# On the added mass effect for porous media

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*Dedicated to Professor Piotr Perzyna*
*on the occasion of his 70th birthday*

CONSIDER A POROUS solid skeleton saturated with $N$ fluid constituents. To describe the saturation condition and the immiscibility of the mixture constituents (phases), $N + 1$ volume fraction parameters are introduced. In the energy equation an added mass effect is incorporated in the form of a constitutive assumption. This allows to include, on the phenomenological level, the influence of the pore structure of the solid constituent on the kinetic energy formulation of the whole mixture. Its consequences are deduced; they lead to a new form of the kinetic energy in the balance law of energy, from which a new form of motion equations are deduced. A particular case of one fluid component in the isotropic case is considered.

## 1. Introduction

IN ANALYSIS of the mechanical behaviour of multiphase media such as porous solids filled with one or a number of fluids, one of the key problems is to establish the proper description of the interphase force (internal body force) accounting for the local interaction of the constituents with each other.

Modelling such materials has been a subject of wide discussion through the last three decades and is based mostly upon the fundamental notions of the *classical mixture theory*, TRUESDELL and TOUPIN [22], BOWEN [8], and its reformulated form – the *theory of interacting continua*, GREEN and NAGHDI [14, 8].

In this case a fluid-filled porous medium is treated as a superposition of two miscible continua: solid and fluid, characterized by two independent velocity fields: $\mathbf{v}^s$ and $\mathbf{v}^f$. In such an approach the microstructure of the solid–fluid mixture is not taken into account in formulating the balance equations and constitutive relations. However, it has been observed that in a number of typical multiphase media, consisting of an identifiable porous matrix and a fluid filling its pores, the internal geometrical pore structure strongly influences the behaviour of the phases, especially the pore fluid phase, inducing an inhomogeneity of the micro-velocity fields. This effect is regarded to be of prime importance in understanding the acoustic properties of porous media saturated with fluids.

In most papers treating the problems of fluid flow through deformable porous solids, the solid–fluid interaction force is taken to depend on the relative macro-velocities of phases (the Stokes drag force) and on respective density gradients. However, a more realistic modelling of such media, in a non-stationary case, should take into account fluctuations of the micro–velocity fields of the fluid phases, which have a strong influence on the solid-fluid force interaction by creating inertial couplings between the constituents represented by virtual or added mass force.

The basic concept of an added mass force can be easily understood by considering the change in kinetic energy of the fluid surrounding an accelerating material object (see the Fig. 1). The classical result, when the effects of any viscous forces are not considered, is that the acceleration of the object induces a resisting force at this object proportional to the mass of the displaced fluid and the acceleration of the object.

Such an approach was applied in some works, mainly for isotropic systems, by introducing on the macroscopic level a suitable form of the kinetic energy (e.g. M. A. Biot [3,4,5], D. Lhuillier [20], O. Coussy [11], A. Bedford et al. [1], B. Yavari and A. Bedford [25], D. Drew et al. [12], G. B. Wallis [23], J. A. Geurst [13]).

The present study is concerned with the added mass force during accelerated flows of $N$ fluids through a porous solid. The problem of dynamic coupling is generalized and extended to saturated porous media with an anisotropic internal pore structure. In the present approach, individual physical properties of immiscible constituents play an important role in transport phenomena.

In previous publications of one of us as well as in a number of other papers (cf. [6, 7, 8, 9, 10, 16, 17, 18, 19, 21]) devoted to modelling fluid-saturated porous solids, the immiscibility effect has been incorporated into the description by introducing a parameter of volume porosity characterizing the volume fraction of the fluid constituent. In the present paper our aim is to state a more fundamental approach, from which the existence of quantities describing the motion of

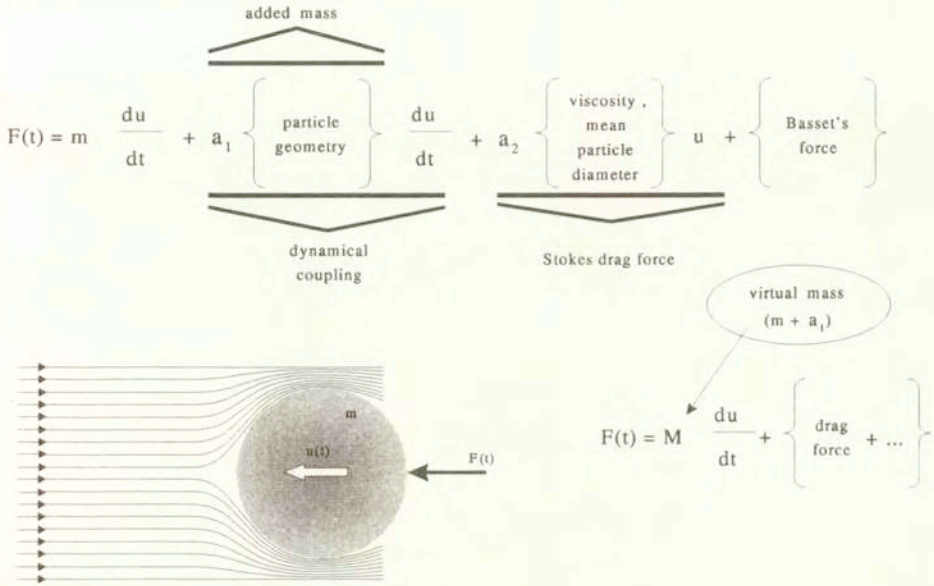the so-called virtual components will follow as a mathematical consequence of a constitutive assumption.



FIG. 1. Added mass and force effect.

## 2. Kinematic relations

In this section we repeat the main model assumptions of the author of the previous paper [18], however, with some generalization by admitting $N$ pore fluid constituents.

Consider a mixture of $N + 1$ constituents and let $s$ and $\alpha = 1, 2, ..., N$, be indices identifying these constituents. Assume that one of the constituents, i.e. $s$, is a solid, while the remaining are of fluid type (liquid or gas). The difficulty in describing the kinematics is that each of the constituents (or phases) performs its own motion that, however, the spatial placement in the present configuration is assumed to be occupied by all constituents. We regard then the physical space as a 3D Euclidean space $\mathbb{E}$ and the domain $B \subset \mathbb{E}$ which is occupied by the constituents as an open set in $\mathbb{E}$ with boundary $\partial B$.

For our considerations at the macro-scale we make use of the local volume average field quantities defined over REV (Representative Elementary Volume – compare, for example [15, 24]) for each phase of the medium.

If $\Omega$ represents the averaging region containing the solid part $\Omega^s$ and the pore region $\Omega^p$ filled with fluid, (see Fig. 2), we have
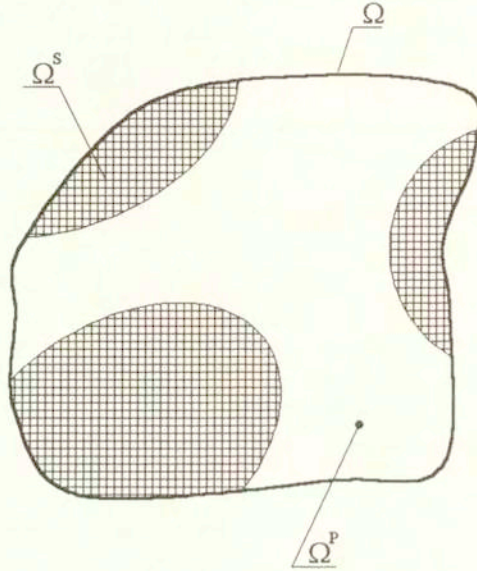
$$(2.1) \qquad \Omega = \Omega^s \cup \Omega^p,$$



FIG. 2. Averaging region of porous medium (REV).

and one can define the volume fraction ratios

$$(2.2) \qquad n^s = \frac{|\Omega^s|}{|\Omega|} \, , \quad n^p = n^f = \frac{|\Omega^p|}{|\Omega|} =: f_v \, ,$$

for the solid and fluid phase, respectively, where $f_v$ is the volume porosity, and the following condition is satisfied

$$n^s + n^p = 1 \, .$$

We consider the porous solid as fully saturated with the fluid phase. The pore-fluid phase may, in general, consist of $\alpha = 1, 2, ..., N$ fluid type components, and therefore we have

$$(2.3) \qquad \Omega^p = \bigcup_{\alpha=1}^{N} \Omega^\alpha \, .$$

In such a case one can introduce volume fraction ratios for each fluid phase, i.e.,

$$(2.4) \qquad \frac{|\Omega^\alpha|}{|\Omega|} = n^\alpha \, .$$

Obviously, $n^\alpha$ is constrained by

$$(2.5) \qquad \sum_{\alpha=1}^{N} n^\alpha = f_v \, ,$$

and introducing the saturation parameters

$$(2.6) \qquad \frac{n^\alpha}{f_v} = s^\alpha \, , \quad \alpha = 1, ..., N,$$

one takes

$$(2.7) \qquad \sum_{\alpha=1}^{N} s^\alpha = 1 \, .$$

## 3. Phase density and linear momentum

We assume for any phase that the microscopic quantities are defined at the pore or grain scale, and we denote them by double upper case indices, say $\rho^{\alpha\alpha}$ and $v^{\alpha\alpha}$ for the mass density and the velocity, respectively. Hence the bulk and effective volume average quantities can be defined with the help of them as their corresponding integral mean values (averages). For the solid phase we may neglect the fluctuations of mass density and velocity from their averages, so that we have for the mass density

$$(3.1) \qquad \rho^s := \langle \rho^{ss} \rangle = \frac{1}{|\Omega^s|} \int_{\Omega^\alpha} \rho^{ss} d\Omega = \rho^{ss}$$

and similarly for the velocity

$$(3.2) \qquad \mathbf{v}^{ss} = \mathbf{v}^s \, .$$

However, for the bulk solid partial density we can write

$$(3.3) \qquad \bar{\rho}^s := \frac{1}{|\Omega|} \int_{\Omega^s} \rho^{ss} d\Omega = n^s \rho^s = (1 - f_v)\rho^s,$$

and the local form of linear momentum for the porous skeleton is

$$(3.4) \qquad \mathbf{l}^s = \frac{1}{|\Omega|} \int_{\Omega^s} \rho^{ss} \mathbf{v}^{ss} d\Omega = n^s \rho^s \mathbf{v}^s = \bar{\rho}^s \mathbf{v}^s \, .$$

For the fluid components their pore velocities are strongly inhomogeneous in $\Omega^p$ (cf. Fig. 3 when one microscopic fluid component velocity $\mathbf{v}^{ff}$ is present) but the mass density fluctuations are small and will be disregarded in the further analysis, i.e.

$$(3.5) \qquad\qquad \rho^{\alpha\alpha} = \rho^{\alpha}, \qquad \alpha = 1, ..., N .$$
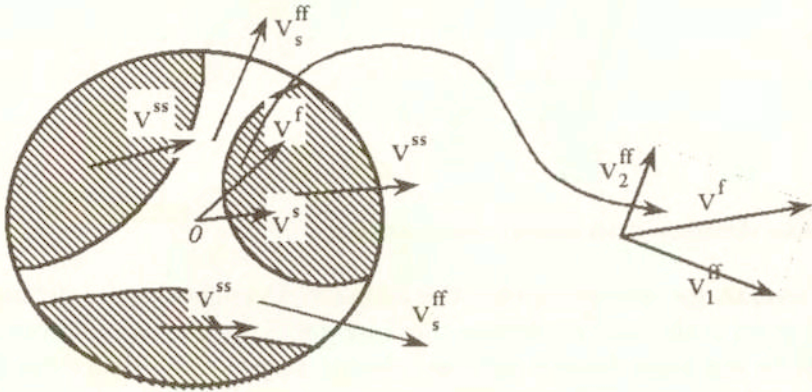


Fig. 3. Velocity scheme for the micro and macro-components.

For each fluid phase the bulk partial mass density $\bar{\rho}^{\alpha}$ and the corresponding saturation parameter $s^{\alpha}$ are coupled in the following relation:

$$\frac{1}{|\Omega|} \int_{\Omega^{\alpha}} \rho^{\alpha\alpha} d\Omega = s^{\alpha} \bar{\rho}^{\alpha} , \qquad \alpha = 1, ..., N$$

$(3.6)$

$$\text{where} \qquad \bar{\rho}^{\alpha} = f_v \rho^{\alpha} = f_v \frac{1}{|\Omega^{\alpha}|} \int_{\Omega^{\alpha}} \rho^{\alpha\alpha} d\Omega.$$

The density of the fluid linear momentum is defined as

$$(3.7) \qquad \mathbf{l}^{\alpha} = \frac{1}{|\Omega|} \int_{\Omega^{\alpha}} \rho^{\alpha\alpha} \mathbf{v}^{\alpha\alpha} d\Omega = \rho^{\alpha} n^{\alpha} \mathbf{v}^{\alpha} = s^{\alpha} \bar{\rho}^{\alpha} \mathbf{v}^{\alpha} .$$

Note that this relation also defines $\mathbf{v}^{\alpha}$ in terms of $\mathbf{v}^{\alpha\alpha}$ and $n^{\alpha}$; $\mathbf{v}^{\alpha}$ is obviously a "barycentric" mean of $\mathbf{v}^{\alpha\alpha}$. If the pores are filled with a one-component fluid, i.e., $N = 1$ and if $\alpha \equiv f$ , then $s^f = 1$ and $n^{\alpha} = n^f = f_v$ , and we arrive at a porous solid saturated with one physical fluid.

The mass density for the whole system is

$$(3.8) \qquad \rho = \bar{\rho}^s + \sum_{\alpha=1}^{N} s^\alpha \bar{\rho}^\alpha \,,$$

and the density of the linear momentum is

$$(3.9) \qquad l = l^s + \sum_\alpha l^\alpha = \bar{\rho}^s \mathbf{v}^s + \sum_\alpha s^\alpha \bar{\rho}^\alpha \mathbf{v}^\alpha \,.$$

## 4. Kinetic energy

The local form of the kinetic energy for the solid phase, when the assumption (3.1) is taken into account, can be exactly described by the average solid velocity $\mathbf{v}^s$ in the form

$$(4.1) \qquad E^s = \frac{1}{2} \frac{1}{|\Omega|} \int\limits_{\Omega^s} \rho^{ss} \mathbf{v}^{ss} \cdot \mathbf{v}^{ss} d\Omega = \frac{1}{2} \bar{\rho}^s \mathbf{v}^s \cdot \mathbf{v}^s,$$

that represents the total kinetic energy of the particles of the solid skeleton in $\Omega$. However, for any fluid component flowing through the pores, the fluctuations of the pore velocity are of the order of the average component velocity as a result of the influence of the pore structure, and the $\alpha$-component of the kinetic energy expressed by the phase velocity $v^\alpha$ only, does not represent the total kinetic energy of that fluid component, i.e.,

$$(4.2) \qquad \frac{1}{2} \frac{1}{|\Omega|} \int\limits_{\Omega^\alpha} \rho^{\alpha\alpha} \mathbf{v}^{\alpha\alpha} \cdot \mathbf{v}^{\alpha\alpha} d\Omega \neq \frac{1}{2} s^\alpha \bar{\rho}^\alpha \mathbf{v}^\alpha \cdot \mathbf{v}^\alpha \,.$$

To describe the total kinetic energy reflecting the real motion of the $\alpha$-component at the pore level, one can follow the approach of BIOT [6], BEDFORD and DRUMHELLER [1] and KUBIK [17] and assume this contribution to the energy to consist of two parts:

1. $K_d$ – the part, responsible for the contribution of the average velocity field of the particular fluid components

$$(4.3) \qquad K_d = \frac{1}{2} \sum_\alpha s^\alpha \bar{\rho}^\alpha \mathbf{v}^\alpha \cdot \mathbf{v}^\alpha \qquad \alpha = 1, ..., N \,,$$

2. $K_n$ – the part, responsible for the inhomogeneity of the velocity fields at the pore level, caused by the pore structure (as well as some other effects of the fluid

components) and expressed by means of the $\alpha$-component velocities relative to the skeleton

$$(4.4) \qquad K_n = \frac{1}{2}\sum_\alpha s^\alpha \bar\rho^\alpha (\mathbf{v}^\alpha - \mathbf{v}^s) \cdot \boldsymbol{A}^\alpha (\mathbf{v}^\alpha - \mathbf{v}^s) \,, \quad \text{where} \quad \alpha = 1, ..., N \,,$$

where the symmetric tensors $\boldsymbol{A}^\alpha$ represent the influence of the geometrical structure of the pores on the kinematics of the flow, and assumed to be positive definite. The non–diagonal part $K_n$ expressed by the tensors $\boldsymbol{A}^\alpha$ has to satisfy the principle of the material objectivity, since their existence is a constitutive assumption.

With the use of the above expressions, the kinetic energy of the system composed of the porous skeleton saturated with $N$ fluid components is

$$(4.5) \quad E = \frac{1}{2}\bar\rho^s \mathbf{v}^s \cdot \mathbf{v}^s + \frac{1}{2}\sum_\alpha s^\alpha \bar\rho^\alpha \mathbf{v}^\alpha \cdot \mathbf{v}^\alpha + \frac{1}{2}\sum_\alpha s^\alpha \bar\rho^\alpha (\mathbf{v}^\alpha - \mathbf{v}^s) \cdot \boldsymbol{A}^\alpha (\mathbf{v}^\alpha - \mathbf{v}^s) \,.$$

Introducing the macroscopic relative velocity $\mathbf{u}^\alpha$ ,

$$(4.6) \qquad\qquad\qquad \mathbf{u}^\alpha = \mathbf{v}^\alpha - \mathbf{v}^s \,,$$

one can reformulate the energy of the considered system as follows

$$(4.7) \quad E = \frac{1}{2}\bar\rho^s \mathbf{v}^s \cdot \mathbf{v}^s$$
$$+ \frac{1}{2}\left\{ \sum_\alpha s^\alpha \bar\rho^\alpha [\mathbf{v}^s \cdot \mathbf{v}^s + 2\mathbf{v}^s \cdot \mathbf{u}^\alpha + \mathbf{u}^\alpha \cdot \mathbf{u}^\alpha + \mathbf{u}^\alpha \cdot \boldsymbol{A}^\alpha \mathbf{u}^\alpha] \right\}$$
$$= \frac{1}{2}\bar\rho^s \mathbf{v}^s \cdot \mathbf{v}^s + \frac{1}{2}\left\{ \sum_\alpha s^\alpha \bar\rho^\alpha [\mathbf{v}^s \cdot \mathbf{v}^s + 2\mathbf{v}^s \cdot \mathbf{u}^\alpha + \mathbf{u}^\alpha (1 + \boldsymbol{A}^\alpha) \mathbf{u}^\alpha] \right\},$$

where the motion of the solid phase is singled out.

Since each tensor $1 + \mathbf{A}^\alpha$ is symmetric and non-singular, we can define a new second order symmetric tensor $\mathbf{P}^\alpha$, for each $\alpha$, such that

$$(4.8) \qquad\qquad 1 + \mathbf{A}^\alpha = (\mathbf{P}^\alpha)^{-1}f_v \quad \text{or} \quad \mathbf{P}^\alpha = (1 + \mathbf{A}^\alpha)^{-1}f_v \,.$$

Using the tensor $\mathbf{P}^\alpha$ we may define a new relative velocity field for the $\alpha$-component

$$(4.9) \qquad\qquad\qquad \overset{\alpha}{\mathbf{u}} = (\mathbf{P}^\alpha)^{-1}\mathbf{u}^\alpha f_v \,.$$

Each $\mathbf{P}^\alpha$ reflects the effect of the tortuosity of the pore structure of the skeleton on the fluid pore velocity.

Applying relation (4.9) in (4.7) yields

$$(4.10) \quad E = \frac{1}{2}\bar\rho^s \mathbf{v}^s \cdot \mathbf{v}^s + \frac{1}{2}\sum_\alpha \bar\rho^\alpha s^\alpha [\mathbf{v}^s \cdot \mathbf{v}^s + 2\frac{1}{f_v}(\mathbf{P}^\alpha \overset{\alpha}{\mathbf{u}}) \cdot \mathbf{v}^s + \frac{1}{f_v}(\mathbf{P}^\alpha \overset{\alpha}{\mathbf{u}}) \cdot \overset{\alpha}{\mathbf{u}}]$$

and after some rearranging, we obtain the canonical representation for the kinetic energy of the system

$$(4.11) \quad E = \frac{1}{2}\bar\rho^s \mathbf{v}^s \cdot \mathbf{v}^s$$

$$+ \frac{1}{2}\sum_\alpha s^\alpha \bar\rho^\alpha \left[ \mathbf{v}^s \cdot (1 - \frac{1}{f_v}\mathbf{P}^\alpha)\mathbf{v}^s + \overset{\alpha}{\mathbf{v}} \cdot (1 - \frac{1}{f_v}\mathbf{P}^\alpha) \overset{\alpha}{\mathbf{v}} \right]$$

$$= \frac{1}{2}\left\{ \mathbf{v}^s \cdot \left[ \bar\rho^s \mathbf{1} + \sum_\alpha s^\alpha \bar\rho^\alpha (1 - \frac{1}{f_v}\mathbf{P}^\alpha) \right] \mathbf{v}^s + \sum_\alpha \overset{\alpha}{\mathbf{v}} \cdot (s^\alpha \bar\rho^\alpha \frac{1}{f_v}\mathbf{P}^\alpha) \overset{\alpha}{\mathbf{v}} \right\},$$

where

$$(4.12) \qquad\qquad \overset{\alpha}{\mathbf{v}} = \mathbf{v}^s + \overset{\alpha}{\mathbf{u}} .$$

Taking the above representation into account one can define the so-called **virtual constituents** of the system. The first virtual constituent is composed of the skeleton and those parts of the fluid components which move at the skeleton velocity $\mathbf{v}^s$, the partial density of which is

$$(4.13) \qquad\qquad \overset{s}{\mathbf{M}} = \left[ \bar\rho^s \mathbf{1} + \sum_\alpha s^\alpha \bar\rho^\alpha (1 - \frac{1}{f_v}\mathbf{P}^\alpha) \right] ;$$

the remaining virtual constituents have partial densities

$$(4.14) \qquad\qquad \overset{\alpha}{\mathbf{M}} = s^\alpha \bar\rho^\alpha \frac{1}{f_v}\mathbf{P}^\alpha , \quad \alpha = 1, ..., N,$$

and they move with the corresponding velocities $\overset{\alpha}{v}$ . Consequently, the linear momentum for the first virtual constituent takes the form

$$(4.15) \qquad\qquad \overset{s}{\mathbf{L}} = \overset{s}{\mathbf{M}} v^s = \left[ \bar\rho^s \mathbf{1} + \sum_\alpha s^\alpha \bar\rho^\alpha (1 - \frac{1}{f_v}\mathbf{P}^\alpha) \right] \mathbf{v}^s ,$$

and the $\alpha$-constituent linear momentum written by means of partial densities and the velocities of the virtual constituents will be

$$(4.16) \qquad\qquad \overset{\alpha}{\mathbf{L}} = s^\alpha \bar\rho^\alpha \frac{1}{f_v}\mathbf{P}^\alpha \overset{\alpha}{\mathbf{v}} .$$

At the same time the canonical representation of the kinetic energy will be

$$(4.17) \qquad E = \frac{1}{2} \overset{s}{\mathbf{v}} \cdot \overset{s}{\mathbf{M}} \overset{s}{\mathbf{v}} + \frac{1}{2} \sum_\alpha \overset{\alpha}{\mathbf{v}} \cdot \overset{\alpha}{\mathbf{M}} \overset{\alpha}{\mathbf{v}} ,$$

where

$$(4.18) \qquad \begin{aligned} \overset{\alpha}{\mathbf{v}} &= \mathbf{v}^s + \overset{\alpha}{\mathbf{u}} , \quad \mathbf{u}^\alpha = (\mathbf{v}^\alpha - \mathbf{v}^s), \\ \overset{\alpha}{\mathbf{u}} &= (1 + \mathbf{A}^\alpha)\mathbf{u}^\alpha = (\mathbf{P}^\alpha)^{-1}\mathbf{u}^\alpha f_v . \end{aligned}$$

One can now prove that the mass densities and the linear momentum densities of the virtual constituents of the porous solid, filled with a multicomponent fluid, satisfy the following conditions:

- for mass densities

$$(4.19) \qquad \overset{s}{\mathbf{M}} + \sum_\alpha \overset{\alpha}{\mathbf{M}} = \bar{\rho}^s \mathbf{1} + \sum_\alpha s^\alpha \bar{\rho}^\alpha \left(1 - \frac{1}{f_v}\mathbf{P}^\alpha\right) + \sum_\alpha s^\alpha \bar{\rho}^\alpha \frac{1}{f_v}\mathbf{P}^\alpha$$

$$= \bar{\rho}^s \mathbf{1} + \sum_\alpha s^\alpha \bar{\rho}^\alpha \mathbf{1} ,$$

- for linear momentum

$$(4.20) \qquad \overset{s}{\mathbf{M}} \mathbf{v}^s + \sum_\alpha \overset{\alpha}{\mathbf{L}}$$

$$= \left[\bar{\rho}^s \mathbf{1} + \sum_\alpha s^\alpha \bar{\rho}^\alpha \left(1 - \frac{1}{f_v}\mathbf{P}^\alpha\right)\right] \mathbf{v}^s + \sum_\alpha s^\alpha \bar{\rho}^\alpha \frac{1}{f_v}\mathbf{P}^\alpha \overset{\alpha}{\mathbf{v}},$$

and since $\overset{\alpha}{\mathbf{v}} = \mathbf{v}^s + \overset{\alpha}{\mathbf{u}}$ , we have

$$(4.21) \qquad \overset{s}{\mathbf{M}} \mathbf{v}^s + \sum_\alpha \overset{\alpha}{\mathbf{L}} = \left[\left(\bar{\rho}^s \mathbf{1} + \sum_\alpha s^\alpha \bar{\rho}^\alpha \mathbf{1}\right)\right] \mathbf{v}^s + \sum_\alpha s^\alpha \bar{\rho}^\alpha \frac{1}{f_v}\mathbf{P}^\alpha \overset{\alpha}{\mathbf{u}}$$

$$= \left[\bar{\rho}^s \mathbf{v}^s + \sum_\alpha s^\alpha \bar{\rho}^\alpha \mathbf{v}^\alpha\right].$$

Notice that for an isotropic pore structure simplifications occur, i.e., $\mathbf{P}^\alpha = \lambda^\alpha \mathbf{1}$, with $\lambda^\alpha$ as a structural permeability parameter for each $\alpha$, and

$$(4.22) \qquad \overset{s}{\mathbf{M}} = \left[\bar{\rho}^s + \sum_\alpha s^\alpha \bar{\rho}^\alpha (1 - \kappa^\alpha)\right]\mathbf{1}, \quad \overset{\alpha}{\mathbf{M}} = s^\alpha \bar{\rho}^\alpha \kappa^\alpha \mathbf{1}, \quad \alpha = 1,...,N,$$

where $\kappa^\alpha = \lambda^\alpha / f_v$ can be called the pore structure parameter corresponding to the $\alpha$-th fluid constituent. The case of $N = 1$ has already been discussed in [9, 17], where for a fluid-saturated porous skeleton the following representation of the velocity of the virtual components was obtained

$$(4.23) \qquad \overset{\alpha}{\mathbf{v}} = \overset{f}{\mathbf{v}} \quad \text{with} \quad \overset{f}{\mathbf{v}} = \mathbf{v}^s + \frac{1}{\kappa}(\mathbf{v}^f - \mathbf{v}^s) \,, \quad \text{or} \quad \mathbf{v}^f = (1 - \kappa)\mathbf{v}^s + \kappa \overset{f}{\mathbf{v}},$$

with only one $\alpha \equiv f$, and $\mathbf{v}^1 = \mathbf{v}^f$, the velocity of a one-component fluid and $\kappa = \kappa^1$, the pore structure parameter, and with the partial mass densities of the virtual components

$$(4.24) \qquad \overset{s}{\mathbf{M}} = \left( \bar{\rho}^s + (1 - \kappa)\,\bar{\rho}^f \right) \mathbf{1} \,, \quad \overset{f}{\mathbf{M}} = \kappa \bar{\rho}^f \, \mathbf{1} \,.$$

## 5. Mass continuity equations

For a porous skeleton and a pore fluid with chemically inert components, the local form of the balance equations of mass can be written in the form

$$(5.1) \qquad \frac{\partial \bar{\rho}^s}{\partial t} + \operatorname{div}(\bar{\rho}^s \mathbf{v}^s) = 0 \,,$$

$$(5.2) \qquad \frac{\partial s^\alpha \bar{\rho}^\alpha}{\partial t} + \operatorname{div}(s^\alpha \bar{\rho}^\alpha \mathbf{v}^\alpha) = 0, \quad \alpha = 1, ..., N.$$

When the virtual division of the system is considered, the local form of the balance equations of mass are

$$(5.3) \qquad \frac{\partial \overset{s}{\mathbf{M}}}{\partial t} + \operatorname{div}(\mathbf{1} \otimes \overset{s}{\mathbf{M}} \mathbf{v}^s) = \overset{s}{\mathbf{H}},$$

$$(5.4) \qquad \frac{\partial \overset{\alpha}{\mathbf{M}}}{\partial t} + \operatorname{div}(\mathbf{1} \otimes \overset{\alpha}{\mathbf{M}} \overset{\alpha}{\mathbf{v}}) = \overset{\alpha}{\mathbf{H}},$$

where $\alpha = 1, 2, ..., N$, while $\overset{s}{\mathbf{H}}$ and $\overset{\alpha}{\mathbf{H}}$ are mass supply functions that will be subsequently identified. Eq. (5.3), when (4.13) and (5.1) are taken into account can be rewritten in the following form

$$(5.5) \qquad \frac{\partial}{\partial t} \left\{ \sum_\alpha s^\alpha \bar{\rho}^\alpha \left( 1 - \frac{1}{f_v} \mathbf{P}^\alpha \right) \right\}$$

$$+ \operatorname{div} \left\{ \mathbf{1} \otimes \left[ \sum_\alpha s^\alpha \bar{\rho}^\alpha \left( 1 - \frac{1}{f_v} \mathbf{P}^\alpha \right) \right] \mathbf{v}^s \right\} = \overset{s}{\mathbf{H}} \,.$$

Equation (5.3), when (4.14) and (5.2) are used yields

$$(5.6) \qquad \frac{\partial}{\partial t}\left\{s^\alpha \bar{\rho}^\alpha \frac{1}{f_v}\mathbf{P}^\alpha\right\} + \operatorname{div}\left\{\mathbf{1}\otimes\left[s^\alpha \bar{\rho}^\alpha \frac{1}{f_v}\mathbf{P}^\alpha \overset{\alpha}{\mathbf{v}}\right]\right\} = \overset{\alpha}{\mathbf{H}},$$

$$(5.7) \qquad \frac{\partial}{\partial t}\left\{s^\alpha \bar{\rho}^\alpha \mathbf{1}\right\} + \operatorname{div}\left\{s^\alpha \bar{\rho}^\alpha\left[\mathbf{1}\otimes\mathbf{v}^s + \frac{1}{f_v}\mathbf{1}\otimes\mathbf{P}^\alpha \overset{\alpha}{\mathbf{u}}\right]\right\} = \mathbf{0}.$$

By subtraction (5.6) from (5.5) one obtains

$$(5.8) \qquad \frac{\partial}{\partial t}\left\{s^\alpha \bar{\rho}^\alpha\left(1 - \frac{1}{f_v}\mathbf{P}^\alpha\right)\right\} + \operatorname{div}\left\{s^\alpha \bar{\rho}^\alpha \mathbf{1}\otimes\left(1 - \frac{1}{f_v}\mathbf{P}^\alpha\right)\mathbf{v}^s\right\} = -\overset{\alpha}{\mathbf{H}},$$

and for any component $\alpha$, but when summing over all $\alpha = 1, 2, ..., N$, we have

$$(5.9) \qquad \frac{\partial}{\partial t}\left\{\sum_\alpha s^\alpha \bar{\rho}^\alpha\left(1 - \frac{1}{f_v}\mathbf{P}^\alpha\right)\right\} + \operatorname{div}\left\{\sum_\alpha s^\alpha \bar{\rho}^\alpha \mathbf{1}\otimes\left(1 - \frac{1}{f_v}\mathbf{P}^\alpha\right)\mathbf{v}^s\right\}$$

$$= -\sum_\alpha \overset{\alpha}{\mathbf{H}}.$$

Comparing (5.9) and (5.5) we find that the mass supplies satisfy the condition

$$(5.10) \qquad \overset{s}{\mathbf{H}} + \sum_\alpha \overset{\alpha}{\mathbf{H}} = 0,$$

which shows that during the motion of the system the first virtual component moving at the skeleton velocity interchanges its mass with the remaining virtual components. It can be shown that the mass supply functions can be written as

$$(5.11) \qquad \overset{s}{\mathbf{H}} = -\sum_\alpha \overset{\alpha}{\mathbf{H}} = \bar{\rho}^s \frac{D^s}{Dt}\left\{\frac{1}{\bar{\rho}^s}\sum_\alpha s^\alpha \bar{\rho}^\alpha\left(1 - \frac{1}{f_v}\mathbf{P}^\alpha\right)\right\},$$

$$\text{where} \quad \frac{D^s}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}^s\cdot\operatorname{grad}.$$

For an isotropic pore structure, for which

$$\mathbf{P}^\alpha = \lambda^\alpha \mathbf{1} = \kappa^\alpha f_v \mathbf{1}$$

we obtain the continuity equations

$$\frac{\partial}{\partial t}\left\{\sum_\alpha s^\alpha \bar{\rho}^\alpha (1-\kappa^\alpha)\right\} + \mathrm{div}\left\{\left[\sum_\alpha s^\alpha \bar{\rho}^\alpha (1-\kappa^\alpha)\right]\mathbf{v}^s\right\} = \mathbf{H}^s$$

$$\frac{\partial}{\partial t}\left\{s^\alpha \bar{\rho}^\alpha \kappa^\alpha\right\} + \mathrm{div}\left\{s^\alpha \bar{\rho}^\alpha \kappa^\alpha \mathbf{v}^\alpha\right\} = \mathbf{H}^\alpha$$

with the condition

$$\overset{s}{\mathbf{H}} + \sum_\alpha \overset{\alpha}{\mathbf{H}} = 0.$$

For the mass supply function we have

$$\overset{s}{\mathbf{H}} = -\sum_\alpha \overset{\alpha}{\mathbf{H}} = \bar{\rho}^s \frac{D^s}{Dt}\left\{\frac{1}{\bar{\rho}^s}\sum_\alpha s^\alpha \bar{\rho}^\alpha (1-\kappa^\alpha)\right\}.$$

## 6. Motion equations for an isotropic porous solid filled with a one-component fluid, $N = 1$

For $N = 1$ a porous solid-fluid composition forms a mixture composed of two virtual components with corresponding velocities $\overset{s}{\mathbf{v}}$ and $\overset{f}{\mathbf{v}}$, (4.23), and densities $\overset{s}{M}$ and $\overset{f}{M}$, (4.24). (Note that now $\overset{s}{\mathbf{M}} = \overset{s}{M}\mathbf{1}$ and $\overset{f}{\mathbf{M}} = \overset{f}{M}\mathbf{1}$.) Now, having defined the virtual components we associate the stress vector $\overset{k}{\mathbf{t}}$ $(k = s, f)$ with each of the components in such a way that the scalar product $\overset{k}{\mathbf{t}} \cdot \overset{k}{\mathbf{v}}$ represents the rate of work of a particular component per unit area of a surface bounding the bulk material. These can be derived from the condition that the total rates of mechanical work for virtual and physical components are equal,

$$(6.1) \qquad \overset{s}{\mathbf{t}} \cdot \overset{s}{\mathbf{v}} + \overset{f}{\mathbf{t}} \cdot \overset{f}{\mathbf{v}} = \mathbf{t}^s \cdot \mathbf{v}^s + \mathbf{t}^f \cdot \mathbf{v}^f .$$

Thus, using (4.23), the relations between $\overset{k}{\mathbf{t}}$ and $\mathbf{t}^\alpha$ are

$$(6.2) \qquad \overset{s}{\mathbf{t}} = \mathbf{t}^s + (1-\kappa)\mathbf{t}^f, \quad \overset{f}{\mathbf{t}} = \kappa \mathbf{t}^f$$

and for the stress tensors we have

$$(6.3) \qquad \overset{s}{\mathbf{T}} = \mathbf{T}^s + (1-\kappa)\mathbf{T}^f, \quad \overset{f}{\mathbf{T}} = \kappa \mathbf{T}^f$$

where the identity $\mathbf{t}^\alpha = \mathbf{T}^\alpha \mathbf{n}$ was used, with $\mathbf{n}$ as the unit normal to a surface element on which the stress tensors are defined.

Now, motion equations for the virtual components may be obtained from the energy balance for the whole solid–fluid composition by applying invariance conditions under the superposed rigid body translations. Therefore we write the energy balance in the following form:

$$(6.4) \quad \int_V \frac{\partial}{\partial t}\left( \bar{\rho}^s e^s + \bar{\rho}^f e^f + \frac{1}{2}\left( \overset{s}{M}\overset{s}{\mathbf{v}}\cdot\overset{s}{\mathbf{v}} + \overset{f}{M}\overset{f}{\mathbf{v}}\cdot\overset{f}{\mathbf{v}} \right)\right) dV$$

$$+ \int_S \left( e^s \bar{\rho}^s \overset{s}{\mathbf{v}} + e^f\left[ \left(\bar{\rho}^f - \overset{s}{M}\right)\overset{s}{\mathbf{v}} + \overset{f}{M}\overset{f}{\mathbf{v}} \right] + \frac{1}{2}\left[ \left( \overset{s}{M}\overset{s}{\mathbf{v}}\cdot\overset{s}{\mathbf{v}} \right)\overset{s}{\mathbf{v}} + \left( \overset{f}{M}\overset{f}{\mathbf{v}}\cdot\overset{f}{\mathbf{v}} \right)\overset{f}{\mathbf{v}} \right] \right)$$

$$\cdot \mathbf{n}\, dS = \int_V \left( \overset{s}{M}\,\mathbf{b}\cdot\overset{s}{\mathbf{v}} + \overset{f}{M}\,\mathbf{b}\cdot\overset{f}{\mathbf{v}} \right) dV + \int_S \left( \overset{s}{\mathbf{t}}\cdot\overset{s}{\mathbf{v}} + \overset{f}{\mathbf{t}}\cdot\overset{f}{\mathbf{v}} \right) dS$$

where $e^s$ and $e^f$ stand for the internal energies per unit mass of the skeleton and the fluid, respectively, and where the external body forces $\overset{s}{M}\,\mathbf{b}$ and $\overset{f}{M}\,\mathbf{b}$ contribute by their rate of work contribution, i.e., $\overset{s}{M}\,\mathbf{b}\cdot\overset{s}{v}$ and $\overset{f}{M}\,\mathbf{b}\cdot\overset{f}{v}$, respectively.

Using (4.24) and continuity equations (5.3) and (5.4) and applying invariance conditions under rigid body translations, we obtain the equations of motion

$$(6.5)$$

$$\operatorname{div} \overset{s}{\mathbf{T}} + \overset{s}{M}\,\mathbf{b} + \overset{s}{\pi} = \overset{s}{M}\frac{\overset{s}{D}}{Dt}\overset{s}{\mathbf{v}} + \frac{1}{2}\overset{s}{M}\left( \overset{f}{\mathbf{v}} - \overset{s}{\mathbf{v}} \right)$$

$$\operatorname{div} \overset{f}{\mathbf{T}} + \overset{f}{M}\,\mathbf{b} + \overset{f}{\pi} = \overset{f}{M}\frac{\overset{f}{D}}{Dt}\overset{f}{\mathbf{v}} + \frac{1}{2}\overset{s}{M}\left( \overset{f}{\mathbf{v}} - \overset{s}{\mathbf{v}} \right)$$

where $\overset{s}{\pi} = -\overset{f}{\pi}$ represents the viscous interaction force and the force $\frac{1}{2}\overset{s}{M}\,\mathbf{u} = \frac{1}{2}\overset{s}{M}\left( \overset{f}{\mathbf{v}} - \overset{s}{\mathbf{v}} \right)$ results from the mass exchange between the virtual components. The equations of motion for the physical components are

$$(6.6)$$

$$\operatorname{div} \mathbf{T}^s + \bar{\rho}^s \mathbf{b} + \pi^s = \bar{\rho}^s \frac{D^s}{Dt}\mathbf{v}^s,$$

$$\operatorname{div} \mathbf{T}^f_c + \bar{\rho}^f \mathbf{b} + \pi^f = \bar{\rho}^f \frac{D^f}{Dt}\mathbf{v}^f,$$

where the interphase interaction force has the form

(6.7)
$$\boldsymbol{\pi}^s = -\boldsymbol{\pi}^f = -\frac{1}{\kappa}\left(\overset{f}{\boldsymbol{\pi}} + \mathbf{T}^f \operatorname{grad}\kappa\right)$$

$$+ (1-\kappa)\,\bar{\rho}^f\left\{\frac{D^f}{Dt}\mathbf{v}^f - \frac{D^s}{Dt}\mathbf{v}^s + \frac{D^f}{Dt}\left[\left(\frac{1}{\kappa}-1\right)\mathbf{u}\right]\right.$$

$$+ \left(\frac{1}{\kappa}-1\right)\mathbf{u}\cdot\operatorname{grad}\left[\mathbf{v}^f + \left(\frac{1}{\kappa}-1\right)\mathbf{u}\right]\right\}$$

$$+ \left(1-\frac{1}{2\kappa}\right)\left\{\frac{1}{\kappa}\mathbf{u}\,\bar{\rho}^s\frac{D^s}{Dt}\left[\frac{\bar{\rho}^f\,(1-\kappa)}{\bar{\rho}^s}\right]\right\},$$

and

$$\mathbf{T}^f_c = \mathbf{T}^f - \left(\frac{1}{\kappa}-1\right)\bar{\varrho}^f\left(\mathbf{v}^s - \mathbf{v}^f\right)\otimes\left(\mathbf{v}^s - \mathbf{v}^f\right)$$

is the so-called complete stress tensor of the fluid phase.

The second and third terms of the right–hand side of (6.7) represent the inertial coupling between the solid and the fluid phases due to fluctuations of the micro-velocity field of the pore fluid that have been taken into account within the description. It is worth noticing that if $\kappa = 1$ the fluctuations are disregarded and the inertial coupling vanishes.

## 7.   Concluding remarks

In this paper we considered a porous solid skeleton saturated with $N$ fluid constituents. In the authors' opinion the individual physical properties of the immiscible constituents as well as the geometrical structure of the pores play an important role in describing transport phenomena through porous materials. The present approach to represent the kinetic energy of the medium at the macro-continuum scale takes into account the immiscibility effect and the non-homogeneity of the fluids' micro-velocities $\mathbf{v}^{ff} - \mathbf{v}^s$. The constitutive assumption regarding the extra contribution of those velocities to the kinetic energy of the fluid flow through the pores at the macro-scale are made in the form of the sum (4.3). In this way the added mass effect appears: the mass densities (4.13) and (4.14) of the virtual components satisfy the mass balance equation with non-vanishing production terms (cf. (5.3)–(5.4)) and the additional inertial solid-fluid interaction in the equations of motion (6.6) is evident. As a consequence we arrived at the total representation of the kinetic energy (4.11).

The tensors $\mathbf{A}^\alpha$ and consequently the tensors $\mathbf{P}^\alpha$ are related to the structure of the pores of the medium – the most important parameter describing the

structure is the tortuosity parameter. Its appearence in the papers of Biot, Bedford *et al.* and Kubik was mostly restricted to the isotropic case. However, the present approach introducing a tensorial characterisation of the structure brings for the first time the consqeuent derivation of the added mass effect without any restrictions concerning the size of strain and the fluid in the pores.

## Acknowledgements

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