

## Planar frictional motion of highly elastic bodies

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*Dedicated to Professor Piotr Perzyna  
on the occasion of his 70<sup>th</sup> birthday*

THE PROBLEM OF MOTION of a beam-like elastic body along a horizontal plane in terms of friction, large displacements and finite strains will be considered. The equations of motion are derived using GIBBS-APPELL approach. Deformations of the body and sliding velocity distribution are presented.

### 1. Introduction

THE MAJORITY of problems considered in dynamics of flexible and movable objects have been solved by using the finite-dimensional approach (systems with finite number of degrees of freedom) and analytical mechanics methods. On the other hand, the strain and stress analysis of flexible elements is based on infinite-dimensional formalism of elastic and inelastic continua. Many flexible dynamical systems considered in structural mechanics, robotics, biomechanics etc. need application of both these descriptors.

In the present paper a problem of planar motion of a highly deformable elastic body resting on a flat, rough and rigid foundation will be considered. The body in form of a beam treated as a three-dimensional continuum (the dimensions of the body are assumed to be arbitrary, thus assumption of the beam theory are neglected) is loaded by its own weight and then, due to application of control torques, starts to move along the plane. We assume very large displacements (movable object) and finite strains (in-plane self-contact may occur). The problem under consideration corresponds to snake-like motion of biologically inspired manipulators. A simplification of this model in the form of a multilink lumped mass system was considered by CHERNOUSKO[1]. The planar contact of beams resting on a rough surface was discussed also by FISCHER and RAMMERSTORFER [2], NIKITIN [7][8], NIKITIN *et al.* [9], MOGILEVSKY and NIKITIN [6] and further by STUPKIEWICZ and MRÓZ [10].

The paper is organized as follows: we begin with a finite-dimensional description of continua using an analytical mechanics approach based on the GIBBS-

APPELL equations, rather seldom applied in continuum mechanics. This will be given in Sec. 2. Next, in Sec. 3, we pass to the statement and solution of the snake-like motion of the beam. Figures with deformations and sliding velocities illustrate the results of the paper.

## 2. GIBBS-APPELL equations for discretized continua

Consider a deformable body  $B$  which, under action of external body forces with density  $\mathbf{b}$  and prescribed surface traction  $\mathbf{p}_R$  starts to move from its reference configuration  $B_R$ , producing contact stresses  $\mathbf{t}_C$ . Assuming large displacements and finite strains let us denote by  $\mathbf{u}(\mathbf{X}, t)$  the displacement, its gradient by  $\mathbf{H}(\mathbf{X}, t)$ , by  $\mathbf{E}(\mathbf{X}, t)$  the Green strain tensor and by  $\mathbf{S}(\mathbf{X}, t)$  the second Piola-Kirchhoff stress tensor. Here  $\mathbf{X}$  and  $t$  stands for the particle and time instant, respectively. Assume that there is a common global reference system  $\{OX^K\}$ ,  $\{Ox^i\}$   $i, K=1,2,3$  for the material as well as for the spatial coordinates, respectively. Assume furthermore that, because of the complexity of the problem we want to pose and solve, a space discretization is preferred. Therefore let us express the function of motion  $\mathbf{u}(X^K, t)$  approximatively in terms of generalized coordinates  $q_\alpha(t)$ ,  $\alpha = 1, 2, \dots, N$  (being the nodal displacements in the Finite Element Technique or time dependent coefficient in series expansion) as follows [4, 5]:

$$(2.1) \quad \mathbf{u} = [u_K(\mathbf{X}, t)] = \left[ \sum_{\alpha=1}^N N_{K\alpha}(\mathbf{X}) q_\alpha(t) \right] = [N_{K\alpha}] [q_\alpha] = \mathbf{N}\mathbf{q},$$

where  $N_{K\alpha}(\mathbf{X})$  are the shape or basic functions, respectively.

Thus, the necessary kinematical quantities of the displacement gradient  $\mathbf{H}$  and strain tensor  $\mathbf{E}$  are of the form:

$$\mathbf{H} = \nabla \mathbf{u} = [u_{K,L}] = [N_{K\alpha,L}] [q_\alpha] = \nabla \mathbf{N}\mathbf{q}, \quad \nabla \mathbf{N} = [N_{K\alpha,L}],$$

$$\begin{aligned} \mathbf{E} = [E_{KL}] &= \frac{1}{2} [\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T \mathbf{H}] \\ &= \left[ \frac{N_{K\alpha,L} + N_{L\alpha,K}}{2} \right] [q_\alpha] + \frac{1}{2} [N_{M\alpha,K} N_{M\beta,L}] [q_\alpha] [q_\beta]. \end{aligned}$$

To derive the equations of motion for a discretized continuum, the GIBBS-APPELL equations will be used [3]. This approach, seldom used in continuum mechanics, is very well known and useful in analytical mechanics, especially in dynamics of nonholonomic systems. The derivation of equations of motion for discretized continua is very simple. Thus introducing the acceleration functional [3]

$$(2.2) \quad G(t) = \frac{1}{2} \int_{V_R} \rho_R (\ddot{\mathbf{u}})^2 dV_R,$$

the principle of motion reads

$$(2.3) \quad \frac{\partial G}{\partial \ddot{q}_\alpha} = Q_\alpha \quad \alpha = 1, 2, \dots, N,$$

where  $Q_\alpha(\mathbf{q})$  denotes the generalized forces. For a discretized continuum these quantities take the form [12]

$$(2.4) \quad \mathbf{Q} = [Q_\alpha] = \int_{V_R} \rho_R \mathbf{b} \frac{\partial \mathbf{u}}{\partial \mathbf{q}} dV_R + \int_{S_R} \mathbf{p}_R \frac{\partial \mathbf{u}}{\partial \mathbf{q}} dS_R + \int_{\Gamma_C} \mathbf{t}_C \frac{\partial \mathbf{u}}{\partial \mathbf{q}} d\Gamma - \int_{V_R} \mathbf{S} (\mathbf{1} + \mathbf{H}) : \frac{\partial \mathbf{H}}{\partial \mathbf{q}} dV_R.$$

Here is

$V_R$  – the volume domain of the body in  $B_R$

$S_R$  – the boundary surface

$\Gamma_C$  – the contact zone.

The left-hand side of (2.3) gives

$$(2.5) \quad \frac{\partial G}{\partial \ddot{\mathbf{q}}} = \int_{V_R} \rho_R \ddot{\mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{q}} dV_R = \int_{V_R} \rho_R \mathbf{N} \mathbf{N}^T dV_R \ddot{\mathbf{q}} = \mathbf{M} \ddot{\mathbf{q}}$$

where  $\mathbf{M} = \int_{V_R} \rho_R \mathbf{N} \mathbf{N}^T dV_R$  – the positive definite mass matrix. Introducing then

(2.1) into the right-hand side of (2.3) and denoting by

$$(2.6) \quad \mathbf{F}^{\text{ext}}(t) = \int_{V_R} \rho_R \mathbf{b} \frac{\partial \mathbf{u}}{\partial \mathbf{q}} dV_R + \int_{S_R} \mathbf{p}_R \frac{\partial \mathbf{u}}{\partial \mathbf{q}} dS_R = \int_{V_R} \mathbf{N} \mathbf{b} dV_R + \int_{S_R} \mathbf{N} \mathbf{p}_R dS_R$$

– the external force vector,

$$\mathbf{F}_C(t) = \int_{\Gamma_C} \mathbf{t}_C \frac{\partial \mathbf{u}}{\partial \mathbf{q}} d\Gamma \quad \text{– the contact force vector}$$

one obtains a nonlinear equation

$$(2.7) \quad \mathbf{M} \ddot{\mathbf{q}} = \mathbf{F}^{\text{ext}}(t) + \mathbf{F}_C(t) - \int_{V_R} \mathbf{S} (\mathbf{1} + \mathbf{H}) : \nabla \mathbf{N} dV_R.$$

The integral term is nonlinear even if the material is linear elastic. Therefore to omit computational difficulties, the incremental approach will be introduced. Thus considering the sequence of configurations  $B_0 = B_R, B_1, \dots, B_N, B_{N+1}, B_t = B_M$  which correspond to partition of the prescribed loads

$$\mathbf{b}_{N+1} = \mathbf{b}_N + \Delta \mathbf{b}, \quad \mathbf{p}_R^{N+1} = \mathbf{p}_R^N + \Delta \mathbf{p}_R, \quad N = 0, 1, \dots, M,$$

one obtains

$$\mathbf{u}_{N+1} = \mathbf{u}_N + \Delta \mathbf{u}, \quad \Delta \mathbf{u} = \mathbf{N} \Delta \mathbf{q}, \quad \mathbf{H}_{N+1} = \mathbf{H}_N + \Delta \mathbf{H}, \quad \Delta \mathbf{H} = \nabla \mathbf{N} \Delta \mathbf{q},$$

$$(2.8) \quad \begin{aligned} \mathbf{E}^{N+1} &= \mathbf{E}^N + \Delta \mathbf{E}, \quad \Delta \mathbf{E} = \frac{1}{2} (\Delta \mathbf{H} + \Delta \mathbf{H}^T + \mathbf{H}^T \Delta \mathbf{H} + \Delta \mathbf{H}^T \mathbf{H}), \\ \mathbf{S}^{N+1} &= \mathbf{S}^N + \Delta \mathbf{S}, \quad \Delta \mathbf{S} = \mathbf{C} \Delta \mathbf{E}, \\ \mathbf{t}_C^{N+1} &= \mathbf{t}_C^N + \Delta \mathbf{t}_C. \end{aligned}$$

Writing (2.7) for the configuration  $B_{N+1}$  and using formula (2.8) we obtain

$$(2.9) \quad \mathbf{M} \Delta \ddot{\mathbf{q}} = \Delta \mathbf{F}^{\text{ext}} + \Delta \mathbf{F}_C - \int_{V_R} [\Delta \mathbf{S} (\mathbf{1} + \mathbf{H}) + \mathbf{S} \Delta \mathbf{H}] : \nabla \mathbf{N} dV_R.$$

To compute the integral it is worth to note that all the terms after simple calculations lead to a symmetric matrix. This fact results immediately when we decompose the matrices  $\nabla \mathbf{N}$  and  $\mathbf{H}$  into their symmetric and skew-symmetric parts, and when we use the known result that the product of symmetric and skew-symmetric matrices is equal to zero. Thus, considering the respective members by using (2.8) it will be:

$$\begin{aligned} \Delta \mathbf{S} \cdot \mathbf{1} : \nabla \mathbf{N} &= [C_{KLMN}] \left[ \frac{N_{M\alpha, N} + N_{N\alpha, M}}{2} + \frac{N_{R\alpha, M} N_{R\gamma, N} + N_{R\gamma, M} N_{R\alpha, N}}{2} \right] \\ &[\Delta q_\alpha] [N_{K\beta, L}] = [C_{KLMN}] [B_{MN\alpha}^0 + B_{MN\alpha\gamma}^1 q_\gamma] [\Delta q_\alpha] [B_{KL\beta}^0] \\ &= \mathbf{B}^0 \mathbf{C} \mathbf{B}^0 \Delta \mathbf{q} + \mathbf{B}^0 \mathbf{C} (\mathbf{B}^1 \mathbf{q}) \Delta \mathbf{q}. \end{aligned}$$

Here for simplicity the symmetric matrices

$$(2.10) \quad \begin{aligned} \mathbf{B}^0 &= [B_{MN\alpha}^0] = \left[ \frac{N_{M\alpha, N} + N_{N\alpha, M}}{2} \right], \\ \mathbf{B}^1 &= [B_{MN\alpha\gamma}^1] = \left[ \frac{N_{R\alpha, M} N_{R\gamma, N} + N_{R\gamma, M} N_{R\alpha, N}}{2} \right], \end{aligned}$$

have been introduced. It is next

$$\begin{aligned} \Delta \mathbf{S} \cdot \mathbf{H} : \nabla \mathbf{N} &= [C_{KLMN}] [B_{MN\alpha}^0 + B_{MN\alpha\gamma}^1 q_\gamma] [\Delta q_\alpha] [N_{i\delta, K}] [q_\delta] [N_{i\beta, L}] \\ &= \mathbf{B}^0 \mathbf{C} \mathbf{B}^0 (\mathbf{B}^0 \mathbf{q}) \Delta \mathbf{q} + \mathbf{B}^0 \mathbf{C} (\mathbf{B}^1 \mathbf{q}) (\mathbf{B}^0 \mathbf{q}) \Delta \mathbf{q}, \end{aligned}$$

$$\mathbf{S}\Delta\mathbf{H} : \nabla\mathbf{N} = \mathbf{S}\nabla\mathbf{N}\Delta\mathbf{q}\nabla\mathbf{N} = [S_{KL}] [N_{i\alpha,K}] [\Delta q_\alpha] [N_{i\beta,L}] = \mathbf{B}^0\mathbf{S}\mathbf{B}^0\Delta\mathbf{q}.$$

Finally, the integral yields

$$\int_{V_R} [\Delta\mathbf{S} (\mathbf{1} + \mathbf{H}) + \mathbf{S}\Delta\mathbf{H}] : \nabla\mathbf{N}dV_R = \int_{V_R} [\mathbf{B}^0\mathbf{C}\mathbf{B}^0 + \mathbf{B}^0\mathbf{C} (\mathbf{B}^1\mathbf{q}) + \mathbf{B}^0\mathbf{C}\mathbf{B}^0 (\mathbf{B}^0\mathbf{q}) + \mathbf{B}^0\mathbf{C} (\mathbf{B}^1\mathbf{q}) (\mathbf{B}^0\mathbf{q}) + \mathbf{B}^0\mathbf{S}\mathbf{B}^0] dV_R\Delta\mathbf{q}.$$

Denoting the matrices

$$\mathbf{K} = \int_{V_R} \mathbf{B}^0\mathbf{C}\mathbf{B}^0dV_R = [K_{\alpha\beta}] = \int_{V_R} [B_{KL\beta}^0] [C_{KLMN}] [B_{MN\alpha}^0] dV_R,$$

$$\mathbf{K}_{NL}(\mathbf{q}) = \int_{V_R} [\mathbf{B}^0\mathbf{C} (\mathbf{B}^1\mathbf{q}) + \mathbf{B}^0\mathbf{C}\mathbf{B}^0 (\mathbf{B}^0\mathbf{q}) + \mathbf{B}^0\mathbf{C} (\mathbf{B}^1\mathbf{q}) (\mathbf{B}^0\mathbf{q})] dV_R$$

$$= [K_{\alpha\beta}^{NL}]$$

$$(2.11) \quad = \int_{V_R} \left\{ [B_{KL\beta}^0] [C_{KLMN}] [B_{MN\alpha\gamma}^1] [q_\gamma] + [B_{MN\alpha}^0] [C_{KLMN}] [B_{SK\gamma}^0] [q_\gamma] [B_{SL\beta}^0] + [B_{SL\alpha}^0] [C_{KLMN}] [B_{MN\beta\gamma}^1] [B_{SK\delta}^0] [q_\gamma] [q_\delta] \right\} dV_R,$$

$$\mathbf{K}_S(\mathbf{q}) = \int_{V_R} \mathbf{B}^0\mathbf{S}(\mathbf{q})\mathbf{B}^0dV_R = [K_{\alpha\beta}^S] = \int_{V_R} [B_{MK\alpha}^0] [S_{KL}] [B_{ML\beta}^0] dV_R$$

we obtain finally the equation of motion

$$(2.12) \quad \mathbf{M}\Delta\ddot{\mathbf{q}} + (\mathbf{K} + \mathbf{K}_{NL} + \mathbf{K}_S) \Delta\mathbf{q} = \Delta\mathbf{F}^{\text{ext}} + \Delta\mathbf{F}_C,$$

which coincides with the form obtained by other methods, e.g. by means of the virtual work principle or by using the Lagrangian equations of second kind. In our opinion, the method based on the GIBBS-APPELL equations seems to be especially preferred in case of very complex systems, e.g. systems composed of rods, beams, plates or shells and three-dimensional blocks. The contact term  $\Delta\mathbf{F}_C$  (following (2.6)<sub>2</sub>) requires separate and careful considerations. This will be the subject of the next section.

### 3. Snake-like motion of a beam

The following problem is under consideration: a highly elastic beam-like body with density  $\rho_R$  in the reference configuration, rests on a fixed, rough and rigid plane (Fig.1). Due to in-plane torques  $M_1(t)$  and  $M_2(t)$  applied to the beam, the body starts to move. Since the dead weight presses the beam onto the rough surface, planar friction occurs. As mentioned in the introduction, a multilink lumped mass system in snake-like motion was considered by CHERNOUSKO [1]. A continuous highly-elastic description (also in discretized version) in terms of large displacements and finite strains is still open. The aim of this chapter is to fill this gap and to show, that by using suitable torques such kind of motion of beam is possible. To realize the motion of the beam in a demanded direction (e.g. a longitudinal or lateral motion of the centre of mass), a control problem must be stated. This will be the subject of a separate paper.

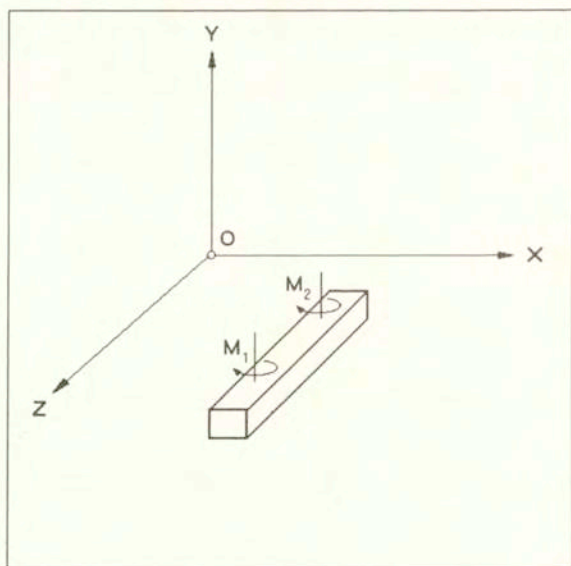


FIG. 1. Scheme of the beam

Consider the contact stress vector  $\mathbf{t}_C$ . Decomposing it into the sum of normal and tangential components we obtain

$$(3.1) \quad \mathbf{t}_C = t_n \mathbf{n} + \mathbf{t}_T = t_n \mathbf{n} - \mu t_n \mathbf{e}_T, \quad \mathbf{e}_T = \frac{\dot{\mathbf{u}}_T}{|\dot{\mathbf{u}}_T|},$$

where

- $\mathbf{n}$  - the outward unit vector, normal to the plane,
- $\mu$  - the friction coefficient,
- $\dot{\mathbf{u}}_T$  - the sliding velocity.

The contact force (2.6)<sub>2</sub> takes the form

$$\mathbf{F}_C = \int_{\Gamma_C} t_n \frac{\partial u_n}{\partial \mathbf{q}} d\Gamma + \int_{\Gamma_C} \mathbf{t}_T \frac{\partial \mathbf{u}_T}{\partial \mathbf{q}} d\Gamma.$$

Here the notations are used

$$\begin{aligned} u_n &= \mathbf{u}\mathbf{n} = \mathbf{N}\mathbf{q}\mathbf{n} = N_{i\alpha}q_\alpha n_i, \\ (3.2) \quad \mathbf{u}_T &= \mathbf{u} - u_n\mathbf{n} = (\mathbf{1} - \mathbf{n} \otimes \mathbf{n}) \mathbf{u} = [\delta_{ij} - n_i n_j] \mathbf{u}, \\ &= [P_{ijN_{i\alpha}q_\alpha}] = \mathbf{P}\mathbf{u} = \mathbf{P}\mathbf{N}\mathbf{q}, \end{aligned}$$

where  $\mathbf{P} = \mathbf{1} - \mathbf{n} \otimes \mathbf{n}$  is the second order projection tensor which maps any vector  $\mathbf{u}$  onto its projection on the plane. Thus it will be further

$$(3.3) \quad \mathbf{F}_C = \int_{\Gamma_C} t_n \mathbf{N}\mathbf{n} d\Gamma + \int_{\Gamma_C} \mathbf{P}\mathbf{N}\mathbf{t}_T d\Gamma.$$

The corresponding increments take the form

$$\begin{aligned} (3.4) \quad \Delta \mathbf{F}_C &= \int_{\Gamma_C} \Delta t_n \mathbf{N}\mathbf{n} d\Gamma + \int_{\Gamma_C} \mathbf{P}\mathbf{N}\Delta \mathbf{t}_T d\Gamma, \\ \Delta \mathbf{t}_T &= \Delta (-\mu t_n \mathbf{e}_T) = \Delta (-\mu t_n \Phi_\epsilon) = -\mu \Delta t_n \Phi_\epsilon - \mu t_n \Delta \Phi_\epsilon, \end{aligned}$$

where for computational reasons, the regularization of the friction law by using the function

$$(3.5) \quad \Phi_\epsilon(\dot{\mathbf{u}}_T) = \begin{cases} \frac{\dot{\mathbf{u}}_T}{\epsilon} = \frac{1}{\epsilon} \mathbf{P}\mathbf{N}\dot{\mathbf{q}} & |\dot{\mathbf{u}}_T| \leq \epsilon \\ \mathbf{e}_T & |\dot{\mathbf{u}}_T| > \epsilon \end{cases}$$

has been introduced. Symbol  $\epsilon$  denotes a positive, sufficiently small number. Hence

$$(3.6) \quad \Delta \Phi_\epsilon = \begin{cases} \frac{\Delta \dot{\mathbf{u}}_T}{\epsilon} = \frac{1}{\epsilon} \mathbf{P}\mathbf{N}\Delta \dot{\mathbf{q}} & |\dot{\mathbf{u}}_T| \leq \epsilon \\ \Delta \mathbf{e}_T & |\dot{\mathbf{u}}_T| > \epsilon \end{cases}$$

where from (3.1)<sub>2</sub> it follows that

$$\begin{aligned}\Delta \mathbf{e}_T &= \Delta \left( \frac{\dot{\mathbf{u}}_T}{|\dot{\mathbf{u}}_T|} \right) = \frac{\Delta \dot{\mathbf{u}}_T |\dot{\mathbf{u}}_T| - \dot{\mathbf{u}}_T |\Delta \dot{\mathbf{u}}_T|}{\dot{\mathbf{u}}_T^2} \\ &= \frac{\Delta \dot{\mathbf{u}}_T}{|\dot{\mathbf{u}}_T|} - \frac{\dot{\mathbf{u}}_T}{|\dot{\mathbf{u}}_T|^2} \frac{\Delta \dot{\mathbf{u}}_T \Delta \dot{\mathbf{u}}_T}{|\dot{\mathbf{u}}_T|} = \frac{\Delta \dot{\mathbf{u}}_T}{|\dot{\mathbf{u}}_T|} - \dot{\mathbf{u}}_T \frac{\mathbf{e}_T \Delta \dot{\mathbf{u}}_T}{|\dot{\mathbf{u}}_T|^2} = (\mathbf{1} - \mathbf{e}_T \otimes \mathbf{e}_T) \frac{\Delta \dot{\mathbf{u}}_T}{|\dot{\mathbf{u}}_T|}.\end{aligned}$$

The first term of  $\Delta \mathbf{e}_T$  demonstrates the local change of the sliding velocity, whereas the second one shows the change of its direction.

Taking (2.1) into account it is generally

$$\begin{aligned}\dot{\mathbf{u}}_T &= \mathbf{PN}\dot{\mathbf{q}}, & \Delta \dot{\mathbf{u}}_T &= \mathbf{PN}\Delta \dot{\mathbf{q}}, \\ |\dot{\mathbf{u}}_T| &= \sqrt{\mathbf{PN}\dot{\mathbf{q}} \cdot \mathbf{PN}\dot{\mathbf{q}}}, & \mathbf{e}_T &= \frac{\mathbf{PN}\dot{\mathbf{q}}}{\sqrt{\mathbf{PN}\dot{\mathbf{q}} \cdot \mathbf{PN}\dot{\mathbf{q}}}},\end{aligned}$$

and then

$$(3.7) \quad \Delta \mathbf{e}_T = \frac{1}{|\dot{\mathbf{u}}_T|} [\mathbf{1} - \mathbf{e}_T \otimes \mathbf{e}_T] \mathbf{PN}\Delta \dot{\mathbf{q}} = \frac{1}{|\dot{\mathbf{u}}_T|} [\delta_{ik} - e_{Ti}e_{Tk}] P_{kj} N_{j\alpha} \Delta \dot{q}_\alpha.$$

Thus finally the regularization function  $\Phi_\varepsilon$  and its increment  $\Delta \Phi_\varepsilon$  yields

$$(3.8) \quad \Phi_\varepsilon(\dot{\mathbf{q}}) = \mathbf{PN}\dot{\mathbf{q}}\phi_\varepsilon \quad \text{where} \quad \phi_\varepsilon = \begin{cases} \frac{1}{\varepsilon} & |\dot{\mathbf{u}}_T| \leq \varepsilon, \\ \frac{1}{|\dot{\mathbf{u}}_T|} & |\dot{\mathbf{u}}_T| > \varepsilon, \end{cases}$$

$$(3.9) \quad \Delta \Phi_\varepsilon(\Delta \dot{\mathbf{q}}) = \mathbf{PN}\Delta \dot{\mathbf{q}}\psi_\varepsilon \quad \text{where} \quad \psi_\varepsilon = \begin{cases} \frac{1}{\varepsilon} & |\dot{\mathbf{u}}_T| \leq \varepsilon, \\ \frac{1}{|\dot{\mathbf{u}}_T|} - \frac{\dot{\mathbf{u}}_T \mathbf{e}_T}{|\dot{\mathbf{u}}_T|^2} & |\dot{\mathbf{u}}_T| > \varepsilon. \end{cases}$$

Substituting these expressions into (3.4) we obtain finally the increment of the contact force

$$(3.10) \quad \Delta \mathbf{F}_C = \int_{\Gamma_C} \left[ \Delta t_n \mathbf{Nn} - \mathbf{PN} (\mu \Delta t_n \phi_\varepsilon \mathbf{PN}\dot{\mathbf{q}} + \mu t_n \psi_\varepsilon \mathbf{PN}\Delta \dot{\mathbf{q}}) \right] d\Gamma.$$



Introducing now the matrices

$$\begin{aligned}
 \Delta \mathbf{F}_n &= \int_{\Gamma_C} \Delta t_n \mathbf{N} n d\Gamma = \int_{\Gamma_C} \Delta t_n N_{i\beta} n_i d\Gamma, \\
 (3.11) \quad \Delta \mathbf{F}_{nT} &= \mu \phi_\varepsilon \int_{\Gamma_C} \Delta t_n \mathbf{P} \mathbf{N} (\mathbf{P} \mathbf{N} \dot{\mathbf{q}}) d\Gamma = \mu \phi_\varepsilon \int_{\Gamma_C} \Delta t_n P_{ik} N_{k\alpha} P_{ij} N_{j\beta} \dot{q}_\beta d\Gamma, \\
 \mathbf{K}_T &= \mu \psi_\varepsilon \int_{\Gamma_C} t_n \mathbf{P} \mathbf{N} (\mathbf{P} \mathbf{N}) d\Gamma \Delta \dot{\mathbf{q}} = \mu \psi_\varepsilon \int_{\Gamma_C} t_n P_{ik} P_{ij} N_{k\alpha} N_{j\beta} d\Gamma \Delta \dot{q}_\beta,
 \end{aligned}$$

and substituting it into (3.10) and next into (2.12), one obtains the final form of the incremental equation

$$(3.12) \quad \mathbf{M} \Delta \ddot{\mathbf{q}} + \mathbf{K}_T \Delta \dot{\mathbf{q}} + (\mathbf{K} + \mathbf{K}_{NL} + \mathbf{K}_S) \Delta \mathbf{q} = \Delta \mathbf{F}^{\text{ext}} + \Delta \mathbf{F}_n + \Delta \mathbf{F}_{nT}$$

It is worth making the following remarks:

- In the case of planar motion along a flat and fixed foundation it is  $\mathbf{P} \dot{\mathbf{u}} = \dot{\mathbf{u}}$  for any  $\dot{\mathbf{u}}$ . Hence the increment (3.7) yields
 
$$\Delta \mathbf{e}_T = \frac{1}{|\dot{\mathbf{u}}_T|} [\mathbf{1} - \mathbf{e}_T \otimes \mathbf{e}_T] \Delta \dot{\mathbf{u}} = \frac{1}{|\dot{\mathbf{u}}_T|} [\delta_{ij} - e_{Ti} e_{Tj}] N_{j\alpha} \Delta \dot{q}_\alpha, \quad i, j = 1, 2.$$
- In our case the contact area  $\Gamma_C$  is known
- The normal contact stresses  $t_n$  resulting from the own weight of the beam are known and are equal to  $\rho \mathbf{g}$  – where  $\mathbf{g}$  means the gravity acceleration. Thus  $\Delta t_n$  is also known and is distributed uniformly
- In a general case the increments  $\Delta t_n = \Delta(\mathbf{T} \mathbf{n}) = \Delta(\mathbf{T} : \mathbf{n} \otimes \mathbf{n})$  (where  $\mathbf{T}$  is the Cauchy stress tensor) are unknown and depend on  $\Delta \mathbf{q}$ . Thus instead of the column matrices  $\Delta \mathbf{F}_n$  and  $\Delta \mathbf{F}_{nT}$  suitable matrices  $\mathbf{K}_{CN}$  and  $\mathbf{K}_{nT}$  appears, being additional terms of the stiffness matrix. The matrix  $\mathbf{K}_{nT}$  is then non-symmetric (see [11]). These circumstances should be taken into account when the beam has dimensions of a slab or of a block.
- Because the only loads applied to the beam consist of normal pressure and in-plane torques producing lateral in-plane bending with planar friction (sliding along the axis of the beam is excluded) – a loss of contact does not appear. It means that the body moves in terms of bilateral contact with the plane.

To illustrate the behaviour of the considered system, numerical examples for a linear elastic isotropic material with the following data are presented (numerical calculations were performed by M. Sci. D. Kedzior of the Institute of Structural Mechanics, CUT):  $\rho_R = 1.7 \cdot 10^3 \text{ kg/m}^3$ , Young's modulus  $E = 1.7 \cdot 10^8 \text{ N/m}^2$ ,  $\nu = 0.48$ , coefficient of friction  $\mu = 1.0$ , cross-section of the beam  $b \times h = 0.005 \times 0.005 \text{ m}$  and its length  $l = 0.1 \text{ m}$ . The torques  $M_1$  and  $M_2$  are modelled as couple

forces with a time program given in Fig. 2. The respective deformation pictures at considered time instants are shown in Fig. 3. Figure 4 shows lateral displacement of the centre of mass of the beam whereas Fig. 5 illustrates its longitudinal motion (in  $z$  direction). The diagram  $u_z$  versus  $u_x$  presented in Fig. 6 illustrates the trajectory of the mass centre. The second example concerns the case  $M_1 = M_2$  with the time program given in Fig. 7. The corresponding deformation pictures with visible snake-like character of motion are given in Figs. 8 and 9.

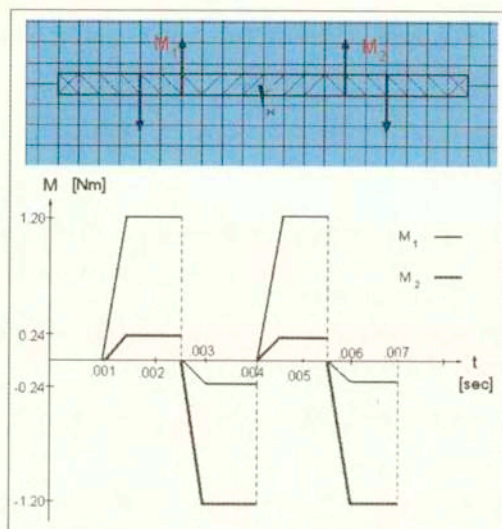


FIG. 2. Time Program I of the applied torques.

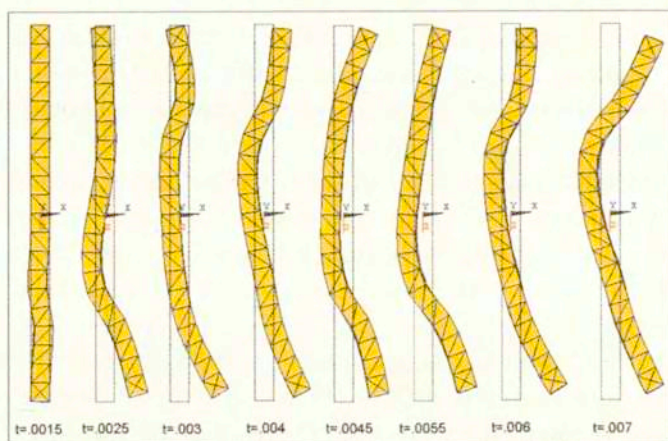


FIG. 3. Deformations caused by Program I.

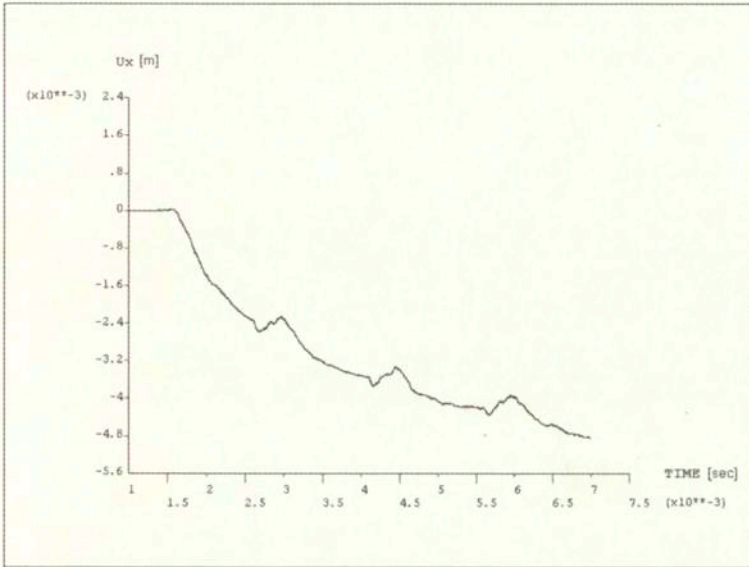


FIG. 4. Lateral displacement of the centre of mass.

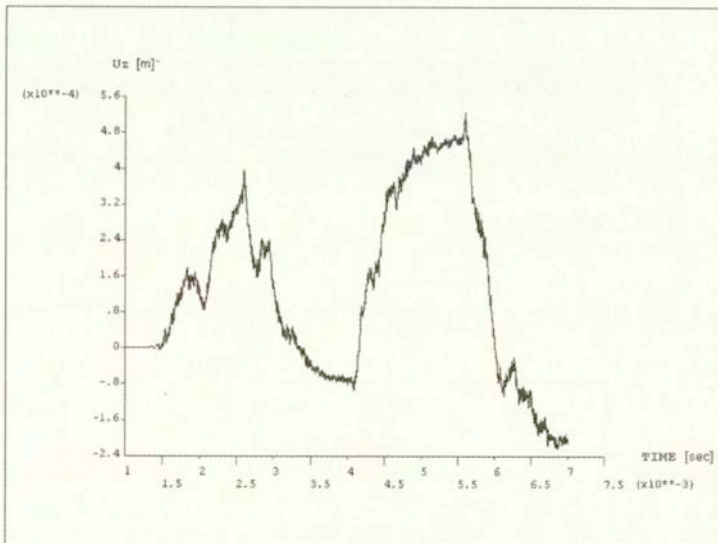


FIG. 5. Longitudinal displacement of the centre of mass.

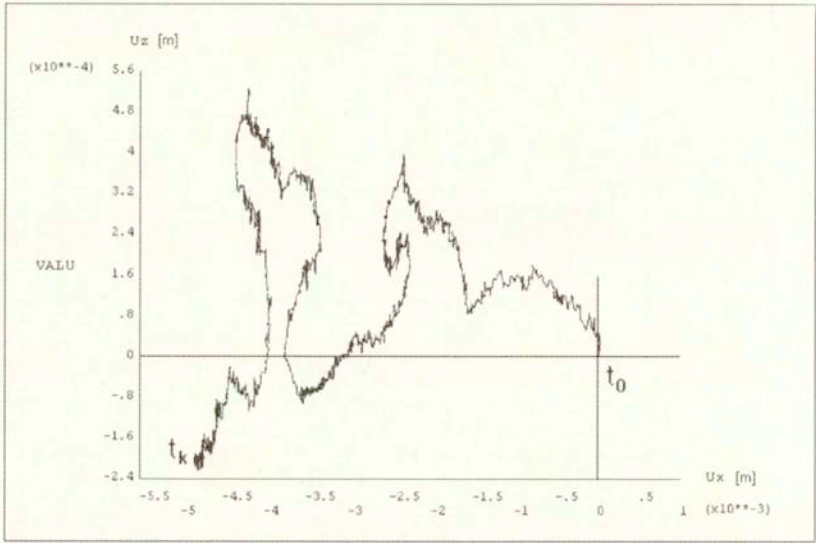


FIG. 6. Trajectory of the centre of mass.

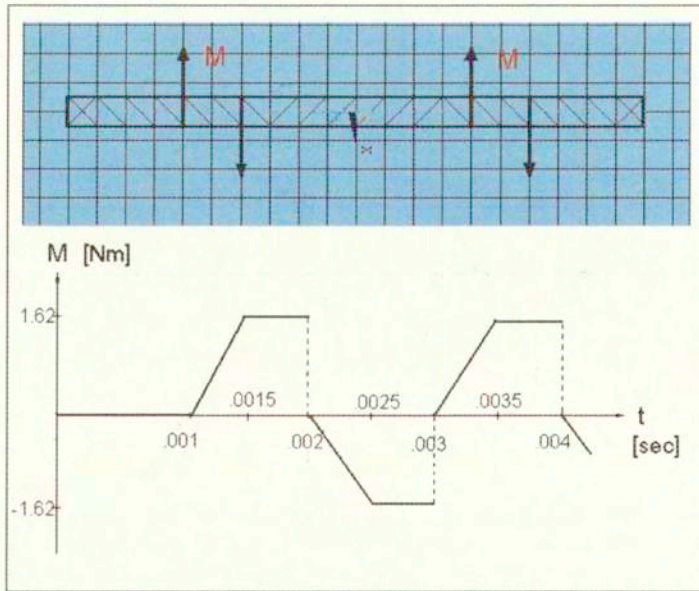


FIG. 7. Time Program II of the torques.

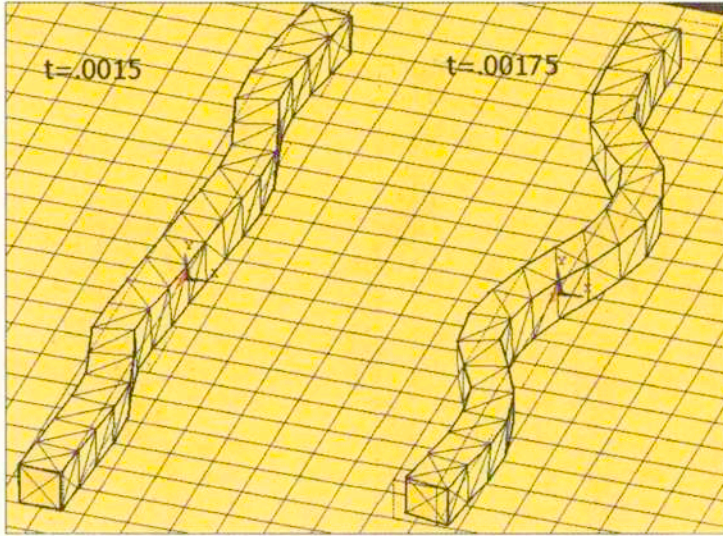


FIG. 8. Deformations caused by Program II.

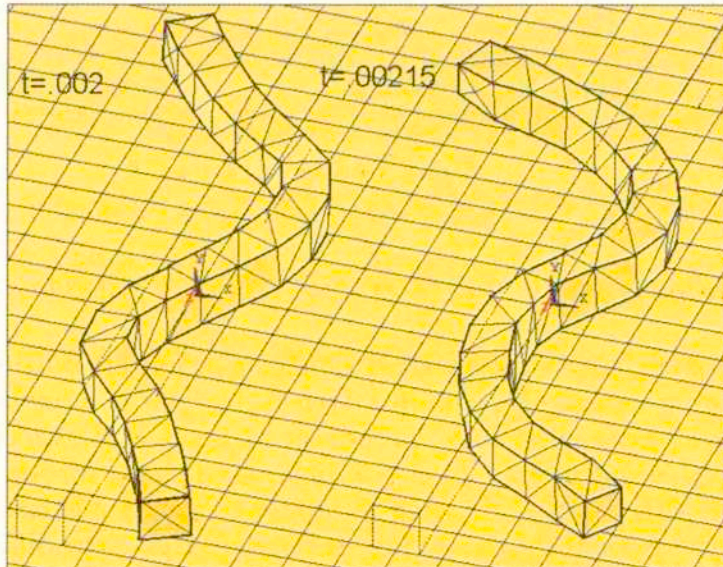


FIG. 9. Deformations caused by Program II.

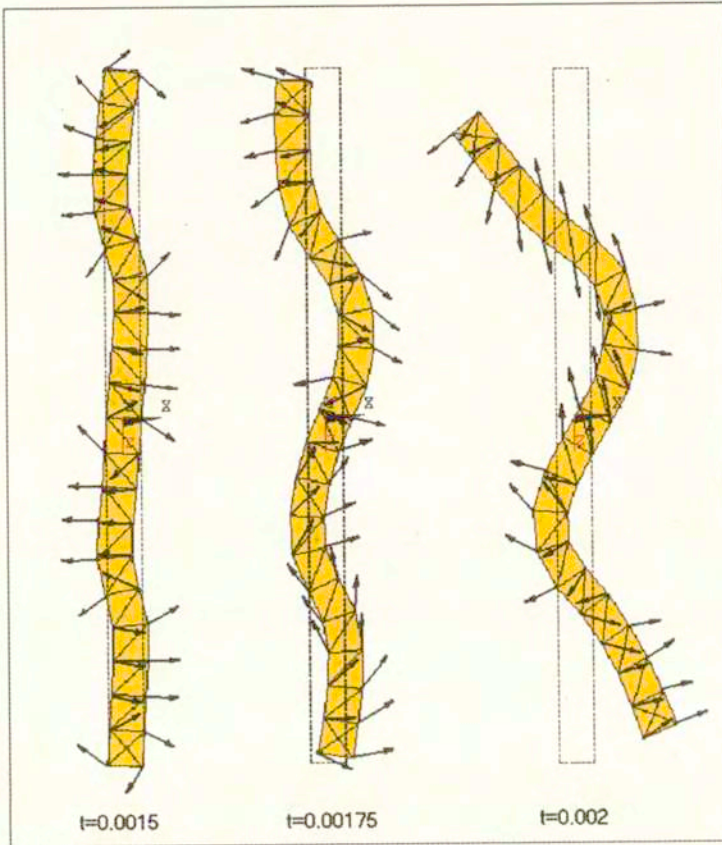


FIG. 10. Sliding velocities for Program II.

#### 4. Conclusions

Equations of motion of a discretized continuum by using the GIBBS-APPELL formalism of analytical mechanics has been derived. As one can see, this proposal is one of the simplest ways to construct equations of motion of discretized continua. A dynamic problem of motion in terms of planar friction was considered. It has been shown that, owing to friction, a high elastic beam can move along a plane under the action of programmable torques perpendicular to the plane of motion. The Figs. 4-6 show the movement of the centre of mass. This kind of motion of the body undergoing in-plane bending and being all the time in contact with the plane, enables us to find such a programme of the control torques, that the centre of mass of the body will be moved in a given direction. Thus an optimal control problem can be formulated.

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