Extremum principles for nonpotential and initial-value problems

J. J. TELEGA

Institute of Fundamental Technological Research,
Polish Academy of Sciences,
Świętokrzyska 21, 00-049 Warsaw, Poland
e-mail: jtelega@ippt.gov.pl

Dedicated to Professor Piotr Perzyna
on the occasion of his 70th birthday

THE AIM of this paper is to derive extremum and saddle-point principles for a class of nonpotential and initial-value problems. The procedure used is based on an extension of the procedure primarily used by BREZIS and EKELAND [7, 8] to classical parabolic equations. In essence, this approach exploits some fundamental notions of convex analysis.

1. Introduction

CONSIDER AN OPERATOR equation, not necessarily linear,

$$(1.1) \qquad N(u) = f.$$

One can distinguish, grosso modo, three possible approaches to variational formulation of (1.1):

(i) The weak formulation, being a rather general form of the virtual work principle

$$(1.2) \qquad < N(u), v > = < f, v > \qquad \forall v \in V.$$

Here $< \cdot, \cdot >$ stands for the duality pairing on $V^* \times V$. The space V is a properly chosen function space, like a Lebesgue or Sobolev space, whilst V^* stands for its dual.

(ii) Stronger is the stationary principle

$$(1.3) \qquad J(u) \to \text{stationary} \quad \text{over} \quad V.$$

Then necessarily

$$\delta J = < N(u) - f, \delta u > = 0 \quad \Rightarrow \quad N(u) = f.$$

(iii) The strongest are extremum principles (minimum or maximum principles and min-max principles or saddle-point principles).
For instance,

$$(1.4) \qquad\qquad J(u) \longrightarrow \min \quad \text{over} \quad V$$

where

$$(1.5) \qquad\qquad J = J_1 + I_C.$$

Here C is a set of constraints and I_C denotes its indicator function [16, 32, 33]:

$$(1.6) \qquad\qquad I_C(v) = \begin{cases} 0 & \text{if} \quad v \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

Min-max principle is formulated as follows:

$$(1.7) \qquad\qquad \min_{u \in V} \max_{p \in Y} L(u, p) = L(\overline{u}, \overline{p}).$$

Here Y is another function space. The point $(\overline{u}, \overline{p})$ may be a saddle-point, see the next section.

It is commonly believed that no general approach allowing for the derivation of extremum principles in the case of nonpotential and initial-value problems is available. For the available results the reader is referred to [5, 17, 19, 20, 34, 35, 37, 38, 40, 41, 42, 48] and the references therein. On the other hand in a series of papers [9–13], extremum and saddle-points principle have been derived in an ad hoc manner for a class of nonpotential and initial-value problems of solid mechanics. In fact, these new principles can be derived by using the general approach developed in the present paper. At the root of our method to the formulation of minimum and saddle-points principles lie the pioneering papers by BREZIS and EKELAND [7, 8]. These authors formulated extremum principles for parabolic heat equation. The approach used exploits fundamental notions of convex analysis. Similar approach was used by NAYROLES [27] and RIOS [29, 31]. Afterwards these papers seem to have been forgotten. Fortunately, AUCHMUTY [3, 4] recalled the ideas due to BREZIS and EKELAND [7, 8] and developed the general framework enabling to derive extremum principles for nonpotential operator equations and parabolic-type problems. Hyperbolic-type (second order in time) equations have not been considered.

The aim of the present paper is threefold. First, inspired by solid mechanics, the approach used by Auchmuty [3, 4] is extended to more general differential inclusions.

Second, having at our disposal the general framework allowing for the formulation of extremum principles of operator and parabolic-type equations, we derive extremum and saddle-points principles for nonlinear, nonconservative and nonpotential elasticity, stationary and nonstationary quasi-linear heat equation.Comments on non-associated plasticity and GAO'S papers [23, 24] are also provided.

Third, a general extremum and saddle-points principles are derived for hyperbolic abstract differential inclusion. This general formulations enables one to derive extremum principles for the Lagrange equations in two important cases: (i) the system studied is subject to conservative forces yet the initial-value problem is to be solved. Usually the boundary value problems are investigated provided that variational principles are used; (ii) the system is subject to nonconservative forces. Extremum principles for the dynamic linear elasticity are also derived.

The approach develop allows for further generalization. For instance, one can develop extremum principle for a coupled system of abstract parabolic-hyperbolic differential inclusion. Then one can derive extremum principles for the nonstationary equations of thermoelasticity. Such coupled problems will be investigated in a separate paper [47]. Thermopiezoelectricity requires still a more general framework: a coupled hyperbolic-parabolic-elliptic differential inclusion, cf. [47].

To facilitate the reading of the paper, in Sec. 2 we introduce the indispensable notions of convex analysis. The approach developed in the present paper or its variants and modifications can likewise be used to the formulation of extremum principles in contact and structural mechanics. We have in mind contact problems with frictions, and beams, plates and shells subject to nonconservating loadings or this type of structures in the dynamic case.

2. Elements of convex analysis

For details the reader is referred to [16, 32, 33]. Let V be a function space and

$$f : V \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$$

a functional. For instance, in the case of finite-dimensional problems $V = \mathbb{R}^m$. In the case of continuous systems V is a suitably chosen function space like the Lebesgue space L^p or Sobolev space $W^{m,p}$.

The conjugate of f is defined by

$$(2.1) \qquad f^*(v^*) = \sup\{< v^*, v > -f(v) \mid v \in V\}, v^* \in V^*.$$

Here V^* denotes the dual space of V, see [16, 32, 33, 49]. For instance, if A is a linear self-adjoint operator and

$$(2.2) \qquad \exists c > 0 \ \forall v \in V, < Av, v > \geq c \parallel v \parallel_V^2$$

then A is invertible. If

$$(2.3) \qquad f(v) = \frac{1}{2} < Av, v >, v \in V$$

then

$$(2.4) \qquad f^*(v^*) = \frac{1}{2} < A^{-1}v^*, v^* >, \quad v^* \in V^*.$$

Let us introduce the *subdifferential*. An element $u^* \in V^*$ is said to be a *subgradient* of f at a point $u \in V$ if

$$(2.5) \qquad f(v) - f(u) \geq < u^*, v - u > \quad \forall v \in V.$$

We denote

$$\partial f(u) = \{\text{the set of all subgradients of } f \text{ at } u\}.$$

The multivalued mapping

$$\partial f : v \longrightarrow \partial f(v)$$

is called the subdifferential of f. The standard example is the subdifferential of the function

$$f(x) = \mid x \mid, \quad x \in \mathbb{R}.$$

It can easily be shown that

$$\partial f(x) = \begin{cases} -1 & \text{if} \quad x < 0, \\ [-1, 1] & \text{if} \quad x = 0, \\ +1 & \text{if} \quad x > 0. \end{cases}$$

The notions of convex and concave functions are elementary and their definitions are well - known, cf. [16, 32, 33].

A convex function f is said to be *proper* if $f \not\equiv +\infty$ and

$$f(v) > -\infty \qquad \forall v \in V.$$

The definition (2.1) of f^* implies that if f is proper then one has

$$(2.6) \qquad f(v) + f^*(v^*) \geq < v^*, v >$$

for each $v \in V$ and each $v^* \in V^*$. Just this inequality will play an essential role in the derivation of minimum principles. In (2.6) the equality holds if and only if $v^* \in \partial f(v)$, or equivalently if and only if $v \in \partial f^*(v^*)$.

Let us pass now to concave functions, since the above notions are typical for convex functions.

The *conjugate* of a concave function g is defined by

$$(2.7) \qquad g^*(v^*) = \inf\{< v^*, v > -g(v) \mid v \in V\}, \quad v^* \in V^*.$$

Caution: in general

$$g^* \neq -(-g^*).$$

For the convex function $f = -g$, one has not $g^*(v^*) = -f^*(v^*)$, but

$$g^*(v^*) = -f^*(-v^*).$$

The set $\partial g(u)$ consists, by definition, of the elements u^* such that

$$g(v) \leq g(u)+ < u^*, v - u > \quad \forall v \in V.$$

We shall call such elements u^* *subgradients* of g at u, and the mapping $u \longrightarrow \partial g(u)$ the subdifferential of g, for simplicity, even though terms like "supergradients" and "superdifferential" might be more appropriate.
One has

$$\partial g(u) = -\partial(-g)(u).$$

If g is proper, i. e. if $(-g)$ is proper, one has

$$(2.8) \qquad g(v) + g^*(v^*) \leq < v^*, v >, \forall v \in V, \forall v^* \in V^*,$$

with equality holding if and only if $v^* \in \partial g(v)$.

Let $\Lambda : V \to Y$ be a linear operator, let g be a function on Y, and let f be a function on V. The functions $g\Lambda$ and Λf defined by

$$(2.9) \qquad (g\Lambda)(v) = g(\Lambda v),$$

$$(2.10) \qquad (\Lambda f)(y) = \inf\{f(v) \mid v \in V, \ \Lambda v = y\}$$

are called the *inverse image* of the function g and the *image* of the function f under the mapping Λ, respectively [25,32,33].

Let us pass to dual functions. The following theorem was proved in IOFFE and TIHOMIROV [25].

THEOREM 1. *Let* $\Lambda : V \to Y$ *be a continuous linear operator. If g is a function on V, and if f is a function on Y then*

$$(\Lambda g)^* = g^* \Lambda^*, \quad (f\Lambda)^* \leq \Lambda^* f^*.$$

If f is a convex function continuous at a point of the set $Im\Lambda$, *then*

$$(f\Lambda)^* = \Lambda^* f^*.$$

Moreover, for every $v^* \in dom(f\Lambda)^*$, *there exists a vector* $y^* \in Y^*$ *such that*

$$v^* = \Lambda^* y^*, \quad (f\Lambda)^*(v^*) = f^*(y^*). \qquad \square$$

The operator Λ^* is defined by

(2.11) $\langle \Lambda^* y^*, v \rangle_{V^* \times V} = \langle y^*, \Lambda v \rangle_{Y^* \times Y}$ for all $y^* \in Y^*$ and all $v \in V$.

We recall that $\Lambda^* : Y^* \to V^*$ stands for the operator adjoint to the operator Λ, $Im\Lambda$ is the image of Λ and $dom f$ is the effective domain of the function f; $dom f = \{v \in V | f(v) < +\infty\}$.

We proceed now to saddle-functions. Let C and C_1 be subsets of V and Y, respectively, and let L be a function from $C \times C_1$ to $[-\infty, +\infty]$. We say that L is a *convex - concave* function if $L(u, z)$ is a convex function of $u \in C$ for each $z \in C_1$ and a concave function of $z \in C_1$ for each $u \in C$. Concave - convex functions are defined similarly. Both kinds of functions are called *saddle-functions*.

A point $(\hat{u}, \hat{z})$ is said to be a saddle-point of L on $C \times C_1$ if

(2.12) $L(\hat{u}, z) \leq L(\hat{u}, \hat{z}) \leq L(u, \hat{z})$ $\forall(u, z) \in C \times C_1$.

Particularly we may have $C = V$, $C_1 = Y$. We observe that saddle - functions are naturally appropriate for piezoelectricity [47].

Given any convex - concave function on $V \times Y$, we define

$$\partial_1 L(u, z) = \partial_u L(u, z)$$

to be the set of all subgradients of the convex function $L(\cdot, z)$ at u, i.e. the set of all $u^* \in V^*$ such that

$$L(u', z) - L(u, z) \geq <u^*, u' - u> \quad \forall u' \in V.$$

Similarly, we define

$$\partial_2 L(u, z) = \partial_z L(u, z)$$

to be the set of all subgradients of the concave function $L(u, \cdot)$ at z, i. e. the set of all $z^* \in Y$ such that

$$L(u, z') - L(u, z) \leq < z^*, z' - z > \quad \forall z' \in Y.$$

The elements (u^*, z^*) of the set

$$\partial L(u, z) = \partial_1 L(u, z) \times \partial_2 L(u, z)$$

are then defined to be the *subgradients* of L at (u, z), and the multivalued mapping

$$\partial L : (u, z) \longrightarrow \partial L(u, z)$$

is called the *subdifferential* of L.

The last notion we need is that of conjugate saddle-functions. We define the *lower conjugate* $\underline{L}^*$ of L by

$$\underline{L}^*(u^*, z^*) = \sup_{u \in V} \inf_{z \in Y} \{< u^*, u > + < z^*, z > -L(u, z)\}$$

and the *upper conjugate* $\overline{L}^*$ of L by

$$\overline{L}^*(u^*, z^*) = \inf_{z \in Y} \sup_{u \in V} \{< u^*, u > + < z^*, z > -L(u, z)\}$$

We have $\underline{L}^* \leq \overline{L}^*$, cf. [32,33]. Both these functions are convex-concave. Under rather weak assumptions, specified in Corollary 37.1.2 by ROCKAFELLAR [32], we have

$$(2.13) \qquad \underline{L}^*(u^*, z^*) = \overline{L}^*(u^*, z^*).$$

For the initial-boundary value problems studied in the paper [47] the property (2.13) holds true since the saddle-functionals assume only finite values on appropriately defined spaces.

Define now the functional $G : V \longrightarrow \overline{\mathbb{R}}$ by

$$(2.14) \qquad G(u) = \sup_{z \in Y} \mathcal{L}(u, z)$$

where $\mathcal{L} : V \times Y \longrightarrow \overline{\mathbb{R}}$ is not necessarily a convex-concave functional. A point $(\hat{u}, \hat{z})$ is a *min-max* point for $\mathcal{L}$ on $V \times Y$ provided $\hat{u}$ minimizes G on V and

$$(2.15) \qquad G(\hat{u}) = \mathcal{L}(\hat{u}, \hat{z}) = \sup_{z \in Y} \mathcal{L}(\hat{u}, z)$$

holds. Consequently, when $(\hat{u}, \hat{z})$ is a min-max point of $\mathcal{L}$ on $V \times Y$, then

$$(2.16) \qquad \mathcal{L}(\hat{u}, z) \leq \mathcal{L}(\hat{u}, \hat{z}) = G(\hat{u}) \leq G(u)$$

for all $(u, z) \in V \times Z$. Since $G(u) \geq \mathcal{L}(u, \hat{z})$ for each $u \in V$, a saddle-point will be a min-max point of $\mathcal{L}$. The converse need not hold. Indeed, take the function $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}$ defined by, cf. [4]

$$\mathcal{L}(x, y) = xy - \frac{1}{3}x^2 - \frac{1}{2}y^2.$$

Then $G(x) = \frac{1}{6}x^2$ and $(0,0)$ is a min-max point of $\mathcal{L}$. It is not a saddle-point in the sense of (2.12) since

$$\mathcal{L}(0, y) \leq \mathcal{L}(0,0) \text{ and } \mathcal{L}(x, 0) \leq \mathcal{L}(0,0)$$

for all (x, y).

3. Minimum principle for nonpotential operator equations

In this section we shall derive minimum principles for a class of boundary-value problems of solid mechanics.

To this end we follow the approach primarily used for the parabolic heat equation by BREZIS and EKELAND [7, 8].

3.1. Abstract framework

It seems that the first attempt to exploit the idea due to Brezis and Ekeland [7, 8] to elliptic-type problems should be attributed to Auchmuty [3], cf. also [4]. Now we shall briefly summarize his results and propose also an extended framework, more suitable to problems of continuum mechanics.

Let V be a locally convex topological space and V^* be its dual space with respect to bilinear pairing $\langle ., \rangle_{V^* \times V}$, usually denoted by $\langle ., . \rangle$.

Auchmuty [3, 4] confines his considerations to V being a Banach space.

Suppose $F : V \longrightarrow V^*$ is a continuous map and $\Phi : V \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be a proper functional on V. Consider the problem of solving the *differential inclusion*

$$(3.1) \qquad\qquad F(u) + f \in \partial\Phi(u)$$

with $f \in V^*$.

We recall that when Φ is proper, its conjugate function Φ^* is convex and l.s.c. (lower semi-continuous) in the weak, weak-* or strong topologies on V^* [16].

Define now the functional J on V by

$$(3.2) \qquad J(u) = \Phi(u) + \Phi^*(F(u) + f) - \langle F(u) + f, u \rangle.$$

The minimum principle associated with (3.1) means evaluating

$$(3.3) \qquad \alpha = \inf\{J(u) | \ u \in V\}. \qquad (\mathcal{P})$$

An element $\hat{u} \in V$ is said to be a minimizer of J on V if $J(\hat{u}) = \alpha$.

The following theorem has been formulated in [3].

THEOREM 2. *Assume that* V, Φ, F *and* f *as above and* J *defined by (3.2) is a proper functional. Then* $\alpha \geq 0$ *and* $J(\hat{u}) = 0$ *if and only if* $\hat{u}$ *is a solution of (3.1).*

P r o o f. The definition of J^* yields

$$\Phi(u) + \Phi^*(u^*) \geq \langle u^*, u \rangle \quad \forall u \in V, \ \forall u^* \in V^*.$$

Equality holds if and only if $u^* \in \partial\Phi(u)$, cf. [16].

Setting now $u^* = F(u) + f$ we conclude that $J(u) \geq 0$ for all $u \in V$, and $J(u) = 0$ if and only if (3.1) holds. $\qquad\square$

Relation (3.2) yields

$$J(u) = \Phi(u) - \langle F(u) + f, u \rangle + \sup\{\langle F(u) + f, v \rangle - \Phi(v) \mid v \in V\} = \sup\{L(u, v) \mid v \in V\},$$

where $L : V \times V \to \mathbb{R}$ is defined by

$$(3.4) \qquad L(u, v) = \langle F(u) + f, v - u \rangle + \Phi(u) - \Phi(v).$$

We observe that L is a Lagrangian of type I for problem $(\mathcal{P})$, cf. [3, 21, 44]. The standard dual principle is to find

$$(3.5) \qquad \alpha^* = \sup\{\mathcal{H}(v) \mid v \in V\}, \qquad (\mathcal{P}^*)$$

where $\mathcal{H} :\longrightarrow \mathbb{R}$ is defined by

$$(3.6) \qquad \mathcal{H}(v) = \inf\{L(u, v) \mid u \in V\}.$$

It is worth noting that the Lagrangian L given by Eq. (3.4) is defined on $V \times V$. Thus its two arguments are of "the same" type.

A point $(\hat{u}, \hat{v}) \in V \times V$ is a saddle point of L if, cf. Sec. 2,

$$(3.7) \qquad L(\hat{u}, v) \leq L(\hat{u}, \hat{v}) \leq L(u, \hat{v}) \ \text{ for all } u, v \text{ in } V.$$

It can easily be shown that:

(i) If $(\mathcal{P})$ and $(\mathcal{P}^*)$ are nontrivial, then α and α^* are finite with $\alpha^* \leq \alpha$. If $\alpha^* > 0$, then (3.1) does not possess a solution.

(ii) If there exists a solution $\hat{v}$ in V of $(\mathcal{P}^*)$ with $\mathcal{H}(\hat{v}) = 0$ and a $\hat{u}$ in V such that $(\hat{u}, \hat{v})$ is a saddle point of L, then $\hat{u}$ will be a solution of (3.1).

(iii) If $(\hat{u}, \hat{v})$ is a saddle point of L, then

$$0 = \mathcal{H}(\hat{v}) = L(\hat{u}, \hat{v}).$$

3.2. Generalized framework

Consider now the following differential inclusion

$$(3.8) \qquad F_1(N(u)) + f \in \partial\Phi_1(\Lambda u)$$

where $\Lambda : V \to Y$ is a continuous linear operator, Y is another locally convex topological space, Y^* denotes its dual, $N : V \to V_1$ is continuous operator, not necessarily linear, and $F_1 : V_1 \to V^*$ is a continuous mapping. Once again, V_1 is a locally convex topological space.

In applications the operator Λ may be a gradient, the linear strain tensor $\mathbf{e}(\mathbf{u})$ or $\Lambda(\mathbf{u}) = (\underline{e}(\underline{u}), \nabla\underline{u})$ in the case of the Green strain tensor $E_{ij}(\mathbf{u}) = \frac{1}{2}(e_{ij}(\mathbf{u}) + u_{i,k}u_{k,j})$.

The minimum principle associated with (3.8) means evaluating

$$(3.9) \qquad \beta = \inf\{K(u) \mid u \in V\} \quad (Q)$$

where, cf. Theorem 2,

$$
\begin{aligned}
K(u) &= \Phi_1(\Lambda u) + (\Phi_1\Lambda)^*[F(N(u)) + f] - \langle F(N(u)) + f, u\rangle \\
&= \Phi_1(\Lambda u) + (\Lambda^*\Phi_1^*)[F(N(u)) + f] - \langle F(N(u)) + f, u\rangle \\
(3.10) \qquad &= \Phi_1(\Lambda u) + inf\{\Phi_1^*(q^*) \mid q^* \in Y^*, \Lambda^*q^* = F(N(u)) + f\} \\
&\quad - \langle F(N(u)) + f, u\rangle.
\end{aligned}
$$

Now the conjugate functional Φ_1^* is defined by

$$\Phi_1^*(q^*) = \sup\{< q^*, q >_{Y^* \times Y} -\Phi_1(q) \mid q \in Y\}.$$

REMARK 1. Formally, the setting of Sec. 3.1 is recovered provided that

$$F(u) = F_1(N(u)), \quad \Phi(u) = \Phi_1(\Lambda u).$$

However, in many applications the form (3.9) is more convenient. For instance, as we shall see below, in the case where $\Phi(u)$ is a quadratic functional in order to find Φ^* we have to calculate an inverse operator.

The minimum principle (Q) can be reformulated as follows

$$
\left|
\begin{array}{l}
\text{Find} \\
\inf\{K_1(u, q^*) \mid u \in V, \ q^* \in Y^*, \ \Lambda^*q^* = F(N(u)) + f\}
\end{array}
\right. \qquad (Q_1)
$$

where

$$(3.11) \qquad K_1(u, q^*) = \Phi_1(\Lambda u) + \Phi_1^*(q^*) - \langle F(N(u)) + f, u\rangle.$$

We observe that the last principle involves the adjoint operator Λ^* which usually can easily be calculated.

A theorem similar to Theorem 2 can readily be formulated and proved by the reader.

Let us pass to the formulation of Lagrangian functional, now denoted by L_1:

$$(3.12) \quad L_1(u, q^*; q) = \Phi_1(\Lambda u) - \Phi_1(q) + \langle q^*, q \rangle_{Y^* \times Y} - \langle F(N(u)) + f, u \rangle_{V^* \times V}.$$

We have

$$K_1(u, q^*) = \sup\{L_1(u, q^*; q) \mid q \in Y\}.$$

The dual problem means evaluating

$$\sup\{\mathcal{H}_1(q) \mid q \in Y\},$$

where

$$(3.13) \quad \mathcal{H}_1(q) = \inf\{L_1(u, q^*; q) \mid u \in V, \ q^* \in Y^*, \ \Lambda^* q^* = F(N(u)) + f\}.$$

One can now readily formulate a counterpart of inequalities (3.7) and properties corresponding to (i)-(iii) formulated in Sec. 3.2.

3.3. Applications

The abstract framework presented in Secs. 3.1 and 3.2 can be applied to a wide range of elliptic-type problems of continuum mechanics. Due to limitation of space only selected applications are studied.

3.3.1. Non-self-adjoint linear elliptic equation. In my paper [39], cf. also [15, 35], the following non-self-adjoint equation was considered

$$(3.14) \qquad\qquad Qu = Lu + Ru = f,$$

where $L = L^*, R = -R$. Moreover, it is assumed that

$$(3.15) \qquad\qquad Lu = P^*CPu, \quad R = Q - L$$

with $C^* = C$.

We take two real Hilbert spaces H_1 and H. Let P be a linear operator $P : H \to H_1$, with the domain $D(P)$ of P dense in H. We assume that $\mathcal{N}(P)$, the null space of P, is $\mathcal{N}(P) = \{0\}$, 0 being the zero element in H. Suppose that $C : H_1 \to H_1$ is linear bounded and self-adjoint, and $C^* = C$, $(Cu_1, u_1)_{H_1} > 0$ for all $u_1 \in H_1$, $u_1 \neq 0$. We have $D(CP) = D(P)$. Furthemore we assume that the adjoint $P^* : H_1 \to H$ of P exists with a domain $D(P^*)$ dense in H_1. Then

the operator $L = P^*CP : H \to H$ is a positive self-adjoint operator with $D(L)$ dense in H.

In [39] a pair of dual extremum principles for Eqs. (3.15), (3.16) was derived. In essence, the basic idea consists in considering the pair of operator equations

(3.16)
$$\begin{cases} Lu + Ru = f \\ Q^*v = Lv - Rv = g \end{cases}$$

with $f, g \in H$. Next we set

(3.17)
$$w_1 = \frac{1}{2}(u + v), \quad w_2 = \frac{1}{2}(u - v)$$

(3.18)
$$F = \frac{1}{2}(f + g), \quad G = \frac{1}{2}(f - g).$$

Thus (3.16) becomes

(3.19)
$$\begin{cases} Lw_1 + Rw_2 = F \\ Lw_2 + Rw_1 = G. \end{cases}$$

Such a formulation was used in [39] to derive dual extremum principles for non-associated plasticity.

Let us briefly show that the formulation proposed in [39] is a specific case of our general setting.

Consider now this setting in the particular case of linear, non-self-adjoint equation (3.14). Any equation of the form

(3.20)
$$Qu = f$$

can be written in the form (3.14). It suffices to set

(3.21)
$$L = \frac{1}{2}(Q + Q^*), \quad R = Q - L.$$

We assume that L is positive definite.

We define the functional Φ by

(3.22)
$$\Phi(u) = \frac{1}{2}(Lu, u)$$

where $(\cdot, \cdot)$ denotes the scalar product in the space H. Then Eq. (3.20) can be written in the form

(3.23)
$$f - Ru \in \partial\Phi(u)$$

where $\partial\Phi(u) = \{Lu\}$. The functional J defined by (3.2) becomes

$$(3.24) \qquad J(u) = \Phi(u) + \Phi^*(f - Ru) - (f, u) = \frac{1}{2}(L^{-1}(Qu - f), Qu - f).$$

Since the operator L is positive therefore J is convex. Theorem 2 implies that $J(\hat{u}) = 0$ if and only if $\hat{u}$ solves (3.20).

From (3.24) we conclude that

$$(3.25) \qquad\qquad Q^*L^{-1}(Qu - f) = 0.$$

The Lagrangian is now given by

$$(3.26) \qquad\qquad L(u, v) = \Phi(u) - \Phi(v) + (f - Ru, v - u).$$

This Lagrangian is obviously a convex-concave functional. Its saddle-point $(\hat{u}, \hat{v})$ obeys

$$(3.27) \qquad\qquad G_u L(\hat{u}, \hat{v}) = G_v L(\hat{u}, \hat{v}) = 0,$$

where G_u and G_v denote partial Gâteux derivatives.
Hence

$$(3.28) \qquad\qquad \begin{cases} L\hat{u} + R\hat{v} = f \\ L\hat{v} + R\hat{u} = f. \end{cases}$$

Since L is positive therefore $\hat{u} = \hat{v}$.

The extremum principles derived in [39] are recovered provided that $g = f$.

The conjugate (dual) functional (3.6) now becomes

$$(3.29) \qquad\qquad \mathcal{H}(v) = (v, f) - \Phi(v) + \Phi^*(f - Rv).$$

REMARK 2. The dual extremum principles derived in [15, 39] are based, ab initio, on the study of Eq. $(3.16)_1$ and the adjoint equation $(3.16)_2$. It means that the following system was considered:

$$(3.30) \qquad\qquad \begin{cases} f - Ru \in \partial\Phi(u) \\ g + Rv \in \partial\Phi(v), \end{cases}$$

or equivalently

$$(3.31) \qquad\qquad \{f - Ru, g + Rv\} \in \partial\Psi(u, v)$$

where

$$(3.32) \qquad\qquad \Psi(u, v) = \Phi(u) + \Phi(v).$$

The functional (3.2) becomes

$$J(u, v) = \Psi(u, v) + \Psi^*(f - Ru, g + Rv) - (f - Ru, u) - (g + Rv, v)$$
$$= \Phi(u) + \Phi(v) + \Phi^*(f - Ru) + \Phi^*(g + Rv) - (f, u) - (g, u).$$

3.3.2. Stationary quasi-linear transport equation. We shall now derive a minimum principle for the following transport equation:

$$(3.33) \qquad \begin{cases} -\mathrm{div}[\mathbf{a}(x, u(x))\nabla u(x)] = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases}$$

Here Ω is a bounded domain in $\mathbb{R}^3$ and $\Gamma = \partial\Omega$ its boundary. For the sake of simplicity only the homogeneous boundary condition is considered. Mixed –boundary condition can also be taken into account.

The problem of solutions to (3.33) was studied by ARTOLA and DUVAUT [1]. For $f \in W^{-1,q}(\Omega)$, $p > 2$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, a solution exists in the space $W_0^{1,p}(\Omega)$. In [1] it is assumed that $a_{ij}(x,r) = a_{ji}(x,r)$. No such symmetry condition is required in our setting.

Suppose that

$$(3.34) \qquad a_{ij}(x,r) = a_{ij}^0(x) + a_{ij}^1(x,r), \quad a_{ij}^0 = a_{ji}^0,$$

where a_{ij}^0 is positive definite for almost every $x \in \Omega$. We assume that $a_{ij}^0 \in L^\infty(\Omega)$. The conditions to be specified by $\mathbf{a}^1$ are specified in [1,22]. Taking account of Eq. (3.31), problem (3.30) is written as follows

$$(3.35) \qquad f + \mathrm{div}(\mathbf{a}^1(x,u)\nabla u) \in \partial\Phi(u)$$

where

$$(3.36) \quad \Phi(u) = \frac{1}{2}\langle -\mathrm{div}\mathbf{a}^0(x)\nabla u, u\rangle_{W^{-1,q}(\Omega)\times W_0^{1,p}(\Omega)}$$

$$= \frac{1}{2}\int_\Omega a_{ij}^0(x)u_{,i}u_{,j}\,dx = \frac{1}{2}\langle \mathbf{a}^0(x)\nabla u, \nabla u\rangle.$$

For a mixed-boundary value problem a boundary term in the functional $J(u)$ will appear. Here $u_{,i} = \partial u/\partial x_i$ and the summation convention has been applied.

The functional (3.2) becomes

$$(3.37) \qquad J(u) = \Phi(u) + \Phi^*\big[f + \mathrm{div}(\mathbf{a}^1(x,u)\nabla u)\big] - \langle f + \mathrm{div}(\mathbf{a}^1(x,u)\nabla u), u\rangle,$$

where Φ is defined by Eq. (3.36), the duality pairing is defined on $W^{-1,q}(\Omega)\times W_0^{1,p}(\Omega)$ and

$$(3.38) \quad \Phi^*(v^*) = \sup\{\langle v^*, v\rangle - \Phi(v) \mid v \in W_0^{1,p}(\Omega)\}$$

$$= \frac{1}{2}\langle L^{-1}v^*, v^*\rangle_{W_0^{1,p}(\Omega)\times W^{-1,q}(\Omega)},$$

where $L^{-1} = (-\mathrm{div}\mathbf{a}\nabla)^{-1}$.

Employing the Lagrangian of type (3.4) we increase the number of variables twice but avoid calculating the inverse operator L^{-1}.

3.3.3. Nonconservative finite elasticity. Let $\mathcal{B}$ denote a hyperelastic solid occupying in its undeformed state the closure $\overline{\Omega}$ of a bounded domain $\Omega \subset \mathbb{R}^3$. Consider the following static problem:

$$(3.39) \qquad \begin{cases} \operatorname{div}\mathbf{T} + \mathbf{f}(x, \mathbf{u}, \mathbf{F}) = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma. \end{cases}$$

with $\det \mathbf{F} > 0$.

Here $\mathbf{T}$ denotes the first Piola-Kirchhoff stress tensor and $\mathbf{F} = \nabla \mathbf{u}$ is the deformation gradient [14, 26, 28]. We recall that $\det \mathbf{F} > 0$, where det denotes the determinant. The hyperelastic constitutive equation is given by

$$(3.40) \qquad \mathbf{T} = \frac{\partial W}{\partial \mathbf{F}}.$$

The stored energy function $W(x, \mathbf{F})$ is nonconvex in $\mathbf{F}$, cf. [14, 26] for a detailed discussion.

The problem of finding a deformation $\mathbf{u} \in W_0^{1,p}(\Omega)^3$, with p sufficiently large [14, 26], such that (3.39) and (3.40) are formally satisfied, is equivalent to

$$(3.41) \qquad \begin{cases} \mathbf{N}(\mathbf{u}, \nabla\mathbf{u}) \in \partial\Phi_1(\nabla\mathbf{u}) & \text{with } \det \nabla\mathbf{u} > 0, \\ \mathbf{u} = 0 & \text{on } \Gamma, \end{cases}$$

where

$$(3.42) \qquad \Phi_1(\nabla\mathbf{u}) = \int_\Omega W(x, \nabla\mathbf{u}) dx$$

and $\mathbf{N}(\mathbf{u}, \nabla\mathbf{u}) = f(x, \mathbf{u}, \nabla\mathbf{u})$ stands for Nemytskii operator, cf. [14, 18].

One could use the variational approach developed in Sec. 3.2 with $\Lambda = \nabla$. However, the functional Φ_1^* is always convex and this facts leads to the conclusion that the broad class of loadings is precluded, cf. [6, 21, 44]. Therefore we propose a different, quite general approach, similar to the one used for the transport equation. Let

$$(3.43) \qquad \mathbf{T} = \frac{\partial \phi}{\partial \mathbf{F}} + \mathbf{T}_1$$

where ϕ is a *convex* function, for instance a quadratic one. ϕ may depend on $x \in \Omega$. If $\mathbf{T}$ is given by (3.40) then

$$(3.44) \qquad \mathbf{T}_1 = \frac{\partial W}{\partial \mathbf{F}} - \frac{\partial \phi}{\partial \mathbf{F}}.$$

Then $(3.41)_1$ is rewritten as follows:

$$(3.45) \quad \begin{cases} \operatorname{div}\left(\dfrac{\partial \phi}{\partial \mathbf{F}}\right) + \operatorname{div}\mathbf{T}_1 + \mathbf{f}(x, \mathbf{u}, \nabla \mathbf{u}) = \mathbf{0} \quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma. \end{cases}$$

We set

$$\Phi(\mathbf{u}) = \int_\Omega \phi(x, \nabla \mathbf{u}) dx.$$

Then (3.35) is written as follows:

$$(3.46) \qquad \mathbf{N}(\mathbf{u}, \nabla \mathbf{u}) + \operatorname{div}\mathbf{T}_1 \in \partial\Phi(\mathbf{u}), \quad \det\nabla\mathbf{u} > 0$$

and the functional (3.2) becomes

$$(3.47) \qquad J(\mathbf{u}) = \Phi(\mathbf{u}) + \Phi^*[\mathbf{N}(\mathbf{u}, \nabla \mathbf{u}) + \operatorname{div}\mathbf{T}_1] - \langle \mathbf{N}(\mathbf{u}, \nabla \mathbf{u}) + \operatorname{div}\mathbf{T}_1, \mathbf{u}\rangle.$$

The minimization problem (3.3) now takes the form

$$\inf\{J(\mathbf{u}) \mid \mathbf{u} \in W_0^{1,p}(\Omega)^3, \ \det\nabla\mathbf{u}(x) > 0\}.$$

REMARK 3.

(i) One can formulate the extremum principle of type (3.9).

(ii) In the case of Cauchy elasticity there exists a response function $\hat{\mathbf{T}}(x, \mathbf{F})$ such that [14, 26]

$$(3.48) \qquad \mathbf{T} = \hat{\mathbf{T}}(x, \mathbf{F}), \quad \det\mathbf{F} > 0.$$

Such a law is in general nonpotential. Employing the results of Secs. 3.1, 3.2, one can formulate extremum principles for Cauchy elastic solids subject to non-conservative forces. The study is left to the reader.

(iii) The minimum and min-max principles proposed by CARINI [11] fall within the general framework considered in Sec. 3.1. This author considered the following nonlinear behaviour:

$$\sigma_{ij}(x, t) = D_{ijkl}(x, t)e_{ij}(\mathbf{u}(x, t)) + \psi_{ij}^n(e(x, t)).$$

The inverse relation has the form

$$e_{ij} = B_{ijkl}(x, t)\sigma_{kl}(x, t) + \Phi_{ij}^n(\sigma(x, t)).$$

Here $\mathbf{e}$ denotes the small strain tensor, σ is the stress tensor t, stands for time (quasi-static evolution), and $\mathbf{B} = \mathbf{B}^T, \mathbf{D} = \mathbf{D}^T$; the subscript T denotes the

transposition. The response function Φ_{ij}^n and ψ_{ij}^n are not necessarily derivable from potential functions.

In the case of nonassociated plasticity we have, cf. [39, 40],

$$\dot{\sigma}_{ij} = E_{ijkl}\dot{e}_{kl}, \quad E_{ijkl} \neq E_{klij}.$$

Here $\dot{e}$ denotes the strain rate tensor and

$$E_{ijkl} = D_{ijkl} - cD_{ijkl}^n$$

with

(3.49)
$$\begin{cases} c = 1 & \text{if } f = 0 \text{ end } \dot{f} = 0 \\ c = 0 & \text{if } f < 0 \text{ or } f = 0 \text{ and } \dot{f} < 0. \end{cases}$$

By $f(\sigma, \alpha) \leq 0$ we denote the yield condition where α stands for a set of internal variables.

The formulation of minimum and saddle point principles for both small deformation and finite nonassociated plasticity deserves a separate study.

REMARK 4. GAO [24] claims to have solved the problem of duality for finite elasticity. Such statement can hardly be taken seriously since his formulation of the primal problem does not take into account the most significant constraint like $\det \mathbf{F} > 0$, where $\mathbf{F}$ stands for the gradient of deformation, cf. [14, 26, 28].

Also in [23] the same author claims that my approach to duality, used in [43] is erroneous. Such a statement is false. It is shown in many papers published mostly with my coworkers that one can use the Rockafellar theory of duality to nonconvex problems, cf. [6, 21, 44, 46] and the references therein. It amounts to finding the dual problem to the convexified or bidual to the primal one. Then, however, additional constraints appear. For instance, in the case of von Kármán plates the membrane forces tensor has to be semi-positive. Otherwise a duality gap arises. It seems that a new framework to coping with nonconvex duality has been proposed in [46]. There an approach developed by ROCKAFELLAR and WETS [33] for finite-dimensional problems has been extended to infinite-dimensional setting. In essence, the approach used exploits augmented Lagrangian method. The study of specific cases has shown that other approaches to nonconvex duality always lead to restrictions on applicability. For instance, Auchmuty's approach to nonconvex duality is not as general as the author believes, cf. [21, 44]. The same pertains to Gao's [24] uncritical statements.

4. Parabolic differential inclusions and extremum principles

4.1. General setting

The aim of this section is to provide a general extremum and saddle point principles for the following problem:

$$(4.1) \qquad \begin{cases} \dot{u}(t) + \partial\Phi(t, u(t)) \ni F(t, u(t)) & \text{on } 0 < t < T, \\ u(0) = u_0 \in X. \end{cases}$$

Here $\dot{u} = \dfrac{du}{dt}$ and B is a real Banach space which is dense in X, being also a Banach space.

The procedure which follows extends the original approach due to BREZIS and EKELAND [7, 8] and is more general than the one studied by AUCHMUTY [3].

We make the following assumptions:

(a) For each $t \in [0, T]$, $\Phi(t, \cdot) : B \to \mathbb{R}$ is a proper, convex and weakly lower semicontinuous function,

(b) $F : [0, T] \times B \to B^*$ is a Nemytskii operator.

Let $Y = L^p(0, T; B)$ be the Lebesgue space of measurable functions $u : [0, T] \to B$ endowed with the norm

$$||u||_Y^p = \int_0^T ||u(t)||_B^p dt.$$

The dual space Y^* of Y is $L^q(0, T; B^*)$, cf. [18]. As usual, $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Let V be the space of all functions v in Y with $\dot{v} \in Y^*$. We recall that $\dot{v}$ is the time derivative of v defined in the distributional sense [18]. V is a Banach space under the norm [18]

$$||u||_V = \left(\int_0^T ||u(t)||_B^p dt \right)^{1/p} + \left(\int_0^T ||\dot{u}(t)||_{B^*}^q dt \right)^{1/q}.$$

Consequently, v is a continuous function from $[0, T]$ to X and the initial condition $(4.1)_2$ is meaningful.

Let

$$(4.2) \qquad \mathcal{K} \equiv \{u \in V | u(0) = u_0 \in X\}.$$

$\mathcal{K}$ is obviously a closed manifold of V.

Generalizing the variational functional introduced by Auchmuty we define $J : \mathcal{K} \to \overline{\mathbb{R}}$ by

(4.3) $J(v)$

$$= \int\limits_0^T \left[\Phi(t, v(t)) + \Phi^* \left[(t, F(t, v(t)) - \dot{v}(t)] - \langle F(t, v(t)) - \dot{v}(t), v(t) \rangle_{B^* \times B} \right] dt.$$

Usually in applications X is a Hilbert space, cf. [18]. The minimum principle associated with problem (4.1) means evaluating

(4.4) $$\alpha = \inf\{ J(v) \,|\, v \in \mathcal{K} \} \qquad (P_t).$$

The functional J assumes either finite or positively infinite values provided that, cf. [3],

(i) if $v \in Y$, $v^* \in Y^*$ then $\Phi(\cdot, v(\cdot))$ and $\Phi^*(\cdot, v(\cdot))$ are Lebesgue measurable on $[0, T]$;

(ii) there exist finite constants c_1, c_2, not necessarily positive, such that $\Phi(t, v) \geq c_1$ and $\Phi^*(t, v^*) \geq c_2$ on $[0, T] \times B$ and $[0, T] \times B^*$ respectively;

(iii) v in $\mathcal{K}$ implies that $F(\cdot, v(\cdot))$ is in Y^*.

Now we can formulate a counterpart of Theorem 2.

THEOREM 3. *Let J, V and K be as above and (i)-(iii) hold. Then $\alpha \geq 0$ and $J(u) = 0$ if and only if u is a solution to (4.1).*

P r o o f. The proof is similar to the elliptic-type problem (3.1). The definition of the conjugate functional yields

$$\Phi(t, v(t)) + \Phi^*(t, v^*(t)) \geq \langle v^*(t), v(t) \rangle$$

for all $v^*(t) \in B$, $v^* \in B^*$ and $t \in [0, T]$.

Integrating over $[0, T]$ we get

(4.5) $$\int\limits_0^T \left[\Phi(t, v(t)) + \Phi^*(t, v^*(t)) \right] dt \geq \int\limits_0^T \langle v^*(t), v(t) \rangle \, dt$$

for all $v \in Y$, $v^*(t) \in Y^*$. In (4.5) equality holds if and only if

$$v(t) \in \partial \Phi(t, v(t))$$

for almost everywhere (a.e.) $t \in [0, T]$. Setting $v^*(t) = F(t, v(t)) - \dot{v}(t)$ we obtain

$$(4.6) \quad \int_0^T \{\Phi(t, v(t)) + \Phi^*[F(t, v(t)) - \dot{v}(t)]\} \, dt \geq \int_0^T \langle F(t, v(t)) - \dot{v}(t), v(t) \rangle \, dt.$$

Moreover, in (4.6) equality holds if and only if $v = u$ obeys (4.1) a.e. on $[0, T]$.

If $p = 2$ and X is a Hilbert space with the norm $\| \cdot \|_X$ induced by the scalar product then we additionally have

$$\int_0^T \langle \dot{v}(t), v(t) \rangle \, dt = \frac{1}{2} \|v(t)\|^2 \Big|_0^T = \frac{1}{2} \left[\|v(T)\|_X^2 - \|v(0)\|_X^2 \right].$$

In this specific, practically important case, the functional J takes the form

$$(4.7) \quad J(v) = \int_0^T \left\{ \Phi(t, v(t)) + \Phi^* \left[t, F(t, v(t)) \right. \right.$$

$$\left. - \dot{v}(t) \right] - \langle F(t, v(t)), v(t) \rangle \left. \right\} dt + \frac{1}{2} \left(\|v(T)\|_X^2 - \|u_0\|_X^2 \right).$$

$\square$

REMARK 5. The functional J involves the conjugate function Φ^*. Similarly to Secs. 3.1 and 3.2 one can formulate Lagrangian and saddle-point principles.

4.2. Applications

Some illustrative examples of application of the extremum principle (P_t) in the case where $p = 2$ and X is a suitably chosen Hilbert space were provided by AUCHMUTY [2,3], cf. also RIOS [29–31]. An alternative approach was used by NAYROLES [27]. More precisely, in [2] an autonomous, nonlinear ordinary differential equation with periodic boundary condition was studied. A general setting for linear initial value problems was elaborated in [3].

The general approach presented in Sec. 4.1 offers many possibilities of finding minimum principles for boundary-initial value problems of solid and fluid mechanics or heat transfer. Particularly, one can formulate minimum and saddle-point principles for heat transfer of biomechanics [36, 45]. To mention but a few other possibilities, we think of applications to quasi-static contact problems with friction, adaptive elasticity (biomechanics), nonassociated plasticity and viscoplasticity, linear and nonlinear heat equations and Navier-Stokes equations.

4.2.1. Nonstationary quasi-linear heat equation. We recall that the stationary transport equation has been investigated in Sec. 3.3.2. Consider now the following parabolic equation:

(4.8)
$$\begin{cases} \kappa \dfrac{d\theta}{dt} - \operatorname{div}\left[\mathbf{a}(x,t,\theta)\nabla\theta\right] = f(x,t), & \text{in } \Omega \times (0,T), \\ \theta(x,t) = 0 & \text{on } \Gamma \times (0,T), \\ \theta(x,0) = \theta^0(x), & x \in \Omega. \end{cases}$$

Here $\theta(x,t)$ denotes the temperature.

Now we set
$$X = L^2(\Omega), \quad B = W_0^{1,p}(\Omega), \quad Y = L^2(0,T;W_0^{1,p}(\Omega)).$$

Hence
$$X^* = X = L^2(\Omega), \quad B^* = W^{-1,q}(\Omega), \quad \frac{1}{p} + \frac{1}{q} = 1.$$

We assume that $\theta^0 \in L^2(\Omega)$. The assumptions on the material coefficients are similar to those specified in [1, 22], except that now they hold for all $t \in [0,T]$ and the function $a_{ij}(x,\cdot,\theta)$ is continuous (in the second argument). Consider two cases.

CASE 1. The coefficients a_{ij} have the form similar to (3.34):

(4.9)
$$a_{ij}(x,t,\theta) = a_{ij}^0(x,t) + a_{ij}^1(x,t,\theta)$$

and the matrix $\mathbf{a}^0(x,t)$ is symmetric and positive definite. Then we set, cf. (3.36),

$$F(t,\theta) = f + \operatorname{div}(\mathbf{a}^1\nabla\theta),$$

$$\Phi(t,\theta) = \frac{1}{2}\left\langle -\operatorname{div}(\mathbf{a}^0\nabla\theta), \theta \right\rangle = \frac{1}{2}\left\langle \mathbf{a}^0\nabla\theta, \nabla\theta \right\rangle.$$

Here we use the notation which is normally used in the study of evolution partial differential equations:

$$\theta(t) = \{\theta(x,t)\,|\,x \in \Omega\}.$$

The minimum principle takes now the form

$$|\ \text{Find } \inf\{J(\theta)\,|\,\theta \in K\}.$$

The variational function J has now the form (4.7) with $v = \theta$ and $\dot{v}$, u_0 being replaced by $\kappa\dot{\theta}$, θ^0 respectively.

Moreover we have

$$K = \left\{\theta \in L^2(0,T;W_0^{1,p}(\Omega)\,\middle|\,\dot{\theta} \in L^2(0,T;W^{-1,q}(\Omega), \theta(0) = \theta^0\right\}.$$

We observe that the coefficient κ may be a function of $x \in \Omega$.

Remark 6. One can consider a slightly more general equation than (4.1):

$$\frac{d}{dt}(a(t)u(t)) + \partial\Phi(t, u(t)) \ni F(t, u(t)).$$

Then the last differential inclusion is written as follows

$$a(t)\dot{u}(t) + \partial\Phi(t, u(t)) \ni F_1(t, u(t))$$

where $F_1(t, u(t)) = F(t, u(t)) - \dot{a}(t)u(t)$.

Case 2. A positive definite, symmetric matrix $b_{ij}(x, t)$ is introduced and we set

$$F(t, \theta) = f - \mathrm{div}\,(\mathbf{b}\nabla\theta) + \mathrm{div}\,(\mathbf{a}\nabla\theta),$$

$$\Phi(t, \theta) = \frac{1}{2}\,\langle\mathbf{b}\nabla\theta, \nabla\theta\rangle\,.$$

For instance, we may take $b_{ij}(x, t) = b(x, t)\delta_{ij}$ where (δ_{ij}) is the Kronecker delta.

5. Second-order differential inclusion

5.1. General setting

Consider the problem of solving

$$(5.1) \qquad \begin{cases} \dfrac{d^2u(t)}{dt^2} + \partial\Phi(t, u(t)) \ni F(t, u(t), \dot{u}(t)) \\ u(0) = u^0, \quad \dot{u}(0) = u^1. \end{cases}$$

Here we confine our considerations to the case where B is a real Banach space which is dense in H, a Hilbert space with the scalar product denoted by $(\cdot, \cdot)$.

The space Y is defined by, cf. Sec. 4.1,

$$Y = L^2(0, T; B).$$

Now the space V is chosen as follows

$$(5.2) \qquad V = \left\{v \in Y \mid \dot{v} \in L^2(0, T; H), \ddot{v} \in Y^*\right\}.$$

We recall that $Y^* = L^2(0, T; B^*)$. The set $\mathcal{K}_h$ of admissible fields is defined by

$$(5.3) \qquad \mathcal{K}_h = \left\{v \in V \mid v(0) = u^0 \in B, \dot{v}(0) = u^1 \in H\right\}.$$

The functions Φ, Φ^*, F satisfy assumptions similar to assumptions (i)—(iii) specified in Sec. 4.1.

We introduce the variational functional

$$(5.4) \quad J(v) = \int_0^T \{\Phi(t, u(t)) + \Phi^*[t, F(t, v(t), \dot{v}(t)) - \ddot{v}(t)]$$

$$- \langle F(t, v(t), \dot{v}(t)), v(t) \rangle_{B^* \times B} \} \, dt - \int_0^T \|\dot{v}(t)\|_H^2 dt + (\dot{v}(T), v(T)) - (u^1, u^0).$$

The minimum principle associated with (5.1) means evaluating

$$\alpha = \inf\{J(v) \,|\, v \in \mathcal{K}_h\}.$$

THEOREM 4. *Assume that* J, V, Φ, Φ^*, F *and* $\mathcal{K}_h$ *are as above. Then* $\alpha \geq 0$ *and* $J(\hat{v}) = 0$ *if, and only if* $\hat{v}$ *is a solution to (5.1).*

P r o o f. We have

$$\Phi(t, v(t)) + \Phi^*(t, v^*(t)) \geq \langle v^*(t), v(t) \rangle$$

for all $v(t) \in B$; $v^*(t) \in B^*$ and $t \in [0, T]$. Integrate over $[0, T]$ and take $v^*(t) = F(t, v(t), \dot{v}(t)) - \ddot{v}(t)$, then

$$(5.5) \quad \int_0^T \{\Phi(t, v(t)) + \Phi^*[t, F(t, v(t), \dot{v}(t)) - \ddot{v}(t)]\} \, dt$$

$$\geq \int_0^T \langle F(t, v(t), \dot{v}(t)) - \ddot{v}(t), v(t) \rangle \, dt.$$

Equality holds here if and only if $\hat{v}$ obeys (5.1) a.e. on $(0, T)$. Moreover, we have

$$(5.6) \quad \int_0^T \langle \ddot{v}(t), v(t) \rangle \, dt = (\dot{v}(T), v(T)) - (u^1, u^0) - \int_0^T (\dot{v}(t), \dot{v}(t)) dt.$$

Substituting (5.6) into (5.5) we get (5.4). □

REMARK 7.

(i) Lagrangian formulation is likewise possible.

(ii) The formulation similar to that presented in Sec. 3.2 will be presented elsewhere both for the parabolic and hyperbolic inclusions.

5.2. Applications

The variational formulation presented in Sec. 5.1 offers a possibility of many applications in analytical, solid and structural mechanics. To mention but a few, we think of linear and nonlinear dynamic elasticity, Lagrange equations in the nonconservative case, vibrating structures like beams, plates and shells.

5.2.1. Dynamic linear elasticity. To provide a nontrivial, illustrative example consider the following system of dynamic elasticity:

$$(5.7) \quad \begin{cases} \rho\ddot{\mathbf{u}} - \operatorname{div}\left[\mathbf{C}(x,t)\mathbf{e}(\mathbf{u}(x,t))\right] = \mathbf{f}(x,t) & \text{in } \Omega \times [0,T], \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma \times [0,T], \\ \mathbf{u}(x,0) = \mathbf{u}^0(x), \quad \dot{\mathbf{u}}(x,0) = \mathbf{u}^1(x) & \text{in } \Omega. \end{cases}$$

Here $\mathbf{u}(x,t)$ denotes the displacement vector, $\mathbf{e}(\mathbf{u})$ is the small strain tensor, ρ denotes the density, not necessarily constant, and C_{ijkl} are components of the elasticity tensor satisfying usual symmetry and coercivity requirements [14, 26].

We set

$$(5.8) \quad \Phi(t,\mathbf{u}) = \frac{1}{2}\langle -\operatorname{div}\mathbf{C}(x,t)\mathbf{e}(\mathbf{u}), \mathbf{u}\rangle$$

$$= \frac{1}{2}\int_\Omega C_{ijkl}(x,t)e_{ij}(\mathbf{u}(x,t))e_{kl}(\mathbf{u}(x,t))dx.$$

The functional spaces are

$$B = H_0^1(\Omega)^3,\ B^* = H^{-1}(\Omega)^3, H = L^2(\Omega)^3,$$

$$(5.9) \quad V = \left\{\mathbf{v} \in L^2(0,T; H_0^1(\Omega))^3 \,|\, \dot{\mathbf{v}} \in L^2(0,T; L^2(\Omega))^3 \,,\right.$$

$$\left.\ddot{\mathbf{v}} \in L^2(0,T; H^{-1}(\Omega)^3)\right\}.$$

Then the variational functional (5.4) becomes

$$(5.10) \quad J(\mathbf{u}) = \int_0^T \left\{\Phi(t,\mathbf{u}(t)) + \Phi^*[t,\mathbf{f}(t)\right.$$

$$\left. -\rho\ddot{\mathbf{u}}(t)] - \langle\mathbf{f}(t), \mathbf{u}(t)\rangle_{H^{-1}(\Omega)^3 \times H_0^1(\Omega)^3}\right\} dt$$

$$-\iint_{0\ \Omega}^{T} \rho\dot{u}_i(x,t)\dot{u}_i(x,t)dxdt + \int_\Omega \rho\dot{u}_i(x,T)\dot{u}_i(x,T)dx - \int_\Omega \rho u_i^1(x)u_i^0(x)dx.$$

6. Final remarks

We have developed a general procedure for finding extremum and saddle-point principles applicable to nonpotential and nonconservative problems of mechanics as well as to first- and second-order differential inclusions. Illustrative examples show the flexibility and versatility of the approach used. This approach allows for: (i) finding minimum and saddle point principles for problems usually believed to possess no such principles, (ii) the study of existence of solutions, cf. RIOS [29–31], (iii) the development of new approximation schemes.

In a separate paper extremum principles for dynamic thermoelasticity and piezoelectricity and thermopiezoelectricity will be derived. Our approach offers also a possibility of the derivation of extremum principles for bio-heat equations, cf. [36]. This problem will be studied in [45]. The papers [45,47] will offer a further development of the general variational approach used in the present paper. Particularly, extremum principles for coupled parabolic-hyperbolic differential inclusion will be derived.

Another field of possible applications is the fluid mechanics, thermodynamics and mechanics of porous media.

I would like to stress that it has not been possible to present in a single paper as many comprehensive examples as I would like to. Also, theoretical considerations have been shortened.

Acknowledgement

References

1. M. ARTOLA, G. DUVAUT, *Un resultat d'homogénéisation pour une classe de problèmes de diffusion non linéaires stationaires*, Ann. Fac. Sci. Toulouse, **4**, 1–27, 1982.

2. G. AUCHMUTY, *Variational principles for periodic solutions of autonomous ordinary differential equations*, in: Proc. Conf. Nonlinear Oscillations in Chemistry and Biology, Lecture Notes in Biomathematics, vol. 66, pp. 252–260, Springer Verlag, New York 1986.

3. G. AUCHMUTY, *Variational principles for operator equations and initial value problems*, Nonlinear Anal., Theory, Methods & Appl., **12**, 531–564, 1988.

4. G. AUCHMUTY, *Min–max problems for non-potential operator equations*, Contemporary Math., **209**, 19–28, 1997.

5. F. BAMPI, A. MORRO, *Topics in the inverse problem of the calculus of variations*, Fisica Matematica, Suppl. B.U.M.I., **5**, 93–115, 1986.

6. W. R. Bielski, J. J. Telega, *The complementary energy principle in finite elasticity,* in : Finite Rotations in Structural Mechanics, W. Pietraszkiewicz, [Ed.], Lecture Notes in Engineering, **19**, pp. 62–81, Springer-Verlag, Berlin 1986.

7. H. Brezis, I. Ekeland, *Un principle variationnel associé à certaines équations paraboliques. Le cas independant du temps,* C. R. Acad. Sci. Paris, **282A**, 971–974, 1976.

8. H. Brezis, I. Ekeland, *Un principle variationnel associé à certaines équations paraboliques. Le cas dependant du temps,* C. R. Acad. Sci. Paris, **282A**, 1197–1198, 1976.

9. A. Carini, *Colonnetti's minimum principle extension to generally nonlinear materials,* Int. J. Solids Structures, **33**, 121–144, 1996.

10. A. Carini, *Saddle-point principles for general nonlinear material continua,* J. Appl. Mech., **64**, 1010–1014, 1997.

11. A. Carini, O. De Donato, *A comprehensive energy formulation for general nonlinear material continua,* J. Appl. Mech., **64**, 353–360, 1997.

12. A. Carini, F. Gena, *Some variational formulation for continuum nonlinear dynamics,* J. Mech. Phys. Solids, **46**, 1253–1277, 1998.

13. A. Carini, G. Maier, *Extremum and saddle-point theorems for elastic solids with dissipative displacement discontinuities,* Arch. Mech., **52**, 523-545.

14. P. G. Ciarlet, *Mathematical Elasticity, vol. I: Three-dimensional Elasticity,* North-Holland, Amsterdam 1988.

15. W. D. Collins, *Dual extremum principles for the heat equation,* Proc. Roy. Soc. Edinb., **77A**, 273–293, 1977.

16. I. Ekeland, R. Temam, *Convex Analysis and Variational Problems,* North-Holland, Amsterdam 1976.

17. V. M. Filippov, V. M. Savchin, S. G. Shorohov, *Variational principles for nonpotential operators,* (in Russian), Itogi Nauki i Tekhniki, Sovriemennye Problemy Matematiki, Vol. **40**, pp. 3–176, VINITI, Moskwa 1992.

18. H. Gajewski, K. Gröger, K. Zacharias, *Nichtlineare Operatorengleichungen und Operatordifferetialgleichungen, Akademie-Verlag,* Berlin 1974.

19. A. Gałka, J. J. Telega, *A variational method for finite elasticity in the case of non - potential loadings I. First Piola - Kirchhoff stress tensor,* Bull. Acad. Pol. Sci., Série sci. tech., **30**, 1982, 121 - 128; II. Symmetric stress tensor and some comments, ibid., pp.129-135.

20. A. Gałka, J. J. Telega, *On variational principles and conservation laws in finite non - potential elasticity,* in: Variational Methods in Engineering, C. A. Brebbia [Ed.], pp. (3–13) – (3–22), Springer - Verlag, Berlin 1985.

21. A. Gałka, J. J. Telega, *Duality and the complementary energy principle for a class of non - linear structures. Part I. Five - parameter shell model,* Arch. Mech., **47**, 1995, 677 - 698; Part II. Anomalous dual variational principles for compressed elastic beams, ibid., pp. 699—724.

22. A. Gałka, J. J. Telega, S. Tokarzewski, *Nonlinear transport equation and macroscopic properties of microheterogeneous media,* Arch. Mech., **49**, 293 - 319, 1997.

23. D. Y. Gao, *Dual extremum principles in finite deformation theory with application in post-buckling analysis of nonlinear beam model*, Appl. Mech. Rev., **50**, 564–571, 1997.

24. D. Y. Gao, *Duality, triality and complementary extremum principles in non-convex parametric variational problems with applications*, IMA J. Appl. Math., **61**, 199–235, 1998.

25. A. D. Ioffe, V. M. Tihomirov, *Theory of Extremal Problems*, North-Holland, Amsterdam, 1979.

26. J. E. Marsden, T. J. R. Hughes, *Mathematical Foundations of Elasticity*, Prentice-Hall, Englewood Cliffs, New Jersey, 1983.

27. B. Nayroles, *Un théorème de minimum pour certains systèms dissipatifs. Variante hilbertienne*, Séminaire d'Analyse Convexe, Exposé No 2, Université de Montpellier, 1976; sea also: C.R. Acad. Sci. Paris, **282A**, 1035–1038, 1976.

28. R. W. Ogden, *Non-linear Elastic Deformations*, Ellis Horwood, Chichester, and John Wiley, 1984.

29. H. Rios, *Étude de la question d'existence pour certains problèmes d'evolution par minimisation d'une fouctionnelle convexe*, C.R. Acad. Sci. Paris, Série A, **283**, 83–86, 1976.

30. H. Rios, *Étude de certains problèmes paraboliques: existence et approximations des solutions*, Séminaire d'Analyse Convexe, Exposé No 1, Thèse de specialité, Université de Montpellier, 1978.

31. H. Rios, *Une étude d'existence sur certains problèmes paraboliques*, Annales Fac. Sci. Toulouse, **1**, 235–255, 1979.

32. R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton 1970.

33. R. T. Rockafellar, R. J.-B. Wets, *Variational Analysis*, Springer, Berlin 1998.

34. R. M. Santilli, *Foundations of Theoretical Mechanics. I. The Inverse Problem in Newtonian Mechanics*, Springer-Verlag, New York 1978; II. Birkhoffian Generalization of Hamiltonian Mechanics, ibid.

35. M. J. Sewell, *Maximum and Minimum Principles: A Unified Approach, with Applications*, Cambridge University Press, Cambridge 1987.

36. M. Stańczyk, J. J. Telega, *Thermal problems in biomechanics: from soft tissues to orthopaedics*, Russian J. Biomechanics, **5**, 30–75, 2001.

37. J. J. Telega, *Variational principles for an incompressible, perfectly plastic material with non - associated flow law*, Bull. Acad. Pol. Sci., Série sci. tech., **25**, 357 - 363, 1977.

38. J. J. Telega, *On variational formulations for non-linear, non-potential operators*, J. Inst. Maths Applics, **24**, 175 - 195, 1979.

39. J. J. Telega, *Dual extremum principles in rate boundary value problems of non - associated plasticity*, Int. J. Eng. Sci., **17**, 215 - 226, 1979.

40. J. J. Telega, *Variational principles for rate boundary-value problems in non-associated plasticity*, Zeitschr. Ang. Math. Mech., **60**, 71-82, 1980.

41. J. J. Telega, *Determination of potential form of operators*, (in Polish), Matematyka Stos., **18**, 1982, 107 - 122, in Polish.

42. J. J. TELEGA, *Derivation of variational principles for rigid - plastic solids obeying non - associated flow laws. I. Further development of the nonlinear method of adding the adjoint operator. Prescribed jumps*, Int. J. Eng. Sci., **20**, 913 - 933, 1982; II. Unprescribed jumps, ibid., pp. 935–945.

43. J. J. TELEGA, *On the complementary energy principle in non-linear elasticity. Part I: von Kármán plates and three-dimensional solids*, C.R. Acad. Sci. Paris, Série II, **308**, 1193-1198, 1989.

44. J. J. TELEGA, *On anomalous dual variational principles and application to compressed nonlinear beams*, Vestnik RUDN, Ser. Mat., **2**, 136 - 147, 1995.

45. J. J. TELEGA, M. STAŃCZYK, *Extremum principles for linear and nonlinear bio-heat equations*, in preparation.

46. J. J. TELEGA, A. GAŁKA, W. BIELSKI, *Augmented Lagrangian methods for a class of convex and nonconvex problems*, J. Theoret. Appl. Mech., **39**, 741–768, 2001.

47. J. J. TELEGA, A. GAŁKA, B. GAMBIN, *Extremum principles for dynamic piezoelectricity, thermoelasticity and thermopiezoelectricity, in preparation.*

48. E. TONTI, *Variational formulations for every nonlinear problem*, Int. J. Eng. Sci., **22**, 1343–1371, 1984.

49. K. YOSIDA, *Functional Analysis*, Springer-Verlag, Berlin 1978.