# Energy-based limit conditions for transversally isotropic solids

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*Dedicated to Professor Piotr Perzyna*
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USING AN EXAMPLE of transversal isotropy, the limit condition having an energy interpretation for anisotropic bodies proposed by J. RYCHLEWSKI [11] has been illustrated. Transversal isotropy is characterized by the highest degree of symmetry, for which the spherical tensor is not any more the eigenstate of the compliance tensor **C**. In the case when the spectral decomposition of the compliance tensor **C** is taken as a main energy-orthogonal decomposition, the limit condition representing a generalization of the Maxwell-Huber-Mises condition is obtained. For a prescribed form of the limit tensor **H**, the Mises condition is presented in the form of a sum of elastic energies corresponding to uniquely defined energy-orthogonal parts of stress with certain weights, representing the limiting values of those energies. The effect of Burzyński's condition on the form of anisotropy and on the limit condition is discussed. Experimental tests are proposed which could be useful in determining the physical parameters describing the transversal isotropy.

## 1. Introduction

IN THE MECHANICS of continuous media, in formulating the constitutive equations, an important role is played by the conditions which limit the region of applicability and validity of these equations. These are usually certain criteria limiting the material strength measures, without any detailed analysis of the state of stress. Hence, it may be the passage from linear to nonlinear elasticity, the limit of appearing of the irreversible deformations (plasticity), appearing of viscosity or other structural changes of the material.

Most of the known limit conditions have a definite energy interpretation, i.e. they are certain limitations imposed on the energy (or its parts).

We are thus discussing the truly classical materials, in which the infinitesimal strain $\varepsilon$ causes the stress $\sigma$ according to Hooke's law

$$(1.1) \qquad \varepsilon = \mathbf{C} \cdot \sigma, \quad \sigma = \mathbf{S} \cdot \varepsilon,$$

$$(1.2) \qquad \mathbf{C} \circ \mathbf{S} = \mathbf{S} \circ \mathbf{C} = \mathbb{I}_S,$$

where the fourth-rank tensors **C**, **S**, $\mathbb{I}_S$ are the compliance, stiffness and unit tensors respectively, of certain symmetry. In indicial notation expressions (1.1) and (1.2) assume the form:

$$(1.3) \qquad \varepsilon_{ij} = C_{ijkl}\sigma_{kl}, \quad \sigma_{mn} = S_{mnij}\varepsilon_{ij},$$

$$(1.4) \qquad S_{ijmn}C_{mnkl} = C_{ijmn}S_{mnkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

From Hooke's law (1.1) it follows that *the elastic energy density* $\Phi$ is given by

$$(1.5) \qquad \Phi(\sigma) \equiv \frac{1}{2}\sigma \cdot \varepsilon = \frac{1}{2}\sigma \cdot \mathbf{C} \cdot \sigma = \frac{1}{2}\varepsilon \cdot \mathbf{S} \cdot \varepsilon.$$

In the case of isotropy, energy $\Phi$ may be presented in the form of a sum of the energy connected with the change of volume $\Phi(\sigma\mathbf{I})$ and the change of shape $\Phi(\mathbf{s})$, namely

$$(1.6) \qquad \Phi(\sigma) = \Phi(\sigma\mathbf{I}) + \Phi(\mathbf{s}) = \frac{1}{2K}\sigma^2 + \frac{1}{4G}\mathbf{s} \cdot \mathbf{s},$$

where $\sigma = \frac{1}{3}\operatorname{tr}\sigma$ and $\mathbf{s} = \sigma - \sigma\mathbf{I}$.

Hooke's law (1.1) describes the behaviour of the material within the elastic range, i.e. as long as the strength condition does not reach the critical value.

The objective of this paper is to formulate the limit condition for anisotropic bodies.

M. T. HUBER [5], on defining the limit criterion for isotropic bodies, assumed that only the distortion energy decided on passing of the material to the plastic state, i.e. only the part $\Phi(\mathbf{s})$ of the elastic energy $\Phi(\sigma)$ (1.6) enters the yield condition. This concept of assuming the distortion energy to be responsible for appearance of the plastic deformations, can be also found in the papers by MISES [16] and HENCKY [2].

The limit condition

$$(1.7) \qquad \frac{1}{h}\Phi(\mathbf{s}) \leq 1, \quad \text{where} \quad h = \frac{k^2}{2G}$$

is equivalent to

$$(1.8) \qquad \mathbf{s} \cdot \mathbf{s} = 2k^2$$

and is well known in the literature as the Huber-Mises-Hencky condition. It is one of the most frequently applied conditions for isotropy.

J. Rychlewski, on preparing the paper [12] for print, found a private letter written by C. Maxwell to Lord Kelvin in 1855 [7], suggesting that the condition of appearing of plastic strains is reaching of a certain limiting value by the distortion energy $\Phi(\mathbf{s})$. Hence Rychlewski, in his paper [12], proposed to call the condition (1.7) *the Maxwell-Huber-Mises limit state condition.*

When the linear-elastic anisotropic bodies are dealt with, we must decide upon a proper generalization of the Maxwell-Huber-Mises condition (1.7).

In case of anisotropic bodies, there is no physical reason to consider the hydrostatic state as a state playing a decisive role in formulation of the strength measures. For bodies with a definite type of anisotropy, the tensor characterizing the anisotropic structure may prove to be the characteristic tensor. The spherical part of the stress tensor may thus enter the limit condition.

When an arbitrary anisotropy (1.1) is considered, the decomposition of energy into the parts connected with the change of volume $\Phi(\sigma\mathbf{I})$ and the change of shape $\Phi(\mathbf{s})$ (1.6) is impossible.

From Eq. (1.5) it follows that for anisotropy (1.1)

$$(1.9) \qquad 2\Phi(\sigma) = \sigma \cdot \mathbf{C} \cdot \sigma = \sigma^2\mathbf{I} \cdot \mathbf{C} \cdot \mathbf{I} + \mathbf{s} \cdot \mathbf{C} \cdot \mathbf{s} + 2\mathbf{I} \cdot \mathbf{C} \cdot \mathbf{s}.$$

When

$$(1.10) \qquad\qquad \mathbf{I} \cdot \mathbf{C} \cdot \mathbf{s} \neq 0,$$

decomposition of the energy $\Phi(\sigma)$ into $\Phi(\sigma\mathbf{I})$ and $\Phi(\mathbf{s})$ is not possible.

In the corresponding literature we can find the attempts of discriminating the spherical parts of a stress tensor also for the anisotropic bodies. Huber's pupil, W. Burzyński, in his Ph.D. dissertation [1], formulated the hypothesis that there is no physical reason against the introduction of the decomposition of the elastic energy into these two components $\Phi(\sigma\mathbf{I})$ and $\Phi(\mathbf{s})$ (1.6) in the case of anisotropic bodies as well.

The Burzynski hypothesis is equivalent to assuming in Eq. (1.9) the condition

$$(1.11) \qquad\qquad \mathbf{I} \cdot \mathbf{C} \cdot \mathbf{s} = 0,$$

what means that all anisotropic bodies are *voluminally isotropic.*

The unit tensor $\mathbf{I}$ is then the eigenstate for the compliance tensor $\mathbf{C}$, i.e.

$$(1.12) \qquad\qquad \mathbf{C} \cdot \mathbf{I} = \lambda\mathbf{I}.$$

In the arbitrary Cartesian coordinate system with orthonormal base $\mathbf{m}_k$, condition (1.12) is equivalent to the set of the equations:

$$(1.13) \qquad C_{1211} + C_{1222} + C_{1233} = 0,$$

$$(1.14) \qquad C_{1311} + C_{1322} + C_{1333} = 0,$$

$$(1.15) \qquad C_{2311} + C_{2322} + C_{2333} = 0,$$

$$(1.16) \qquad C_{1111} - C_{2222} = C_{2233} - C_{1133},$$

$$(1.17) \qquad C_{1111} - C_{3333} = C_{2233} - C_{1122}.$$

Equations (1.13)–(1.17) represent certain constraints imposed on the type of anisotropy. The number of independent components of the compliance tensor $\mathbf{C}$ is then reduced from 21 to 16.

The Burzynski conditions (1.13)–(1.17) are satisfied identically in cases of isotropy and in materials with cubic symmetry. In other cases the conditions introduce certain additional limitations.

Certain attempts of formulating the limit criteria for some classes of anisotropy were made in papers [8, 9]. The problem has been solved completely by J. Rychlewski in the paper [12].

Rychlewski, looking for the limit condition in the form proposed by Mises [17]

$$(1.18) \qquad \sigma \cdot \mathbf{H} \cdot \sigma \le 1,$$

introduced the notion of *energy-orthogonal* states of stress and proved that every condition of the form (1.18) had an energetic sense. It means that each quadratic criterion (1.18) has a definite energy-based interpretation.

It is a pity that paper [12] of such a fundamental importance, has not appeared in English translation till now. The paper [12] in its present form doesn't contain any examples of application of the obtained results to the derivation of the limit conditions for some types of anisotropy.

This paper is aimed at following the way of reasoning of Rychlewski [12] in formulating the limit condition (1.18) for the case of transversal isotropy.

Transversal isotropy is selected because it is the type of anisotropy characterized by the highest symmetry properties, for which the spherical tensor is no more a proper elastic state.

## 2. Main energy-orthogonal decomposition

According to the definition given in paper [12] two states of stress $\alpha, \beta \in S$ are called *energy-orthogonal* if

$$(2.1) \qquad \alpha \times \beta \equiv \alpha \cdot \mathbf{C} \cdot \beta = 0.$$

Equality (2.1) means that the state of stress $\boldsymbol{\alpha}$ does not perform any work on the deformations produced by the state of stress $\boldsymbol{\beta}$ and vice versa.

It is easy to prove that the proper elastic states of the compliance tensor $\mathbf{C}$ are energy-orthogonal as well (see also [13, 14]).

It is well known that the proper states of $\mathbf{C}$ [11] corresponding to various elastic moduli (Kelvin moduli) are orthonormal

$$(2.2) \qquad \boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = \delta_{KL}.$$

If

$$(2.3) \qquad \mathbf{C} \cdot \boldsymbol{\omega}_K = \lambda^{-1} \boldsymbol{\omega}_K$$

then

$$(2.4) \qquad \boldsymbol{\omega}_L \cdot \mathbf{C} \cdot \boldsymbol{\omega}_K = \boldsymbol{\omega}_L \cdot \lambda^{-1} \boldsymbol{\omega}_K = \lambda^{-1} \delta_{LK}$$

and for $K \neq L$

$$(2.5) \qquad \boldsymbol{\omega}_L \cdot \mathbf{C} \cdot \boldsymbol{\omega}_K = 0.$$

Equation (2.5) means that the spectral decomposition [11] is at the same time the energy-orthogonal decomposition. It is not true inversely. The spectral decomposition in RYCHLEWSKI's paper [12] was called *the main energy-orthogonal decomposition.*

The approach to the problem of formulation of a quadratic limit condition (1.18) will be illustrated here by an example of transversal isotropy.

In order to find the main energy-orthogonal decomposition, the spectral decomposition must be determinated first.

Transversal isotropy was already considered in papers [9] and [11]. In those papers the main energy-orthogonal decomposition was used. It means that elastic and plastic properties were dependent.

The material is called *transversally isotropic* if it contains a certain direction $\mathbf{k} \otimes \mathbf{k}$ such that all shearings of the type

$$(2.6) \qquad \begin{aligned} \boldsymbol{\tau} &= \mathbf{a} \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{a}, \\ \boldsymbol{\tau} &= \mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a} \end{aligned} \quad , \quad \text{where} \quad \mathbf{a} \cdot \mathbf{k} = \mathbf{b} \cdot \mathbf{k} = \mathbf{a} \cdot \mathbf{b} = 0$$

are the eigenstates for the compliance tensor $\mathbf{C}$.

The anisotropic elastic properties are represented by the fourth-rank tensor $\mathbf{C}$. In this paper we will follow the notation representing the elastic coefficients as a second-rank tensor in a six-dimensional space. Stress and strain are considered then as vectors in a six-dimensional Cartesian space as well as second-rank tensors in three-dimensional Cartesian reference system.

The space $S$ of symmetric tensors of second-rank is six-dimensional. Base (polybase) of this space will be created by six linearly independent tensors. If in a physical space the base has the form of a orthonormal triad of vectors $\mathbf{m}_k$ selected so that $\mathbf{m}_3 = \mathbf{k}$, then the bases in $S$ are formed by dyads $\mathbf{m}_k \otimes \mathbf{m}_l$.

There are infinite number of polybases in $S$. As the most natural base in $S$ we consider the orthonormal polybase $\mathbf{e}_\beta \in S$ $(\beta = I, II, ..., VI)$ of the form

$$(2.7) \qquad \mathbf{e}_I = \mathbf{m}_1 \otimes \mathbf{m}_1 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$(2.8) \qquad \mathbf{e}_{II} = \mathbf{m}_2 \otimes \mathbf{m}_2 \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$(2.9) \qquad \mathbf{e}_{III} = \mathbf{k} \otimes \mathbf{k} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$(2.10) \qquad \mathbf{e}_{IV} = \frac{1}{\sqrt{2}}(\mathbf{m}_2 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{m}_2) \sim \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$(2.11) \qquad \mathbf{e}_V = \frac{1}{\sqrt{2}}(\mathbf{k} \otimes \mathbf{m}_1 + \mathbf{m}_1 \otimes \mathbf{k}) \sim \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$(2.12) \qquad \mathbf{e}_{VI} = \frac{1}{\sqrt{2}}(\mathbf{m}_1 \otimes \mathbf{m}_2 + \mathbf{m}_2 \otimes \mathbf{m}_1) \sim \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Three-by-three matrices representing the components of the tensors $\mathbf{e}_\beta$ are taken in the base $\mathbf{m}_k$.

In order to discuss the Burzynski conditions $(1.13)-(1.17)$ it will be more convenient to consider another polybase $\mathbf{a}_\alpha \in S$ in which the spherical tensor is discriminated. The remaining tensors are deviators.

Tensors $\mathbf{a}_\alpha \in S$ $(\alpha = I, II, ..., VI)$ also form the orthonormal base in $S$ and can be written as follows

$$(2.13) \qquad \mathbf{a}_I = \frac{1}{\sqrt{3}}(\mathbf{m}_1 \otimes \mathbf{m}_1 + \mathbf{m}_2 \otimes \mathbf{m}_2 + \mathbf{k} \otimes \mathbf{k}) \sim \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$(2.14) \quad \mathbf{a}_{II} = \frac{1}{\sqrt{2}}(\mathbf{m}_1 \otimes \mathbf{m}_1 - \mathbf{m}_2 \otimes \mathbf{m}_2) \sim \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$(2.15) \quad \mathbf{a}_{III} = \frac{1}{\sqrt{6}}(\mathbf{m}_1 \otimes \mathbf{m}_1 + \mathbf{m}_2 \otimes \mathbf{m}_2 - 2\mathbf{k} \otimes \mathbf{k}) \sim \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

$$(2.16) \quad \mathbf{a}_{IV} = \mathbf{e}_{IV},$$

$$(2.17) \quad \mathbf{a}_V = \mathbf{e}_V,$$

$$(2.18) \quad \mathbf{a}_{VI} = \mathbf{e}_{VI}.$$

The basis $\mathbf{a}_\alpha$ and $\mathbf{e}_\beta$ are connected by the rotation $Q_{\alpha\beta}$ in six-dimensional space, namely

$$(2.19) \quad \mathbf{a}_\alpha = Q_{\alpha\beta}\mathbf{e}_\beta,$$

where the matrix $Q_{\alpha\beta}$ has the form

$$(2.20) \quad Q_{KL} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In the case of transversal isotropy the fourth-rank compliance tensor $\mathbf{C}$, which has components $C_{ijkl}$ relative to the base $\mathbf{m}_k$, is represented in the polybase $\mathbf{e}_\beta$ by the following six-by-six matrix [15]:

$$(2.21) \quad \mathbf{C} \sim C_{\alpha\beta} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{1122} & C_{1111} & C_{1133} & 0 & 0 & 0 \\ C_{1133} & C_{1133} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C_{1313} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1111} - C_{1122} \end{bmatrix}.$$

It means that transversal isotropy is described by only five independent components of **C**, namely

(2.22) $\qquad C_{1111}, \quad C_{3333}, \quad C_{1122}, \quad C_{1133}, \quad C_{1313}.$

The matrix $C_{\alpha\beta}$ (2.21) after rotation (2.20) will change its form according to the formula

(2.23) $\qquad \tilde{C}_{MN} = Q_{M\alpha} C_{\alpha\beta} Q_{N\beta}.$

The matrix $\tilde{C}_{MN}$ represents the tensor **C** in the polybase $\mathbf{a}_\alpha$ and can be expressed as

(2.24) $\mathbf{C} \sim \tilde{C}_{MN}$

$$
= \begin{bmatrix}
A & 0 & B & 0 & 0 & 0 \\
0 & C_{1111} - C_{1122} & 0 & 0 & 0 & 0 \\
B & 0 & D & 0 & 0 & 0 \\
0 & 0 & 0 & 2C_{1313} & 0 & 0 \\
0 & 0 & 0 & 0 & 2C_{1313} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{1111} - C_{1122}
\end{bmatrix}.
$$

The following notations are introduced:

(2.25) $\qquad A = \dfrac{1}{3}(2C_{1111} + 2C_{1122} + 4C_{1133} + C_{3333}),$

(2.26) $\qquad B = \dfrac{\sqrt{2}}{3}(C_{1111} + C_{1122} - C_{1133} - C_{3333}),$

(2.27) $\qquad D = \dfrac{1}{3}(C_{1111} + C_{1122} - 4C_{1133} + 2C_{3333}).$

In order to construct the main energy-orthogonal decomposition of the space $S$ for transversal isotropy, spectral decomposition of elasticity tensor has to be found out.

Spectral decomposition of elasticity tensor [11] opens completely new possibilities for comparing elastic materials. The spectral decomposition of the compliance tensor **C** is known if there are known all eigenvalues $\dfrac{1}{\lambda_K}$ and eigenstates $\omega_K$ of **C**.

Elastic moduli $\lambda_K$ (eigenvalues of the stiffness tensor **S**) will be called after Rychlewski – *the Kelvin moduli*. For the eigenvalue $\dfrac{1}{\lambda^*}$ of multiplicity one,

the proper state $\omega^*$ – eigentensor corresponding to it, is given uniquely by the following relation

$$(2.28) \qquad \mathbf{C} \cdot \omega^* = \frac{1}{\lambda^*} \omega^*.$$

In this case the orthogonal projector $\mathbf{P}^*$ has the form

$$(2.29) \qquad \mathbf{P}^* = \omega^* \otimes \omega^*.$$

When Kelvin moduli are not distinct, i.e. there are some eigenvalues of multiplicity two, three or more, then there exist infinite number of possible eigentensors from which the basic eigentensors may be selected. They create the proper subspaces $P_K$ [11]. The subspace $P_K$ contains all proper elastic states corresponding to the elastic modulus $\lambda_K$. The orthogonal projectors $\mathbf{P}_K$ defined uniquely, map the space $S$ onto subspaces $P_K$.

$$(2.30) \qquad \mathbf{P}_K \cdot \sigma = \sigma_K \in P_K.$$

In the polybase $\omega_K$ – eigentensors, which are selected so that the form an orthonormal set in six-dimensional space, the matrix six-by-six for $\mathbf{C}$ has a diagonal form with eigenvalues on the diagonal.

From (2.24) it is implied that for transversal isotropy, the following components

$$(2.31) \qquad 2C_{1313}, \quad C_{1111} - C_{1122}$$

are eigenvalues for $\mathbf{C}$ of multiplicity two. If we denote respectively by

$$(2.32) \qquad \frac{1}{\lambda_3} = 2C_{1313}, \quad \frac{1}{\lambda_4} = C_{1111} - C_{1122},$$

then the proper subspaces $P_3$ and $P_4$ corresponding to them are two-dimensional. They are created by tensors of the following form:

$$(2.33) \qquad \sigma_3 \sim \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & q \\ p & q & 0 \end{pmatrix} \in P_3, \quad \sigma_4 \sim \begin{pmatrix} u & v & 0 \\ v & -u & 0 \\ 0 & 0 & 0 \end{pmatrix} \in P_4.$$

An orthonormal base in the subspace $P_3$ may be taken as follows:

$$(2.34) \qquad \omega_{III} = \mathbf{e}_V = \frac{1}{\sqrt{2}} (\mathbf{m}_1 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{m}_1),$$

$$(2.35) \qquad \omega_{IV} = \mathbf{e}_{IV} = \frac{1}{\sqrt{2}} (\mathbf{m}_2 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{m}_2).$$

Two tensors of the form

$$(2.36) \qquad \boldsymbol{\omega}_V = \mathbf{a}_{II} = \frac{1}{\sqrt{2}}(\mathbf{e}_I - \mathbf{e}_{II}) = \frac{1}{\sqrt{2}}(\mathbf{m}_1 \otimes \mathbf{m}_1 - \mathbf{m}_2 \otimes \mathbf{m}_2),$$

$$(2.37) \qquad \boldsymbol{\omega}_{VI} = \mathbf{e}_{VI} = \frac{1}{\sqrt{2}}(\mathbf{m}_1 \otimes \mathbf{m}_2 + \mathbf{m}_2 \otimes \mathbf{m}_1)$$

may be selected as a base in $P_4$ .

Two distinct Kelvin moduli $\lambda_1$ and $\lambda_2$ are obtained from the characteristic equation

$$(2.38) \qquad \det \begin{pmatrix} A - \dfrac{1}{\lambda} & B \\[2mm] B & D - \dfrac{1}{\lambda} \end{pmatrix} = \left(\dfrac{1}{\lambda}\right)^2 - (A+D)\dfrac{1}{\lambda} + AD - B^2 = 0.$$

Both moduli $\lambda_1$ and $\lambda_2$ are of multiplicity one and have the form

$$(2.39) \qquad \lambda_1^{-1} = \frac{1}{2}\left[A + D - \sqrt{(A-D)^2 + 4B^2}\right],$$

$$(2.40) \qquad \lambda_2^{-1} = \frac{1}{2}\left[A + D + \sqrt{(A-D)^2 + 4B^2}\right].$$

The parameters A, B, D are described by relations (2.25)–(2.27).

The proper states corresponding to $\lambda_1$ and $\lambda_2$ (2.39)–(2.40) create two one-dimensional subspaces $P_1$ and $P_2$. They are orthogonal to each other and orthogonal to $P_3$ and $P_4$.

From (2.24) it is implied that the eigentensors, proper states $\boldsymbol{\omega}_I$ and $\boldsymbol{\omega}_{II}$ corresponding to $\lambda_1$ and $\lambda_2$, may be obtained from the tensors $\mathbf{a}_I$ (2.13) and $\mathbf{a}_{III}$ (2.15), by rotation, namely

$$(2.41) \qquad \boldsymbol{\omega}_I = \cos(\aleph - \aleph_0)\mathbf{a}_I + \sin(\aleph - \aleph_0)\mathbf{a}_{III},$$

$$(2.42) \qquad \boldsymbol{\omega}_{II} = -\sin(\aleph - \aleph_0)\mathbf{a}_I + \cos(\aleph - \aleph_0)\mathbf{a}_{III},$$

where

$$(2.43) \qquad \aleph_0 \Rightarrow \tan \aleph_0 = \sqrt{2}$$

and

$$(2.44) \qquad \tan 2(\aleph - \aleph_0) = \frac{2B}{A - D} \quad (A \neq D).$$

Thus substituting values of $\mathbf{a}_I$ and $\mathbf{a}_{III}$ into (2.41) – (2.42) we immediately arrive at the result

$$(2.45) \qquad \boldsymbol{\omega}_I = \frac{1}{\sqrt{2}}\left[\sin\aleph\mathbf{I} + \sqrt{3}\sin(\aleph_0 - \aleph)\mathbf{k}\otimes\mathbf{k}\right],$$

$$(2.46) \qquad \boldsymbol{\omega}_{II} = \frac{1}{\sqrt{2}}\left[\cos\aleph\mathbf{I} - \sqrt{3}\cos(\aleph_0 - \aleph)\mathbf{k}\otimes\mathbf{k}\right].$$

Tensors $\boldsymbol{\omega}_I$ and $\boldsymbol{\omega}_{II}$ in the base $\mathbf{m}_k$ have the following matrix representations:

$$(2.47) \qquad
\begin{aligned}
\boldsymbol{\omega}_I &\sim \frac{1}{\sqrt{2}}\begin{pmatrix} \sin\aleph & 0 & 0 \\ 0 & \sin\aleph & 0 \\ 0 & 0 & \sqrt{2}\cos\aleph \end{pmatrix}, \\[2ex]
\boldsymbol{\omega}_{II} &\sim \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\aleph & 0 & 0 \\ 0 & \cos\aleph & 0 \\ 0 & 0 & -\sqrt{2}\sin\aleph \end{pmatrix}.
\end{aligned}$$

Finally the spectral decomposition of the compliance tensor $\mathbf{C}$ (1.1) for the transversal isotropy has the form:

$$(2.48) \quad \begin{aligned}
\mathbf{C} &= \frac{1}{\lambda_1}\mathbf{P}_1 + \frac{1}{\lambda_2}\mathbf{P}_2 + \frac{1}{\lambda_3}\mathbf{P}_3 + \frac{1}{\lambda_4}\mathbf{P}_4 \\
&= \frac{1}{\lambda_1}\boldsymbol{\omega}_I \otimes \boldsymbol{\omega}_I + \frac{1}{\lambda_2}\boldsymbol{\omega}_{II} \otimes \boldsymbol{\omega}_{II} + \frac{1}{\lambda_3}(\boldsymbol{\omega}_{III}\otimes\boldsymbol{\omega}_{III} + \boldsymbol{\omega}_{IV}\otimes\boldsymbol{\omega}_{IV}) \\
&\qquad + \frac{1}{\lambda_4}(\boldsymbol{\omega}_V\otimes\boldsymbol{\omega}_V + \boldsymbol{\omega}_{VI}\otimes\boldsymbol{\omega}_{VI}).
\end{aligned}$$

The Kelvin moduli $\lambda_1$, $\lambda_2$, $\lambda_3$ i $\lambda_4$ are given by formulae (2.32) and (2.39) – (2.40).

The spectral decomposition of the space $S = P_1 \oplus P_2 \oplus P_3 \oplus P_4$ is in the same time the main energy-orthogonal decomposition. Hence transversal isotropy is described now by $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $\aleph$ instead of the parameters (2.22).

Decomposing a stress $\boldsymbol{\sigma} \in S$ into the parts $\boldsymbol{\sigma}_K$, $\boldsymbol{\sigma}_K = \mathbf{P}_K \cdot \boldsymbol{\sigma}$ in the proper subspaces $P_K$ we obtain that

$$(2.49) \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_4, \quad \boldsymbol{\sigma}_K \in P_K.$$

The above decomposition has a unique form. Since this decomposition is energy-orthogonal as well, then the elastic energy $\Phi(\boldsymbol{\sigma})$ (1.5) may be written as the following sum:

$$(2.50) \qquad \Phi(\boldsymbol{\sigma}) = \Phi(\boldsymbol{\sigma}_1) + \Phi(\boldsymbol{\sigma}_2) + \Phi(\boldsymbol{\sigma}_3) + \Phi(\boldsymbol{\sigma}_4)$$

where

$$(2.51) \quad \Phi(\sigma_1) = \frac{1}{2}\sigma_1 \cdot \mathbf{C} \cdot \sigma_1$$
$$= \frac{1}{2\lambda_1}\sigma_1 \cdot \sigma_1 = \frac{1}{4\lambda_1}\left[\sin\aleph \mathrm{tr}\sigma + \sqrt{3}\sin(\aleph_0 - \aleph)\mathbf{k}\cdot\sigma\cdot\mathbf{k}\right]^2,$$

$$(2.52) \quad \Phi(\sigma_2) = \frac{1}{2}\sigma_2 \cdot \mathbf{C} \cdot \sigma_2$$
$$= \frac{1}{2\lambda_2}\sigma_2 \cdot \sigma_2 = \frac{1}{4\lambda_2}\left[\cos\aleph \mathrm{tr}\sigma - \sqrt{3}\cos(\aleph_0 - \aleph)\mathbf{k}\cdot\sigma\cdot\mathbf{k}\right]^2,$$

$$(2.53) \quad \Phi(\sigma_3) = \frac{1}{2}\sigma_3 \cdot \mathbf{C} \cdot \sigma_3$$
$$= \frac{1}{2\lambda_3}\sigma_3 \cdot \sigma_3 = \frac{1}{2\lambda_3}\left[(\mathbf{m}_1 \cdot \sigma \cdot \mathbf{k})^2 + (\mathbf{m}_2 \cdot \sigma \cdot \mathbf{k})^2\right],$$

$$(2.54) \quad \Phi(\sigma_4) = \frac{1}{2}\sigma_4 \cdot \mathbf{C} \cdot \sigma_4$$
$$= \frac{1}{2\lambda_4}\sigma_4 \cdot \sigma_4 = \frac{1}{2\lambda_4}\left\{[(\mathbf{m}_1\cdot\sigma\cdot\mathbf{m}_1) - (\mathbf{m}_2\cdot\sigma\cdot\mathbf{m}_2)]^2 + 4(\mathbf{m}_1\cdot\sigma\cdot\mathbf{m}_2)^2\right\}.$$

From (2.50) – (2.54) it is implied that

$$(2.55) \quad 2\Phi(\sigma) = \sigma \cdot \mathbf{C} \cdot \sigma = \frac{1}{2\lambda_1}\left[\sin\aleph \mathrm{tr}\sigma + \sqrt{3}\sin(\aleph_0 - \aleph)\mathbf{k}\sigma\mathbf{k}\right]^2$$
$$+ \frac{1}{2\lambda_2}\left[\cos\aleph \mathrm{tr}\sigma - \sqrt{3}\cos(\aleph_0 - \aleph)\mathbf{k}\sigma\mathbf{k}\right]^2$$
$$+ \frac{1}{\lambda_3}\left[(\mathbf{m}_1 \cdot \sigma \cdot \mathbf{k})^2 + (\mathbf{m}_2 \cdot \sigma \cdot \mathbf{k})^2\right]$$
$$+ \frac{1}{\lambda_4}\left\{[(\mathbf{m}_1 \cdot \sigma \cdot \mathbf{m}_1) - (\mathbf{m}_2 \cdot \sigma \cdot \mathbf{m}_2)]^2 + 4(\mathbf{m}_1 \cdot \sigma \cdot \mathbf{m}_2)^2\right\}.$$

The limit condition of the Mises type (1.18) representing a generalization of the Maxwell-Huber-Mises condition for transversal isotropy may be taken in the form [9]:

$$(2.56) \quad \frac{1}{h_1}\Phi(\sigma_1) + \frac{1}{h_2}\Phi(\sigma_2) + \frac{1}{h_3}\Phi(\sigma_3) + \frac{1}{h_4}\Phi(\sigma_4) \leq 1$$

where $h_\alpha$ are energy limits of elasticity $\Phi(\sigma_\alpha)$ (2.51) – (2.54).

It means that the limit criterion (2.56) bounds the weighted sum of stored energies, corresponding to uniquely defined, energy–orthogonal parts of stress. Taking the yield condition in the form (2.56) we assumed that the tensors $\mathbf{C}$ and $\mathbf{H}$ were coaxial.

Let us denote by $\mathbf{M}$ the dyad $\mathbf{k} \otimes \mathbf{k}$ i.e.

$$(2.57) \qquad \mathbf{M} = \mathbf{k} \otimes \mathbf{k},$$

then the elastic energies $\Phi(\boldsymbol{\sigma}_K)$ may be expressed in the invariant form [10]:

$$(2.58) \quad \Phi(\boldsymbol{\sigma}_1) = \frac{1}{4\lambda_1} \left[ \sqrt{2} \cos \aleph \left( \mathrm{tr}\mathbf{M}s + \frac{1}{3}\mathrm{tr}\boldsymbol{\sigma} \right) - \sin \aleph \left( \mathrm{tr}\mathbf{M}s - \frac{2}{3}\mathrm{tr}\boldsymbol{\sigma} \right) \right]^2,$$

$$(2.59) \quad \Phi(\boldsymbol{\sigma}_2) = \frac{1}{4\lambda_2} \left[ \cos \aleph \left( \frac{2}{3}\mathrm{tr}\boldsymbol{\sigma} - \mathrm{tr}\mathbf{M}s \right) - \sqrt{2} \sin \aleph \left( \mathrm{tr}\mathbf{M}s + \frac{1}{3}\mathrm{tr}\boldsymbol{\sigma} \right) \right]^2,$$

$$(2.60) \quad \Phi(\boldsymbol{\sigma}_3) = \frac{1}{\lambda_3} \left[ \mathrm{tr}\mathbf{M}s^2 - (\mathrm{tr}\mathbf{M}s)^2 \right],$$

$$(2.61) \quad \Phi(\boldsymbol{\sigma}_4) = \frac{1}{2\lambda_4} \left[ \mathrm{tr}s^2 - 2\mathrm{tr}\mathbf{M}s^2 + \frac{1}{2}(\mathrm{tr}\mathbf{M}s)^2 \right].$$

Denoting by $\sigma_{ij}$ components of a stress tensor $\boldsymbol{\sigma}$ in the base $\mathbf{m}_k$, the following symbols can be introduced:

$$(2.62) \qquad r = \frac{1}{2 + (1-\gamma)^2} \left( \sigma_{11} + \sigma_{22} + (1-\gamma)\sigma_{33} \right),$$

$$(2.63) \qquad s = \frac{1-\gamma}{2(2 + (1-\gamma)^2)} \left( \sigma_{11} + \sigma_{22} - \frac{2}{1-\gamma}\sigma_{33} \right),$$

$$(2.64) \qquad u = \frac{1}{2} \left( \sigma_{11} - \sigma_{22} \right),$$

$$(2.65) \qquad v = \sigma_{12}, \quad p = \sigma_{13}, \quad q = \sigma_{23},$$

and

$$(2.66) \qquad 1 - \gamma = \sqrt{2} \cot \aleph.$$

The graphical illustration of the parts of stress $\boldsymbol{\sigma}_K \in P_K$ (2.49) is presented in Fig. 1.
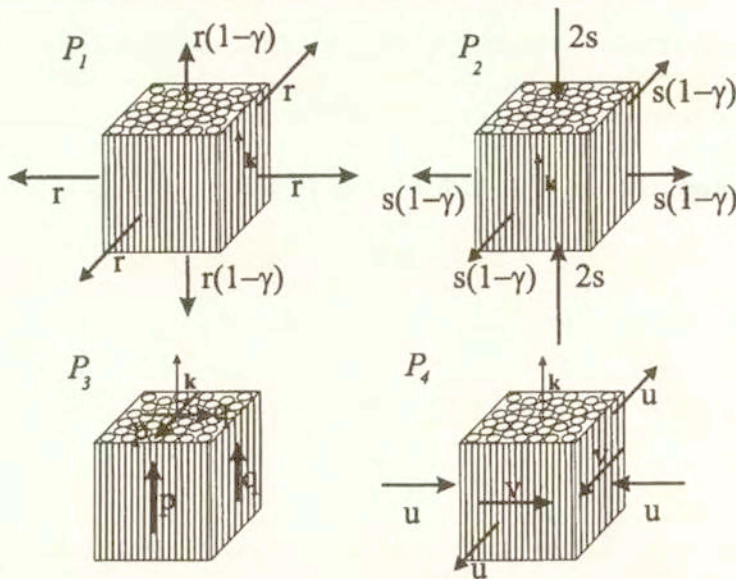
FIG. 1.  Graphical illustration of subspaces $P_K$ of the proper states of **C** for transversally isotropic solids.

## 3. Safe states of stress

The energy is the most universal physical notion. The idea that the stored elastic energy is an appropriate measure of the mechanical behaviour of the elastic material is quite widely acceptable.

The classical Maxwell-Huber-Mises condition (1.7) limits the distortion energy only. Consequently, plastic deformations occur in the plastic shaping process and are not accompanied by volume changes. It means that the spherical parts of stress tensors are safe.

In case of anisotropic bodies, there is no physical reason to consider the hydrostatic state as a safe stress. Depending on the type of anisotropy, different states can be taken as safe stresses. Using an example of transversal isotropy we assume that two different states of stresses are safe.

The limit condition (2.56) which is based on the main energy-orthogonal decomposition is discussed. If energy limit of elasticity $h_K$ tends to infinity ($h_K \to \infty$) then the subspace $P_K$ consists of safe stresses. In contrast, while a state of stress $\sigma_K^* \in P_K$ ($\sigma_K^*$ is a proper state) is a safe stress then the energy $\Phi(\sigma_K^*)$ does not enter the limit condition. It means that $h_K$, an elasticity limit for $\Phi(\sigma_K^*)$, tends to infinity.

The situation becomes more complicated if the proposed safe stress is not a proper state for the compliance tensor $\mathbf{C}$. Still the limit conditions (2.56) is under consideration.

Now let us assume that the hydrostatic pressure is the safe stress as it is for isotropy. We remind that the spherical tensor is not a proper state for transversal isotropy. Now, we consider two different states of stress

$$(3.1) \qquad \boldsymbol{\sigma}^a = \sigma_a \mathbf{I} + \mathbf{s}, \quad \boldsymbol{\sigma}^b = \sigma_b \mathbf{I} + \mathbf{s}, \quad \sigma_a \neq \sigma_b,$$

with the same deviatoric parts $\mathbf{s}$ and different isotropic parts.

Since it has been assumed that the hydrostatic pressure is safe, the function (2.56) in the limit state should have the same value for both states of stress, namely

$$(3.2) \qquad \frac{1}{h_1}\Phi(\sigma_1^a) + \frac{1}{h_2}\Phi(\sigma_2^a) + \frac{1}{h_3}\Phi(\sigma_3^a) + \frac{1}{h_4}\Phi(\sigma_4^a)$$

$$= \frac{1}{h_1}\Phi(\sigma_1^b) + \frac{1}{h_2}\Phi(\sigma_2^b) + \frac{1}{h_3}\Phi(\sigma_3^b) + \frac{1}{h_4}\Phi(\sigma_4^b),$$

where $\sigma_K^a = \mathbf{P}_K \cdot \boldsymbol{\sigma}^a$ and $\sigma_K^b = \mathbf{P}_K \cdot \boldsymbol{\sigma}^b$. The above condition, after using Eqs. (2.51)–(2.52) can be rewritten in the form

$$(3.3) \qquad \frac{1}{h_1}(\Phi(\sigma_1^a) - \Phi(\sigma_1^b)) + \frac{1}{h_2}(\Phi(\sigma_2^a) - \Phi(\sigma_2^b)) = 0.$$

Finally, after substituting (2.51)–(2.52) into (3.3) and taking advantage of the fact that $\sigma_a \neq \sigma_b$ we obtain the following equation:

$$(3.4) \quad \frac{1}{6}(\sigma_a + \sigma_b)\left[\frac{\cos^2(\aleph - \aleph_0)}{\lambda_1 h_1} + \frac{\sin^2(\aleph - \aleph_0)}{\lambda_2 h_2}\right]$$

$$+ \frac{\sin 2(\aleph - \aleph_0)}{2\sqrt{3}}(\mathbf{s} \cdot \mathbf{a}_{III})\left[\frac{1}{\lambda_1 h_1} - \frac{1}{\lambda_2 h_2}\right] = 0,$$

which should be satisfied by any $\sigma_a$, $\sigma_b$ and $\mathbf{s}$. In particular it is convenient to assume that for $\mathbf{s} = 0$, two following cases are considered:

$$(3.5) \qquad \sigma_b = -\sigma_a \quad \text{and} \quad \sigma_b \neq -\sigma_a.$$

By combining the above assumptions we obtain from (3.4) the following conditions:

- $\dfrac{1}{h_1} = \dfrac{1}{h_2} = 0, \quad$ when $\aleph \neq \aleph_0$

- $\dfrac{1}{h_1} = 0$ and $\dfrac{1}{h_2} \neq 0$, when $\aleph = \aleph_0$ (the spherical tensor is the proper state of $\mathbf{C}$).

We note that for $\aleph \neq \aleph_0$ the spherical tensor can be a safe state only then when the subspace $P_1 \oplus P_2 \in S$ is the subspace of safe stresses.

As an example of transversally isotropic materials, the fibre-reinforced composites are considered.

Let us suppose that tensions in the privileged fiber direction $\mathbf{k}$ are safe stresses. It means that the fibres are inextensible or they are much stronger than the matrix material. In a general case the tension in the $\mathbf{k}$ direction is not a proper state for transversal isotropy.

Now, we consider the following two states of stress

$$(3.6) \qquad \boldsymbol{\sigma}^c = \sigma_c \mathbf{k} \otimes \mathbf{k} + \mathbf{p}, \quad \boldsymbol{\sigma}^d = \sigma_d + \mathbf{p}, \quad \sigma_c \neq \sigma_d \quad \text{and} \quad \mathbf{k} \cdot \mathbf{p} \cdot \mathbf{k} = 0$$

only with different projectors on the $\mathbf{k}$ direction.

Using the analogous way as above we conclude that the state $a\mathbf{k} \otimes \mathbf{k}$ is safe only when:

- $\dfrac{1}{h_1} = \dfrac{1}{h_2} = 0$, for $\aleph \neq 0$,

- $\dfrac{1}{h_1} = 0$ and $\dfrac{1}{h_2} \neq 0$, for $\aleph = 0$ (it means that the state $a\mathbf{k} \otimes \mathbf{k}$ is the proper state for $\mathbf{C}$).

We note that for $\aleph \neq 0$ the obtained conditions are, at the same time, the conditions which are satisfied when all stress states from the subspace $P_1 \oplus P_2$ are safe, including the states for which $\mathbf{k} \cdot \boldsymbol{\sigma} \cdot \mathbf{k} = 0$.

In the real material the assumptions that the spherical tensor and the tension in the $\mathbf{k}$ direction are safe states do not necessary implicate that any state of stress $\boldsymbol{\sigma} \in P_1 \oplus P_2$ for which $\mathrm{tr}\boldsymbol{\sigma} = 0$ or $\mathbf{k} \cdot \boldsymbol{\sigma} \cdot \mathbf{k} = 0$ is safe.

The proposed yield condition (2.56) is one of the possible generalizations of the Maxwell-Huber-Mises condition for the case of anisotropy. We have established it for anisotropic bodies for which elastic and plastic properties are dependent, namely the tensors $\mathbf{C}$ and $\mathbf{H}$ are coaxial. In a general case the fourth-rank tensor $\mathbf{H}$ proposed by Mises [17] can be assumed to be arbitrary.

## 4. Physical interpretation of material parameters for transversally isotropic solids

Transversal isotropy is described by the following five parameters: four Kelvin moduli – $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and one stiffness distributor $\aleph$.

In order to determine these parameters some experimental tests should be proposed. The values of the moduli $\lambda_3$ and $\lambda_4$ can be determined without special

difficulties. Taking for each of them two states of stress from the subspace $P_3$ or $P_4$ respectively, we find relations between stresses and strains (see Fig.2). Hence, it is implied that $\lambda_3$ and $\lambda_4$ are as follows:

$$(4.1) \qquad\qquad \tan\varphi_1 = \lambda_3, \quad \tan\varphi_2 = \lambda_4$$

where the angles $\varphi_1$ and $\varphi_2$ are shown in Fig. 2.
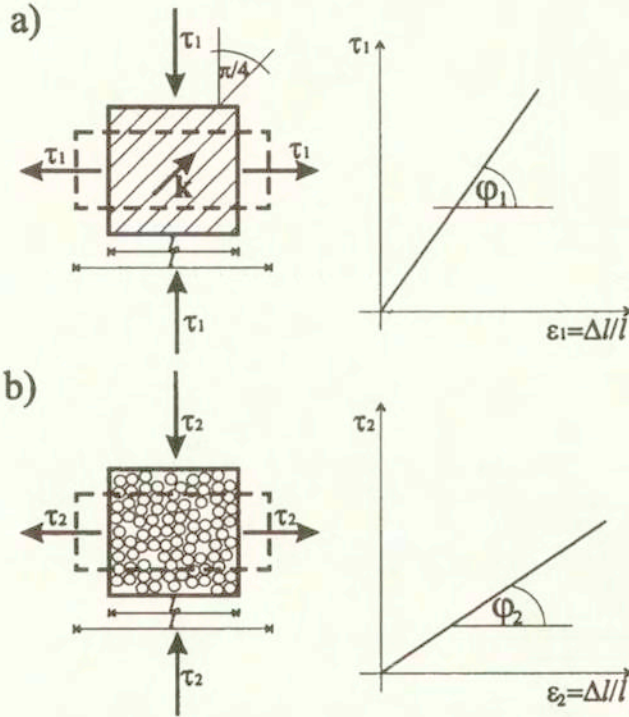


FIG. 2. Proposed experimental tests useful in determining the Kelvin moduli $\lambda_3$(a) and $\lambda_4$(b).

Calculations of the remaining three parameters are more complicated. We propose the experimental tests illustrated in Fig. 3 together with the obtained stress-strain relations.

Both the proposed states of stress have the orthogonal projections onto subspaces $P_1$ and $P_2$.

For simplicity, the following notations are introduced:

$$(4.2) \quad r_3 = \frac{\varepsilon_l}{\varepsilon_a} = \frac{\varepsilon_{33}}{\varepsilon_{11}}, \quad \text{(Fig. 3a)}; \quad r_4 = \frac{\varepsilon_b}{\varepsilon_h} = \frac{\varepsilon_{11}}{\varepsilon_{33}}, \quad \text{(Fig. 3b)}; \quad \bar{r} = r_3 - 2r_4.$$

If the value of $\bar{r}$ is known then the stiffness distributor $\aleph$ can be found from the relation

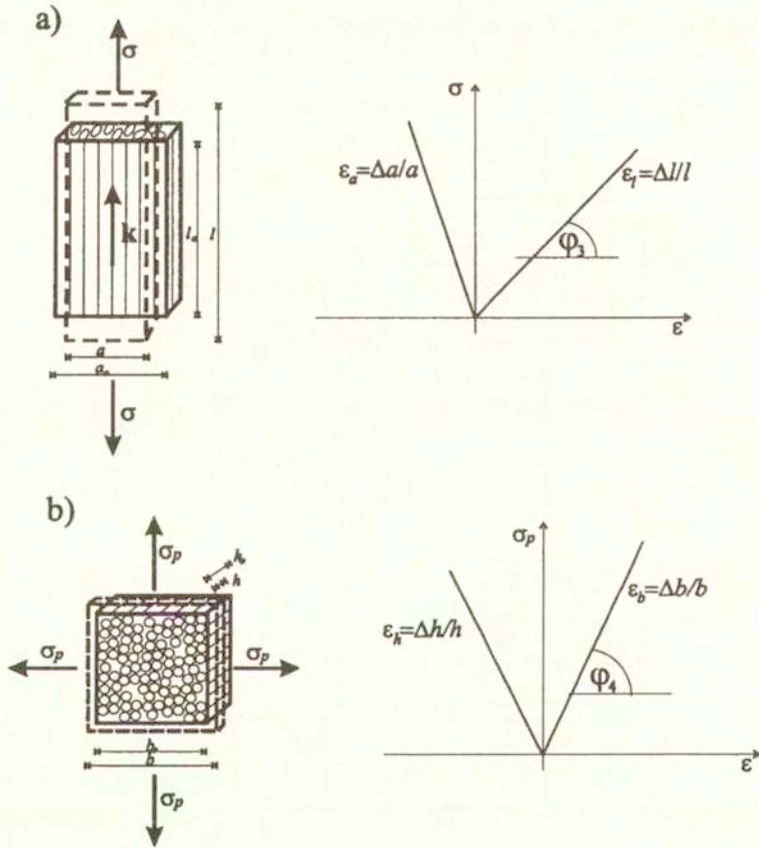$$(4.3) \qquad \cot 2\aleph = \frac{\sqrt{2}}{4}\bar{r}.$$



FIG. 3. Proposed experimental tests useful in determining the Kelvin moduli $\lambda_1$ and $\lambda_2$ as well as the stiffness distributor $\aleph$.

We note that $\bar{r} = -1$ for cubic symmetry and isotropy.

The Kelvin moduli $\lambda_1$ and $\lambda_2$ for a known value of the stiffness distributor $\aleph$ are given by the formulae

$$(4.4) \qquad \lambda_1 = \frac{(1 - \cot^2 \aleph)\tan\varphi_3\tan\varphi_4}{\tan\varphi_3 - \cot^2 \aleph \tan\varphi_4},$$

$$(4.5) \qquad \lambda_2 = \frac{(1 - \cot^2 \aleph)\tan\varphi_3\tan\varphi_4}{\tan\varphi_4 - \cot^2 \aleph \tan\varphi_3}.$$

Let us consider two special cases of transversal isotropy, depending on the value of $\aleph$, namely:

- $\aleph = \aleph_0$,
- $\aleph = 0$.

From Eqs. (2.44)–(2.45) it is implied that for $\aleph = \aleph_0$ the spherical tensor is a proper state and $B = 0$. Consequently we conclude that the Burzynski conditions (1.13)–(1.17) are satisfied. In this case the number of independent material parameters reduces to four.

On the other hand when $\aleph = 0$, tension in the direction $\mathbf{k}$ is a proper state (see Fig. 3a) and we have $\varepsilon_a = \varepsilon_{11} = \varepsilon_{22} = 0$. From Eqs. (2.44) and (2.25) it is implied that $C_{1133} = 0$. We should emphasize that in both cases some extra constraints are imposed on the material because the stiffness distributor is determined.

Carrying on the proposed experimental tests (see Fig. 2 and Fig. 3) until permanent deformation or damage appear, we can determine the values of elasticity limits $h_K$ in the limit condition (2.56). The elastic energies in the limit state for each of the tests can be expressed as follows:

$$
(4.6) \qquad \Phi_1 = \tau_1^* \varepsilon_1^*, \quad \Phi_2 = \tau_2^* \varepsilon_2^*, \quad \Phi_3 = \frac{1}{2}\sigma^* \varepsilon_l^*, \quad \Phi_4 = \sigma_p^* \varepsilon_b^*,
$$

where $(.)^*$ denote the values of the stresses and strains in the limit state (see Fig. 2 and Fig. 3).

Thus substituting the found stiffness distributor value into the formulas (2.51)–(2.54) we obtain

$$
(4.7) \qquad
\begin{aligned}
h_1 &= \frac{(k_1 - k_2)\Phi_3\Phi_4}{4(2k_1 - k_2 + 1)\Phi_4 - (2 - k_1)\Phi_3}, \\[2mm]
h_2 &= \frac{(k_1 - k_2)\Phi_3\Phi_4}{4(2k_1 - k_2 - 1)\Phi_4 + (2 - k_1)\Phi_3},
\end{aligned}
$$

$$
(4.8) \qquad h_3 = \Phi_1, \quad h_4 = \Phi_2,
$$

where the following parameters are introduced:

$$
(4.9) \qquad k_1 = \cos 2\aleph \left(1 + \frac{4}{\bar{r}r_3}\right), \quad k_2 = \cos 2\aleph \left(1 + \frac{2}{\bar{r}r_4}\right).
$$

We note that the state of stress is safe if for sufficiently high value of it the limit condition is not reached.

## 5.  The Mises limit condition for transversal isotropy

The limit condition proposed in Sec. 2 is based on the main energy-orthogonal decomposition. It is in some sense a generalization of the Maxwell-Huber-Mises yield condition for anisotropic materials. Formulating that condition we have assumed that elastic and limit properties are mutually dependent. Tensor $\mathbf{C}$ and $\mathbf{H}$ are coaxial. In real materials they can be arbitrary.

MISES [16, 17] proposed the limit condition in the form

$$(5.1) \qquad \qquad \mathbf{s} \cdot \mathbf{H} \cdot \mathbf{s} \leq 1.$$

It means that he introduced the fourth rank limit tensor $\mathbf{H}$ responsible for limit properties. Besides, he assumed that the spherical part of stress tensor is safe.

When anisotropic materials are considered, there is no physical reason to privilege the spherical tensor. This assumption is in common use in hydrodynamics and isotropy.

Rychlewski [12] introduced the limit condition (1.18) which is in some sense a generalization of Mises one (5.1). He considered the quadratic form $\sigma \cdot \mathbf{H} \cdot \sigma \geq 0$ instead of (5.1). According to the proposed condition, the stress $\hat{\sigma}$ is safe if the following condition

$$(5.2) \qquad \qquad \hat{\sigma} \cdot \mathbf{H} \cdot \hat{\sigma} = 0$$

is satisfied.

We should emphasize that MISES did not bind up the condition (5.2) with stored elastic energy. Rychlewski [12], considering two quadratic forms (1.5) and (1.8), i.e.

$$(5.3) \qquad \qquad \sigma \cdot \mathbf{C} \cdot \sigma \quad \text{and} \quad \sigma \cdot \mathbf{H} \cdot \sigma,$$

proved the theorem that any stress measures of the form (5.3) have uniquely determined energy-based interpretation and they may be decomposed into the following sums:

$$(5.4) \qquad \frac{1}{2}\sigma \cdot \mathbf{C} \cdot \sigma \;=\; \Phi(\sigma) = \Phi(\tilde{\sigma}_1) + \Phi(\tilde{\sigma}_2) + ... + \Phi(\tilde{\sigma}_\chi),$$

$$(5.5) \qquad \sigma \cdot \mathbf{H} \cdot \sigma \;=\; \frac{1}{h_1}\Phi(\tilde{\sigma}_1) + \frac{1}{h_2}\Phi(\tilde{\sigma}_2) + ... + \frac{1}{h_\chi}\Phi(\tilde{\sigma}_\chi),$$

where

$$(5.6) \qquad \sigma = \tilde{\sigma}_1 + ... + \tilde{\sigma}_\chi, \quad \chi \leq 6 \quad \text{and} \quad \tilde{\sigma}_\alpha \times \tilde{\sigma}_\beta = \begin{cases} 0, & \alpha \neq \beta \\ 2\Phi(\tilde{\sigma}_\alpha), & \alpha = \beta. \end{cases}$$

It means that $\tilde{\sigma}_\alpha$ are energy-orthogonal states of stress (2.1), while the parameters $\tilde{h}_\alpha$ are energy limits of elasticity $\Phi(\tilde{\sigma}_\alpha)$. We should emphasize that the states of stress (5.6) need not be orthogonal, i.e. $\tilde{\sigma}_\alpha \cdot \tilde{\sigma}_\beta \neq 0$ for $\alpha \neq \beta$.

The purpose of this work is to demonstrate the limit condition of the form (5.5) for transversal isotropy. Using Rychlewski's approach proposed in paper [12] we will find the energy-orthogonal decomposition of the space $S$ (5.6) and moduli $\tilde{h}_\alpha$.

The quadratic form (5.2)b may be rewritten in the form

$$(5.7) \qquad \sigma \cdot \mathbf{H} \cdot \sigma = \sigma \times (\mathbf{S} \circ \mathbf{H} \circ \mathbf{S}) \times \sigma$$

where the definition of the energy scalar product (2.1) and the condition (1.2) were used.

Fourth-rank tensor $\mathbf{S} \circ \mathbf{H} \circ \mathbf{S}$ realizes a symmetric linear transformation of the space of symmetric second-rank tensors $S$ into itself, i.e.

$$(5.8) \qquad (\mathbf{S} \circ \mathbf{H} \circ \mathbf{S}) \times \alpha = \beta, \quad \alpha, \beta \in S.$$

Tensor $\kappa$ is a proper state of the operator (5.8) if

$$(5.9) \qquad (\mathbf{S} \circ \mathbf{H} \circ \mathbf{S}) \times \kappa = \frac{1}{2\tilde{h}}\kappa.$$

The operator $\mathbf{S} \circ \mathbf{H} \circ \mathbf{S}$ is a symmetric one, thus there is a set of energy-orthogonal tensors $\kappa_\alpha$ corresponding to the values $\dfrac{1}{2\tilde{h}_\alpha}$

$$(5.10) \qquad \kappa_\alpha \times \kappa_\beta = \delta_{\alpha\beta}.$$

Equation (5.9) after using the definition of energy scalar product (2.1) and multiplying left-sided by $\mathbf{C}$, takes the form

$$(5.11) \qquad (\mathbf{H} - \frac{1}{2\tilde{h}}\mathbf{C}) \cdot \kappa = 0.$$

States $\kappa$ create the kernel of the operator $\mathbf{H} - \dfrac{1}{2\tilde{h}}\mathbf{C}$ but moduli $\tilde{h}$ are determined from the equation

$$(5.12) \qquad \det(\mathbf{H} - \frac{1}{2\tilde{h}}\mathbf{C}) = 0.$$

If the moduli $\tilde{h}_\alpha$ are distinct, the energy-orthogonal states corresponding to them have unique form. For multiple moduli $\tilde{h}_\alpha$ the energy-orthogonal states $\kappa_\alpha$ form the subspaces $H_\alpha \subset S$.

Let us denote by $\mathbf{P}_\alpha^H$ ($\alpha = 1, ..., \chi$) the projectors of a stress tensor $\boldsymbol{\sigma}$ onto the energy-orthogonal subspaces $H_\alpha$; then

$$(5.13) \qquad \tilde{\sigma}_\alpha = \mathbf{P}_\alpha^H \times \boldsymbol{\sigma}, \quad \tilde{\sigma}_\alpha \in H_\alpha.$$

We want to emphasize that the tensors $\mathbf{C}$ and $\mathbf{H}$ in (5.11) are mutually independent.

Usually, some coupling of elastic properties with the limit ones is observed. For instance, it can be assumed that $\mathbf{C}$ and $\mathbf{H}$ are co-axial [12] that is they have the same eigentensors. In this case solution of Eq. (5.11) becomes simplified. For transversally isotropic materials it leads to Eq. (2.56) given in the Sec. 2.

In this section we will focus on the case of transversally isotropic material for which tensors $\mathbf{C}$ and $\mathbf{H}$ are not coaxial. We only assume that the orientation of the preference direction $\mathbf{k}$ is the same for material in the elastic range as well as in the limit state. The matrix representation of the compliance tensor $\mathbf{C}$ has the form (2.21) or (2.24) according to the basis.

HILL in 1948 [3, 4] proposed the yield condition of the form (5.1) for material with orthotropic symmetry. It was expressed by six independent components of the limit tensor $\mathbf{H}$. When transversal isotropy is considered, the number of independent components of $\mathbf{H}$ reduces to three [6]. Therefore the matrix representations of $\mathbf{H}$ in the polybases (2.7) – (2.12) and (2.13) – (2.18) respectively are as follows:

$$(5.14) \qquad \mathbf{H} \sim \frac{1}{2} \begin{bmatrix} g+n & g-n & -2g & 0 & 0 & 0 \\ g-n & g+n & -2g & 0 & 0 & 0 \\ -2g & -2g & 4g & 0 & 0 & 0 \\ 0 & 0 & 0 & 2m & 0 & 0 \\ 0 & 0 & 0 & 0 & 2m & 0 \\ 0 & 0 & 0 & 0 & 0 & 2n \end{bmatrix}$$

and

$$(5.15) \qquad \mathbf{H} \sim \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & n & 0 & 0 & 0 & 0 \\ 0 & 0 & 3g & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & n \end{bmatrix}.$$

Notations are taken from the HILL paper [3].

It is easy to prove that the spherical tensor is a safe state. Then the equation (3.2) is satisfied. We note that the matrix representation of $\mathbf{H}$ (5.15) in the base $\mathbf{a}_k$ has a diagonal from. It means that the tensors $\mathbf{a}_k$ are the proper states for $\mathbf{H}$

and the componenets $3g$, $m$ and $n$ on the diagonal are the proper values. For the spherical state the corresponding proper value is equal to 0.

In order to obtain the energy-orthogonal decomposition (5.6), the matrix representations of $\mathbf{C}$ and $\mathbf{H}$ in the base $\mathbf{a}_K$ are used, Eq. (5.12) reduces to the form

$$(5.16) \quad \frac{1}{2\tilde{h}}\left[A\left(3g - \frac{1}{2\tilde{h}}D\right) + \frac{1}{2\tilde{h}}B^2\right]\left[m - \frac{1}{\tilde{h}}C_{1313}\right]^2$$

$$\left[n - \frac{1}{2\tilde{h}}(C_{1111} - C_{1122})\right]^2 = 0.$$

The solution of the above equation is created by two roots of multiplicity two and two single ones. The double roots have the form:

$$(5.17) \qquad \tilde{h}_3 = \frac{C_{1313}}{m} = \frac{1}{2\lambda_3 m}, \quad \tilde{h}_4 = \frac{C_{1111} - C_{1122}}{2n} = \frac{1}{2\lambda_4 n}.$$

The proper states $\kappa$ corresponding to the proper values (5.17) form two subspaces $H_3 = P_3$ and $H_4 = P_4$, respectively. Both subspaces are two-dimensional. They consist of the tensors of the form (2.33).

An energy-orthogonal base in the subspace $H_3$ may be taken as follows

$$(5.18) \qquad \boldsymbol{\kappa}_{III} = \sqrt{\lambda_3}\boldsymbol{\omega}_{III}, \quad \boldsymbol{\kappa}_{IV} = \sqrt{\lambda_3}\boldsymbol{\omega}_{IV},$$

and in the subspace $H_4$ the base may be selected as

$$(5.19) \qquad \boldsymbol{\kappa}_V = \sqrt{\lambda_4}\boldsymbol{\omega}_V, \quad \boldsymbol{\kappa}_{VI} = \sqrt{\lambda_4}\boldsymbol{\omega}_{VI}.$$

Two single roots of Eq. (5.16) have the following form:

$$(5.20) \qquad \tilde{h}_1 \to \infty, \quad \tilde{h}_2 = \frac{AD - B^2}{6Ag} = \frac{1}{6\lambda_1\lambda_2 Ag}.$$

The energy-orthogonal states $\kappa$ corresponding to them are given by formulae

$$(5.21) \qquad \boldsymbol{\kappa}_1 = \frac{1}{\sqrt{A}}\mathbf{a}_I, \quad \boldsymbol{\kappa}_2 = \sqrt{\frac{B^2}{A(AD - B^2)}}\left(\mathbf{a}_I - \frac{A}{B}\mathbf{a}_{III}\right).$$

The subspaces $H_1$ and $H_2$ are one-dimensional.

The projectors $\mathbf{P}_\alpha^H$ of stress tensor $\boldsymbol{\sigma}$ onto the subspaces $H_\alpha$ can be expressed as

$$\mathbf{P}_1^H = \frac{1}{A}\mathbf{a}_I \otimes \mathbf{a}_I,$$

$$(5.22) \quad \mathbf{P}_2^H = \frac{B^2}{A(AD - B^2)}\left(\mathbf{a}_I - \frac{A}{B}\mathbf{a}_{III}\right) \otimes \left(\mathbf{a}_I - \frac{A}{B}\mathbf{a}_{III}\right),$$

$$\mathbf{P}_3^H = \lambda_3\mathbf{P}_3, \quad \mathbf{P}_4^H = \lambda_4\mathbf{P}_4$$

On substituting (5.22) into (5.13) we obtain the energy-orthogonal stresses $\tilde{\boldsymbol{\sigma}}_\alpha$. We see that $\tilde{\sigma}_3 = \sigma_3$ and $\tilde{\sigma}_4 = \sigma_4$. It means that they have the same form as for the main energy-orthogonal decomposition (see Fig. 1c, d and formulae (2.64 – 2.65)).

In order to find the stresses $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$, the following scalar parameters are introduced

$$(5.23) \qquad \tilde{s} = \frac{1}{6}(\sigma_{11} + \sigma_{22} - 2\sigma_{33}) \quad \text{and} \quad \mu = \frac{\sqrt{2}B}{A}.$$

The stresses $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are illustrated in Fig. 4 (we remind that $\sigma = \frac{1}{3}\operatorname{tr}\sigma$).
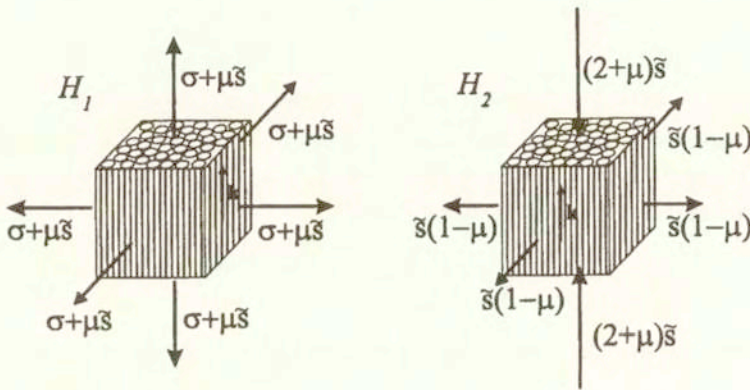


FIG. 4. Subspaces $H_1$ and $H_2$ of energy-orthogonal states $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ for transversally isotropic material connected with the limit tensor **H** (5.14)-(5.15).

While the moduli $\tilde{h}_\alpha$ and stresses $\tilde{\sigma}_\alpha$ are given the limit condition of Mises type (5.5) for the limit tensor **H** (5.15) takes the form

$$(5.24) \qquad \frac{1}{\tilde{h}_2}\Phi(\tilde{\sigma}_2) + \frac{1}{\tilde{h}_3}\Phi(\tilde{\sigma}_3) + \frac{1}{\tilde{h}_4}\Phi(\tilde{\sigma}_4) \le 1.$$

Since $\tilde{h}_1 \to \infty$, the stress $\tilde{\sigma}_1 \in H_1$ is safe and the part of energy $\Phi(\tilde{\sigma}_1)$ has no influence on the limit condition (5.24).

The energies $\Phi(\tilde{\sigma}_3)$ and $\Phi(\tilde{\sigma}_4)$ are given by formulae (2.53) and (2.54). However, the energies $\Phi(\tilde{\sigma}_1)$ and $\Phi(\tilde{\sigma}_2)$ may be expressed as follows:

$$(5.25) \qquad 2\Phi(\tilde{\sigma}_1) = \tilde{\sigma}_1 \times \tilde{\sigma}_1 = 3A(\sigma + \mu\tilde{s})^2,$$

$$(5.26) \qquad 2\Phi(\tilde{\sigma}_2) = \tilde{\sigma}_2 \times \tilde{\sigma}_2 = \frac{6(AD - B^2)}{A}\tilde{s}^2.$$

Analysing the obtained result we note that the stresses $\tilde{\sigma}_2$, $\tilde{\sigma}_3$ and $\tilde{\sigma}_4$ do not cause elastic volume changes.

If the hydrostatic pressure is the only acting stress then the corresponding stress tensor $\boldsymbol{\sigma} = \tilde{\sigma}_1 \in H_1$ is safe.

On the other hand, assuming that the considered state of stress $\boldsymbol{\sigma}$ has the following properties $\sigma = u = v = p = q = 0$ and $\tilde{s} \neq 0$ (see $(2.64) - (2.65)$ and $(5.23)$), we obtain that $\boldsymbol{\sigma} \in H_1 \oplus H_2$, and the part of energy $\Phi(\tilde{\sigma}_1)$ has no influence on the limit condition.

Considering the state of stress $\boldsymbol{\sigma}$ for which

$$(5.27) \qquad\qquad \sigma = -\mu\tilde{s},$$

we note that no part of the stress tensor is a safe state. In spite of $\mathrm{tr}\boldsymbol{\sigma} \neq 0$ we have that $\Phi(\tilde{\sigma}_1) = 0$ and the total elastic energy $\Phi(\boldsymbol{\sigma})$ enters the limit condition.

We should emphasize that for two states of stress with different spherical parts only, the values of the quadratic form $(5.3)_2$ are equal.

The tensors $\mathbf{C}$ and $\mathbf{H}$ are coaxial if the stiffness distributor $\aleph = \aleph_0$ ($B = 0$). Then the spherical tensor is a proper state for both tensors.

## 6. Summary

Using the example of transversal isotropy, the limit condition of the Mises type having an energy-based interpretation has been obtained. Following the approach proposed by J. Rychlewski, the limit condition was given in the form of weighted sum of stored energies corresponding to uniquely defined energy-orthogonal parts of stress.

Without special difficulties the presented approach may be adopted to the problem of formulating the limit condition for other types of anisotropy.

In the general case tensors $\mathbf{C}$ and $\mathbf{H}$ are independent. This property allows to consider different types of symmetry in elastic and limit states. Assuming some form of coupling of elastic and plastic properties, the solution of the problem becomes simplified.

We note that the presented approach may be applied to the problem of describing the plastic anisotropy evolution in the material, which changes its properties when passing from elastic to plastic state. Then we may assume that the evolution of moduli $h_\alpha$ or $\tilde{h}_\alpha$ depends on dissipation of energy connected with irreversible deformations.

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