

Material instabilities in fiber-reinforced nonlinearly elastic solids under plane deformation

J. MERODIO⁽¹⁾ and R. W. OGDEN⁽²⁾

⁽¹⁾ *Department of Structural and Mechanical Engineering,
E.T.S. Ing. Industriales y Tel.,
University of Cantabria, 39005, Santander, Spain
e-mail: merodioj@unican.es*

⁽²⁾ *Department of Mathematics, University of Glasgow
Glasgow G12 8QW, UK
e-mail: rwo@maths.gla.ac.uk*

*Dedicated to Professor Piotr Perzyna
on the occasion of his 70th birthday*

MATERIAL INSTABILITIES in fiber-reinforced nonlinearly elastic solids are examined under plane deformation. In particular, the materials under consideration are isotropic nonlinearly elastic models augmented by a function that accounts for the existence of a unidirectional reinforcing. This function describes the anisotropic (transversely isotropic) character of the material and is referred to as a *reinforcing model*. The onset of failure is signalled by the loss of ellipticity of the governing differential equations. Previous work has dealt with the analysis of specific reinforcing models and has established that the loss of ellipticity for such augmented isotropic materials requires *contraction* in the reinforcing direction. The loss of ellipticity was related to fiber kinking. Here we generalize these results and establish *sufficient* conditions for the ellipticity of the governing equations of equilibrium for more general reinforcing models to be guaranteed. We also establish *necessary* conditions for failure of ellipticity. The incipient loss of ellipticity is interpreted in terms of fiber kinking, fiber de-bonding, fiber splitting and matrix failure in fiber-reinforced composite materials. Attention is restricted to incompressible materials in this paper.

Key words: Fiber failure, fiber kinking, fiber de-bonding, fiber splitting, matrix failure, loss of ellipticity, reinforcing models, anisotropy.

1. Introduction

FAILURE MECHANISMS in composite materials which consist of an isotropic base material with unidirectional reinforcement have received increased attention in the last few years. These failure mechanisms include fiber kinking [1-8], fiber splitting [9], fiber de-bonding [10] and matrix failure [11-12]. These analyses provide different theories to capture and explain the failure modes for the mate-

rials under consideration. However, a unified approach to the prediction of fiber instability or fiber failure in fiber-reinforced composite materials is lacking.

In this paper, our objective is to present a continuum-mechanical model in the setting of nonlinear elasticity theory that captures and predicts the material instabilities mentioned above for particular fiber-reinforced materials. For this purpose, a sufficiently general (transversely isotropic) strain energy depending on deformation invariants that penalize deformation in a particular, direction, serves as the material model. The onset of failure is heralded by the loss of ellipticity of the governing differential equations [6–8].

For a given strain-energy function the loss of ellipticity condition determines both the *deformation* associated with the existence of surfaces of weak discontinuity and the *direction of the normal* to that surface. Surfaces of weak discontinuity (or weak surfaces) are surfaces across which the second derivative of the deformation field is discontinuous, while across a fully developed (or strong) surface of discontinuity the first derivative (i.e. the deformation gradient) suffers a finite jump. In the present analysis we relate the angle between the weak surface normal and the fiber-reinforcement direction to a particular failure mechanism. The argument is summarized as follows. Under fiber *contraction* the onset of fiber kinking is associated with weak surfaces that lie close to the normal to the direction of fiber reinforcement [1]. Thus, if the loss of ellipticity analysis yields a weak surface perpendicular to the fiber under fiber contraction, the associated fiber failure is identified as fiber kinking. By contrast, for fiber de-bonding the angle between the weak surface and the fiber reinforcement is close to zero [10]. For fiber kinking combined with fiber splitting, the simultaneous existence of weak surfaces close to and normal to the fiber direction is required [9]. Matrix failure arises under fiber *extension* and is associated with weak surfaces perpendicular to the fiber reinforcement [11–12]. These various possibilities are depicted in Fig. 1.

Constitutive equations that suffer a loss of ellipticity have been studied in a variety of contexts (see, for example, [6–8] [13–20]). In particular, the loss of ellipticity of some particular transversely isotropic nonlinear elastic materials under *plane* deformations has been examined in [7, 8, 18, 19]. The procedure used in these analyses is the following. An isotropic base material is augmented by a uniaxial reinforcement in what is referred to as the *fiber direction*. The plane of deformation contains the fiber reinforcement. In [7–8] and [18] the isotropic base material considered is a neo-Hookean material (incompressible), while in [19] it is the special Blatz-Ko material (compressible). In each case the same reinforcing model was used to characterize the anisotropy of the constitutive equation: the so-called *standard reinforcing model*. As is well known, the neo-Hookean model retains ellipticity at all deformations. By contrast, the Blatz-Ko material loses ellipticity at sufficiently large deformations both in tension and compression

(see, for example, [16]). Nevertheless, these papers conclude that the standard fiber reinforcement “weakens” the material in fiber compression since the loss of ellipticity involves fiber contraction, while it “strengthens” the material in fiber tension. In tension the loss of ellipticity can be avoided for a reinforcement of sufficient strength. Furthermore, the analysis of [7–8] interpreted the loss of ellipticity in terms of kink-band phenomena for fiber-reinforced materials. Here, we follow the same procedure and define the strain energy as consisting of an isotropic base material augmented by a reinforcing model. For the latter, two general classes of functions are examined.

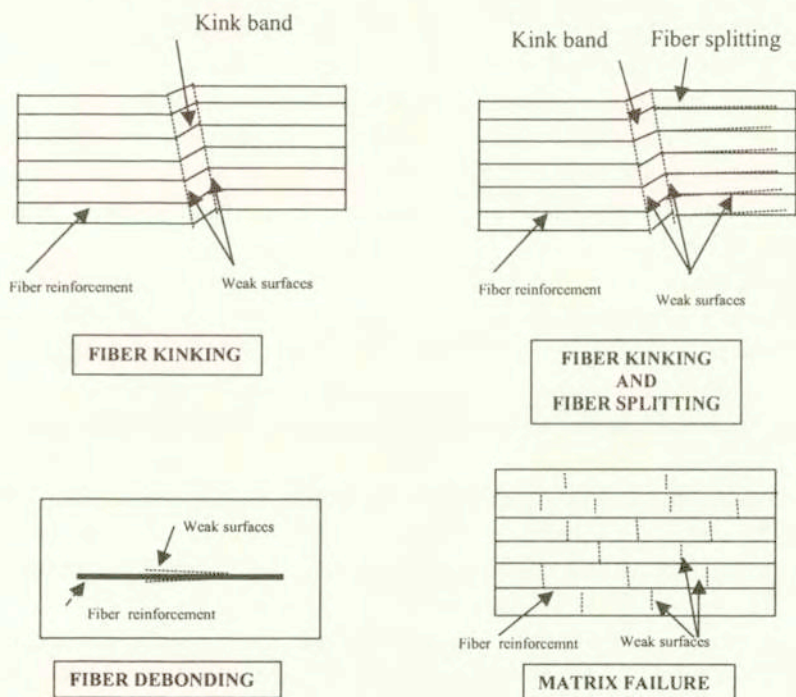


FIG. 1. Kinematics of fiber kinking, fiber kinking with fiber splitting, fiber de-bonding and matrix failure in fiber reinforced composite materials. The boundary of the kink band in the incipient fiber kinking mechanism is interpreted as a weak surface and is close to the normal direction of the fiber reinforcement (upper left figure). In the fiber kinking combined with fiber splitting there is also a weak surface in the direction of the fiber reinforcement (upper right figure). Fiber de-bonding is associated with weak surfaces close to the fiber reinforcement direction (lower left figure). Matrix failure is associated with weak surfaces normal to the fiber reinforcement (lower right figure)

In three dimensions, two independent deformation invariants, denoted I_4 and I_5 , are sufficient to characterize the anisotropic nature of a transversely isotropic material. These are additional to the usual three invariants I_1, I_2, I_3 of

the Cauchy-Green deformation tensors required for isotropy in a compressible material (for an incompressible material $I_3 \equiv 1$). The invariant I_4 represents the square of the stretch in the direction of the fiber reinforcement. The standard reinforcing model is a quadratic function that depends only on I_4 . The invariant I_5 is also related to the fiber stretch but additionally registers the reaction of the reinforcement to shear deformations and to deformations of surface area elements normal to the fiber direction. Under plane deformations with the fiber direction in the considered plane I_4 and I_5 are no longer independent and the material response depends only on $I_1 (= I_2)$ and I_4 (in the case of incompressibility). The ellipticity analysis for a general strain-energy function restricted to the plane in question then depends on only one anisotropic invariant. Nevertheless, each of I_4 and I_5 will be considered separately in the reinforcement model since each adds a distinct anisotropic character to the isotropic base material.

The paper is organized as follows. In Sec. 2, the material model is introduced and the ellipticity, strong ellipticity and loss of ellipticity conditions for the governing differential equations are summarized. Specialization to plane strain is discussed in Sec. 2.5. In Sec. 3, the ellipticity status of a general reinforcing model depending on I_4 is established. It is shown that failure of ellipticity is to be expected in fiber compression. In particular, under fiber contraction the incipient loss of ellipticity is interpreted in terms of fiber kinking. Failure can also occur in fiber extension if the reinforcing model loses convexity, in which case fiber de-bonding is an appropriate interpretation of the associated failure mode. Convex reinforcing models are discussed briefly in Sec. 3.2. The analysis in Sec. 3 is carried out for a general fiber-reinforcement orientation within the plane of deformation. This allows us, additionally, to make a qualitative analysis of the ellipticity status of a reinforcement consisting of two fiber families in the plane of deformation. This is discussed in Sec. 3.3.

In Sec. 4, our study focuses briefly on the invariant I_5 . Under fiber contraction it is found that failure of ellipticity may occur in two different modes, which may be associated with fiber kinking and fiber splitting. In fiber extension under a suitable simple shear deformation, de-bonding is again a possible failure mode if the reinforcing model is non-convex. A weak surface may also arise perpendicular to the fiber direction and this is interpreted as matrix failure. These examples of failure modes are not exhaustive. Fiber de-bonding and matrix failure are also possible failure modes under fiber extension if the base material loses ellipticity, whether or not the reinforcing model is convex. In Sec. 5 we summarize and discuss briefly the results obtained in the previous sections.

2. The material model and ellipticity

2.1. Description of the deformation

Let \mathbf{X} denote the position vector of a material particle in the stress-free reference configuration and let \mathbf{x} denote the location of the particle in the deformed configuration. The deformation gradient tensor $\partial\mathbf{x}/\partial\mathbf{X}$ is denoted \mathbf{F} . The left and right Cauchy-Green deformation tensors, respectively \mathbf{B} and \mathbf{C} , are given by

$$(2.1) \quad \mathbf{B} = \mathbf{F}\mathbf{F}^T, \quad \mathbf{C} = \mathbf{F}^T\mathbf{F},$$

and the principal (isotropic) invariants of \mathbf{C} (equivalently of \mathbf{B}) are defined by

$$(2.2) \quad I_1 = \text{tr } \mathbf{C}, \quad I_2 = I_3 \text{tr } (\mathbf{C}^{-1}), \quad I_3 = \det \mathbf{C}.$$

Let the unit vector \mathbf{A} define the direction of fiber reinforcement in the undeformed configuration. The combination of \mathbf{A} and \mathbf{C} introduces two additional (in general independent) invariants, denoted I_4 and I_5 , which are defined by

$$(2.3) \quad I_4 = \mathbf{A} \cdot (\mathbf{C}\mathbf{A}), \quad I_5 = \mathbf{A} \cdot (\mathbf{C}^2\mathbf{A}).$$

Let the vector \mathbf{a} result from the action of \mathbf{F} on \mathbf{A} , so that

$$(2.4) \quad \mathbf{a} = \mathbf{F}\mathbf{A}.$$

For a homogeneous deformation \mathbf{a} is the image of \mathbf{A} in the deformed configuration. On use of (2.4) and (2.1) we may therefore write (2.3) as

$$(2.5) \quad I_4 = \mathbf{a} \cdot \mathbf{a}, \quad I_5 = \mathbf{a} \cdot (\mathbf{B}\mathbf{a}).$$

In terms of the principal stretches $(\lambda_1, \lambda_2, \lambda_3)$ of the deformation we have

$$(2.6) \quad I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = I_3(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}), \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2,$$

$$(2.7) \quad I_4 = \lambda_1^2 A_1^2 + \lambda_2^2 A_2^2 + \lambda_3^2 A_3^2 = a_1^2 + a_2^2 + a_3^2,$$

$$(2.8) \quad I_5 = \lambda_1^4 A_1^2 + \lambda_2^4 A_2^2 + \lambda_3^4 A_3^2 = \lambda_1^2 a_1^2 + \lambda_2^2 a_2^2 + \lambda_3^2 a_3^2$$

where (A_1, A_2, A_3) are the components of \mathbf{A} referred to the principal axes of \mathbf{C} , and (a_1, a_2, a_3) those of \mathbf{a} referred to the principal axes of \mathbf{B} . It is clear from the above that $\sqrt{I_4}$ is the stretch in the direction \mathbf{A} of the fiber reinforcement. Therefore the invariant I_4 registers deformations that modify the length of the fiber. The invariant I_5 has no similar simple interpretation in general and it

depends on both changes in the fiber length and shearing strains. However, the following connection is of interest. From the Cayley-Hamilton theorem for \mathbf{C} , namely

$$(2.9) \quad \mathbf{C}^3 - I_1 \mathbf{C}^2 + I_2 \mathbf{C} - I_3 \mathbf{I} = \mathbf{0},$$

we obtain

$$(2.10) \quad I_5 = I_1 I_4 - I_2 + \mathbf{A} \cdot (\mathbf{C}^* \mathbf{A}),$$

where \mathbf{I} is the identity tensor and $\mathbf{C}^* = I_3 \mathbf{C}^{-1}$ is the adjugate of \mathbf{C} . Since a reference surface area element of unit magnitude with normal in the direction \mathbf{A} transforms to $\sqrt{I_3} \mathbf{F}^{-T} \mathbf{A}$ (Nanson's formula), the final term in (2.10) is interpreted as the square of the ratio of deformed to undeformed surface area elements and could be used as an alternative to I_5 as a measure of the influence of reinforcement.

2.2. Strain energy and stress

According to SPENCER [21], for an elastic material without internal constraints the most general strain-energy function for a homogeneous transversely isotropic nonlinear elastic solid depends only on the invariants (I_1, I_2, I_3, I_4, I_5). In this paper we focus on *incompressible* elastic materials, so that $I_3 \equiv 1$ and hence

$$(2.11) \quad \lambda_1 \lambda_2 \lambda_3 = 1.$$

As a result only four independent invariants remain, and we write the strain energy per unit reference volume as

$$(2.12) \quad W = W(I_1, I_2, I_4, I_5).$$

The nominal stress tensor \mathbf{S} is calculated from the strain energy W in the form

$$(2.13) \quad \mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - p \mathbf{F}^{-1},$$

where p is the Lagrange multiplier associated with the incompressibility constraint $\det \mathbf{F} = 1$. To make this explicit in respect of (2.12) we require the formulas

$$(2.14) \quad \frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^T, \quad \frac{\partial I_2}{\partial \mathbf{F}} = 2I_1 \mathbf{F}^T - 2\mathbf{F}^T \mathbf{F} \mathbf{F}^T,$$

$$(2.15) \quad \frac{\partial I_4}{\partial \mathbf{F}} = 2\mathbf{A} \otimes \mathbf{F} \mathbf{A}, \quad \frac{\partial I_5}{\partial \mathbf{F}} = 2(\mathbf{A} \otimes \mathbf{F} \mathbf{C} \mathbf{A} + \mathbf{C} \mathbf{A} \otimes \mathbf{F} \mathbf{A}),$$

and hence

$$(2.16) \quad \mathbf{S} = 2W_1\mathbf{F}^T + 2W_2(I_1\mathbf{I} - \mathbf{C})\mathbf{F}^T + 2W_4\mathbf{A} \otimes \mathbf{FA} \\ + 2W_5(\mathbf{A} \otimes \mathbf{FCA} + \mathbf{CA} \otimes \mathbf{FA}) - p\mathbf{F}^{-1},$$

where the subscripts 1, 2, 4, 5 on W indicate differentiation with respect to I_1, I_2, I_4, I_5 , respectively, and \mathbf{I} is again the identity tensor.

The corresponding expression for the Cauchy stress tensor $\sigma = \mathbf{FS}$ is

$$(2.17) \quad \sigma = 2W_1\mathbf{B} + 2W_2(I_1\mathbf{I} - \mathbf{B})\mathbf{B} + 2W_4\mathbf{a} \otimes \mathbf{a} \\ + 2W_5(\mathbf{a} \otimes \mathbf{Ba} + \mathbf{Ba} \otimes \mathbf{a}) - p\mathbf{I},$$

The energy function and the stress must vanish in the reference configuration (where $I_1 = I_2 = 3$ and $I_4 = I_5 = 1$) and it therefore follows that

$$(2.18) \quad W(3, 3, 1, 1) = 0, \quad 2W_1(3, 3, 1, 1) + 4W_2(3, 3, 1, 1) - p_0 = 0,$$

$$(2.19) \quad W_4(3, 3, 1, 1) + 2W_5(3, 3, 1, 1) = 0,$$

where p_0 is the value of p in that configuration. Conditions on the second derivatives of W at $(3, 3, 1, 1)$ for consistency with the classical linear theory of transversely isotropic elasticity may be obtained but we omit the details here for the three-dimensional case. For the simpler case of plane strain the appropriate connections will be noted in Sec. 2.5.

2.3. Equilibrium and ellipticity

The equation of equilibrium in the absence of body forces has the form $\text{Div } \mathbf{S} = \mathbf{0}$ and may be written in the component form

$$(2.20) \quad \mathcal{A}_{\alpha i \beta j} x_{j, \alpha \beta} - p_{, i} = 0,$$

where

$$(2.21) \quad \mathcal{A}_{\alpha i \beta j} = \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}},$$

Greek indices being associated with the components of \mathbf{X} and Roman indices with those of \mathbf{x} . The subscripts following a comma indicate differentiation with respect to the relevant coordinate.

The linearized equations governing a small incremental deformation superimposed on a homogeneous finite deformation have a similar structure to (2.20) and may be written

$$(2.22) \quad \mathcal{A}_{\alpha i \beta j} u_{j, \alpha \beta} - \bar{p}_{, i} = 0,$$

where \mathbf{u} , with components (u_1, u_2, u_3) , is the incremental displacement and \bar{p} is the corresponding increment in p . The incremental incompressibility condition is

$$(2.23) \quad \operatorname{div} \mathbf{u} = 0.$$

If we regard \mathbf{u} as a function of the deformed position \mathbf{x} and we introduce the updated version \mathcal{A}_{0piqj} of the components $\mathcal{A}_{\alpha i \beta j}$, then the incremental equations may be written

$$(2.24) \quad \mathcal{A}_{0piqj} u_{j, pq} - \bar{p}_{, i} = 0,$$

where (see, for example, [23])

$$(2.25) \quad \mathcal{A}_{0piqj} = F_{p\alpha} F_{q\beta} \mathcal{A}_{\alpha i \beta j}.$$

Now consider incremental deformations of the form

$$(2.26) \quad \mathbf{u} = \mathbf{m} e^{ik\mathbf{n} \cdot \mathbf{x}}, \quad \bar{p} = q e^{ik\mathbf{n} \cdot \mathbf{x}},$$

where \mathbf{m} is the amplitude vector, k is the 'wave' number and \mathbf{n} is a constant unit vector. On substitution into the Eq. (2.24) this leads to

$$(2.27) \quad \mathbf{Q}(\mathbf{n})\mathbf{m} + iq\mathbf{n} = \mathbf{0},$$

where the *acoustic tensor* $\mathbf{Q}(\mathbf{n})$ has components defined by

$$(2.28) \quad Q_{ij} = \mathcal{A}_{0piqj} n_p n_q,$$

and the vectors \mathbf{m} and \mathbf{n} satisfy the orthogonality condition

$$(2.29) \quad \mathbf{m} \cdot \mathbf{n} = 0$$

resulting from the incompressibility constraint (2.23)

It follows that for an incremental deformation of the form (2.26) to be admissible the equality

$$(2.30) \quad \mathcal{A}_{0piqj} n_p n_q m_i m_j \equiv [\mathbf{Q}(\mathbf{n})\mathbf{m}] \cdot \mathbf{m} = 0$$

must hold, where, without loss of generality, \mathbf{m} has been taken to be a unit vector. For a non-trivial solution this equation, together with (2.24), defines a pair of (unit) vectors \mathbf{m} and \mathbf{n} .

If the Eqs. (2.20) (or 2.24)) are *elliptic* then no such solutions exist. The condition for ellipticity is that

$$(2.31) \quad \mathcal{A}_{0piqj}n_p n_q m_i m_j \neq 0$$

for all vectors $\mathbf{m} \neq \mathbf{0}$, $\mathbf{n} \neq \mathbf{0}$ such that $\mathbf{m} \cdot \mathbf{n} = 0$.

A stronger requirement is the *strong-ellipticity condition*

$$(2.32) \quad \mathcal{A}_{0piqj}n_p n_q m_i m_j > 0 \quad \mathbf{m} \neq \mathbf{0}, \mathbf{n} \neq \mathbf{0}, \mathbf{m} \cdot \mathbf{n} = 0.$$

The analysis of Eq. (2.31) for specific forms of the energy function W furnishes the ellipticity status of that particular strain energy. A deformation gradient \mathbf{F} satisfying (2.31) for every pair of unit vectors \mathbf{m} and \mathbf{n} such that $\mathbf{m} \cdot \mathbf{n} = 0$ is said to be an *elliptic deformation* for that W . If all possible deformations for a particular material are elliptic then the material itself is referred to as an *elliptic material* (the isotropic neo-Hookean material is an example of an elliptic material). On the other hand, if, for some pair of orthogonal unit vectors \mathbf{m} and \mathbf{n} , a deformation gradient \mathbf{F} satisfies Eq. (2.30), then the deformation is said to be non-elliptic for that material model. Furthermore, the unit vector \mathbf{n} is identified as the normal vector to a surface (in the deformed configuration), referred to as a *weak surface*, across which some of the differentiability properties required in the derivation of the equilibrium equations are not satisfied by some or all the variables involved. The pre-image of \mathbf{n} is $\mathbf{N} = \mathbf{F}^T \mathbf{n}$, which is not

2.4. Reinforcing model

If an incompressible isotropic elastic material is reinforced with unidirectional reinforcing then the augmented strain-energy function may be written

$$(2.33) \quad W = W(I_1, I_2, I_4, I_5) = W_{\text{iso}}(I_1, I_2) + W_{\text{fib}}(I_4, I_5).$$

The first term in (2.33) represents the *isotropic base material*, while the second term is the so-called *reinforcing model*, the subscript standing for „fiber” reinforcement. This strain energy must be consistent with the conditions (2.18) and (2.19).

In what follows we shall restrict W_{fib} to functions that depend only on one invariant. Section 3 will be concerned with I_4 reinforcement and it will be convenient to write $W_{\text{fib}}(I_4, I_5) = F(I_4)$, while in Sec. 4 the focus will be on I_5 reinforcement and we will write $W_{\text{fib}}(I_4, I_5) = G(I_5)$.

In the literature (see [18] and [19]) use has been made of the so-called *standard reinforcing model* defined by the function

$$(2.34) \quad F(I_4) = \alpha(I_4 - 1)^2, \quad F'(I_4) = 2\alpha(I_4 - 1), \quad F''(I_4) = 2\alpha,$$

where $\alpha > 0$ is an anisotropy parameter which is a measure of the strength (or degree) of anisotropy. The standard reinforcing model penalizes deformation in the fiber direction and is a convex function of I_4 . In [18]–[19], for α sufficiently large, loss of ellipticity was found in fiber compression, i.e. for $I_4 < 1$. On the other hand, the considered materials gain stability in fiber extension. In Sec. 3 we generalize these results and provide a unified derivation of necessary and sufficient conditions for the ellipticity status of $F(I_4)$, regardless of the fiber orientation in the plane of deformation.

At this point we note that the contribution of the term W_4 to the Cauchy stress (2.17) gives a traction component $2I_4W_4$ in the deformed fibre direction. Thus, for the reinforcing model $F(I_4)$ this contribution is positive (negative) in fiber extension (contraction) provided

$$(2.35) \quad F'(I_4) > 0 (< 0) \quad \text{for} \quad I_4 > 1 (< 1), \quad F'(1) = 0.$$

It may also be appropriate to take

$$(2.36) \quad F'(I_4) \rightarrow -\infty (\infty) \quad \text{as} \quad I_4 \rightarrow 0 (\infty),$$

although we note that the standard model Eq. (2.29) does not satisfy the lower of these limits. Similarly, the contribution of the term W_5 to the Cauchy stress gives a traction component $4I_5W_5$ and hence, for the reinforcing model $G(I_5)$, the traction in the fiber direction is positive (negative) according to whether I_5 is greater than or less than unity, provided

$$(2.37) \quad G'(I_5) > 0 (< 0) \quad \text{for} \quad I_5 > 1 (< 1), \quad G'(1) = 0.$$

Analogously to (2.36) we take

$$(2.38) \quad G'(I_5) \rightarrow -\infty (\infty) \quad \text{as} \quad I_5 \rightarrow 0 (\infty).$$

We emphasize that $I_5 > 1$ does not in general correspond to fiber extension. In what follows we shall adopt the inequalities (2.35)–(2.38).

2.5. Restriction to plane strain

Our concern in this paper is the ellipticity analysis of the materials introduced above under the plane strain restriction, with the fiber reinforcement lying in the considered plane. We aim to derive conditions on $F(I_4)$ and $G(I_5)$ that provide a qualitative understanding of the ellipticity status of the model (2.33).

We take the plane in question to correspond to the (X_1, X_2) coordinate plane so that the basic finite deformation is such that $x_3 = X_3$ with (x_1, x_2) independent of X_3 . The incremental displacement field \mathbf{u} is then such that $u_3 = 0$,

with (u_1, u_2) depending only on x_1 and x_2 . It follows that $F_{13} = F_{23} = F_{31} = F_{32} = 0$ and $F_{33} = 1$, and, for the components of \mathbf{C} , $C_{13} = C_{23} = 0$ and $C_{33} = 1$.

The out-of-plane principal stretch is now $\lambda_3 = 1$ and, by incompressibility, $\lambda_1 \lambda_2 = 1$. Hence, the invariants (2.6) reduce to

$$(2.39) \quad I_2 = I_1 = \lambda_1^2 + \lambda_2^2 + 1, \quad I_3 = 1.$$

The fiber direction \mathbf{A} lies in the (X_1, X_2) plane, and therefore

$$(2.40) \quad I_4 = \lambda_1^2 A_1^2 + \lambda_2^2 A_2^2, \quad I_5 = \lambda_1^4 A_1^2 + \lambda_2^4 A_2^2.$$

The important connection

$$(2.41) \quad I_5 = (I_1 - 1)I_4 - 1$$

then follows, while the specialization of (2.10) leads to

$$(2.42) \quad \mathbf{A} \cdot (\mathbf{C}^* \mathbf{A}) = I_1 - I_4 - 1.$$

Thus, when restricted to plane strain, the strain, energy $W(I_1, I_2, I_4, I_5)$ of a fiber-reinforced incompressible elastic material (i.e. a transversely isotropic incompressible elastic material) can be represented in terms of two independent invariants, and we write

$$(2.43) \quad \hat{W}(I_1, I_4) = W(I_1, I_1, I_4, (I_1 - 1)I_4 - 1).$$

Let \mathbf{F} now denote the in-plane restriction of the deformation gradient. We then have

$$(2.44) \quad \frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^T, \quad \frac{\partial I_4}{\partial \mathbf{F}} = 2\mathbf{A} \otimes \mathbf{F}\mathbf{A},$$

specializing (2.14)₁ and (2.15)₁. The corresponding plane restriction of the nominal stress tensor is then given by

$$(2.45) \quad \mathbf{S} = 2\hat{W}_1 \mathbf{F}^T + 2\hat{W}_4 \mathbf{A} \otimes \mathbf{F}\mathbf{A} - \hat{p}\mathbf{F}^{-1},$$

where, in general, \hat{p} differs from the p in (2.16). Note that the only out-of-plane component of nominal stress (S_{33}) has to be calculated from (2.16) and is not given by (2.45).

Restrictions on \hat{W} in the reference configuration analogous to those given for W in (2.18) and (2.19) are

$$(2.46) \quad \hat{W}(3, 1) = 0, \quad 2\hat{W}_1(3, 1) - \hat{p}_0 = 0, \quad \hat{W}_4(3, 1) = 0,$$

where \hat{p}_0 is the value of \hat{p} in the reference configuration.

Comparison with the corresponding classical linear theory (see, for example, [22], p. 160) shows that

$$(2.47) \quad 2\hat{W}_1(3, 1) + \hat{W}_{44}(3, 1) = (c_{11} + c_{33} - 2c_{13})/4, \quad \hat{W}_1(3, 1) = c_{44}/2,$$

where $c_{11}, c_{13}, c_{33}, c_{44}$ are the constants arising in the classical theory (this notation being appropriate for reinforcement aligned in the x_3 direction).

For the \hat{W} defined above the components of $\mathcal{A}_{\alpha i \beta j}$ are explicitly

$$(2.48) \quad \mathcal{A}_{\alpha i \beta j} = 4\hat{W}_{11}F_{i\alpha}F_{j\beta} + 2\hat{W}_1\delta_{ij}\delta_{\alpha\beta} + 4\hat{W}_{14}(F_{i\alpha}F_{j\gamma}A_\beta + F_{j\beta}F_{i\gamma}A_\alpha)A_\gamma \\ + 4\hat{W}_{44}F_{i\gamma}F_{j\delta}A_\alpha A_\beta A_\gamma A_\delta + 2\hat{W}_4A_\alpha A_\beta \delta_{ij},$$

and, by use of (2.25), the updated version of this is given by

$$(2.49) \quad \mathcal{A}_{0piqj} = 4\hat{W}_{11}B_{pi}B_{qj} + 2\hat{W}_1\delta_{ij}B_{pq} + 4\hat{W}_{14}(B_{pi}a_j a_q + B_{qj}a_i a_p) \\ + 4\hat{W}_{44}a_p a_q a_i a_j + 2\hat{W}_4 a_p a_q \delta_{ij}.$$

In (2.48) and (2.49) and henceforth, the indices take the values 1 and 2 only.

In terms of components of \mathbf{m} and \mathbf{n} referred to the principal axes of \mathbf{B} , the strong ellipticity condition (2.32), specialized to two dimensions, becomes

$$(2.50) \quad 2\hat{W}_{11}(\lambda_1^2 - \lambda_2^2)^2 n_1^2 n_2^2 + \hat{W}_1(\lambda_1^2 n_1^2 + \lambda_2^2 n_2^2) \\ + 4\hat{W}_{14}(\lambda_1^2 - \lambda_2^2)n_1 n_2 (n_1 a_1 + n_2 a_2)(n_2 a_1 - n_1 a_2) \\ + 2\hat{W}_{44}(n_1 a_1 + n_2 a_2)^2 (n_2 a_1 - n_1 a_2)^2 + \hat{W}_4 (n_1 a_1 + n_2 a_2)^2 > 0,$$

where the orthogonality $\mathbf{m} \cdot \mathbf{n} = 0$ has been used to write $m_1 = n_2, m_2 = -n_1$.

The inequality (2.50) must hold for all (n_1, n_2) such that $n_1^2 + n_2^2 = 1$. For the special case of an *isotropic* material this inequality reduces to

$$(2.51) \quad 2\hat{W}_{11}(\lambda_1^2 - \lambda_2^2)^2 n_1^2 n_2^2 + \hat{W}_1(\lambda_1^2 n_1^2 + \lambda_2^2 n_2^2) > 0,$$

for all considered (n_1, n_2) , and this can be rearranged as

$$(2.52) \quad (I_1 + 1)[\hat{W}_1 + 2(I_1 - 3)\hat{W}_{11}]n_1^2 n_2^2 + \hat{W}_1(\lambda_1 n_1^2 - \lambda_2 n_2^2)^2 > 0.$$

It then follows immediately that necessary and sufficient conditions for this to hold are

$$(2.53) \quad \hat{W}_1 > 0, \quad 2(I_1 - 3)\hat{W}_{11} + \hat{W}_1 > 0$$

(see, for example, [20]; alternative (and equivalent) inequalities in terms of the stretches can be found in [23]). In general, however, the inequalities (2.53) are not necessary, and certainly not sufficient, for (2.50) to hold.

It is interesting to note that when evaluated in the reference configuration, the inequality (2.50) reduces to

$$(2.54) \quad \hat{W}_1(3, 1) + 2\hat{W}_{44}(3, 1)(n_1a_1 + n_2a_2)^2(n_2a_1 - n_1a_2)^2 > 0$$

for all unit vectors (n_1, n_2) , with $\mathbf{a} = \mathbf{A}$. For this to hold, the necessary and sufficient conditions are easily seen to be

$$(2.55) \quad \hat{W}_1(3, 1) > 0, \quad 2\hat{W}_1(3, 1) + \hat{W}_{44}(3, 1) > 0.$$

We assume that the inequalities (2.55) hold. Thus, by continuity, strong ellipticity holds in some neighbourhood of the reference configuration and on any path of deformation from the reference configuration strong ellipticity holds until a deformation is met at which strong ellipticity just fails. This happens (if at all) when a point is reached at which strict inequality is replaced by

$$(2.56) \quad 2\hat{W}_{11}(\lambda_1^2 - \lambda_2^2)^2n_1^2n_2^2 + \hat{W}_1(\lambda_1^2n_1^2 + \lambda_2^2n_2^2) \\ + 4\hat{W}_{14}(\lambda_1^2 - \lambda_2^2)n_1n_2(n_1a_1 + n_2a_2)(n_2a_1 - n_1a_2) \\ + 2\hat{W}_{44}(n_1a_1 + n_2a_2)^2(n_2a_1 - n_1a_2)^2 + \hat{W}_4(n_1a_1 + n_2a_2)^2 \geq 0$$

with equality holding for one or more unit vectors (n_1, n_2) .

3. The effect of I_4 reinforcement

3.1. Reinforcing model

With the restriction to plane strain we now consider the strain energy

$$(3.1) \quad \hat{W}(I_1, I_4) = W_{\text{iso}}(I_1) + W_{\text{fib}}(I_4)$$

in which an isotropic base material with strain energy $W_{\text{iso}}(I_1)$ is augmented by the reinforcing model $W_{\text{fib}}(I_4) = F(I_4)$. This is the plane strain specialization of (2.39) with I_5 omitted. For this separable form of energy (the dependence of \hat{W} on I_1 and I_4 being decoupled) the strong ellipticity condition (2.50) reduces to

$$(3.2) \quad 2\hat{W}_{11}(\lambda_1^2 - \lambda_2^2)^2n_1^2n_2^2 + \hat{W}_1(\lambda_1^2n_1^2 + \lambda_2^2n_2^2) \\ + (\mathbf{a} \cdot \mathbf{n})^2[\hat{W}_4 + 2(\mathbf{a} \times \mathbf{n})^2\hat{W}_{44}] > 0.$$

We note that $(\lambda_1^2 - \lambda_2^2)^2 = (I_1 - 3)(I_1 + 1)$ and that the first two terms in (3.2) are independent of \mathbf{A} and I_4 . The third and fourth terms depend on the deformation through $\mathbf{a} = \mathbf{F}\mathbf{A}$ and I_4 .

We now assume that the isotropic base material satisfies the strong ellipticity inequalities (2.53). (The effect of relaxation of one or more of these inequalities will be discussed later.) Then

$$(3.3) \quad W'_{\text{iso}}(I_1) > 0, \quad W'_{\text{iso}}(I_1) + 2(I_1 - 3)W''_{\text{iso}}(I_1) > 0,$$

the prime indicating differentiation with respect to I_1 . Note that in the reference configuration the inequalities (3.3) reduce to the single inequality $W'_{\text{iso}}(3) > 0$.

With reference to (3.2) we see that since \mathbf{n} may be chosen so that $\mathbf{a} \cdot \mathbf{n} = 0$, the ellipticity status of the model (3.1) depends on the sign of

$$(3.4) \quad \hat{W}_4 + 2(\mathbf{a} \times \mathbf{n})^2 \hat{W}_{44} \equiv F'(I_4) + 2(\mathbf{a} \times \mathbf{n})^2 F''(I_4),$$

where a prime denotes differentiation with respect to I_4 . In view of (2.48)₃ we have $F'(1) = 0$. Since we may choose \mathbf{n} so that $\mathbf{a} \times \mathbf{n} = \mathbf{0}$, it is clear that for (3.4) to be non-negative it is necessary that $F'(I_4) \geq 0$. If also $F''(I_4) \geq 0$ then (3.4) is non-negative for all (n_1, n_2) . If, on the other hand, $F''(I_4) < 0$ then

$$F'(I_4) + 2(\mathbf{a} \times \mathbf{n})^2 F''(I_4) \geq F'(I_4) + 2I_4 F''(I_4).$$

It follows that (3.4) is non-negative if and only if

$$(3.5) \quad F'(I_4) \geq 0, \quad F'(I_4) + 2I_4 F''(I_4) \geq 0.$$

Thus, sufficient conditions for (3.2) are clearly (3.5) together with (3.3).

3.1.1. The ellipticity status of $F(I_4)$. Here we are concerned with the ellipticity status of the reinforcing model $F(I_4)$ and its influence on the overall ellipticity of the energy function (3.1). Without loss of generality we may take $F(1) = 0$. Hence, recalling (2.35), the restrictions on F in the reference configuration are

$$(3.6) \quad F(1) = 0, \quad F'(1) = 0, \quad 2W'_{\text{iso}}(3) + F''(1) > 0,$$

the latter following from (2.55). This is certainly satisfied if $F''(1) \geq 0$, which, in fact, follows from (2.35).

Because of the factor $(\mathbf{n} \cdot \mathbf{a})^2$ in (3.2), in isolation from the isotropic base material, $F(I_4)$ always loses ellipticity since \mathbf{n} may be chosen so that $\mathbf{n} \cdot \mathbf{a} = 0$. For all other \mathbf{n} , the contribution of F to (3.2) is strictly positive if and only if

$$(3.7) \quad F'(I_4) > 0, \quad F'(I_4) + 2I_4 F''(I_4) > 0.$$

Of course, the first of these inequalities fails in the reference configuration, while strict inequality in the second is also lost in the reference configuration if $F''(1) = 0$.

We note here that the terms involving I_4 in (3.2) may be written as

$$(3.8) \quad I_4\{(\hat{\mathbf{a}} \cdot \mathbf{n})^4 F'(I_4) + (\hat{\mathbf{a}} \cdot \mathbf{n})^2 (\hat{\mathbf{a}} \times \mathbf{n})^2 [F'(I_4) + 2I_4 F''(I_4)]\},$$

where $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}|$. It is useful to consider (3.8) as quadratic in $x = (\mathbf{n} \cdot \hat{\mathbf{a}})^2$ with $0 \leq x \leq 1$. Then, (3.8) is written simply as

$$(3.9) \quad f(x) \equiv -ax^2 + (a+b)x,$$

where

$$(3.10) \quad a = 2I_4^2 F''(I_4), \quad b = I_4 F'(I_4).$$

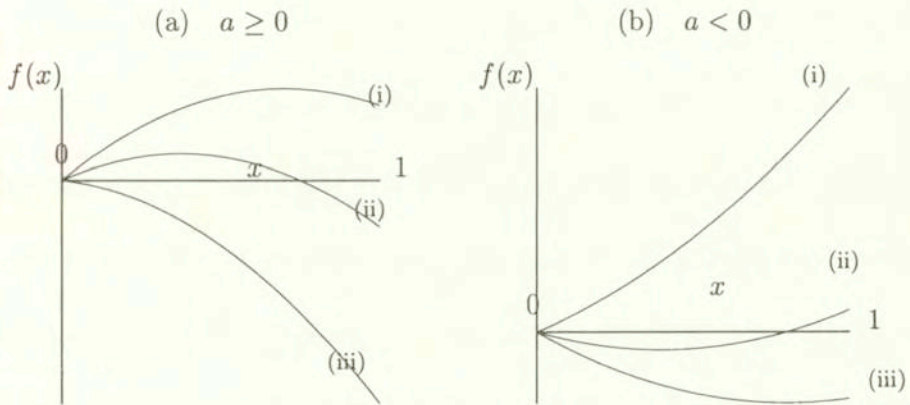


FIG. 2. Properties of the function $f(x)$ for $x = (\mathbf{n} \cdot \hat{\mathbf{a}})^2$: (a) $a \geq 0$ with (i) $b > 0$, (ii) $b < 0, a + b > 0$, (iii) $a + b < 0$; (b) $a < 0$ with (i) $a + b > 0$, (ii) $b > 0, a + b < 0$, (iii) $b < 0$.

Noting that $f(0) = 0, f(1) = b$ and $f'(0) = a + b$ we show the behaviour of $f(x)$ in Figs. 2(a) and 2(b) for $a \geq 0$ and $a < 0$ respectively. It is clear from Fig. 2(a) that for $F''(I_4) \geq 0$ the expression (3.8) first becomes negative as soon as $F'(I_4)$ becomes negative, and that it is negative for a range of values of x near 1 (i.e. where \mathbf{n} is nearly parallel to \mathbf{a}) provided $F'(I_4) + 2I_4 F''(I_4) > 0$. For $F'(I_4) + 2I_4 F''(I_4) \leq 0$ it is negative for all $x \in (0, 1]$. Figure 2(b) is applicable

for fiber extension. The expression (3.8) first becomes negative (near $x = 0$) as $F'(I_4) + 2I_4F''(I_4)$ changes from positive to negative, which is relevant for functions F that are non-convex ($F''(I_4) < 0$ for some I_4).

For the standard reinforcing model (2.34), $F'(I_4) > 0$ if and only if $I_4 > 1$, while $F'(I_4) + 2I_4F''(I_4) > 0$ if and only if $I_4 > 1/3$. Deformation gradients \mathbf{F} satisfying $I_4 < 1/3$ are not of interest since ellipticity will be lost at a larger value of $I_4 < 1$ on a path from $I_4 = 1$.

3.1.2. Overall ellipticity. Our goal is to determine the set of deformation gradients, denoted \mathbf{E} , containing the undeformed configuration for which it is possible to construct a parametrized family of (plane) deformation gradients \mathbf{F} such that ellipticity of \hat{W} is not lost at an intermediate deformation on a path of deformation from the undeformed configuration. We refer to \mathbf{E} as the *effective elliptic region* for \hat{W} . Since we assume that strong ellipticity holds in the reference configuration it follows that strong ellipticity holds within \mathbf{E} . The boundary of \mathbf{E} , denoted $\partial\mathbf{E}$, is defined by the loss of strong ellipticity condition, i.e. by the set of deformation gradients \mathbf{F} for which (2.56) holds for one or more unit vectors \mathbf{n} with (2.50) holding for all other \mathbf{n} . This boundary is therefore associated with breakdown of (strong) ellipticity. Since the isotropic base material is assumed to be strongly elliptic it is clear that a necessary condition for the breakdown of ellipticity of an elliptic isotropic nonlinearly elastic solid augmented with $F(I_4)$ is that, for $\mathbf{F} \in \mathbf{E}$, either $F'(I_4) < 0$ or $F'(I_4) + 2I_4F''(I_4) < 0$ on some path of deformation before the boundary $\partial\mathbf{E}$ is reached.

It is worth pointing out here that (3.2) is *quartic* in the components (n_1, n_2) and can be rearranged as a quartic in a single variable (e.g., n_1/n_2) with values between $-\infty$ and $+\infty$. Necessary and sufficient conditions for such a quartic to be positive can be written down explicitly, but they are extremely complicated and not easy to interpret. It is therefore appropriate to examine the influence of (3.8) on the inequality (3.2). This is particularly important for strong reinforcement in which the magnitude of (3.8) dominates (3.2).

At this point it is appropriate to consider the ellipticity status of W_{iso} on the same basis as that of F and we write the left-hand side of (2.51) as

$$(3.11) \quad i(x) \equiv -ax^2 + (a + b - c)x + c,$$

where again $x = n_1^2$ and the notation a, b, c is defined by

$$(3.12) \quad a = 2(I_1 + 1)(I_1 - 3)W''_{\text{iso}}(I_1), \quad b = \lambda_1^2 W'_{\text{iso}}(I_1), \quad c = \lambda_2^2 W'_{\text{iso}}(I_1),$$

the definitions of a and b being different from those appearing in (3.10). Figure 3 shows the behaviour of $i(x)$. In Fig. 3(a) $b > 0$ (and hence $c > 0$). Strong ellipticity holds for the upper curve, corresponding to $a > 0$ (i.e. $W''_{\text{iso}}(I_1) > 0$),

and the middle curve, for which $a < 0$ but $W'_{iso}(I_1) + 2(I_1 - 3)W''_{iso}(I_1)$ is positive. Ellipticity is lost as the latter term passes through zero, and the lower curve corresponds to $W'_{iso}(I_1) + 2(I_1 - 3)W''_{iso}(I_1) < 0$. In Fig. 3 (b) we have $b < 0$ (and $c < 0$) and $i(x)$ is negative except for an intermediate range of values of x when $a > 0$ and $W'_{iso}(I_1) + 2(I_1 - 3)W''_{iso}(I_1) \geq 0$.

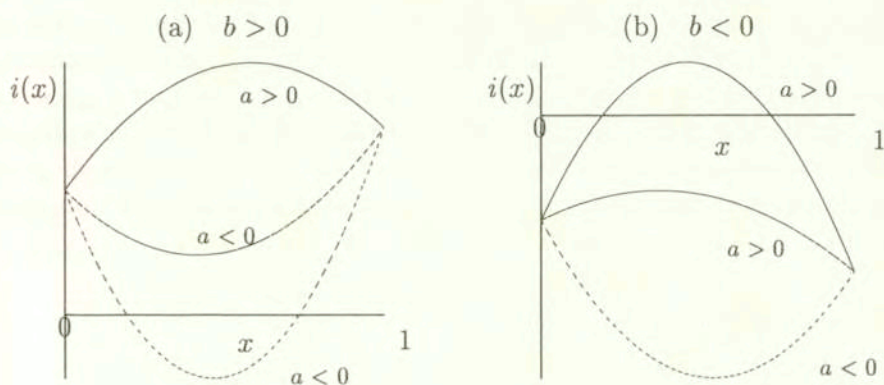


FIG. 3. Properties of the function $i(x)$ for $x = n^2$: (a) $b > 0$; (b) $b < 0$. In each case there is a maximum if $a > 0$ and a minimum if $a < 0$.

REMARK 1. Weak surfaces cannot be aligned with the fiber reinforcement axis since then we would have $\mathbf{n} \cdot \mathbf{a} = 0$ and, because of the assumed strong ellipticity of the isotropic base material, the inequality (3.2) holds, as indeed does (2.50). Weak surfaces are the only possible carriers of discontinuity for the equilibrium Eqs. (2.22) or (2.24). Therefore, no surface of discontinuity, either weak or fully developed (i.e. strong [7, 8]) can be aligned with the fiber direction. In [7], for the standard reinforcing model, this result was established for weak and strong surfaces with a particular deformation on one side of the surface, namely a deformation for which the (reference) fiber direction is a Lagrangian principal direction (i.e. an eigenvector of \mathbf{C}).

REMARK 2. If $F''(I_4) \geq 0$, *fiber kinking*, is the relevant failure mechanism under compressive strain in the fiber direction ($I_4 < 1$). It suffices to show that the weak surface at breakdown of ellipticity is normal to the fiber. If \mathbf{n} is parallel to \mathbf{a} then (3.8) reduces to

$$(3.13) \quad I_4 F'(I_4),$$

which is easily shown to be the least value of (3.8) whatever the sign of $F'(I_4) + 2I_4F''(I_4)$. However, with reference to Fig. 2 (a) we see that (3.8) first becomes negative on a path of deformation from the reference configuration (where $I_4 = 1$) as I_4 decreases with $F'(I_4) + 2I_4F''(I_4) > 0$ and this negative value decreases with I_4 . Thus, breakdown of ellipticity occurs as I_4 decreases from 1 when the negative value of $I_4F'(I_4)$ balances the positive value of the first pair of terms in (3.2) with $\mathbf{n} = \hat{\mathbf{a}}$. If \mathbf{a} is an eigenvector of \mathbf{B} then this happens for $n_2 = 0$ and (3.8) reduces to $I_4F'(I_4) + \lambda_1^2W'_{\text{iso}}(I_1)$. Since, by strong ellipticity of the base material, $W'_{\text{iso}}(I_1) > 0$, this will vanish for some $I_4 < 1$ even for reinforcements of moderate strength. For very strong reinforcement it will vanish for I_4 close to 1. Suppose the deformation consists of a simple shear of amount γ in a direction normal to the reference direction of the fiber superimposed on a pure shear with stretch λ in the same direction. For simplicity, let $\mathbf{A} = \mathbf{e}_1$. Then, the components of the deformation gradient are

$$(3.14) \quad \begin{pmatrix} \lambda & 0 \\ \gamma\lambda & \lambda^{-1} \end{pmatrix},$$

and

$$(3.15) \quad \mathbf{a} = \mathbf{FA} = \begin{pmatrix} \lambda \\ \gamma\lambda \end{pmatrix}.$$

Since $\mathbf{n} = \hat{\mathbf{a}}$ the angle, θ say, that the weak surface makes with the \mathbf{e}_1 axis (measured counterclockwise) is given by $\tan \theta = -1/\gamma$, while $I_4 = \lambda^2(1 + \gamma^2)$. Clearly, as γ increases λ must decrease in order to maintain $I_4 < 1$. If $\gamma > 0$ then $\pi/2 < \theta < \pi$ while if $\gamma < 0$ we have $0 < \theta < \pi/2$. Thus, as γ increases from 0 the weak surface rotates from the vertical (aligned with \mathbf{e}_2) counterclockwise (as does the fiber with changing deformation), and if γ decreases from 0 the surface rotates clockwise. We emphasize that larger values of $|\gamma|$ require smaller values of λ for loss of ellipticity.

REMARK 3. If $F''(I_4) < 0$ and $F'(I_4) + 2I_4F''(I_4) > 0$ then there can be no loss of ellipticity, but if $F'(I_4) + 2I_4F''(I_4) < 0$ then the (negative) minimum value of (3.8) is

$$(3.16) \quad \frac{(F' + 2I_4F'')^2}{8I_4F''}$$

whether $F' > 0$ or $F' < 0$, and it occurs for

$$(3.17) \quad (\mathbf{n} \cdot \mathbf{a})^2 = \frac{F' + 2I_4F''}{4F''}$$

(see Fig. 2 (b)). Loss of ellipticity occurs first, however, in fiber extension with $F'(I_4) > 0$ when $F'(I_4) + 2I_4F''(I_4)$ passes from positive to negative. This, of course, requires loss of convexity of F .

Thus, in fiber *extension* ($I_4 > 1$) ellipticity can fail, if again \mathbf{a} is an eigenvector of \mathbf{B} , when $\mathbf{n} \cdot \mathbf{a}$ is small since the negative contribution to (2.45) then balances the positive contribution due to W'_{iso} provided the reinforcement is sufficiently strong. In this case the weak surface is close to parallel to the fiber direction and the relevant failure mechanism can be interpreted as de-bonding.

It is interesting to note that in this case the contribution of $F(I_4)$ to the component of nominal traction, s say, in the fiber direction is, from (2.45), $2I_4^{1/2}F'(I_4)$. Hence, $ds/dI_4 = I_4^{-1/2}[F'(I_4) + 2I_4F''(I_4)]$ and thus failure of ellipticity is associated with s passing through a maximum during fiber extension.

Figure 4 shows a schematic of the possible failure mechanisms for $I_4 < 1$ and $I_4 > 1$ and the associated properties of $F(I_4)$.

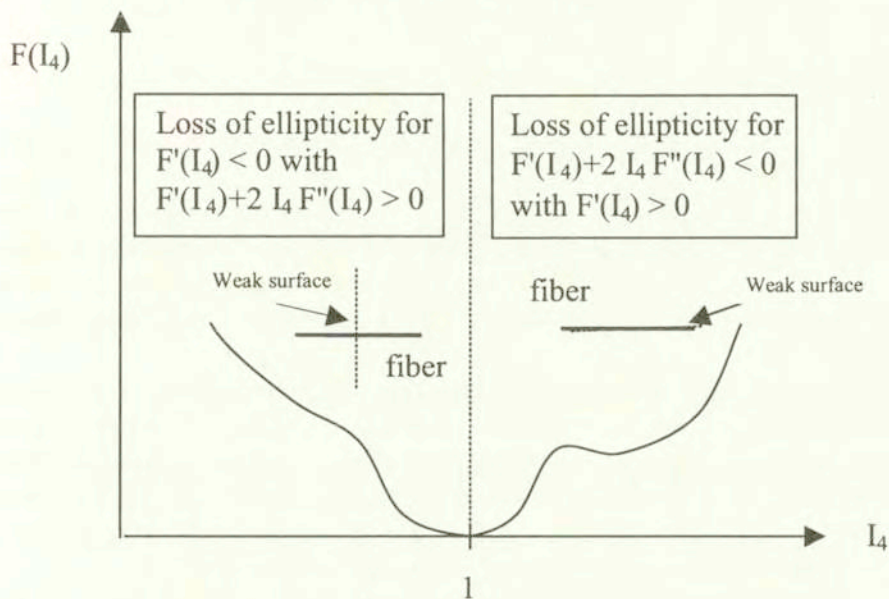


FIG. 4. Loss of ellipticity associated with the properties of $F(I_4)$ in the case of a strongly elliptic isotropic base material. Under compression in the fiber direction, the associated weak surface is normal to the fiber as appropriate for the fiber kinking mechanism. Under fiber extension, the weak surface is close to the fiber direction, as in fiber de-bonding.

REMARK 4. As discussed previously, for $\mathbf{n} \cdot \mathbf{a} = 0$ ellipticity cannot fail if W_{iso} is strongly elliptic. However, it is worth noting here that if W_{iso} is allowed to lose ellipticity then this can happen for \mathbf{n} such that $\mathbf{n} \cdot \mathbf{a} = 0$, i.e. when the weak

surface coincides with the fiber direction. This is independent of the properties of the reinforcing model $F(I_4)$ and is not therefore depicted in Fig. 4.

3.2. Ellipticity of convex I_4 reinforcement

In this section we are concerned with convex reinforcing models, so that $F''(I_4) \geq 0$. Suppose that $F(I_4) = \alpha f(I_4)$, where $\alpha (> 0)$ is an anisotropy parameter, as in the standard reinforcing model (2.34), then loss of ellipticity requires fiber contraction since $F'(I_4) \geq 0$ and $F''(I_4) \geq 0$ in fiber extension. Furthermore, the breakdown of ellipticity for the considered materials, i.e. models with an elliptic isotropic base material, satisfies a nesting property with respect to the parameter α . The result is stated as follows.

PROPOSITION. If \mathbf{F} is on the ellipticity boundary $\partial\mathbf{E}$ for α_1 and $\alpha_2 > \alpha_1$ then $\mathbf{F} \notin \mathbf{E}$ for α_2 .

P r o o f. This follows easily from (3.2), which we now write as

$$(3.18) \quad 2W''_{\text{iso}}(I_1)(\lambda_1^2 - \lambda_2^2)^2 n_1^2 n_2^2 + W'_{\text{iso}}(I_1)(\lambda_1^2 n_1^2 + \lambda_2^2 n_2^2) \\ + \alpha(\mathbf{a} \cdot \mathbf{n})^2 [f'(I_4) + 2(\mathbf{a} \times \mathbf{n})^2 f''(I_4)] > 0.$$

By hypothesis $2W''_{\text{iso}}(I_1)(\lambda_1^2 - \lambda_2^2)^2 n_1^2 n_2^2 + W'_{\text{iso}}(I_1)(\lambda_1^2 n_1^2 + \lambda_2^2 n_2^2) > 0$. Now consider that the left-hand side of (3.17) vanishes for the deformation gradient \mathbf{F} when $\alpha = \alpha_1$ and for a specific \mathbf{n} , but is otherwise non-negative. It follows that for this \mathbf{F} and \mathbf{n} the left-hand side of (3.17) is negative for $\alpha_2 > \alpha_1$. Hence, $\mathbf{F} \notin \mathbf{E}$ for $\alpha = \alpha_2$.

In [18] this nesting property of \mathbf{F} giving rise to the breakdown of ellipticity was illustrated for the standard reinforcing model (2.34) by reference to plots in (C_{11}, C_{12}, α) -space. Furthermore, it may be shown that the asymptotic form of the breakdown of ellipticity curves in (C_{11}, C_{12}, α) -space as α and $C_{12} \rightarrow \infty$ is $I_4 = C_{11} \rightarrow 1/3$. We recall that in Sec. 3.1 it was noted that $I_4 = 1/3$ is associated with vanishing of $F'(I_4) + 2I_4 F''(I_4)$ in respect of (2.34).

If the isotropic base material is non-elliptic, however, then the effect of the anisotropic parameter is as follows. Two possibilities have to be considered depending on the fiber stretch. If the fiber is under contraction then the nesting property is as in the case of an elliptic base material. If the fiber is subject to extension with a deformation gradient \mathbf{F} , then if $\mathbf{F} \in \mathbf{E}$ for α_1 , then $\mathbf{F} \in \mathbf{E}$ is elliptic for $\alpha_2 > \alpha_1$. Therefore, deformation gradients giving rise to breakdown of ellipticity are nested with respect to α in fiber contraction, while the elliptic regions are nested with respect to α in fiber extension. This allows us to conclude that an elliptic isotropic base material augmented with a convex reinforcing model gains stability in fiber extension while it is weakened in fiber contraction. Similarly,

as α increases, i.e. as the degree of anisotropy increases, the solid becomes more stable in fiber extension, but less stable in fiber compression.

3.3. The case of two reinforcing models

The general analysis above does not depend on the fiber orientation in the plane of deformation. This allows us to consider simultaneously the qualitative effect of more than one fiber direction on the ellipticity status of an augmented isotropic base material. The undeformed angles of the fibers have an important role since a deformation gradient \mathbf{F} may generate contraction in one of the fibers and extension in the other, for example. The analysis could include several fibers and several reinforcing models. However, here we focus on the simple case of two fibers and the influence of their relative orientation under a deformation gradient corresponding to (plane) pure homogeneous strain. Thus,

$$(3.19) \quad \mathbf{F} = \lambda \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda^{-1} \mathbf{e}_2 \otimes \mathbf{e}_2,$$

where $\mathbf{e}_1, \mathbf{e}_2$ are the (in-plane) Cartesian unit basis vectors.

Let the fibers be defined by the convex reinforcing models $F_1(I_4^{(1)})$ and $F_2(I_4^{(2)})$ so that $W_{\text{fib}} = F_1(I_4^{(1)}) + F_2(I_4^{(2)})$. Without loss of generality we take fiber 1 to be aligned with \mathbf{e}_1 in the undeformed configuration and hence $I_4^{(1)} = \lambda^2$. Let the direction of fiber 2 be given by the angle ϕ relative to the \mathbf{e}_1 direction in the undeformed configuration. Because of symmetry it is sufficient to consider $0 < \phi < \pi/2$. Then, we have

$$(3.20) \quad I_4^{(2)} = \lambda^2 \cos^2 \phi + \lambda^{-2} \sin^2 \phi.$$

A preliminary step is to consider for which angles ϕ fiber 2 is in contraction or extension for a given λ . We consider separately the ranges of angles $0 < \phi < \pi/4$ and $\pi/4 < \phi < \pi/2$. Then it is easily seen that, for $0 < \phi < \pi/4$,

$$(3.21) \quad \begin{aligned} &\text{if } 0 < \lambda < \tan \phi \text{ or } \lambda > 1 \text{ then } I_4^{(2)} > 1, \\ &\text{if } \tan \phi < \lambda < 1 \text{ then } I_4^{(2)} < 1, \end{aligned}$$

with $I_4^{(2)} = 1$ corresponding to $\tan \phi = \lambda$. For $\pi/4 < \phi < \pi/2$ the corresponding inequalities are

$$(3.22) \quad \begin{aligned} &\text{if } 0 < \lambda < 1 \text{ or } \tan \phi < \lambda \text{ then } I_4^{(2)} > 1, \\ &\text{if } 1 < \lambda < \tan \phi \text{ then } I_4^{(2)} < 1, \end{aligned}$$

with again $I_4^{(2)} = 1$ corresponding to $\tan \phi = \lambda$.

For this double reinforcement the strong ellipticity inequality (3.2) becomes

$$(3.23) \quad 2W''_{\text{iso}}(I_1)(\lambda_1^2 - \lambda_2^2)^2 n_1^2 n_2^2 + W'_{\text{iso}}(I_1)(\lambda_1^2 n_1^2 + \lambda_2^2 n_2^2) \\ + (\mathbf{a} \cdot \mathbf{n})^2 [W'_{\text{fib}}(I_4) + 2(\mathbf{a} \times \mathbf{n})^2 W''_{\text{fib}}(I_4)] > 0.$$

If $\lambda < 1$ then fiber 1 is under contraction and $F_1(I_4^{(1)})$ is non-elliptic and contributes a negative quantity $I_4 F'_1(I_4^{(1)})$ to (3.23) when $\mathbf{n} \times \mathbf{e}_1 = \mathbf{0}$. If fiber 2 is in extension it contributes a positive term to (3.23) and counteracts the effect of fiber 1. If it is of sufficient magnitude this may have the effect of restoring strong ellipticity. On the other hand, if fiber 2 is under contraction then it contributes a negative term to (3.23) and enhances the prospects of loss of ellipticity. It follows that compared with the material with a single reinforcement, the doubly-reinforced, material gains stability if fiber 2 is such that $\pi/4 < \phi < \pi/2$. For $\lambda > 1$, the opposite state of affairs applies: fiber 1 is in extension and contributes a positive term to (3.23). Then, if fiber 2 is under contraction its contribution to W_{fib} is such that ellipticity may be lost, but if fiber 2 is extended then W_{fib} is strongly elliptic. The loss of ellipticity can be avoided if fiber 2 is such that $0 < \phi < \pi/4$.

4. The effect of I_5 reinforcement

In this section we consider the reinforcing model

$$(4.1) \quad \hat{W}(I_1, I_4) = W_{\text{iso}}(I_1) + W_{\text{fib}}(I_5),$$

where I_4 has been replaced by I_5 in (3.1) and we recall that $I_5 = (I_1 - 1)I_4 - 1$. Thus, while in (3.1) I_1 and I_4 are decoupled in (4.1) there is a coupling of I_1 and I_4 through I_5 . For convenience, we write $G(I_5) = W_{\text{fib}}(I_5)$ and we analyze the reinforcing model $G(I_5)$, again with the restriction to plane deformations. We will show that no particular property of the reinforcing model enables loss of ellipticity of $G(I_5)$ to be avoided, unlike the situation for $F(I_4)$.

The domain for I_5 is $0 < I_5 < \infty$ and the condition $I_5 = 1$ is satisfied in many configurations (in addition to the undeformed configuration) depending on the fiber orientation in the undeformed configuration. It necessarily entails fiber contraction since, without loss of generality, if we consider $\mathbf{A} = \mathbf{e}_1$ then, $I_5 = C_{11}^2 + C_{12}^2 = 1$ if and only if $I_4 = C_{11} < 1$ provided $C_{12} \neq 0$ and therefore necessarily involves the shearing indicator C_{12} . It is worth noting that in general (in plane strain) it follows from the connection (2.41) that $I_4 \geq 1$ implies $I_5 \geq 1$ while $I_5 \leq 1$ implies $I_4 \leq 1$ (in particular, note that $I_5 = 1$ implies $I_4 \leq 1$, with equality if and only if the material is undeformed). These implications do not go

the other way. This can be seen by noting that the counterpart of the expression (3.20) for I_5 , obtained by replacing λ by λ^2 , is

$$(4.2) \quad I_5 = \lambda^4 \cos^2 \phi + \lambda^{-4} \sin^2 \phi.$$

Thus, by reference to (3.21) and (3.22), for $0 < \phi < \pi/4$,

$$(4.3) \quad \begin{aligned} &\text{if } 0 < \lambda < \sqrt{\tan \phi} \text{ or } \lambda > 1 \text{ then } I_5 > 1, \\ &\text{if } \sqrt{\tan \phi} < \lambda < 1 \text{ then } I_5 < 1, \end{aligned}$$

and for $\pi/4 < \phi < \pi/2$:

$$(4.4) \quad \begin{aligned} &\text{if } 0 < \lambda < 1 \text{ or } \sqrt{\tan \phi} < \lambda \text{ then } I_5 > 1, \\ &\text{if } 1 < \lambda < \sqrt{\tan \phi} \text{ then } I_5 < 1, \end{aligned}$$

with $I_5 = 1$ corresponding to $\tan \phi = \lambda^2$ in each case.

On substitution of (4.1) into (3.2) we obtain

$$(4.5) \quad \begin{aligned} 2W''_{\text{iso}}(I_1)(\lambda_1^2 - \lambda_2^2)^2 n_1^2 n_2^2 + W'_{\text{iso}}(I_1)(\lambda_1^2 n_1^2 + \lambda_2^2 n_2^2) \\ + 2G''(I_5) [I_4(\lambda_1^2 - \lambda_2^2)n_1 n_2 + (I_1 - 1)(\mathbf{n} \cdot \mathbf{a})(\mathbf{a} \times \mathbf{n})_3]^2 \\ + G'(I_5) [(I_1 - 1)(\mathbf{n} \cdot \mathbf{a})^2 + I_4(\lambda_1^2 n_1^2 + \lambda_2^2 n_2^2) \\ + 4(\lambda_1^2 - \lambda_2^2)n_1 n_2(\mathbf{n} \cdot \mathbf{a})(\mathbf{a} \times \mathbf{n})_3] > 0, \end{aligned}$$

where $(\mathbf{a} \times \mathbf{n})_3 = a_1 n_2 - a_2 n_1$.

We note two special cases of (4.5). First, we note that if $\mathbf{n} \cdot \mathbf{a} = 0$ then the terms in G in (4.5) reduce to

$$(4.6) \quad 2G''(I_5) [I_4(\lambda_1^2 - \lambda_2^2)n_1 n_2]^2 + I_4 G'(I_5)(\lambda_1^2 n_1^2 + \lambda_2^2 n_2^2),$$

while if $\mathbf{n} \times \mathbf{a} = \mathbf{0}$ they reduce to

$$(4.7) \quad 2G''(I_5) [I_4(\lambda_1^2 - \lambda_2^2)n_1 n_2]^2 + I_4 G'(I_5)(\lambda_1^2 n_1^2 + \lambda_2^2 n_2^2 + \lambda_1^2 + \lambda_2^2).$$

REMARK 5. In Sec. 3.1 it was pointed out that $F(I_4)$ does not admit a weak surface aligned with the fiber direction. This is not the case for $G(I_5)$, as we now show. We recall from (2.37) that $G'(I_5) < 0$ for $I_5 < 1$. Thus, if either $n_1 = 0$ or $n_2 = 0$, for example, the expression (4.6) is negative when $I_5 < 1$, in which case $I_4 < 1$ and the fiber is under contraction. Thus, ellipticity can fail for $I_5 < 1$. If $n_1 = 0$ ($a_2 = 0$) this corresponds to a weak surface parallel to the fiber direction and may be associated with fiber splitting [9].

REMARK 6. The case of (4.7) with $n_2 = 0$ and $a_2 = 0$ may be associated with a weak surface normal to the fiber direction. Thus, failure of ellipticity under fiber contraction can correspond to fiber kinking, as for the $F(I_4)$ reinforcement.

REMARK 7. If the degree of anisotropy is sufficiently strong then the terms in G dominate the left-hand side of (4.5) and hence loss of ellipticity cannot be avoided under contraction if I_5 is sufficiently small. Deformations satisfying $I_5 = 1$ involve $C_{11} < 1$ and $C_{12} \neq 0$ simultaneously. We can therefore conclude that for deformation gradients satisfying $I_4 = C_{11} < 1$ and $I_5 \leq 1$ the material is expected to lose ellipticity.

REMARK 8. Note that the coefficient of $G'(I_5)$ in (4.5) is not sign definite, so that failure of ellipticity can occur even if $G''(I_5) \geq 0$, i.e. if G is convex.

For the special case when \mathbf{a} is an eigenvector of \mathbf{B} the behaviour of the terms in G in (4.5) can be seen as follows. Let $\hat{\mathbf{a}}$ be the first eigenvector of \mathbf{B} . Then, $\mathbf{n} \cdot \hat{\mathbf{a}} = n_1$ and $(\hat{\mathbf{a}} \times \mathbf{n})_3 = n_2$ and the terms in G may be written as

$$(4.8) \quad g(x) \equiv -ax^2 + (a+b)x + c,$$

where $x = n_1^2$ and

$$(4.9) \quad a = 4I_4[2\lambda_1^4 I_4 G''(I_5) + (\lambda_1^2 - \lambda_2^2)G'(I_5)],$$

$$(4.10) \quad b = 2I_4\lambda_1^2 G'(I_5), \quad c = I_4\lambda_2^2 G'(I_5).$$

Note that the definitions of a, b and c differ from those in (3.12).

Figure 5 shows the behaviour of $g(x)$, which is very similar to that of $i(x)$ shown in Fig. 3 except that a, b, c are different. Figure 5(a) corresponds to fiber contraction and it is clear that the negative contribution of the terms in G to (4.5) near $x = 0$ and $x = 1$ will balance the positive contribution from strongly elliptic W_{iso} whenever the reinforcement is sufficiently strong. In fiber extension, corresponding to Fig. 5(b), loss of ellipticity requires $a < 0$, but this is not sufficient since the minimum value of $g(x)$ must be negative. If this is the case then loss of ellipticity occurs at an intermediate value of x . It follows from (4.6) and (4.7) that the weak surface is neither close to the fiber axis nor close to the normal to the fiber. It is not clear how to interpret the associated failure mechanism in this situation.

An illustration of a possible situation in which \mathbf{a} is not an eigenvector is provided by simple shear. Consider, in particular, a simple shear deformation in which the direction of shear is perpendicular to the undeformed fiber direction. For definiteness we take $\mathbf{A} = \mathbf{e}_2$ and consider the simple shear with amount of shear γ so that $\mathbf{a} = \gamma\mathbf{e}_1 + \mathbf{e}_2$ and hence $I_4 = 1 + \gamma^2$, $I_5 = 1 + 3\gamma^2 + \gamma^4$. If $\mathbf{a} \cdot \mathbf{n} = 0$

then $\mathbf{n} = (-\mathbf{e}_1 + \gamma\mathbf{e}_2)/\sqrt{I_4}$ and the contribution of the terms in G to (4.5) is then

$$(4.11) \quad 2I_4 G''(I_5) \gamma^4 (\gamma^2 + 4) + G'(I_5) (\lambda^2 + \gamma^2 \lambda^{-2}).$$

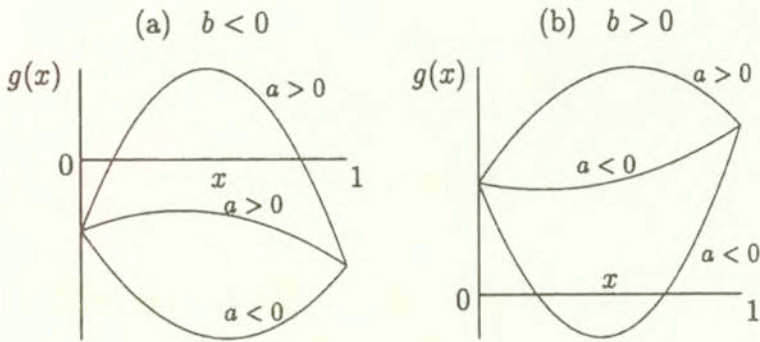


FIG. 5. Properties of the function $g(x)$ for $x = n_1^2$.

REMARK 9. It follows from (4.11) that for a simple shear deformation in which the direction of shear is not parallel to the fiber direction (and, in particular, when it is perpendicular to the fiber direction) a necessary condition for loss of ellipticity (if the base material is strongly elliptic) is $G'''(I_5) < 0$. In this case the weak surface is parallel to the fiber direction and de-bonding is the appropriate failure mechanism.

REMARK 10. If, instead of $\mathbf{a} \cdot \mathbf{n} = 0$, \mathbf{n} is parallel to \mathbf{a} then a similar situation to that described in Remark 9 ensues. In this case the weak surface is perpendicular to the fiber direction and the appropriate failure mechanism is matrix failure.

REMARK 11. If the isotropic base material loses ellipticity with W'_{iso} becoming negative then overall ellipticity can fail either for $\mathbf{n} \cdot \mathbf{a} = 0$ or $\mathbf{n} \times \mathbf{a} = \mathbf{0}$. With reference to (4.5), it can be seen that this can occur for $G'(I_5)$ and $G''(I_5)$ with appropriate signs.

Figure 6 shows a schematic of the possible failure mechanisms for $I_5 < 1$ and $I_5 > 1$ and the associated properties of $G(I_5)$.

5. Discussion and Summary

This analysis has been motivated by instability phenomena in fiber-reinforced composite materials and has focused on failure prediction. The materials considered are isotropic base materials augmented by a function that accounts for

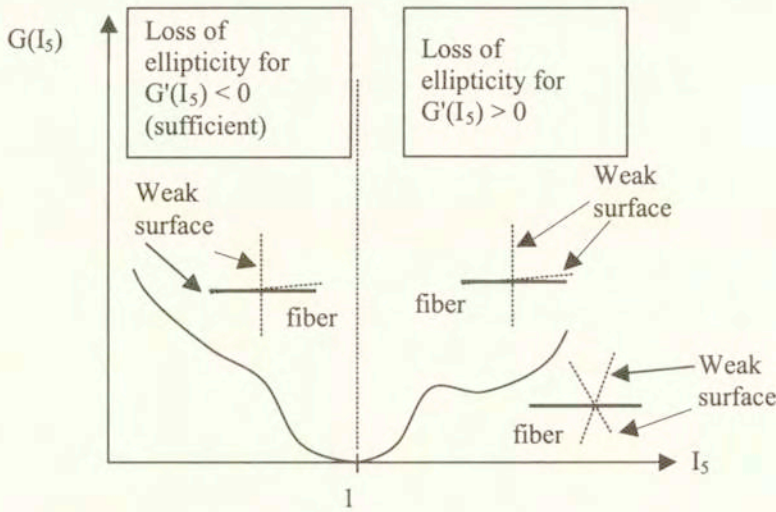


FIG. 6. Loss of ellipticity associated with $G(I_5)$ for the case of a strongly elliptic isotropic base material.

the existence of fiber reinforcement (the reinforcing model). The onset of failure is associated with loss of ellipticity of the governing differential equations. A detailed analysis of the ellipticity status of the I_4 reinforcing model has been given. In particular, in Sec. 3 simple conditions that guarantees the ellipticity of the I_4 reinforcing model has been determined. It was found that loss of ellipticity (and hence fiber failure) is expected under fiber contraction. Fiber failure may also occur under fiber extension if the reinforcing model is non-convex. In Sec. 4, the I_5 reinforcing model has been considered briefly and its effect on the loss of ellipticity has been illustrated in some simple cases. We have indicated how the breakdown of ellipticity may be related to different fiber failure mechanisms.

It should be emphasized that we have focused on instabilities associated with loss of ellipticity in a homogeneous material homogeneously deformed so that boundary conditions are not involved. We have not considered other types of instability such as buckling, which, under appropriate boundary conditions, may be initiated prior to loss of ellipticity.

References

1. B. BUDIANSKY and N. A. FLECK, *Compressive failure of fiber composites*, J. Mech. Phys. Solids, **41**, 183–211, 1993.
2. B. BUDIANSKY, N. A. FLECK and J. C. AMAZIGO, *On kink-band propagation in fiber composites*, J. Mech. Phys. Solids, **9**, 1637–1653, 1998.

3. P. M. MORAN and C. F. SHIH, *Kink band formation and band broadening in ductile matrix fiber composites: experiments and analysis*, Int. J. Solids Structures, **35**, 1709–1722, 1998.
4. S. KYRIAKIDES, R. ARSECULERANE, E. PERRY and K. M. LIECHTI, *On the compressive failure of fiber reinforced composites*, Int. J. Solids Structures, **32**, 689–738, 1995.
5. T. J. VOGLER and S. KYRIAKIDES, *Initiation and axial propagation of kink bands in fiber composites*, Acta Metall. Mater., **45**, 2443–2454, 1997.
6. H. M. JENSEN and J. CHRISTOFFERSEN, *Kink band formation in fiber reinforced materials*, J. Mech. Phys. Solids, **45**, 1121–1136, 1997.
7. J. MERODIO and T. J. PENCE, *Kink surfaces in a directionally reinforced neo-Hookean material under plane deformation: I. Mechanical equilibrium*, J. Elasticity, **62**, 119–144, 2001.
8. J. MERODIO and T. J. PENCE, *Kink surfaces in a directionally reinforced neo-Hookean material under plane deformation: II. Kink band stability and maximally dissipative band broadening*, J. Elasticity, **62**, 145–170, 2001.
9. S. H. LEE, C. S. YERRAMALLI, A. M. WAAS, *Compressive splitting response of glass-fiber reinforced unidirectional composites*, Composites Science and Technology, **60**, 2957–2966, 2000.
10. M. R. PIGGOTT, *Why interface testing by single-fibre methods can be misleading*, Composites Science and Technology, **57**, 965–974, 1997.
11. T. OKABE, J. KOMOTORI, M. SHIMIZU, N. TAKEDA, *Mechanical behavior of SiC fiber reinforced brittle-matrix composites*, Journal of Material Science, **34**, 3405–3412, 1999.
12. K. LIAO and K. L. REIFSNIDER, *A tensile model for unidirectional fiber-reinforced brittle matrix composite*, Int. J. Fracture, **106**, 95–115, 2000.
13. J. K. KNOWLES and E. STERNBERG, *On the ellipticity of the equations of nonlinear elastostatics for a special material*, J. Elasticity, **5**, 341–361, 1975.
14. L. ZEE and E. STERNBERG, *Ordinary and strong ellipticity in the equilibrium theory of incompressible hyperelastic solids*, Arch. Rat. Mech. Anal., **83**, 50–90, 1983.
15. P. ROSAKIS, *Ellipticity and deformations with discontinuous gradients in plane elastostatics of compressible solids*, Arch. Rat. Mech. Anal., **109**, 1–37, 1990.
16. C. O. HORGAN, *Remarks on ellipticity for the generalized Blatz-Ko constitutive model for a compressible nonlinearly elastic solid*, J. Elasticity, **42**, 165–176, 1996.
17. J. K. KNOWLES and E. STERNBERG, *On the failure of ellipticity and the emergence of discontinuous deformation gradients in plane finite elastostatics*, J. Elasticity, **8**, 329–379, 1978.
18. G. Y. QIU and T. J. PENCE, *Loss of ellipticity in plane deformations of a simple directionally reinforced incompressible nonlinearly elastic solid*, J. Elasticity, **49**, 31–63, 1997.
19. N. TRIANTAFYLIDIS and R. C. ABAYARATNE, *Instability of a finitely deformed fiber-reinforced elastic material*, J. Appl. Mech., **50**, 149–156, 1983.
20. R. C. ABAYARATNE, *Discontinuous deformation gradients in plane finite elastostatics of incompressible materials*, J. Elasticity, **10**, 255–293, 1980.
21. A. J. M. SPENCER, *Deformations of fibre-reinforced materials*, Oxford University Press, 1972.

22. A. E. H. LOVE, *A treatise on the mathematical theory of elasticity*, 4th Edition, Dover Publications, New York 1944.
23. R. W. OGDEN, *Non-linear elastic deformations*, Ellis Horwood, Chichester 1984.

Received July 23, 2002.
