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Thermomechanical coupled waves in a viscoplastic medium

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Dedicated to Professor Piotr Perzyna
on the occasion of his 70^{th} birthday

IDEAS OF PERZYNA'S theory of material systems in the cases of viscoplastic materials and heat conductors are elaborated. Both theories are based on the notion of a *method of preparation*. They are combined into one coupled model for an elasto-viscoplastic heat conductor. We study its behavior by numerically calculating solutions for plane waves passing through a body.

Key words: visco-plastic materials, hyperbolic systems, nonlinear wave propagation, method of preparation

1. Introduction

FOR AN ELASTO-VISCOPLASTIC body, stress depends on the present *configuration*, and additionally, on *irreversible deformations* accumulated in the past. Perzyna introduced the notion of a *method of preparation* to summarize all information that is given on top of the actual configuration, which allows to determine the present response of the studied material [1].

The description of the method of preparation may contain information that is eventually not needed – as opposed to the notion of a *state* – which by definition must be minimal [2, 3]. Thus a description by the method of preparation may be more convenient. In particular, the method of preparation may be given in terms of the whole *history* of a sample, or comprised in form of just a finite number of essential quantities, the *internal variables* or *state parameters*, cf. [3]. Sometimes it may be useful to have access to both descriptions, cf. [5, 6 7].

We demonstrate the concept of a method of preparation in the case of an elasto-viscoplastic material, for a rigid heat conductor and for a coupled thermomechanical model. For the sake of simplicity we refrain from technicalities and,

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initially, we confine ourselves to 1D models. Then a 3D generalization will be done together with some thermodynamic background for the model equations introduced.

2. General setup of the theory of materials

Constructing mathematical models of real behavior of material objects we use a framework which has its source in the system theory. The accurate setup of such a framework one can find in the papers of W. NOLL [8.9]. There the concepts of state, configuration and evolution function as well as response functions have been used. The approach presented by Noll was *intrinsic*, in the sense that the concepts are independent of the observer and of the description used (i.e. material or spatial). Such a level of abstraction is suitable when a global system approach is used, which is open for further specification.

It was Perzyna who almost at the same time introduced the concept of a *method of preparation* to the theory of materials. The method of preparation together with the configuration form the state of a material element in Noll's approach. *Response functions* are defined on the collection of all states. Inputs (*processes* in Noll's terminology) applied to a material element (or a material system) being at a given state, result in an evolution of the state, and consequently in a new response. The evolution of states forced by inputs (processes), governed in Noll's approach by his *evolution function*, is described in Perzyna's approach by a particular operator responsible for the evolution of the method of preparation, while the actual configuration is contained in the value of the input (process) at the actual time.

A second substantial contribution of P. Perzyna to the theory of materials was the first application of the concept of *internal state variables* (*parameters*) to viscoplasticity theory. After the pioneering paper of K. C. VALANIS who for the first time in the world literature in 1967 formulated the thermodynamic setup of internal state variables for inelestic materials in [10] (later on a celebrated paper by COLEMAN and GURTIN [11] appeared), PERZYNA with his coworker W. WOJNO introduced in [12] the inelastic strain tensor $\boldsymbol{C}^i$ for the description of viscoplastic flow in the complete nonlinear setup of a thermodynamic theory.

The method of preparation of the actual configuration contains all information necessary to give at the given state a unique response value of the material element. The choice of a particular form of the description – or better to say, a particular class of response functions – is crucial for the choice of the method of preparation. Sometimes it can be the whole past (or summed) history of the configuration, represented by a function defined on the infinite domain $[0, \infty)$, in other cases just the rate of change of configuration is taken into account.

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In particular, the above approach can be applied to mechanics – the configuration being a tensorial measure of deformation, to thermodynamics – the configuration being the collection of a tensorial measure of deformation, the absolute temperature and possibly its gradient. Further applications can be found in electro-magnetic fields, diffusion, filtration and so on, where configurations may contain more quantities. In each case the pure *constitutive theory* has to be compatible with *invariancy requirements*. This has to be studied in the context of appropriate *balance equations*. For a one-dimensional setting, this will be demonstrated on the examples of the next three sections. In the framework we are going to use the method of preparation given by two quantities: a tensorial and scalar one (in the 3D thermodynamic case), both are called internal state variables. The first one will represent a measure of inelastic (permanent) deformation of the material element, as in Perzyna's and Wojno's approach, while the second one is a thermal state variable, called in our previous paper a *new temperature scale*, which represents a summed history of the temperature. For both variables *evolution equations* are introduced in the form of ordinary differential equations (ODEs) for each fixed material point (element).

3. Elasto-viscoplastic material

For simplicity as well as for the purpose of the present paper, first the 1D case will be considered. Introducing u as the displacement of a material point X in the case of the mechanical model we identify the configuration with the partial derivative $\partial u/\partial X =: m$ and introduce C_p as a method of preparation. The scalar quantity C_p is an internal state variable, identified with plastic, better to say, inelastic strain, which requires an ODE to govern its evolution. In the context of Noll's theory the pair (m, C_p) can be regarded as a state. Then the *response* function $\hat{S}$ is introduced giving the stress S of the material element (point)

$$S = \hat{S}(m, C_p) \,. \tag{3.1}$$

The evolution of the method of preparation is defined by the so-called kinetic function Ω appearing on the right-hand side of the evolution equation for the internal state variable C_p. We have the Cauchy problem

$$\dot{C}_p = \Omega(m, C_p) \,, \quad C_p(0) = C_p^0 \,, \tag{3.2}$$

where $\dot{C}_p$ denotes the partial derivative with respect to time t at the constant particle X. A particular solution to the equation can be found as a function of time t if an input (a process) in form of a time-dependent configuration $m(t)$ is given as well as an initial condition C_p^0. For the internal variable C_p the kinetic function Ω needs to be identified on the basis of experimental observations and suitable physical identifications. Usually the following is assumed [13]:

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- *elastic range:* if the actual stress is small there is no *yielding*, the state variable C_p does not change, i.e. Ω vanishes;
- *yielding:* if the stress reaches the *plastic limit* κ_0 (or *yield stress*), C_p changes, increases or decreases according to the sign of S. While in the pure plasticity (i.e. in the rate-independent case), the yield stress may never be exceeded, in viscoplasticity we may have *over-stress*, and there is a relation between this over-stress and the rate of plastic flow, i.e.

$$\Omega(m, C_p) = \Phi(\hat{S}(m, C_p); \kappa_0) \tag{3.3}$$

  with some function Φ and a parametric dependence on κ_0;
- *viscosity:* we have rate-dependence due to the evolution equation for the plastic strain.

If we assume vanishing body forces, the first balance law, called the conservation of momentum $\rho_0 v$, where $v = \dot{u}$ denotes the velocity and ρ_0 the reference mass density, together with the geometric and evolution equations, may be expressed by

$$\dot{m} = v_{,X} , \tag{3.4}$$

$$\dot{v} = \frac{1}{\rho_0} \hat{S}(m, C_p)_{,X} , \tag{3.5}$$

$$\dot{C}_p = \Omega(m, C_p) . \tag{3.6}$$

This system (3.4 – 3.6) is suitable for time-integration. Note that here all right-hand sides depend only on the *local state* $U = (v, m, C_p)^T$, and on its first partial derivatives $U_{,X}$. This can be brought to the general form

$$\dot{U} + f(U)_{,X} = b(U) \tag{3.7}$$

with the flux term f and the source term b. In this form we will solve this system in Sec. 7.2 and its thermo-mechanical generalization in Sec. 7.3. Note that for the hyperbolic system (3.4 – 3.6) the *local state* is the *state* extended by the *local velocity*.

As an easy example, consider the balance equations with constitutive equations for stress S and plastic flow speed Ω determined by

$$S = E(m - C_p), \tag{3.8}$$

$$\dot{C}_p = \Omega = \Phi(S; \kappa_0), \tag{3.9}$$

where E is Young's modulus and Φ is a (regularized) indicator function of the interval $[-\kappa_0, \kappa_0]$. Here we denote by κ_0 the yield stress, for smaller values of S

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there is practically no flow, above we have a dramatic increase of flow speed. As an example we take

$$\Phi(S, \kappa_0) = \text{sign}\,(S)\text{pos}\,abs\,((S - \kappa_0)^n)$$

with pos$(\cdot)$ denoting the positive part of the real input argument. For implementation, the absolute value, sign and positive part have been regularized in the way proposed in [14].

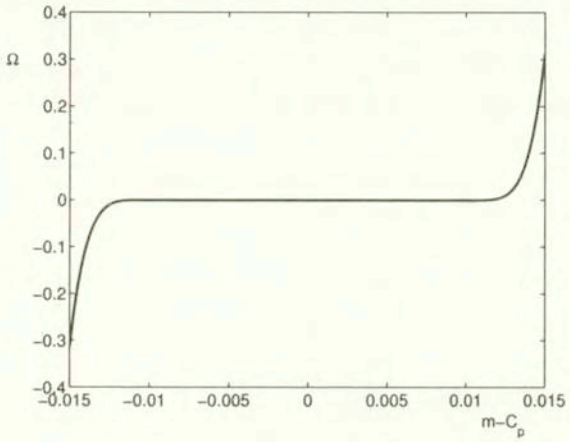

FIG. 1. Flow speed as function of $m - C_p$.

It is worthwhile to note that in the plasticity theory usually *incompressibility* is assumed. However, for the considered 1D theory this is of no consequence.

4. Heat conduction

Several technological situations at moderate and high temperatures (such as distribution of temperature around propagating cracks [15], and temperature distribution in solid materials due to laser pulse train of a very short duration [16]) and many physical experiments at low temperatures (cf. [17, 18, 19, 20, 21, 22]), show the necessity of taking into account the wave structure of the heat transport.

There are several phenomenological approaches aimed at a new heat conduction equation that could describe the mentioned phenomena. Some of them are reviewed in the papers [23, 24, 25]. The present approach keeps a proportionality law ([26]) between the heat flux q and the spatial gradient of a thermal internal state variable β in the form

$$q = -\alpha\beta_{,x}\ , \tag{4.1}$$

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with a proportionality coefficient α which may depend on ϑ. Here the lower case x denotes the spatial position of the material point X. (Notice that in the case of a deformable material body x is the position after the displacement u, i.e. $x = u + X$). The classical Fourier law, on the other hand, is

$$q = -k(\vartheta)\vartheta_{,x} \tag{4.2}$$

where the heat conduction coefficient $k(\vartheta)$ can depend, in general, on the absolute temperature ϑ. The scalar internal state variable β will represent the history of the temperature in the sense that its value at the present time will be a solution to an initial-value problem for a suitable ODE in which the right-hand side is a function of (at least) ϑ and β. Let us notice that in the linear case such an ODE can have the form:

$$\dot{\beta} = \frac{1}{\tau}(\vartheta - \beta), \text{ with initial condition } \beta(0) = \beta^0, \tag{4.3}$$

where the coefficient τ is called the *thermal relaxation time*. If we perform spatial differentiation in (4.3) and make use of (4.1) we get

$$\tau\dot{q} + q = -\alpha\vartheta_{,x} . \tag{4.4}$$

The last equation can be regarded as a modification of the Fourier law if the heat conductivity coefficient $k(\vartheta)$ is identified with α and one has added to (4.2) a term in which the time derivative of the heat flux q is multiplied by the coefficient τ. If one omits introducing the thermal internal state variable β, the last equation (4.4) becomes the well-know *telegraph equation* which is the governing equation of the so-called *rate type approach* modelling heat conduction, and is called the MAXWELL-CATTANEO-VERNOTTE-KALISKI equation ([27, 28, 29, 30, 31]). It is interesting to add that the approach based on the last equation can be obtained as a particular case of the summed temperature–gradient history approach proposed by the authors of [32].

There are some drawbacks in applying the equation (4.4) to the thermodynamic theory of thermomechanics, especially because of the internal dissipation inequality. The present author (W. K.) with PERZYNA in [33] (and later in [34] in a more general setup) calculated for the first time finite speeds of thermal (and thermo-mechanical) waves in the framework of a thermodynamic theory with internal state variables. Later on COLEMAN *et al.* [35] repeated the first results of Kosiński and Perzyna from 1972 in a slightly different set-up, without referring to the internal state variable approach and to the original paper [33].

The model proposed in [26] is based on a different procedure than the derivation of the wave-type heat propagation equation; in fact it is somehow a next generation of [33], [34]. In that model a scalar internal state variable β has

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been introduced to represent at the same time a history of the thermodynamic temperature and the potential for its gradient. In the course of obtaining the consequences for the laws of thermodynamics, a modified Fourier-type law was found, leading to finite speeds of propagation of thermal and thermo-mechanical waves. The model has been mostly applied to heat conductors in 1D and 3D cases and to thermo-elastic solids (cf. [36, 37], and the literature given in [38]). Its similarities to other models are shown in [39, 40] (cf. also [23, 25]). In terms of the material description in the last approach, the state is formed by the configuration, composed of the temperature ϑ and the method of preparation. The latter is composed of the (thermal) internal state variable β, and its spatial gradient $\partial\beta/\partial x = p$. The state is a triple (ϑ, β, p), while the response is composed of two quantities: the internal energy e and the heat flux q, i.e. the pair (e, q). For the evolution of the method of preparation we have at our disposal two differential equations, written in a general form as:

$$\dot{\beta} = F(\vartheta, \beta) , \tag{4.5}$$

$$\dot{p} = F_1(\vartheta)_{,x} + F_2'(\beta)p \tag{4.6}$$

with

$$F(\vartheta, \beta) = F_1(\vartheta) + F_2(\beta) \text{ and } F_2'(\beta) = dF_2(\beta)/d\beta .$$

For the response function we may write

$$e = \hat{e}(\vartheta, \beta, p) \text{ and } q = \hat{q}(\vartheta, \beta, p) . \tag{4.7}$$

Note that both variants have a drawback: the classical one based on (4.4) needs an initial condition for the heat flux q, the semi-empirical ([39]) theory ([14–15]) requires initial data for β. (Notice that the initial condition for p can be calculated from the initial condition for β by simple differentiation.) We conclude this subsection with a formulation of the first order system for the one-dimensional case. We introduce the vector of thermodynamic unknowns as $V = (e, \beta, p)^T$ with the *internal energy* e, the *semi-empirical temperature* β and the gradient of the latter $p = \nabla\beta$, in the one-dimensional case $p = \beta_{,x}$.

We assume a one-to-one relation between absolute temperature ϑ and internal energy e, e.g. given by Debye's law $e = e_4\vartheta^4 + e_0\vartheta$ or simple proportionality, $e = c_v\vartheta$. Together with the previous constitutive relation (4.1) in the form $q = -\alpha(\vartheta)p$, and by adding the energy balance $\dot{e} = -q_{,x} + r$, we obtain the system

$$\begin{pmatrix} \dot{e} \\ \dot{\beta} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} -q_{,x} \\ 0 \\ F_{1,x} \end{pmatrix} + \begin{pmatrix} r \\ F_1 + F_2 \\ F_2'p \end{pmatrix} . \tag{4.8}$$

Here r denotes the known external heat source density, and $F_1 = F_1(\vartheta), F_2 = F_2(\beta)$.

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5. Elasto-viscoplastic heat conductor

In this section the governing system of equations is formulated for the 1D case. The theory is based on thermodynamics of heat conduction in a viscoplastic medium with over-stress function. Such a function is known from Perzyna's model of viscoplasticity (cf. [3, 4, 13]. We assume here that this function is independent of the gradient of β. Due to thermodynamics, the stress appearing in it is not the total second Piola-Kirchhoff stress tensor as calculated from the free energy function by the potential relation, but its so-called *instantaneous part* $\boldsymbol{S}_{RI}$, in which the gradient of β has been neglected (no coupling with $\nabla\beta$). Consequently, we assume that the evolution equation for the inelastic strain tensor $\boldsymbol{C}_p$ is given by the polynomial over-stress function

$$\Omega(\boldsymbol{S}_{RI}, \vartheta) = \gamma^* \left(\frac{\sqrt{J_2}}{\kappa(\vartheta)} - 1 \right)^n \frac{\partial \sqrt{J_2}}{\partial \boldsymbol{S}^D_{RI}} = \gamma^* \left(\frac{\sqrt{J_2}}{\kappa(\vartheta)} - 1 \right)^n \frac{\boldsymbol{S}^D_{RI}}{\sqrt{J_2}} \tag{5.1}$$

in which the yield stress κ as well as the viscosity coefficient γ^* may depend on the temperature ϑ, and $\boldsymbol{S}^D_{RI}$ is the deviatoric part of the instantaneous stress tensor $\boldsymbol{S}_{RI}$, with J_2 as its second invariant. In the present 1D case $J_2 = \sqrt{\frac{2}{3}}\, abs(S_R)$.

Finally, we collect all equations and formulate them as a system of PDEs which a vector of unknowns $\boldsymbol{U}$ has to satisfy. Restricting ourselves to the 1D case we assume the notations: S as S^1_1 component of the first Piola-Kirchhoff stress tensor $\boldsymbol{S}$ and E for the component E_{11} of the elastic part of the Lagrange strain tensor $\boldsymbol{E}^e$, writing C_p for the component C_{11p} of the tensor internal state variable $\boldsymbol{C}_p$, and we put S_R for the S^{11}_{RI} component of the stress tensor $\boldsymbol{S}_{RI}$ appearing in the definition of Perzyna's over-stress function, q_κ as the only component of the reference heat flux vector $\boldsymbol{q}_\kappa$ and p for the first component of the spatial gradient of β, $p = \nabla\beta$. Since $1 + m$ is the F_{11} component of the deformation gradient $\boldsymbol{F}$, we get

$$E = \frac{(1+m)^2 - C_p}{2}\,. \tag{5.2}$$

Let us assume that the free energy ψ is (an isotropic, in 3 D case) function of the strain m, inelastic strain (i.e. internal state variable) C_p, gradient of β and temperature ϑ , i.e.

$$\psi = \hat{\psi}(m, \vartheta, \beta_{,x}, C_p)\,. \tag{5.3}$$

Repeating our assumptions from [38, 41] telling that:

(1) the heat flux is linear in the gradient of β, cf. (4.1),

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(2) the specific heat is independent of the gradient of β, and adding two more assumptions here, namely:

(3) elastic properties are independent of the inelastic strain and are expressed by the strain-temperature-stress relations of a neo-Hookean material,

(4) the specific heat is of 3rd order in temperature,

we end up with the following partition of the free energy function:

$$\hat{\psi}(m, \vartheta, p, C_p) = \psi_1(E, \vartheta) + \psi_2(m, p, \vartheta) \tag{5.4}$$

where the first mechanical part of ψ_1 is quadratic in E and bilinear in $\vartheta - \vartheta_0$ and E, while ψ_2 must possess the form

$$\psi_2(m, p, \vartheta) = 0.5\psi_{20}\,\vartheta\, p^2\, m, \tag{5.5}$$

where the factor 0.5 is introduced for convenience only. The linearity of the heat flux in p(cf.(4.1)), and the independence of the coefficient α of the strain m together with the above assumptions lead to the specification of the coefficient and the form of ψ_1

$$\psi_1(E, \vartheta) = c_1^2 E^2 - \frac{\gamma}{\rho_0}(\vartheta - \vartheta_0)E - c_{v0}\,\vartheta\,(\ln\frac{\vartheta}{\vartheta_0} - 1) - \frac{1}{12}\,c_{v3}\frac{\vartheta^4}{\vartheta_0^3} - \eta_0\,(\vartheta - \vartheta_0)\,, \tag{5.6}$$

with $\alpha(\vartheta) = \rho_0\,\psi_{20}\vartheta^2 F'(\vartheta)$ and constants $\psi_{20}, \gamma, \vartheta_0, c_{v0}, c_{v3}, \eta_0$. The dependence of α on F_1' turns out to be a consequence of the second law of thermodynamics, together with the following stress-temperature-strain relation

$$\begin{aligned} S &= \rho_0 \frac{\partial \psi}{\partial m} - \rho_0 p \frac{\partial \psi_2}{\partial p} m^{-1} \\ &= \left[\rho_0 c_1^2 \frac{(1+m)^2 - C_p}{2} - \gamma(\vartheta - \vartheta_0)\right](1+m) - \frac{1}{2}\rho_0\,\vartheta\,\psi_{20}\,p^2\,, \end{aligned} \tag{5.7}$$

where $c_1^2 = (\lambda + 2\mu)/\rho_0$, with λ and μ as Lamé's constants.

On the other hand, the thermodynamic identity between the internal energy (per unit mass) ϵ, the free energy ψ and the entropy $\eta = -\partial\psi/\partial\vartheta$, which is of the form $\epsilon = \psi + \vartheta\eta$, applied to (5.4–5.6) gives the following expression for ϵ

$$\hat{\epsilon}(E, \vartheta) = \frac{c_1^2}{2}E^2 + \frac{\gamma}{\rho_0}\vartheta_0 E + c_{v0}\,\vartheta + \frac{1}{4}\,c_{v3}\frac{\vartheta^4}{\vartheta_0^3} + \eta_0\,\vartheta_0\,. \tag{5.8}$$

For the first component of the instantaneous stress $\boldsymbol{S}_{RI}$ we have

$$S_R = \rho_0(1+m)^{-1}\frac{\partial \psi_1}{\partial m} = \left[\rho_0 c_1^2 \frac{(1+m)^2 - C_p}{2} - \gamma(\vartheta - \vartheta_0)\right]. \tag{5.9}$$

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For the inelastic strain we have (cf. (5.1))

$$\dot{C}_p = \gamma^* \left(\frac{abs(S_R)}{\sqrt{\frac{3}{2}}\,\kappa} - 1 \right)^n \frac{S_R}{\sqrt{\frac{2}{3}}\,abs(S_R)} \,. \tag{5.10}$$

For the only nontrivial heat flux component q_κ we have

$$q_\kappa = -\rho_0 \psi_{20} \vartheta^2 \frac{\partial F(\beta, \vartheta)}{\partial \vartheta} p \,. \tag{5.11}$$

For β we have the evolution equation

$$\dot{\beta} = F, \quad \text{with} \quad F(\vartheta, \beta) = F_1(\vartheta) + F_2(\beta) \tag{5.12}$$

and for its material gradient $\pi := \beta_{,X} = p(1+m)$, the so-called prolongated equation (cf. (4.6)),

$$\dot{\pi} = F_1(\vartheta)_{,X} + F_2'(\beta) p (1+m) \,. \tag{5.13}$$

The above constitutive functions are now restricted by the second law of thermodynamics expressed in terms of the *residual dissipation inequality*

$$-\frac{\rho_0}{\vartheta} \left[\frac{\partial F}{\partial \beta} \frac{\partial \psi}{\partial p} p + \frac{\partial \psi}{\partial C_P} \Omega(S_R, \vartheta) \right] \geq 0 \,, \tag{5.14}$$

in which the function Ω is the right-hand side of Eq. (5.10). The balance equations, in this case of linear momentum and energy, will be

$$\dot{v} = \frac{1}{\rho_0} S_{,X} + b \,, \tag{5.15}$$

$$\dot{\left(\rho_0 \epsilon + \frac{\rho_0}{2} v^2\right)} + (q_\kappa - vS)_{,X} = \rho_0 r + \rho_0 v\, b \,, \tag{5.16}$$

where r and b represent heat sources and body forces, respectively. The equations are accompanied by the geometrical compatibility equation

$$\dot{m} = v_{,X} \,. \tag{5.17}$$

6. Numerical results

For hyperbolic systems, we may apply the well-known explicit time-stepping schemes due to Lax-Friedrich, Osher or Lax-Wendroff. While the first mentioned

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methods have the advantage of monotonicity, and hence converge to entropy solutions, the Lax-Wendroff scheme is second order accurate and converges faster – at the price of possible violations of monotonicity and artificial oscillations near the wave fronts. We solve the sample problems by an *ad hoc* hybrid combination of the mentioned methods to find a compromise between speed and stability.

We present here in Fig. 2 the resulting data for the mechanical quantities, velocity and strain. There is a wave running from left to right which results from applying a short pressure impulse at the left boundary, which results in a compression wave.

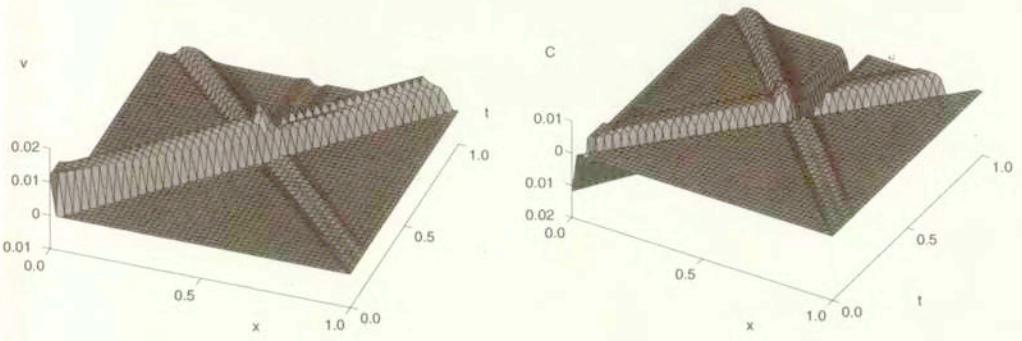

FIG. 2. Velocity and strain.

At the same time, a heat impulse occurs at the right boundary. It runs as a wave from right to left, and it it affects also visibly the mechanical submodel. In the Fig. 3, we depict the temperature increment together with the heat flux. Here the hyperbolic character of the heat transition model becomes obvious.

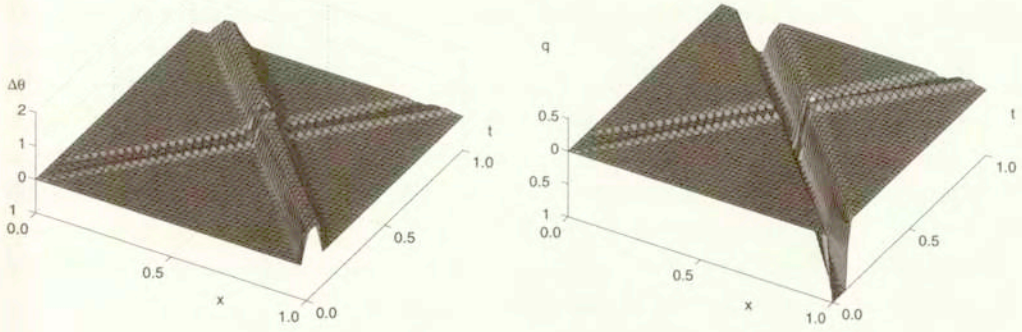

FIG. 3. Temperature and heat flux.

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Finally, in Fig. 4 we show the residual deformations, i.e. the internal variable C_p. Note that we have permanent deformations first where the mechanical wave originates. However, due to the nonlinear coupling with the thermal wave, at elevated temperature the state re-enters the plastic range after the crossing point.

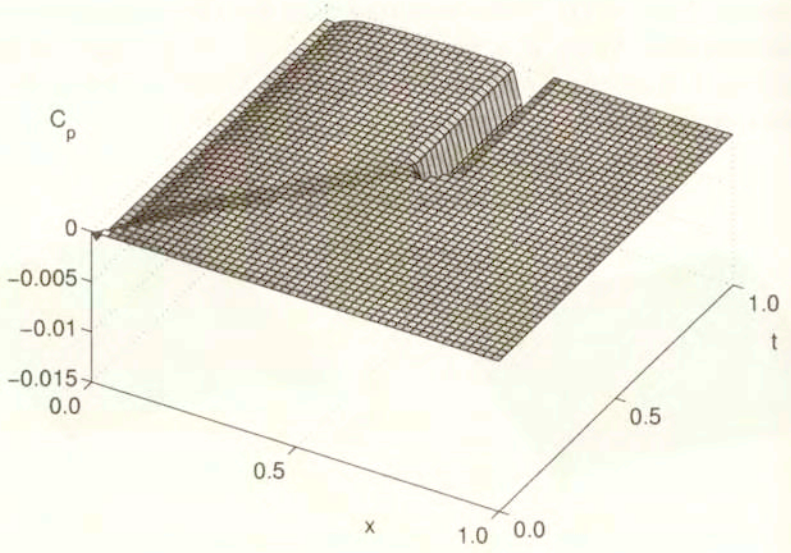

FIG. 4. Permanent deformation.

7. Generalizations

We have presented just the one-dimensional implementations of the general theory. Further, we have limited ourselves to the minimum number of couplings to obtain a non-trivial thermo-mechanical theory. That way it was possible to present the ideas behind the general theory in a particularly clear manner. At this stage, we want to give just some hints as to the full three-dimensional formulation of the theory. However, it has to be mentioned that a major problem is the identification of material parameters/functions for each real material at hand. To keep this paper self-contained, we quote some of the foundations of thermodynamics of an inelastic material with the modified Fourier law presented in [42] (cf. also [38], [43]) and repeat the most important consequences of that presentation. The case of 3D thermo-mechanics at finite strains requires to introduce instead of scalar quantities the following variables:

- $\boldsymbol{F}$ – the deformation gradient tensor;
- the Lagrange strain tensor $\boldsymbol{E}$ related to $\boldsymbol{F}$ by $\boldsymbol{E} = 0.5(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$;

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- the actual mass density ρ related to the reference mass density ρ_0 by the law $\rho_0 = J\rho$, with $J = \det \boldsymbol{F}$;
- the symmetric Cauchy stress tensor $\boldsymbol{T}$ related to the first Piola-Kirchhoff stress tensor $\boldsymbol{S}$ by the identity $\boldsymbol{S} = J\boldsymbol{T}\boldsymbol{F}^{-T}$ while the
- second Piola-Kirchhoff stress tensor $\boldsymbol{S}_R$ is given by the identity $\boldsymbol{S}_R = \boldsymbol{F}^{-1}\boldsymbol{S}$;
- the particle velocity $\boldsymbol{v}$,
- the heat flux vector $\boldsymbol{q}$, related to the reference heat flux $\boldsymbol{q}_\kappa$ by the identity $\boldsymbol{q}_\kappa = J\boldsymbol{q}\boldsymbol{F}^{-T}$.

We assume that the spatial gradient grad $\beta = \nabla\beta$ and the inelastic strain tensor $\boldsymbol{C}_p$ appears in the constitutive equations for free energy ψ, entropy η, stress $\boldsymbol{S}$ and heat flux $\boldsymbol{q}_\kappa$ as the method of preparation:

$$
\begin{aligned}
\psi &= \psi^*(\boldsymbol{F}, \vartheta, \nabla\vartheta, \nabla\beta, \boldsymbol{C}_p), \qquad \eta = \eta^*(\boldsymbol{F}, \vartheta, \nabla\vartheta, \nabla\beta, \boldsymbol{C}_p),\\
\boldsymbol{S} &= \boldsymbol{S}^*(\boldsymbol{F}, \vartheta, \nabla\vartheta, \nabla\beta, \boldsymbol{C}_p), \qquad \boldsymbol{q}_\kappa = \boldsymbol{Q}^*(\boldsymbol{F}, \vartheta, \nabla\beta, \boldsymbol{C}_p).
\end{aligned} \tag{7.1}
$$

For the internal state variables – the scalar β and the tensor $\boldsymbol{C}_p$ – the evolution equations have the form (cf. (3.6), (2.61))

$$
\dot{\beta} = F(\vartheta, \beta)\,, \tag{7.2}
$$

$$
\dot{\boldsymbol{C}}_p = \Omega^*(\boldsymbol{F}, \vartheta, \boldsymbol{C}_p)\,. \tag{7.3}
$$

Notice that the independence of the heat flux of the actual value of the temperature gradient is crucial for the development of a hyperbolic model. Now the second law of thermodynamics

$$
\rho_0(\dot{\eta}\vartheta - \dot{\epsilon}) + \boldsymbol{S}\cdot\dot{\boldsymbol{F}} - \vartheta^{-1}\boldsymbol{q}_\kappa\cdot \mathrm{Grad}\vartheta \geq 0, \tag{7.4}
$$

is used to get the following potential–type consequences :

$$
\begin{aligned}
\boldsymbol{0} &= \partial\psi^*/\partial\nabla\vartheta, \qquad \boldsymbol{S} = \rho_0\left[\partial\psi^*/\partial\boldsymbol{F} - \nabla\beta\otimes(\partial\psi^*/\partial\nabla\beta)\boldsymbol{F}^{-T}\right],\\
\eta &= -\partial\psi^*/\partial\vartheta, \qquad \boldsymbol{q}_\kappa = -\rho_0(\vartheta\partial F/\partial\vartheta)(\partial\psi^*/\partial\nabla\beta)\boldsymbol{F}^{-T},
\end{aligned} \tag{7.5}
$$

and a reduced inequality, with F substituted from (7.2) and Ω^* from (7.3),

$$
-\rho_0\left[(\partial F/\partial\beta)(\partial\psi^*/\partial\nabla\beta)\cdot\nabla\beta + (\partial\psi^*/\partial\boldsymbol{C}_p)\cdot\Omega^*\right] \geq 0. \tag{7.6}
$$

The first three identities of (7.5) are well-known, the last one is rather not typical (cf. [34]). The stress potential relation in $(7.5)_2$ contains two components: the first component is rather classical, while the second one represents the direct

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coupling between mechanical and thermal fields in which an extra stress term appears. Substituting $\boldsymbol{q}_\kappa$ into the stress relation in (7.5) and (7.6) we get a new potential relation for the Piola-Kirchhoff stress (in terms of the heat flux) and a residual inequality, both in terms of the heat flux. Here this potential relation is presented together with the relation for the second Piola-Kirchhoff stress $\boldsymbol{S}_R$ in terms of $\nabla\beta$ and the actual heat flux vector, all in a general anisotropic case.

$$
\begin{aligned}
\boldsymbol{S} &= \rho_0 \partial\psi^*/\partial\boldsymbol{F} + (\vartheta \partial F/\partial\vartheta)^{-1}(\nabla\beta \otimes \boldsymbol{q}_\kappa), \\
\boldsymbol{S}_R &= \rho_0 \boldsymbol{F}^{-1}\left[\partial\psi^*/\partial\boldsymbol{F} - \nabla\beta \otimes (\partial\psi^*/\partial\nabla\beta)\boldsymbol{F}^{-T}\right], \\
\boldsymbol{q} &= -\rho\vartheta(\partial\psi^*/\partial\nabla\beta)(\partial F/\partial\vartheta)\ .
\end{aligned} \tag{7.7}
$$

Let us assume in $(7.7)_3$ a linear relation with respect to $\nabla\beta$. Then due to the principle of material frame indifference this will be true if the free energy function is of the form

$$
\psi^* = \psi_1^*(\boldsymbol{B}, \vartheta, \boldsymbol{C}_p) + 0.5\psi_2^*(\boldsymbol{B}, \vartheta, \boldsymbol{C}_p)\nabla\beta \cdot \nabla\beta, \tag{7.8}
$$

where the factor 0.5 has been assumed for convenience only, with $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$; assuming this, Eq. $(7.7)_3$ turns into

$$
\boldsymbol{q} = -\alpha^*(\boldsymbol{B}, \vartheta, \boldsymbol{C}_p)\nabla\beta \ \text{ with } \ \alpha^*(\boldsymbol{B}, \vartheta, \boldsymbol{C}_p) = \rho\vartheta \partial F/\partial\vartheta \ \psi_2^*(\boldsymbol{B}, \vartheta, \boldsymbol{C}_p)\ . \tag{7.9}
$$

In the present case, the independence of the free energy function of β leads to splitting the function $F(\vartheta, \beta)$ into two independent terms

$$
F(\vartheta, \beta) = F_1(\vartheta) + F_2(\beta). \tag{7.10}
$$

Moreover, in the linear case (7.9) the proportionality material coefficient α^* has to be independent of both strains, i.e.

$$
\alpha^*(\boldsymbol{B}, \vartheta, \boldsymbol{C}_p) = \alpha^*(\vartheta), \quad \psi_2^*(\boldsymbol{B}, \vartheta, \boldsymbol{C}_p) = \psi_{21}^*(\vartheta)J \ \text{ and } \ \boldsymbol{q} = -\alpha^*(\vartheta)\ \nabla\beta, \tag{7.11}
$$

with $\psi_{21}^* \geq 0$.

If the internal energy $\epsilon^* = \psi^* + \vartheta\eta^*$ is independent of $\nabla\beta$ then the function ψ_{21}^* is linear in ϑ, i.e. $\psi_{21}^*(\vartheta) = \psi_{20}\vartheta$. In this case Eq. $(7.7)_1$ due to (7.8) will take the form

$$
\boldsymbol{S} = 2\rho_0(\partial\psi_1^*/\partial\boldsymbol{B})\boldsymbol{F} + \rho_0\psi_{20}\vartheta J(0.5|\nabla\beta|^2\boldsymbol{I} - \nabla\beta \otimes \nabla\beta)\boldsymbol{F}^{-T}, \tag{7.12}
$$

or equivalently, using $(7.11)_3$,

$$
\boldsymbol{S} = 2\rho_0(\partial\psi_1^*/\partial\boldsymbol{B})\boldsymbol{F} + \rho_0\frac{\psi_{20}\vartheta J}{\alpha^*(\vartheta)^2}(0.5|\boldsymbol{q}|^2\boldsymbol{I} - \boldsymbol{q} \otimes \boldsymbol{q})\boldsymbol{F}^{-T}\ , \tag{7.13}
$$

with $\alpha^*(\vartheta) = \rho_0\psi_{20}^*\vartheta^2 F_1'(\vartheta)$.

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Let us notice that even in this simplified case the stress–strain-temperature relation has an extra term due to the thermo-mechanical coupling; this term can be called an *extra thermal stress*. This contribution can have a substantial meaning in describing thermo-mechanical coupling phenomena in viscoplastic materials. We assume for the material the property of *isotropy*. Hence, the free energy function can be expressed in terms of the right Cauchy-Green stress tensor $\boldsymbol{C}$ instead of the left one $\boldsymbol{B}$. Moreover, for the mechanical response a neo-Hookean type of behavior is assumed, however, in terms of the so-called elastic strain tensor $\boldsymbol{E}^e$ defined by

$$\boldsymbol{E}^e = 0.5(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{C}_P)\ . \tag{7.14}$$

Then for the free energy function we will have

$$\psi(\boldsymbol{F}, \vartheta, \operatorname{grad}\beta, \boldsymbol{C}_P) = \psi_1(\boldsymbol{E}^e, \vartheta) + \psi_2(\boldsymbol{F}, \nabla\beta, \vartheta), \tag{7.15}$$

in which in ψ_1 the first part is quadratic and isotropic in $\boldsymbol{E}^e$:

$$\frac{\lambda}{2\rho_0}(\operatorname{tr}\boldsymbol{E}^e)^2 + \frac{\mu}{\rho_0}\operatorname{tr}(\boldsymbol{E}^e)^2 \tag{7.16}$$

while the thermo-mechanical coupling is bilinear in both variables, i.e.

$$-\frac{\gamma}{\rho_0}(\vartheta - \vartheta_0)\operatorname{tr}\boldsymbol{E}^e\,. \tag{7.17}$$

Assuming, moreover, that the specific heat is of 3rd order in temperature and independent of $\nabla\beta$, we can write

$$\begin{aligned}\psi_1(\boldsymbol{E}^e, \vartheta) = {} & \frac{\lambda}{2\rho_0}(\operatorname{tr}\boldsymbol{E}^e)^2 + \frac{\mu}{\rho_0}\operatorname{tr}(\boldsymbol{E}^e)^2 - \frac{\gamma}{\rho_0}(\vartheta - \vartheta_0)\operatorname{tr}\boldsymbol{E}^e \\ & - c_{v0}\,\vartheta\,(\ln\frac{\vartheta}{\vartheta_0} - 1)\ - \frac{1}{12}\,c_{v3}\frac{\vartheta^4}{\vartheta_0^3} - \eta_0\,(\vartheta - \vartheta_0).\end{aligned} \tag{7.18}$$

The second part we assume in the form already used in our previous paper (cf. [41])

$$\psi_2(\boldsymbol{F}, \nabla\beta, \vartheta) = 0.5\psi_{20}\vartheta\nabla\beta\cdot\nabla\beta\det\boldsymbol{F}. \tag{7.19}$$

Due to the potential relation $\eta = -\partial\psi/\partial\vartheta$ and the relation between the internal and free energies $\epsilon^* = \psi + \eta\vartheta$, the internal energy is independent of $\nabla\beta$ and has the form

$$\begin{aligned}\epsilon^*(\boldsymbol{E}^e, \vartheta) = {} & \frac{\lambda}{2\rho_0}(\operatorname{tr}\boldsymbol{E}^e)^2 \\ & + \frac{\mu}{\rho_0}\operatorname{tr}(\boldsymbol{E}^e)^2 + \frac{\gamma}{\rho_0}\vartheta_0\operatorname{tr}\boldsymbol{E}^e + c_{v0}\vartheta + \frac{1}{4}\,c_{v3}\frac{\vartheta^4}{\vartheta_0^3} + \eta_0\,\vartheta_0.\end{aligned} \tag{7.20}$$

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Due to thermodynamical consequences for the first Piola-Kirchhoff stress tensor $\boldsymbol{S}$ we obtain the following stress–strain-temperature relation

$$\boldsymbol{S} = \lambda\ \mathrm{tr}\boldsymbol{E}^e\boldsymbol{F} + 2\mu\boldsymbol{F}\boldsymbol{E}^e - \gamma(\vartheta - \vartheta_0)\boldsymbol{F} + \rho_0\ \psi_{20}\vartheta\ J\ (0.5|\nabla\beta|^2\boldsymbol{I} - \nabla\beta\otimes\nabla\beta)\boldsymbol{F}^{-T}. \tag{7.21}$$

For the instantaneous stress tensor $\boldsymbol{S}_{RI}$ being equal to $\boldsymbol{S}_R = \boldsymbol{F}^{-1}\boldsymbol{S}$ at vanishing $\nabla\beta$ we get from (7.21)

$$\boldsymbol{S}_{RI} = \rho_0\ \boldsymbol{F}^{-1}\frac{\partial\psi_1(\boldsymbol{F},\vartheta)}{\partial\boldsymbol{F}} = \lambda\mathrm{tr}\boldsymbol{E}^e\boldsymbol{I} + 2\mu\boldsymbol{E}^e - \gamma(\vartheta - \vartheta_0)\boldsymbol{I}\ . \tag{7.22}$$

This stress tensor is used in the defintion of the over-stress function and the evolution equation for the inelastic strain (cf. (5.1) and (7.3)).

8. Conclusions

More than 20 years after the publication of Perzyna's book [3] on viscoplasticity, the theory laid down there is still alive, and there are still challenging new problems in this field.

Thanks to tremendous progress in computational methods, today it is possible to realize numerically some of the concepts developed theoretically in the period of rapid development of the viscoplastic material models.

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