even less, they can be derived by any averaging procedure for a single real microscopic component. Macroscopic constitutive properties reflect for each component microscopic properties of both real components as well as microscopic interactions between them.

In the next subsection we consider a higher gradient model which allows for interactions in constitutive relations for partial stresses.

2.2.3. $\mathcal{C}^{(2)}$ -models. We proceed to consider the model based on constitutive variables (2.16). However we limit the attention to the simplest case in which the model is *linear* with respect to the gradient of porosity. In such a case it may appear solely in the constitutive relation for the source $\hat{\mathbf{p}}$ because this is the only vectorial constitutive function. We simplify the model even further and assume linearity of the source with respect to both the filter velocity \mathbf{w} and $\text{grad}n$. Consequently

$$(2.32) \quad \hat{\mathbf{p}} = \pi \mathbf{w} - N \text{grad}n,$$

where material parameters π and N may still depend on ρ^F and \mathbf{F}^S . We have left out a possible nonlinear contribution proportional to the vector product $\mathbf{w} \times \text{grad}n$ which would appear in a general nonlinear isotropic model. In contrast to linear contributions, such a term would be nondissipative. The minus sign in (2.32) is related to the property of incompressible models (e.g. [10]) in which N coincides with the pore pressure. In general it may not be the case. This parameter seems to be always positive. However, such a property does not follow from the second law of thermodynamics. We assume also the relation (2.26) to hold.

Bearing the relation (2.6) in mind we obtain the following contribution of the source to the inequality (2.20):

$$(2.33) \quad \begin{aligned} (I) - \text{models} : \hat{\mathbf{p}} \cdot \mathbf{w} \\ = \pi \mathbf{w} \cdot \mathbf{w} - N \frac{1}{\rho^{FR} J^S} \{ \text{Grad} \rho^F - \rho^F \mathbf{F}^{ST} \text{Div} \mathbf{F}^{S-1} \} \cdot \dot{\mathbf{X}}^F, \\ (C) - \text{models} : \hat{\mathbf{p}} \cdot \mathbf{w} = \pi \mathbf{w} \cdot \mathbf{w} - N \frac{1}{\rho_0^{FR}} \text{Grad} \rho^F \cdot \dot{\mathbf{X}}^F, \quad \mathbf{w} \equiv \mathbf{F}^S \dot{\mathbf{X}}^S. \end{aligned}$$

Contributions appearing with the parameter N in these relations change the identities of the previous subsection following from coefficients of spatial derivatives. Namely relations (2.24)₁ and (2.24)₂ will be influenced. We collect all these results of the second law of thermodynamics in the juxtaposition.

After easy manipulations we obtain the following relations for the partial Piola-Kirchhoff stresses in both classes of models

$$(2.34) \quad \begin{aligned} \text{(I)-models : } \mathbf{P}^S &= \rho^S \frac{\partial \psi^S}{\partial \mathbf{F}^S} + nN \mathbf{F}^{S-T}, \\ \text{(C)-models : } \mathbf{P}^S &= \rho^S \frac{\partial \psi^S}{\partial \mathbf{F}^S}, \end{aligned}$$

$$(2.35) \quad \text{(I)-models and (C)-models: } \mathbf{P}^F = - \left(\rho^{F2} \frac{\partial \psi^F}{\partial \rho^F} + nN \right) \mathbf{F}^{S-T},$$

as well as the following identities for partial free energy functions

$$(2.36) \quad \begin{aligned} \text{(I) - models : } \frac{\partial \psi^F}{\partial \mathbf{F}^S} &= -\rho^F \frac{\partial \psi^F}{\partial \rho^F} \mathbf{F}^{S-T}, \quad \rho^S \frac{\partial \psi^S}{\partial \rho^F} = \frac{N}{\rho^{FR} J^S}, \\ \text{(C) - models : } \frac{\partial \psi^F}{\partial \mathbf{F}^S} &= - \left(\rho^F \frac{\partial \psi^F}{\partial \rho^F} + \frac{N}{\rho_0^{FR}} \right) \mathbf{F}^{S-T}, \quad \rho^S \frac{\partial \psi^S}{\partial \rho^F} = \frac{N}{\rho_0^{FR}}. \end{aligned}$$

Table 1.

(I)-models	(C)-models
$\Lambda^n = \rho^S \frac{\partial \psi^S}{\partial \rho^F} + \rho^F \frac{\partial \psi^F}{\partial \rho^F},$	$\Lambda^n = \rho^S \frac{\partial \psi^S}{\partial \rho^F} + \rho^F \frac{\partial \psi^F}{\partial \rho^F},$
$\mathbf{P}^S = \rho^S \frac{\partial \psi^S}{\partial \mathbf{F}^S} + \rho^F \frac{\partial \psi^F}{\partial \mathbf{F}^S} - \mathbf{P}^F,$	$\mathbf{P}^S = \rho^S \frac{\partial \psi^S}{\partial \mathbf{F}^S} + \rho^F \frac{\partial \psi^F}{\partial \mathbf{F}^S} - \mathbf{P}^F,$
$\mathbf{P}^F = -\rho^F \Lambda^n \mathbf{F}^{S-T},$	$\mathbf{P}^F = -\rho^F \Lambda^n \mathbf{F}^{S-T},$
$\rho^S \frac{\partial \psi^S}{\partial \rho^F} - \frac{N}{\rho^{FR}} J^{S-1} = 0, \quad \rho^{FR} = \text{const}$	$\rho^S \frac{\partial \psi^S}{\partial \rho^F} - \frac{N}{\rho_0^{FR}} = 0,$
$\frac{\partial \psi^F}{\partial \mathbf{F}^S} + \rho^F \frac{\partial \psi^F}{\partial \rho^F} \mathbf{F}^{S-T} = 0,$	$\frac{\partial \psi^F}{\partial \mathbf{F}^S} + \left(\rho^F \frac{\partial \psi^F}{\partial \rho^F} + \frac{N}{\rho_0^{FR}} \right) \mathbf{F}^{S-T} = 0.$

Then for the partial Cauchy stresses follow the relations of the form

(I)-models :

$$\mathbf{T}^S = \rho^S J^{S-1} \frac{\partial \psi^S}{\partial \mathbf{F}^S} \mathbf{F}^{ST} + nN J^{S-1} \mathbf{1}, \quad \mathbf{T}^F = -p^F \mathbf{1},$$

$$p^F := \rho^{F2} J^{S-1} \frac{\partial \psi^F}{\partial \rho^F} + nN J^{S-1},$$

(2.37)

(C)-models :

$$\mathbf{T}^S = \rho^S J^{S-1} \frac{\partial \psi^S}{\partial \mathbf{F}^S} \mathbf{F}^{ST}, \quad \mathbf{T}^F = -p^F \mathbf{1},$$

$$p^F := \rho^{F2} J^{S-1} \frac{\partial \psi^F}{\partial \rho^F} + nN J^{S-1}.$$

It is seen that both models contain couplings of stresses which may lead to Biot's constitutive relations for stresses (e.g. [2]) of the linear model. We proceed to investigate this question.

In order to simplify the construction of the linear model we evaluate the above nonlinear relations for isotropic materials. In such a case free energies ψ^S, ψ^F satisfying the principle of material objectivity depend on the deformation gradient \mathbf{F}^S solely through the invariants of its symmetric part. For instance we can choose the invariants of the right Cauchy - Green deformation tensor \mathbf{C}^S

$$\mathbf{C}^S := \mathbf{F}^{ST} \mathbf{F}^S, \quad I := \text{tr} \mathbf{C}^S,$$

(2.38)

$$II := \frac{1}{2} (I^2 - \text{tr} \mathbf{C}^{S2}), \quad III \equiv J^{S2} := \det \mathbf{C}^S.$$

Then

$$(2.39) \quad \psi^S = \psi^S(I, II, III, \rho^F), \quad \psi^F = \psi^F(I, II, III, \rho^F).$$

Let us exploit the identity (2.36)₁ under the above assumption. Bearing the following relation in mind

$$(2.40) \quad \frac{\partial \psi^F}{\partial \mathbf{F}^S} = 2\mathbf{F}^S \frac{\partial \psi^F}{\partial \mathbf{C}^S},$$

we obtain

$$(2.41) \quad \frac{\partial \psi^F}{\partial \mathbf{F}^S} \mathbf{F}^{ST} = 2 \left(\frac{\partial \psi^F}{\partial I} \mathbf{B}^S + \frac{\partial \psi^F}{\partial II} I \mathbf{B}^S - \frac{\partial \psi^F}{\partial II} \mathbf{B}^{S2} + \frac{\partial \psi^F}{\partial III} III \mathbf{1} \right),$$

$$\mathbf{B}^S := \mathbf{F}^S \mathbf{F}^{ST}, \quad III \equiv J^{S2},$$

where the symmetric tensor \mathbf{B}^S is called the left Cauchy-Green deformation tensor. Now we can write (2.36)₁ in the form

(I)-models :

$$\left(2III \frac{\partial \psi^F}{\partial III} + \rho^F \frac{\partial \psi^F}{\partial \rho^F} \right) \mathbf{1} + 2 \left(\frac{\partial \psi^F}{\partial I} + \frac{\partial \psi^F}{\partial II} I \right) \mathbf{B}^S - 2 \frac{\partial \psi^F}{\partial II} \mathbf{B}^{S2} = 0,$$

(2.42) (C)-models :

$$\left(2III \frac{\partial \psi^F}{\partial III} + \rho^F \frac{\partial \psi^F}{\partial \rho^F} + \frac{N}{\rho_0^{FR}} \right) \mathbf{1} + 2 \left(\frac{\partial \psi^F}{\partial I} + \frac{\partial \psi^F}{\partial II} I \right) \mathbf{B}^S - 2 \frac{\partial \psi^F}{\partial II} \mathbf{B}^{S2} = 0.$$

These relations must hold for arbitrary deformations \mathbf{B}^S . Consequently, according to the corollaries of the Cayley-Hamilton theorem in the tensor analysis, coefficients of tensors $\mathbf{1}, \mathbf{B}^S, \mathbf{B}^{S2}$ have to vanish independently. Hence in both cases the free energy ψ^F must be independent of invariants I, II , and, consequently (comp. relations for N in Table. 1) N must be independent of these two invariants in (C)-models but not necessarily in (I)-models. In addition we have

(2.43) (I)-models : $\psi^F = \psi^F (J^S, \rho^F), \quad J^S \frac{\partial \psi^F}{\partial J^S} + \rho^F \frac{\partial \psi^F}{\partial \rho^F} = 0;$

(2.44) (C)-models : $\psi^F = \psi^F (J^S, \rho^F), \quad J^S \frac{\partial \psi^F}{\partial J^S} + \rho^F \frac{\partial \psi^F}{\partial \rho^F} = -\frac{N}{\rho_0^{FR}},$

$$N = N (J^S, \rho^F).$$

The differential equation (2.43)₂ for (I)-models can be easily solved by the method of characteristics. If we denote the variable along the characteristic by ξ then we can write this equation in the characteristic form

(2.45) $\frac{dJ^S}{d\xi} = J^S, \quad \frac{d\rho^F}{d\xi} = \rho^F, \quad \frac{d\psi^F}{d\xi} = 0.$

Consequently ψ^F is constant along the characteristics, and these are labelled by the following initial values

(2.46) $\rho^F J^{S-1} = \text{const},$

which follows from (2.45)_{1,2}. It means that the solution of the identity (2.36)₁ for isotropic materials with incompressible real fluid is of the form

(2.47) (I)-models: $\psi^F = \psi^F (\rho_t^F).$

The problem is more complicated for compressible materials. However we can simplify the equation (2.44)₂ if we change the variables in a way suggested by the above solution for the (I)-model. Namely we obtain

$$(2.48) \quad \text{(C)-models: } (\rho^F, J^S) \rightarrow (\rho_t^F, J^S) \implies \psi^F = \psi^F(\rho_t^F, J^S)$$

and

$$(2.49) \quad J^S \frac{\partial \psi^F}{\partial J^S} = -\frac{N}{\rho_0^{FR}} \implies \psi^F = \psi_{ideal}^F(\rho_t^F) - \frac{1}{\rho_0^{FR}} \int_1^{J^S} \frac{N(\rho_t^F, \xi)}{\xi} d\xi.$$

It is instructive to apply the relation (2.36)₂ in the formulae (2.37) for Cauchy stresses. For isotropic materials we obtain immediately

(I)-models:

$$(2.50) \quad \mathbf{T}^S = \rho^S J^{S-1} \left\{ \left(J^S \frac{\partial \psi^S}{\partial J^S} + \rho^F \frac{\partial \psi^S}{\partial \rho^F} \right) \mathbf{1} + 2 \left(\frac{\partial \psi^S}{\partial I} + I \frac{\partial \psi^S}{\partial II} \right) \mathbf{B}^S - 2 \frac{\partial \psi^S}{\partial II} \mathbf{B}^{S2} \right\},$$

$$\mathbf{T}^F = -\rho^F J^{S-1} \left(\rho^F \frac{\partial \psi^F}{\partial \rho^F} + \rho^S \frac{\partial \psi^S}{\partial \rho^F} \right) \mathbf{1},$$

(C)-models:

$$(2.51) \quad \mathbf{T}^S = \rho^S J^{S-1} \left\{ J^S \frac{\partial \psi^S}{\partial J^S} \mathbf{1} + 2 \left(\frac{\partial \psi^S}{\partial I} + I \frac{\partial \psi^S}{\partial II} \right) \mathbf{B}^S - 2 \frac{\partial \psi^S}{\partial II} \mathbf{B}^{S2} \right\},$$

$$\mathbf{T}^F = -\rho^F J^{S-1} \left(\rho^F \frac{\partial \psi^F}{\partial \rho^F} + \rho^S \frac{\partial \psi^S}{\partial \rho^F} \right) \mathbf{1}.$$

Let us discuss first the structure of stress relations for (I)-models. As indicated by (2.47), the first contribution to the partial stress \mathbf{T}^F cannot contain any coupling to the deformation of the skeleton. Consequently it is the derivative of the partial Helmholtz free energy ψ^S with respect to ρ_t^F which relates \mathbf{T}^F to the deformation of the skeleton. According to the relation (2.36)₂ for the existence of this coupling, the coefficient N must be different from zero. For the symmetry required in the Biot's model it is necessary to introduce a rather complicated dependence of the partial free energy function ψ^S on the mass density ρ^F which would create a term in $\rho^S \frac{\partial \psi^S}{\partial J^S}$ canceling out the contribution $\rho^S \rho^F J^{S-1} \frac{\partial \psi^S}{\partial \rho^F}$ to the stress \mathbf{T}^S (the wrong sign!) and simultaneously produces another one introducing the coupling to the deformation of the fluid. Even though it is possible

in principle, such a model does not seem to be very plausible and we do not investigate it any further.

Let us mention in passing that in a particular case of the constant partial free energy ψ^F , the relation (2.37)₂ for incompressible real fluids yields $N = \frac{p^F}{n}$ which is the pore pressure of classical models. Hence in this particular case the coefficient in the diffusion force (2.32) coincides with that of classical models of consolidation. This structure has been indicated in the work [14] on nonlinear sources.

The structure of stress relations for (C)-models is simpler. As thermodynamical requirements do not lead to any restrictions of the free energy ψ^S on ρ_t^F and the free energy ψ^F on J^S , we may produce as a particular case a desired dependence and symmetry. We shall do so for the linear model.

Concluding the above thermodynamical considerations we see that the Biot's constitutive relations for stresses can be derived from nonlinear $\mathcal{C}^{(2)}$ -models by a specific choice of partial free energies. It means that such a transition requires a higher gradient model as a background.

2.3. Linear models

The relations derived in the previous subsection yield immediately the constitutive relations of the fully linear model. Let us introduce the Almansi-Hamel deformation measure \mathbf{e}^S commonly used in the linear theory of elasticity (e.g. [7])

$$(2.52) \quad \mathbf{e}^S := \frac{1}{2} (\mathbf{1} - \mathbf{B}^{S-1}).$$

The linearity follows from the assumption that deformations of the skeleton are small, and that changes of porosity are small

$$(2.53) \quad \max \left\{ \left| \lambda^{(\alpha)} \right| \right\}_{\alpha=1,2,3} \ll 1, \quad \det (\mathbf{e}^S - \lambda^{(\alpha)} \mathbf{1}) = 0, \\ |\varsigma| \ll 1, \quad \varsigma := \frac{n - n_0}{n_0},$$

where $\lambda^{(\alpha)}$ are eigenvalues of \mathbf{e}^S , and they are called principal deformations while n_0 denotes the reference constant value of porosity.

We construct the linear version of the (C)-model. In this model, also in its fully nonlinear version, the porosity is related to the mass density by the relation (2.9)₂, i.e.

$$(2.54) \quad n = \frac{\rho_t^F J^S}{\rho_0^{FR}}.$$

The variable ς can be now coupled to changes of the partial mass density ρ_t^F and to volume changes of the skeleton $J^S \approx 1 + \text{tr} \mathbf{e}^S$. We have

$$(2.55) \quad \varsigma = \frac{n}{n_0} - 1 \approx \frac{\rho_t^F - \rho_0^F}{\rho_0^F} + \text{tr} \mathbf{e}^S, \quad \rho_0^F \equiv n_0 \rho_0^{FR}.$$

The first contribution describes the macroscopic volume changes of the fluid component, i.e. it is the (macroscopic) negative fluid *dilation* ϵ while the second contribution describes changes of the macroscopic volume e . Hence

$$(2.56) \quad \varsigma \approx e - \epsilon, \quad e := \text{tr} \mathbf{e}^S, \quad \epsilon := -\frac{\rho_t^F - \rho_0^F}{\rho_0^F}.$$

Let us note that the macroscopic model constructed in this way does not require any reference to the microscopic description.

Before we proceed with the linearization of constitutive relations it is convenient to change the variables in (2.51) from (ρ^F, J^S) to (ρ_t^F, J^S) . Bearing identities of the previous section in mind we obtain

$$(2.57) \quad \begin{aligned} \mathbf{T}^S &= \rho_t^S \left\{ J^S \frac{\partial \psi^S}{\partial J^S} + 2 \left(\frac{\partial \psi^S}{\partial I} + I \frac{\partial \psi^S}{\partial II} \right) \mathbf{B}^S - 2 \frac{\partial \psi^S}{\partial II} \mathbf{B}^S \right\} - n_0 N \frac{\rho_t^F}{\rho_0^F} \mathbf{1}, \\ \mathbf{T}^F &= - \left(\rho_t^{F2} \frac{\partial \psi^F}{\partial \rho_t^F} + n_0 N \frac{\rho_t^F}{\rho_0^F} \right) \mathbf{1}. \end{aligned}$$

Obviously the stress tensor in the skeleton possesses already the structure desired in the comparison to Biot's model. The last contribution can be written in terms of the variable ϵ which reflects the coupling with the fluid. Such a contribution does not appear in the stress tensor for the fluid. However the free energy ψ^F may be still dependent on both variables (ρ_t^F, J^S) . Consequently we can choose this dependence in such a way that the symmetry required by the Biot's model remains preserved. It is easy to check that the following choice:

$$(2.58) \quad \psi^F = \psi_{\text{ideal}}^F(\rho_t^F) + n_0 N \frac{e}{\rho_t^F}, \quad N = \text{const}, \quad J^S \equiv 1 + e,$$

indeed yields the desired coupling and it is compatible with the relation (2.49) provided the coupling coefficient N in the fully nonlinear model possesses a specific dependence on volume changes of the skeleton J^S

$$(2.59) \quad N = \frac{N_0}{J^S}, \quad N_0 = \text{const}.$$

This yields indeed the relation (2.58) because in the linear model we do not have to distinguish between N and N_0 in the above relation. Such a dependence is

solely motivated by the requirement of the linear model and cannot be derived from any thermodynamical relations.

Let us make an important observation that the coupling in the partial stress tensor \mathbf{T}^F cannot be corrected any more because the contribution ψ_{ideal}^F is independent of volume changes of the skeleton. Hence the coupling constant must be of the order of the constant N .

The standard linearization procedure leads now to the following linear constitutive relations for partial stresses:

$$(2.60) \quad \begin{aligned} \mathbf{T}^S &= \mathbf{T}_0^S + \lambda^S e \mathbf{1} + 2G^S \mathbf{e}^S + n_0 N \epsilon \mathbf{1}, \\ \mathbf{T}^F &= -p^F \mathbf{1}, \quad p^F := p_0^F - (R\epsilon + n_0 N e), \end{aligned}$$

where \mathbf{T}_0^S, p_0^F are initial values of the stress in the skeleton: $\mathbf{F}^S = \mathbf{1}$, or equivalently $\mathbf{e}^S = \mathbf{0}$, as well as $\epsilon = 0$, and of the partial pressure in the fluid, respectively. The material parameters λ^S, G^S, R, N may depend solely on the initial porosity n_0 .

In order to obtain Biot's stress relations (1.2) we have to identify $\lambda^S = A$, $G^S = G$, $Q = n_0 N$ and to assume that initial stresses are zero (stresses in Biot's relations are the so-called excess stresses):

In contrast to customary **macroscopic** thermodynamical models of multi-component systems, the deformation measures \mathbf{e}^S and ϵ appearing in the Biot's model do not have the meaning of macroscopic quantities. They represent some macroscopic averages of real **microscopic** deformations which are not clearly specified for the Biot's model. One can solely presume their interpretation from Gedankenexperiments provided by BIOT and WILLIS [2] and quoted later on in many works on this subject. Some rather vague remarks²⁾ about the effect of microscopic interactions on such an interpretation do not influence the construction of the Biot's model.

Let us mention that the physical interpretation of Biot's displacement fields \mathbf{u} for the skeleton and \mathbf{U} for the fluid which define the variables \mathbf{e}^S and ϵ do not clear the interpretation of the latter either. The reason is that they were introduced for small deformations of the skeleton, again without any clear distinction between real microscopic deformations and macroscopic deformations. Simultaneously, a motion of the fluid component has never been considered in a manner similar to that which is customary in the fluid mechanics. Such a

²⁾e.g.: "... It should be pointed out that this expression (i.e. defining ϵ by the divergence of the fluid displacement \mathbf{U} ; K.W.) is not the actual strain in the fluid but simply the divergence of the fluid-displacement field which itself is derived from the average volume flow through the pores..." or "... the dry specimen may not exhibit the same properties as the saturated one. As an example of this we may cite the case where the elastic properties result from surface forces of a capillary nature at the interface of the fluid and the solid" ([2], p. 595).

quantity as the displacement of the fluid \mathbf{U} can be solely introduced locally in time by integrating the ordinary differential equation $\frac{\partial \mathbf{x}}{\partial t} = \mathbf{v}^F(\mathbf{x}, t)$ with an initial condition $\mathbf{x}(t = t_0) = \mathbf{x}_0$ and $\mathbf{U} = \mathbf{x} - \mathbf{x}_0$. t_0 should not be too far from the present instant of time t . This is an unnecessary complication of the model. In addition, the microstructural variable ζ in the Biot's model is sometimes defined in a different way than this presented above. It may be introduced by the following definition (e.g. [15])

$$(2.61) \quad \zeta := n_0 \operatorname{div}(\mathbf{u} - \mathbf{U}),$$

where \mathbf{u}, \mathbf{U} denote *microscopic average* displacements of the skeleton and the fluid, respectively. Even though a motivation of this formula and an interpretation of \mathbf{u} and \mathbf{U} is missing it seems to be based on the argument that the following relations hold true:

$$(2.62) \quad e \equiv \operatorname{tr} e^S = \operatorname{div} \mathbf{u}, \quad \epsilon = \operatorname{div} \mathbf{U} \implies \zeta = \frac{\zeta}{n_0}.$$

Using these deformation measures, Biot and Willis introduce an alternative description by conjugate dynamical quantities: the bulk stress tensor \mathbf{T} and the pore pressure p for which the constitutive relations (1.2) transform as follows:

$$(2.63) \quad \begin{aligned} \mathbf{T} &:= \mathbf{T}^S + \mathbf{T}^F = (\lambda + \alpha^2 M) e \mathbf{1} + 2G e^S - \alpha M \zeta \mathbf{1}, \\ p &:= \frac{p^F}{n_0} = -\alpha M e + M \zeta, \end{aligned}$$

where³⁾

$$(2.64) \quad \begin{aligned} \lambda &:= A - \frac{Q^2}{R}, \quad G := G^S, \quad A \equiv \lambda^S, \quad Q \equiv n_0 N, \\ M &:= \frac{R}{n_0^2}, \quad \alpha := n_0 \frac{Q + R}{R}. \end{aligned}$$

In the literature on Biot's model it is not always clear which stresses and variables are being used.

³⁾In [15] the following notation is used

$$H := \lambda + \alpha^2 M + 2G, \quad C := \alpha M.$$

It is rather common for the literature on Biot's model that the notation varies from paper to paper.

We have considered in this section the simplest version of a linear higher gradient model but it is almost obvious that neither the contributions of relative velocities to free energies nor the nonlinear contributions of the gradient of porosity can change anything in the structure of a linearized model.

Field equations of the model follow from partial balance equations of mass and momentum. For the linear model considered in this section they have the form

$$(2.65) \quad \begin{aligned} \frac{\partial \rho_t^S}{\partial t} + \rho^S \operatorname{div} \mathbf{v}^S &= 0, & \rho^S \frac{\partial \mathbf{v}^S}{\partial t} &= \operatorname{div} \mathbf{T}^S + \pi (\mathbf{v}^F - \mathbf{v}^S) - n_0 N \operatorname{grad} \zeta, \\ \frac{\partial \rho_t^F}{\partial t} + \rho_0^F \operatorname{div} \mathbf{v}^F &= 0, & \rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} &= \operatorname{div} \mathbf{T}^F - \pi (\mathbf{v}^F - \mathbf{v}^S) + n_0 N \operatorname{grad} \zeta. \end{aligned}$$

These equations differ from the equations used in the Biot's model due to the presence of the source term with the gradient of porosity. However in the linear model in which the coefficient N is **constant** and the relation (2.54) holds true, the contribution of the source can be formally incorporated into the constitutive relations for partial stresses. After the transformation

$$(2.66) \quad \mathbf{T}^S \rightarrow \mathbf{T}^S - n_0 N \zeta \mathbf{1}, \quad \mathbf{T}^F \rightarrow \mathbf{T}^F + n_0 N \zeta \mathbf{1},$$

we obtain new constitutive laws similar to the relations derived by CIESZKO and KUBIK in [18], and repeated in [19], which possess the same symmetry as the original Biot relations and the field equations do not contain sources. Instead of the coupling coefficient $n_0 N$ we have then $2n_0 N$ and, of course, the elastic coefficients λ^S and R must be corrected on $-n_0 N$. We investigate in the next section the influence of these relations on the propagation of sound waves in porelastic materials.

3. Fundamentals of acoustic waves in linear poroelastic media

3.1. Wave front

Let us remind a few elementary properties of the description of wave fronts in continua. Acoustic waves in continua are related to the propagation of a nonmaterial singular surface – the *wave front* σ – on which acceleration fields are discontinuous but the velocity fields remain continuous. If the latter does not hold, we deal either with shock waves or with vortex sheets.

Let us assume that the instantaneous geometry of the front σ is given by the following equation

$$(3.1) \quad f(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \mathcal{B}_t \subset \mathbb{R}^3, \quad t \in \mathcal{T},$$

which is at least of the class \mathcal{C}^2 with respect to \mathbf{x} , and of the class \mathcal{C}^1 with respect to time t . \mathcal{B}_t denotes the current configuration of the skeleton. The smoothness requirements mean that changes of the normal vector to the surface are differentiable, and changes of curvatures are continuous. Simultaneously there exists a smooth speed of propagation of the surface. In order to see these properties we use the identity

$$(3.2) \quad df \equiv d\mathbf{x} \cdot \text{grad } f + dt \frac{\partial f}{\partial t} = 0.$$

As the gradient of f is orthogonal to the surface (f is constant along the surface, i.e. the vector $\text{grad } f$ may possess solely an orthogonal component), we can define a unit normal vector by the relation

$$(3.3) \quad \mathbf{n} := \frac{\text{grad } f}{|\text{grad } f|}.$$

The second gradient of f , i.e. a quantity proportional to $\text{grad } \mathbf{n}$ is related to curvatures. Bearing the relation (3.2) in mind we obtain for the speed of propagation

$$(3.4) \quad v_n := \frac{d\mathbf{x}}{dt} \cdot \mathbf{n} = - \frac{\frac{\partial f}{\partial t}}{|\text{grad } f|}.$$

It is easy to see that the relation (3.1) does not impose any conditions on the tangential component of the velocity of the surface. This means that kinematics of slip motions cannot be described by such a relation. However this is immaterial in the theory of waves.

If the speed of propagation v_n is given then the relation (3.4) is the nonlinear differential equation for the function f

$$(3.5) \quad \frac{\partial f}{\partial t} + v_n |\text{grad } f| = 0.$$

With an appropriate initial condition for the position of the front (i.e. $f(\mathbf{x}, t = 0) = \text{given}$) this equation forms a nonlinear Cauchy problem.

3.2. Kinematic and dynamic compatibility conditions, speeds of propagation

The behaviour of various kinematic quantities on singular surfaces has been studied since 150 years and the modern theory follows the way proposed by Hadamard. An excellent presentation of this subject can be found in the classical book of C. TRUESDELL and R. A. TOUPIN [16] (Chapter IV). We use here a

particular case of these kinematic compatibility conditions following under the assumption of continuity of both motions and velocities.

We have for the skeleton

$$(3.6) \quad \begin{aligned} [[\rho_t^S]] &= 0, \quad [[\mathbf{v}^S]] = 0, \quad [[\mathbf{e}^S]] = 0, \\ [[\dots]] &:= \sigma^+ \lim(\dots) - \sigma^- \lim(\dots), \end{aligned}$$

where limits are evaluated on the positive and negative sides of the front.

Introducing the notation

$$(3.7) \quad R^S := \left[\left[\frac{\partial \rho_t^S}{\partial t} \right] \right], \quad \mathbf{A}^S := \left[\left[\frac{\partial \mathbf{v}^S}{\partial t} \right] \right],$$

we arrive at the following relations for discontinuities of various derivatives appearing in field equations of the linear model.

$$(3.8) \quad \begin{aligned} [[\text{grad } \rho_t^S]] &= -\frac{1}{v_n} R^S \mathbf{n}, \quad [[\text{grad } \mathbf{v}^S]] = -\frac{1}{v_n} \mathbf{A}^S \otimes \mathbf{n}, \\ [[\text{grade}^S]] &= -\frac{1}{v_n} \left[\left[\frac{\partial \mathbf{e}^S}{\partial t} \right] \right] \otimes \mathbf{n}. \end{aligned}$$

It follows from the kinematic relation between \mathbf{v}^S and \mathbf{e}^S (compare (2.10) for such a relation in the nonlinear model)

$$(3.9) \quad \frac{\partial \mathbf{e}^S}{\partial t} = \frac{1}{2} \left(\text{grad } \mathbf{v}^S + (\text{grad } \mathbf{v}^S)^T \right),$$

and this yields

$$(3.10) \quad [[\text{grad } \mathbf{e}^S]] = \frac{1}{2v_n^2} (\mathbf{A}^S \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{A}^S) \otimes \mathbf{n}.$$

Similarly we have on the wave front σ for the fluid component

$$(3.11) \quad [[\rho_t^F]] = 0, \quad [[\mathbf{v}^F]] = 0 \quad \Rightarrow \quad [[\varsigma]] = 0.$$

With the following notation

$$(3.12) \quad R^F := \left[\left[\frac{\partial \rho_t^F}{\partial t} \right] \right], \quad \mathbf{A}^F := \left[\left[\frac{\partial \mathbf{v}^F}{\partial t} \right] \right],$$

we obtain the kinematic compatibility conditions for the fluid component in the form

$$(3.13) \quad \begin{aligned} [[\text{grad } \rho_t^F]] &= -\frac{1}{v_n} R^F \mathbf{n}, \quad [[\text{grad } \mathbf{v}^F]] = -\frac{1}{v_n} \mathbf{A}^F \otimes \mathbf{n} \quad \Rightarrow \\ [[\text{grad } \varsigma]] &= \left(-\frac{1}{\rho_0^F v_n} R^F + \frac{1}{v_n^2} \mathbf{A}^S \cdot \mathbf{n} \right) \mathbf{n}, \quad [[\text{grad } \epsilon]] = \frac{1}{\rho_0^F v_n} R^F \mathbf{n}. \end{aligned}$$

These relations yield the following relations for jumps of partial stress gradients implied by constitutive relations (2.60)

$$(3.14) \quad \begin{aligned} [[\text{grad } \mathbf{T}^S]] &= \lambda^S \frac{1}{v_n^2} (\mathbf{A}^S \cdot \mathbf{n}) \mathbf{1} \otimes \mathbf{n} + G^S \frac{1}{v_n^2} (\mathbf{A}^S \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{A}^S) \otimes \mathbf{n} \\ &\quad + n_0 N \frac{1}{\rho_0^F v_n} R^F \mathbf{1} \otimes \mathbf{n}, \\ [[\text{grad } \mathbf{T}^F]] &= R \frac{1}{\rho_0^F v_n} R^F \mathbf{1} \otimes \mathbf{n} + n_0 N \frac{1}{v_n^2} (\mathbf{A}^S \cdot \mathbf{n}) \mathbf{1} \otimes \mathbf{n}, \end{aligned}$$

where material constants λ^S, G^S, N, R are assumed to be continuous across the front σ .

We use these relations in the balance equations of the linear model (2.65). Evaluation of jumps of these equations on the wave front σ gives rise to the conditions

$$(3.15) \quad R^S = \rho^S \frac{1}{v_n} \mathbf{A}^S \cdot \mathbf{n}, \quad R^F = \rho_0^F \frac{1}{v_n} \mathbf{A}^F \cdot \mathbf{n},$$

$$(3.16) \quad \begin{aligned} v_n^2 \mathbf{A}^S &= \left(\frac{\lambda^S + G^S - n_0 N}{\rho^S} \right) (\mathbf{A}^S \cdot \mathbf{n}) \mathbf{n} + \frac{G^S}{\rho^S} \mathbf{A}^S + \frac{2n_0 N}{\rho^S} (\mathbf{A}^F \cdot \mathbf{n}) \mathbf{n}, \\ v_n^2 \mathbf{A}^F &= \frac{R - n_0 N}{\rho_0^F} (\mathbf{A}^F \cdot \mathbf{n}) \mathbf{n} + \frac{2n_0 N}{\rho_0^F} (\mathbf{A}^S \cdot \mathbf{n}) \mathbf{n}. \end{aligned}$$

Relations (3.15) imply that neither mass density of the skeleton ρ_i^S nor mass density of the fluid ρ_i^F yield their own modes of propagation. Rather their amplitudes R^S, R^F are determined by the normal component of the acceleration discontinuity in the skeleton $\mathbf{A}^S \cdot \mathbf{n}$, and by the normal component of the corresponding acceleration in the fluid $\mathbf{A}^F \cdot \mathbf{n}$, respectively.

The second relation (3.16) shows that the amplitude \mathbf{A}^F possesses solely the component in the direction of propagation \mathbf{n} . It means that this is related to a longitudinal wave.

The amplitude \mathbf{A}^S possesses both the normal component as well as the transversal component. Separating these contributions we can write (3.16) in the the following form:

$$(3.17) \quad \left(\begin{array}{cc} \frac{\lambda^S + 2G^S - n_0 N}{\rho^S} - v_n^2 & \frac{2n_0 N}{\rho^S} \\ \frac{2n_0 N}{\rho_0^F} & \frac{R - n_0 N}{\rho_0^F} - v_n^2 \end{array} \right) \begin{pmatrix} \mathbf{A}^S \cdot \mathbf{n} \\ \mathbf{A}^F \cdot \mathbf{n} \end{pmatrix} = 0,$$

$$(3.18) \quad \left(v_n^2 - \frac{G^S}{\rho^S} \right) \mathbf{A}_\perp^S = 0, \quad \mathbf{A}_\perp^S := \mathbf{A}^S - (\mathbf{A}^S \cdot \mathbf{n}) \mathbf{n}, \quad \text{i.e. } \mathbf{A}_\perp^S \cdot \mathbf{n} = 0.$$

Relation (3.18) means that the discontinuity \mathbf{A}_\perp^S may be different from zero solely on the front σ which propagates with the speed

$$(3.19) \quad v_n^S = \sqrt{\frac{G^S}{\rho^S}}.$$

Certainly, this is the classical relation for the transversal wave in a solid. The corresponding wave is called S-wave in geophysics. Obviously in the model we consider in this section this speed of propagation of the S-wave which is not influenced by the presence of the fluid component.

Relation (3.17) forms the eigenvalue problem typical for problems of wave propagations. It yields the following dispersion relation for the speeds of propagation:

$$(3.20) \quad \left(\frac{\lambda^S + 2G^S - n_0 N}{\rho^S} - v_n^2 \right) \left(\frac{R - n_0 N}{\rho_0^F} - v_n^2 \right) - \frac{4n_0^2 N^2}{\rho^S \rho_0^F} = 0.$$

Eigenvalues v_n^2 determine the speeds of propagation, and the eigenvectors – amplitudes of the corresponding modes of propagation. We obtain two solutions of this problem:

$$(3.21) \quad v_n^2 = \frac{1}{2} \left\{ (c^{S2} + c^{F2}) \pm \sqrt{(c^{S2} - c^{F2})^2 + 16 \frac{n_0^2 N^2}{\rho^S \rho_0^F}} \right\},$$

$$(3.22) \quad c^S := \sqrt{\frac{\lambda^S + 2G^S - n_0 N}{\rho^S}}, \quad c^F := \sqrt{\frac{R - n_0 N}{\rho_0^F}}.$$

The first one – v_n^{P1} – corresponds to the P1 longitudinal wave, while the second one – v_n^{P2} – to the P2 longitudinal wave (Biot's wave).

In the limit case $N = 0$ the speed of the P1-wave is equal to $c^S = \sqrt{\frac{\lambda^S + 2G^S}{\rho^S}}$,

and the speed of the P2-wave is equal to $c^F = \sqrt{\frac{R}{\rho_0^F}}$. These are the relations

which follow from the linearized $\mathcal{C}^{(1)}$ -model without contributions of porosity gradient (comp. [17]).

Let us mention that in spite of constitutive relations identical with those of Biot the relations (3.21), (3.22) are different from the results of Biot's propagation condition due to the influence of the gradient of porosity on the source of momentum.

The above presented analysis of propagation of wave fronts may deviate considerably from the observations of waves in soils. Apart from flaws of the model, the discrepancies may result from the fact that *in situ* measurements are made usually in the range of low frequencies (a few Hz) where the attenuation of waves is relatively small. In addition, one measures the phase velocities rather than speeds of propagation which correspond to phase velocities in the limit of very high (theoretically infinite) frequencies. These may differ by 5–10%.

4. Conclusion

The thermodynamical analysis presented in this work seems to close the issue of thermodynamical admissibility of the classical Biot's model of saturated poroelastic materials. We conclude in general that Biot's field equations violate the second law of thermodynamics. A part of the model – constitutive relations for stresses – may contain a coupling between deformations of components described by the material parameter Q provided we introduce a constitutive dependence on higher gradients. However this correction yields a change in the structure of momentum source appearing in partial momentum balance equations which has been ignored by Biot.

Simultaneously we have shown that the original Biot's model as well as a corrected version with the gradient of porosity lead to the same modes of propagation of acoustic bulk waves. There appear numerical discrepancies in the values of speeds of propagations which may be of the order of accuracy of the *in situ* measurements.

The subject which has not been discussed in any details in this work but seems to worry people working on granular materials is the structure of the speed of propagation of shear waves. Both the Biot's model and the model discussed in this paper yield the speed of shear waves given by the formula (3.19). This is questioned by experimentalists investigating S-waves in soils. This seems to be the main issue requiring a correction of a linear model independently of its thermodynamical admissibility.

References

1. I. TOLSTOY, *Acoustics, elasticity, and thermodynamics of porous media: Twenty-one Papers by M. A. Biot*, Acoustical Society of America, 1991.
2. M. A. BIOT, D. G. WILLIS, *The elastic coefficients of the theory of consolidation*, Jour. Appl. Mech., **24**, 594–601, 1957.

3. K. WILMAŃSKI, *Some questions on material objectivity arising in models of porous materials*, [in:] M. BROCATO [Ed.], *Rational Continua, classical and new*, Springer-Verlag, Italia Srl, Milano, 149–161, 2001.
4. I. EDELMAN, K. WILMAŃSKI, *Asymptotic analysis of a surface waves at vacuum/porous medium and liquid/porous medium interfaces*, *Continuum Mech. Thermodyn.*, **14**: 25–44, 2002.
5. M. A. GOODMAN, S. C. COWIN, *A continuum theory for granular materials*, *Arch. Rat. Mech. Anal.*, **44**, 249–266, 1972.
6. R. M. BOWEN, *Compressible porous media models by use of the theory of mixtures*, *Int. J. Engn. Sci.*, **20**, 697–763 1982.
7. K. WILMAŃSKI, *Thermomechanics of continua*, Springer, Berlin (1998).
8. K. WILMAŃSKI, *Lagrangian Model of two-phase porous material*, *J. Non-Equilib. Thermodyn.*, **20**, 50–77, 1995.
9. K. WILMAŃSKI, *Mass exchange, diffusion and large deformations of poroelastic materials*, [in:] *Modeling and mechanics of granular and porous materials*, G. CAPRIZ, V. N. GHIONNA, P. GIOVINE [Eds.], 213–244, Birkhäuser 2002.
10. K. WILMAŃSKI, *Note on the notion of incompressibility in theories of porous and granular materials*, *ZAMM*, **81**, 37–42, 2001.
11. K. WILMAŃSKI, *Thermodynamics of multicomponent continua*, [in:] *Earthquake thermodynamics and phase transformations in the earth's interior*, J. MAJEWSKI, R. TEISSEYRE [Eds.], 567–655, Academic Press, San Diego 2001.
12. I. MÜLLER, *Thermodynamics*, Pitman 1985.
13. W. G. GRAY, *Elements of a systematic procedure for the derivation of macroscale conservation equations for multiphase flow in porous media*, [in:] *Kinetic and continuum theories of granular and porous media*, K. HUTTER, K. WILMAŃSKI [Eds.], CISM Courses and Lectures No. 400, Springer WienNewYork, 67–130, 1999.
14. T. WILHELM, K. WILMAŃSKI, *On the Onset of Flow Instabilities in granular media due to porosity inhomogeneities*, *Int. Jour. Multiphase Flows*, in print 2002.
15. R. D. STOLL, *Sediment acoustics*, *Lecture Notes in Earth Sciences*, vol. 26, Springer-Verlag, Berlin 1989.
16. C. TRUESDELL, R. D. TOUPIN, *The Classical field theories*, [in:] *Handbuch der Physik*, S. FLÜGGE [Ed.], Springer-Verlag, Berlin, 226–793, 1960.
17. K. WILMAŃSKI, *Waves in porous and granular materials*, [in:] *Kinetic and continuum theories of granular and porous media*, K. HUTTER, K. WILMAŃSKI [Eds.], CISM Courses and Lectures No. 400, Springer WienNewYork, 131–186, 1999.

18. M. CIESZKO, J. KUBIK, *Constitutive description of fluid-saturated porous media with immiscible elastic constituents*, [in:] *Problems of environmental and damage mechanics* Proc. of the XXXI Polish Solid Mechanics Conference SolMec'96, Mierki near Olsztyn, September 9-14, 1996, W. KOSIŃSKI, R. DE BOER and D. GROSS [Eds.], IPPT PAN, Warszawa, 39-57 1997.
19. M. CIESZKO, J. KUBIK, M. KACZMAREK, (in Polish) 2000, *Fundamentals of dynamics of saturated porous media*, IPPT PAN, Warszawa, 2000.

Received May 10, 2002; revised version November 14, 2002.
