Arch. Mech., **54**, 5-6, pp. 635–646, Warszawa 2002

# Some effects of phase transitions on heat propagation

K. SAXTON[(1)] and R. SAXTON[(2)]

[(1)] *Department of Mathematics and Computer Science*
*Loyola University*
*New Orleans, LA 70118, USA*

[(2)] *Department of Mathematics*
*University of New Orleans*
*New Orleans, LA 70148, USA*

*Dedicated to Professor Piotr Perzyna*
*on the occasion of his* $70^{th}$ *birthday*

We consider phase transitions in solids due to heat propagating through crystalline materials at low temperatures. These are considered in a steady state context where, at the transition temperature, the specific heat becomes singular and the heat conductivity has a maximum. Several consequences are found for the heat capacity having finite or infinite jump discontinuities.

## 1. Introduction

In this introduction, we outline the main features of the low-temperature heat propagation model found in [6, 12] and [13]. An important aspect of the model is a hyperbolic to parabolic change of type which occurs at the temperature of maximum heat conductivity, $\vartheta_\lambda$. This is associated with the appearance (as temperature decreases) of an internal variable acting as an order parameter. In the steady state limit this change of type disappears, however a second order phase transition takes place, with the specific heat of the material undergoing an abrupt change at $\vartheta_\lambda$. The model is based strongly on experimental results of [4, 5, 9] and [10] in the context of thermodynamics with internal state variables, [13]. The experimental results give evidence of second sound - hyperbolic or wavelike thermal effects - in crystals of sodium fluoride and bismuth, as has been observed previously in liquid helium, [1]. Significantly, these features are only present at temperatures below which the materials reach their peak thermal conductivities (approximately 18.5 K and 4.5 K for NaF and Bi, respectively). Wavelike thermal phenomena are not seen at higher temperatures, where only diffusive heat propagation is found. We represent this as follows.

Heat conduction in rigid solids is governed by balance of energy

$$\varepsilon_\lambda + q_x = 0, \quad \varepsilon'(\vartheta) = c_v(\vartheta) \tag{1.1}$$

where $\epsilon$ is the internal energy per unit volume, $c_v$ is the specific heat at constant volume and $q$ is the heat flux. In the region $U_+$ in $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ where $\vartheta > \vartheta_\lambda$, heat propagation is understood using (1.1) together with the constitutive relation

$$q = k(\vartheta)\vartheta_x\,, \tag{1.2}$$

while in the region $U_-$ where $\vartheta < \vartheta_\lambda$, heat propagation is instead controlled by (1.1) together with the system

$$p_t = g_1(\vartheta)\vartheta_x + g_2(\vartheta)p, \tag{1.3}$$

$$q = -\alpha(\vartheta)p. \tag{1.4}$$

Here $\vartheta$ and $\vartheta_\lambda$ are absolute temperatures, $p$ is the internal state variable, $\varepsilon$, $\alpha$, $k$, $g_1$ and $g_2$ are constitutive functions with $\alpha$, $k$, $g_1 : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$, $g_2 : \mathbb{R}^+ \to \mathbb{R}^-$.

Let $\psi = \varepsilon - \eta\vartheta$ where $\psi$ and $\eta$ represent the Helmholtz free energy and the entropy density per unit volume. We will assume that the Helmholtz free energy function takes the form

$$\psi(\vartheta, p) = \psi_1(\vartheta) + \frac{1}{2}\psi_{20}\vartheta p^2. \tag{1.5}$$

The constitutive functions $\alpha, g_1, g_2, k$ are then subject to restrictions arising from the second law of thermodynamics,

$$\eta_t + (q/\vartheta)_x \geq 0. \tag{1.6}$$

These are found to be

$$\alpha(\vartheta) = \psi_{20}\vartheta^2 g_1(\vartheta), \quad g_2(\vartheta) \leq 0. \tag{1.7}$$

The function $g_1$ can be determined from the speed of second sound pulses while $g_2$ can be found by steady state conductivity measurements, [12], [13]. Since the presence of low temperature wavelike features is a relatively short time effect, [8], we are interested in pursuing steady state features further here.

The steady state condition is defined by $p_t = 0$ in (1.3), which gives

$$g_1(\vartheta)\vartheta_x = -g_2(\vartheta)p, \tag{1.8}$$

and the steady-state conductivity coefficient, $\mathcal{K}(\vartheta)$, is given by

$$q(\vartheta) = \Big(\frac{\psi_{20}\vartheta^2 g_1^2(\vartheta)}{g_2(\vartheta)}\Big)\vartheta_x = -\mathcal{K}(\vartheta)\vartheta_x. \tag{1.9}$$

We will make the following hypotheses concerning the constitutive functions, $g_1, g_2 \in C(\mathbb{R}^+)$,

$$-\infty < \lim_{\vartheta\to\vartheta-} \frac{g_1^2(\vartheta)}{g_2(\vartheta)} < 0 \quad \text{and} \quad g_i(\vartheta) = 0, \;\; i = 1,2, \;\; \vartheta \geq \vartheta_\lambda, \tag{1.10}$$

to allow the possibility of a conductivity peak for $\mathcal{K}(\vartheta)$ as $\vartheta \to \vartheta-$. Examples of $g_1$ and $g_2$ include

$$g_1(\vartheta) = a\vartheta^{\frac{1}{2}}(\vartheta - \vartheta)_+^{r_1}, \;\; a > 0, \tag{1.11}$$

and

$$g_2(\vartheta) = -b(1 + \epsilon\vartheta^4)(\vartheta_\lambda - \vartheta)_+^{r_2}, \;\; b > 0, \;\; |\epsilon| << 1, \tag{1.12}$$

with $2r_1 = r_2 > 0$, where $z_+^r \equiv z^r H(z)$ and $H(z)$ denotes the step function.

Let us define the steady state conductivity, $K(\vartheta)$, for all temperatures, as

$$K(\vartheta) = \begin{cases} \mathcal{K}(\vartheta) & \text{if} \quad \vartheta < \vartheta_\lambda, \\ k(\vartheta) & \text{if} \quad \vartheta \geq \vartheta_\lambda. \end{cases} \tag{1.13}$$

Experimental observations reveal $K(\vartheta)$ to be continuous, in particular across $\vartheta = \vartheta_\lambda$, from which it follows $k(\vartheta_\lambda+) = \mathcal{K}(\vartheta_\lambda-)$. We will assume that $K'(\vartheta_\lambda) = 0$. Reasonable choices in (1.11), (1.12) are $r_1 = 1/5$ for NaF ($\vartheta_\lambda = 18.5K$), $r_1 = 1/4$ for Bi ($\vartheta_\lambda = 4K$), and $\epsilon = 3/\vartheta_\lambda^4$, with a useful empirical example of $K(\vartheta)$ given by

$$K_{\text{emp}}(\vartheta) = \frac{\psi_{20}a^2}{b}\frac{\vartheta^3}{1 + 3\vartheta^4/\vartheta_\lambda^4}. \tag{1.14}$$

The aim of this paper is to investigate some properties of phase transitions connecting the states $U_-$ and $U_+$ under the steady state condition $p_t = 0$. Let $\Gamma$ denote a curve $x = \varphi(t)$ in $\mathbb{R} \times \mathbb{R}^+$ separating the regions and consider the equations

$$\varepsilon(\vartheta)_t - (\mathcal{K}(\vartheta)\vartheta_x)_x = 0, \;\; \text{in } U_- \tag{1.15}$$

and

$$\varepsilon(\vartheta)_t - (k(\vartheta)\vartheta_x)_x = 0, \;\; \text{in } U_+. \tag{1.16}$$

Our interest in these equations comes from the jump in the specific heat, $c_v(\vartheta) = \varepsilon'(\vartheta)$, across $\Gamma$. We are unaware of observational indications for a latent heat contribution in the present context, but it is important that we allow the possibility of $c_v$ becoming unbounded, at least locally, in $U_-$. Letting $u$ be a generic function, we denote limits of $u$, as $x \to \varphi(t)$ from below and above $\vartheta_\lambda$, by $u|_{\Gamma-}$ and $u|_{\Gamma+}$ respectively, and write the jump $u|_{\Gamma+} - u|_{\Gamma-}$ across $\Gamma$ as $[u]$. This means that if $[\vartheta] = 0$ then $[\varepsilon] = 0$. We will however have a second order phase transition, $[c_v] \neq 0$, and to examine this we list some simple consequences.

Equation (1.1) implies the jump relation

$$-s[\varepsilon] + [q] = 0, \tag{1.17}$$

where $s = \dot{\varepsilon}(t)$.

If $[\vartheta] = 0$, then $[q] = 0$ and so $[K(\vartheta)\vartheta_x] = 0$. By the continuity of $K(\vartheta)$ then, assuming $k(\vartheta_\lambda) > 0$, $[\vartheta_x] = 0$, and so $[\vartheta_t] = 0$ because $[\vartheta] = 0$ implies $[\vartheta_t] + s[\vartheta_x] = 0$. Therefore

$$[\varepsilon_t] = [c_v]\vartheta_t|_\Gamma. \tag{1.18}$$

Similarly, $[\varepsilon_t] + s[\varepsilon_x] = 0$ because $[\varepsilon] = 0$. Combining this with the jump of Eq. (1.1), $[\varepsilon_t] + [q_x] = 0$, implies $[q_x] = s[\varepsilon_x]$ or

$$[q_x] = s[c_v]\vartheta_x|_\Gamma. \tag{1.19}$$

Since $\vartheta_\lambda$ is the temperature of maximum heat conductivity $(K'(\vartheta_\lambda) = 0)$, (1.19) shows

$$[\vartheta_{xx}] = -\frac{s}{k(\vartheta_\lambda)}[c_v]\vartheta_x|_\Gamma. \tag{1.20}$$

If we more generally allow $[\vartheta] \neq 0$, we have from (1.17)

$$s[\varepsilon(\vartheta)] + [K(\vartheta)\vartheta_x] = 0. \tag{1.21}$$

An appropriate interpretation of (1.21) is important also when $[\vartheta] = [\varepsilon(\vartheta)] = 0$ but $[c_v]$ is undefined in (1.18). In this case $s$ may be defined by computing the ratio of the jumps in (1.21) in terms of limits. For example, (1.21) implies that if one state, say $\vartheta_x|_{\Gamma+}$, is zero, then

$$s = k(\vartheta_\lambda) \lim_{\delta\to 0+} \frac{\vartheta_x(\varphi(t)-\delta, t)}{\varepsilon(\varphi(t)+\delta, t)) - \varepsilon(\vartheta(\varphi(t)-\delta, t))} \tag{1.22}$$

while if either state of $\vartheta_x$ is nonzero then in order for $s$ to be finite, the solution must cross $\vartheta_\lambda$.

We will examine several forms of discontinuity in $c_v$ since it is hard to obtain empirical evidence to determine whether or not specific heat contains a genuine singularity at $\vartheta_\lambda$. Our aim is to derive mathematical consequences of these assumptions.

In Sec. 2, we begin by considering the case of $[c_v]$ being finite, with $c_v$ piecewise constant (a second order phase transition) and $K$ constant, then allow $c_v$ to become infinite within $U_-$. Section 3 deals with nonlinear constitutive laws having infinite, but locally integrable $c_v$ ('lambda' phase transitions), following which we examine the speed of propagation of solutions with compactly supported data about $\vartheta = 0$ and $\vartheta = \vartheta_\lambda$.

## 2. Piecewise constant constitutive terms

In this section we examine the case $K(\vartheta) \equiv 1$. It is convenient to introduce the normalized temperature

$$T = \frac{\vartheta}{\vartheta_\lambda} - 1, \tag{2.1}$$

with equations (1.15) and (1.16) taking the form

$$T_t - \frac{1}{\tilde{c}_v} T_{xx} = 0, \tag{2.2}$$

where

$$\tilde{c}_v(T) = \begin{cases} c_-, & \text{if} \quad T < 0, \\ c_+, & \text{if} \quad T \geq 0, \end{cases} \tag{2.3}$$

and $c_- > c_+ > 0$. We take initial conditions

$$T(x,0) = \begin{cases} T_c, & \text{if} \quad x < 0, \\ T_h, & \text{if} \quad x \geq 0, \end{cases} \tag{2.4}$$

with $T_c \leq 0\,(\vartheta(x,0) \leq \vartheta_\lambda, x < 0)$ and $T_h \geq 0\,(\vartheta(x,0) \geq \vartheta_\lambda, x \geq 0)$, together with the conditions

$$T(\varphi(t),t) = 0, \quad [T_x(\varphi(t),t)] = 0, \quad [T_{xx}(\varphi(t),t)] = -s[\tilde{c}_v] T_x(\varphi(t),t), \tag{2.5}$$

(see (1.20).

### 2.1. Phase transitions

Consider similarity solutions of the form

$$T(x,t) = \begin{cases} f(\eta), & \text{if} \quad (x,t) \in U_-, \\ g(\eta), & \text{if} \quad (x,t) \in U_+, \end{cases} \tag{2.6}$$

where $\eta = \dfrac{x}{\sqrt{t}}$. Since $\dfrac{dT}{dt}\Big|_\Gamma = 0$, clearly

$$\varphi(t) = \gamma\sqrt{t}, \quad s = \frac{\gamma}{2\sqrt{t}} \tag{2.7}$$

for some value of $\gamma$.

We write (2.3), (2.4), (2.5) and (2.6) as

$$f''(\eta) + \frac{1}{2}c_- f'(\eta)\eta = 0, \quad -\infty < \eta < \gamma, \tag{2.8}$$

$$g''(\eta) + \frac{1}{2}c_+ g'(\eta)\eta = 0, \quad \gamma < \eta < \infty, \tag{2.9}$$

$$f(-\infty) = T_c, \quad g(\infty) = T_h, \quad f(\gamma) = g(\gamma) = 0, \quad f'(\gamma) = g'(\gamma), \tag{2.10}$$

$$f''(\gamma) - g''(\gamma) = -\frac{1}{2}\gamma f'(\gamma)(c_- - c_+). \tag{2.11}$$

Solving, we obtain

$$f(\eta) = T_c\Big(1 - \frac{1 + erf(\sqrt{c}\_\eta/2)}{1 + erf(\sqrt{c}\_\gamma/2)}\Big), \quad -\infty < \eta < \gamma, \tag{2.12}$$

and

$$g(\eta) = T_h\Big(1 - \frac{1 - erf(\sqrt{c}_+\eta/2)}{1 - erf(\sqrt{c}_+\gamma/2)}\Big), \quad \gamma < \eta < \infty, \tag{2.13}$$

where $erf(z) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z \exp(-x^2)\,dx$. Consequently, the location of $\Gamma$ is found via $(2.10)_4$,

$$-\sqrt{c}_- \frac{T_c}{1 + erf(\sqrt{c}\_\gamma/2)} e^{-c_-\gamma^2/4} = \sqrt{c}_+ \frac{T_h}{1 - erf(\sqrt{c}_+\gamma/2)} e^{-c_+\gamma^2/4} \tag{2.14}$$

from which, having used (2.7), (2.11) is identically satisfied. We remark on two limiting cases of (2.14).

a) If $T_c \to 0$ $(\vartheta(x,0) \to \vartheta_\lambda, x < 0)$ and $T_h > 0$, then $\gamma \to -\infty$.

b) If $T_h \to 0$ $(\vartheta(x,0) \to \vartheta_\lambda, x \geq 0)$ and $T_c < 0$, then $\gamma \to \infty$.

**2.2. The unbounded limit**

Now we examine the case $c_- \to \infty$, with $c_+$ constant, with $T_c < 0 < T_h$. Although this form of $c_v$ is clearly not integrable, it is instructive to compare the features of the solutions to those in the following sections.

Rewriting (2.14) as

$$-\frac{T_c}{T_h}\frac{e^{-c_-\gamma^2/4}}{e^{-c_+\gamma^2/4}} = \frac{\sqrt{c}_+}{\sqrt{c}_-}\left(\frac{1+erf(\sqrt{c}_-\gamma/2)}{1-erf(\sqrt{c}_+\gamma/2)}\right) \tag{2.15}$$

it is easy to see that as $c_- \to \infty$, $c_-\gamma^2 \to \infty$ while $\gamma \to 0$, the phase transition becoming stationary. (A little further investigation shows that, asymptotically, $\gamma \sim \dfrac{2}{\sqrt{c}_-}\sqrt{\ln\sqrt{c}_- - \ln\left|\dfrac{2\sqrt{c}_+T_h}{T_c}\right|}$.)

In particular, we observe from (2.12) that in the limit

$$f(\eta) = T_c, \quad -\infty < \eta < 0, \tag{2.16}$$

while (2.13) becomes

$$g(\eta) = T_h\, erf(\sqrt{c}_+\eta/2), \quad 0 < \eta < \infty. \tag{2.17}$$

Thus we obtain a jump of $[T] = -T_c$ across $\Gamma$. Noting that this limiting solution no longer satisfies $(2.10)_{2,3}$, we remark that it may be considered consistent with (1.21) in a sense provided $s = 0$, $[\epsilon]$ not being defined but the second term being finite by (2.16), (2.17).

## 3. Locally integrable specific heat

In this section, we employ nonlinear constitutive relations which allow analysis using similarity solutions. For this reason we will assume that, for $\vartheta$ in a neighbourhood of $\vartheta_\lambda$, all functions can be represented in terms of the normalized temperature $T$ (this will however not be assumed when we examine $\vartheta \to 0$ at the end of the final section). We will also assume that $c_v$ is unbounded at, but locally integrable about $T = 0$, monotone increasing for $T < 0$ and monotone decreasing for $T > 0$. Since $\vartheta_\lambda$ is a maximum for the continuous function $K(\vartheta)$, $\tilde{K}(T) = K(\vartheta_\lambda(T+1))$ is similarly monotone increasing below, and monotone decreasing above $T = 0$. $c_v$ and $K$ are both considered to be positive for all

$\vartheta > 0$ and so $\varepsilon$ and $W$, defined by $W'(T) = \tilde{K}(T), W(0) = 0$, are both invertible on their domain. Setting $\tilde{c}_v(T) = c_v(\vartheta_\lambda(T+1))$ we will, for simplicity, adopt the power law form $\tilde{c}_v(T) = c|T|^{-\nu}$, $c > 0$ with $0 < \nu < 1$ for $T \in (T_c, T_h)$, $-\delta < T_c \leq 0 \leq T_h < \delta$ and $\delta > 0$ sufficiently small. (1.13), (1.14), (1.16) may be rewritten as

$$\tilde{\varepsilon}(T)_t - W(T)_{xx} = 0. \tag{3.1}$$

Since $W(T) = w$ is an invertible function of $T$, (3.1) can similarly be rewritten in the form

$$e(w)_t - w_{xx} = 0, \tag{3.2}$$

where $e = \tilde{\varepsilon} \circ W^{-1}$. For simplicity, we now use the fact that $K(T) \approx K(0) \equiv 1$ (in normalized units) for $T_c \leq 0 \leq T_h$ and small $\delta$, and employ the power law hypothesis to express (3.2) as

$$|w|^{-\nu} w_t - w_{xx} = 0, \quad 0 < \nu < 1, \tag{3.3}$$

where we have set $c = 1$ for convenience. (3.3) is a slow-diffusion porous medium equations ([2, 3, 7, 11]) as can be seen by substituting $w = (1-\nu)^{1/1-\nu}|e|^{\nu/1-\nu}e$, which gives

$$e_t - (1-\nu)^{\nu/1-\nu}(|e|^{\nu/1-\nu}e_x)_x = 0. \tag{3.4}$$

We will consider self-similar solutions to (3.3) of the form $w(x,t) = f(t)g(xh(t))$. Substituting into (3.3) and assuming $f(t) > 0, h(t) > 0, g = g(z)$ and $z = xh(t)$, implies that for certain constants $\lambda, \mu$, we have

$$\lambda|g|^{-\nu}g + \mu|g|^{-\nu}g'z - g'' = 0, \tag{3.5}$$

where

$$\frac{\dot{f}}{f^{\nu+1}h^2} = \lambda, \quad \frac{\dot{h}}{f^\nu h^3} = \mu, \tag{3.6}$$

and up to a constant factor,

$$f(t) = h(t)^{\lambda/\mu}. \tag{3.7}$$

### 3.1. Examples of continuous solutions and blowup

First consider the case $\lambda = 0, f(t) = 1$. Equations (3.5), (3.6) then give

$$\mu|g(z)|^{-\nu}g'(z)z - g''(z) = 0, \quad \frac{dh(t)}{dt} = \mu h^3(t). \tag{3.8}$$

Considering monotone increasing solutions to $(3.8)_1$ of the form $g(z) = a|z|^{\beta-1}z$ with $a > 0$ gives $\mu \geq 0, a = (\mu/(\beta-1))^{1/\nu}$ and $\beta = 2/\nu$, where we take the particular solution $h(t) = (1-2\mu t)^{-1/2}$ for $(3.8)_2$ so that

(3.9) $$g(xh(t)) = ax|x|^{2/\nu-1}(1-2\mu t)^{-1/\nu}, \quad 0 \leq t < 1/2\mu.$$

This allows us to construct two families of solutions, $w(x,t)$ :

(3.10) $$w_-(x,t) = \begin{cases} a_-\dfrac{x|x|^{2/\nu-1}}{(1-2\mu_- t)^{1/\nu}}, & x < 0, \\ 0, & x \geq 0, \end{cases}$$

(3.11) $$w_+(x,t) = \begin{cases} 0, & x < 0, \\ a_+\dfrac{x|x|^{2/\nu-1}}{(1-2\mu_+ t)^{1/\nu}}, & x \geq 0. \end{cases}$$

These (unbounded) solutions have a maximal time of existence, $t < 1/2\mu_\pm$, at which point they develop infinite jump discontinuities. Both satisfy $s = 0$, corresponding to (1.21) in the sense of (1.22). In view of the smallness of $\delta$ discussed above, this class of solution can only be considered as a first approximation to solutions of the full model since, as $x$ leaves the vicinity of the origin, the solutions leave the region where $K \approx 1$.

Another class of solutions exists for the case $\mu = 0, h(t) = 1$, which is bounded. Here (3.5) and (3.6) become

(3.12) $$\lambda|g(x)|^{-\nu}g(x) - g''(x) = 0, \quad \frac{df(t)}{dt} = \lambda f^{\nu+1}(t).$$

For $\lambda < 0, g(x)$ is periodic since the quantity $E = \frac{1}{2}g'^2 - \frac{\lambda}{2-\nu}|g|^{2-\nu}$ is a data-dependent constant in $x$. This delivers an $x$-periodic solution

(3.13) $$w(x,t) = ag(x)(1-\nu a^\nu \lambda t)^{-1/\nu},$$

where we can choose $0 < a = f(0) < \delta/E$ to meet the smallness requirement.

In the following, we will only consider solutions which are both a priori bounded and have compact support.

### 3.2. Finite and infinite speeds of propagation

An important motivation for introducing hyperbolicity into heat conduction models is that of finite speed of propagation. Since the linear heat equation violates this condition, one can attempt to correct the situation by more detailed modelling, for instance as sketched earlier. In the steady state regime under consideration here, hyperbolic effects are no longer a feature and one might expect propagation speed to be infinite again. The fact that this is not entirely the case turns out to be a result of the discontinuity in $c_v$.

Consider again the selfsimilar solution $w(x,t) = f(t)g(xh(t))$ to (3.3), now with $\mu/\lambda = 1-\nu$. Equation (3.5) then gives

$$\lambda(|g|^{-\nu}gz)' = g'' \tag{3.14}$$

from which we have either $g(z) = 0$ or, specializing to $w(x,t)$ lying in $U_- \cup \{0\}$,

$$g(z) = -\nu^{1/\nu}(\frac{\lambda}{2}z^2 + b)^{1/\nu}, \quad b > 0, \tag{3.15}$$

where we choose $\lambda < 0$. Since (3.7) gives $f(t) = h(t)^{1/(1-\nu)}$, (3.6) can be solved to give

$$f(t) = ((2-\nu)(d-\lambda t))^{-\frac{1}{2-\nu}}, \quad d > 0, \tag{3.16}$$

and

$$h(t) = ((2-\nu)(d-\lambda t))^{-\frac{1-\nu}{2-\nu}}. \tag{3.17}$$

This implies

$$w(x,t) = \begin{cases} -f(t)\nu^{1/\nu}\Big(\dfrac{\lambda}{2}x^2h(t)^2 + b\Big)^{1/\nu}, & |x| < \left|\dfrac{2b}{\lambda h^2(t)}\right|^{1/2}, \\ 0, & |x| \geq \left|\dfrac{2b}{\lambda h^2(t)}\right|^{1/2}, \end{cases} \tag{3.18}$$

which is a compact support Barenblatt-Pattle solution. Thus, given an initial 'cold pulse' $(\vartheta(x,0) \leq \vartheta_\lambda)$ coming directly below the temperature of the phase transition, with

$$w(x,0) = \begin{cases} -f(0)\nu^{1/\nu}\Big(\frac{\lambda}{2}x^2h(0)^2 + b\Big)^{1/\nu}, & |x| < \left|\dfrac{2b}{\lambda h^2(0)}\right|^{1/2}, \\ 0, & |x| \geq \left|\dfrac{2b}{\lambda h^2(0)}\right|^{1/2}, \end{cases} \tag{3.19}$$

we obtain an expanding cold region whose support about $\vartheta = \vartheta_\lambda$ never becomes unbounded.

Finally, for $\mu/\lambda = (1-\nu)/2$, we remark on a 'dipole' solution (cf. [3]) which changes sign once, going from cold to hot temperatures or vice-versa. Here $g(z) = 0$, or

$$g(z) = \pm(1-\nu)^{\frac{1}{1-\nu}}z(c - \frac{\nu}{2(2-\nu)}|z|^{2-\nu})^{1/\nu}, c > 0, \tag{3.20}$$

while

$$f(t) = (h(t))^{\frac{2}{1-\nu}} = (1-\lambda t)^{-1}, \; \lambda = -(1-\nu)^{\frac{\nu}{1-\nu}}, \tag{3.21}$$

giving

$$w(x,t) = \begin{cases} \pm(1-\nu)^{\frac{1}{1-\nu}} x f(t) h(t) \Big( c - \dfrac{\nu}{2(2-\nu)} |x h(t)|^{2-\nu} \Big)^{1/\nu}, & \\ & |x| < \dfrac{(2c(2-\nu)/\nu)^{\frac{1}{2-\nu}}}{h(t)}, \\ 0, & |x| \geq \dfrac{(2c(2-\nu)/\nu)^{\frac{1}{2-\nu}}}{h(t)}. \end{cases} \tag{3.22}$$

We have tried to capture finite speed of propagation as well as other features of the physics for temperatures below $\vartheta_\lambda$, but we have been less motivated in doing so elsewhere due to the fact that wavelike features have only been observed clearly in this one region. We have however considered only a simple model here, for which we have taken $c_v$ to be a symmetric function about $\vartheta_\lambda$, which need not generally be the case. Consequently, the behaviour of the periodic and dipole solutions may be somewhat different to that which might be found experimentally. All of these results should be contrasted with the behaviour of solutions at temperatures well below $\vartheta_\lambda$. If we consider similar (small) data to that in (3.19), except with a 'cold pulse' below $\vartheta = \vartheta_\lambda$ being replaced by a 'hot pulse' above $\vartheta = 0$, we may use, for example, the empirical form (1.14) to find that close to $\vartheta = 0, K(\vartheta) \approx \vartheta^3$, where we have dropped inessential constants. For many materials including those under consideration, Debye's law has, similarly, $c_v(\vartheta) \approx \vartheta^3$. Therefore, (1.15) takes the form

$$(\vartheta^4)_t - (\vartheta^4)_{xx} = 0, \tag{3.23}$$

a linear parabolic equation in $u = \vartheta^4$, with the usual infinite speed of propagation.

## References


1. C. C. ACKERMAN, B. BERTMAN, H. A. FAIRBANK, and R. A. GUYER, *Second sound in solid helium*, Phys. Rev. Letters, **16**, 789–791, 1966.

2. M. BERTSCH, and D. HILHORST, *The interface between regions where $u < 0$ and $u > 0$ in the porous medium equation*, Appl. Anal., **41**, 111–130, 1991.

3. J. HULSHOF, *Similarity solutions of the porous medium equation with sign changes*, J. Math. Anal. App., **157**, 75–111, 1991.

4. H. E. JACKSON, C. T. WALKER, and T. F. MCNELLY, *Second sound in NaF*, Phys. Rev. Letters, **25**, 1, 26–28, 1970.

5. H. E. JACKSON, and C. T. WALKER, *Thermal conductivity, second sound, and phonon-phonon interactions in NaF*, Physical Review B, **3**, 4, 1428–1439, 1971.

6. W. KOSIŃSKI, K. SAXTON, and R. SAXTON *Second sound speed in a crystal of NaF at low temperature*, Arch. Mech., **49**, 1, 189–196, 1997.

7. A. A. LACEY, J. R. OCKENDON, and A. B. TAYLOR, *"Waiting-time" solutions of a non-linear diffusion equation*, SIAM J. Appl. Math, **42**, 6, 1252–1264, 1982.

8. H. LI and K. SAXTON, *Large-asymptotic behaviour of solutions to quasilinear hyperbolic equations with nonlinear damping*, Q. App. Math., to appear.

9. T. F. MCNELLY,S. J. ROGERS, D. J. CHAMIN, R. J. ROLLEFSON, W. M. GOUBAU, G. E. SCHMIDT, J. A. KRUMHANSI, and R. O. POHL, *Heat pulses in NaF: onset of second sound*, Phys. Rev. Letters, **24**, 3, 100–102, 1970.

10. V. NARAYANAMURTI, and R. C. DYNES, *Observation of second sound in bismuth*, Phys. Rev. Letters, **28**, 22, 1461–1465, 1972.

11. S. SAKAGUCHI, *Regularity of the interfaces with sign changes of solutions of the one-dimensional porous medium equation*, J. Diff. Equations, **178**, 1–59, 2002.

12. K. SAXTON, R. SAXTON, and W. KOSIŃSKI, *On second sound at the critical temperature*, Q. App. Math., 57, 723–740, 1999.

13. K. SAXTON, and R. SAXTON, *Nonlinearity and memory effects in low temperature heat propagation*, Arch. Mech., 52, 127–142, 2000.

*Received June 7, 2002, revised version September 11, 2002..*