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On the reduction of constants in plane elasticity with eigenstrains

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Dedicated to Professor Piotr Perzyna
on the occasion of his 70^{th} *birthday*

IN THIS PAPER the reduced parameter dependence in linear plane elasticity with eigenstrains (transformation strains) is studied. The focus is on simply connected inhomogeneous materials and two-phase materials with perfectly bonded interfaces. In the analysis we rely on the result of CHERKAEV, LURIE and MILTON (Proc. Roy. Soc. Lond. A 438, 519-529, 1992), and we show that the stress field is invariant under a shift in area bulk and shear compliances, if the eigenstrains obey certain conditions. The analysis can be extended to multiply connected inhomogeneous materials and materials with slipping interfaces.

1. Introduction

IN THIS PAPER we focus on linear plane elasticity with eigenstrains to study a reduced parameter dependence. In the terminology of MURA [1] eigenstrains may represent nonelastic strains such as thermal expansion, plastic strain, phase transformation, initial strain, and other. The classical work pointing out the reduced dependence of stresses on elastic constants in plane elasticity was performed by MICHELL [2]. He showed that for materials with holes, the in-plane stress fields are independent of elastic constants, provided that the loading is in terms of prescribed tractions and that there are no net forces on internal boundaries. This result was utilized in an experimental technique called photoelasticity. DUNDURS [3–4] extended Michell's result to planar two-phase materials and showed that stress fields depend on only two non-dimensional constants, instead of three, if the composite material is subjected to tractions. This concept was generalized to multi-phase materials by NEUMEISTER [5]. CHERKAEV, LURIE and MILTON [6] showed that the stress field in two-dimensional (planar) elasticity is invariant under a shift in elastic bulk and shear compliances, which is directly related to the Dundurs result, and they extended the concept of reduced parameter dependence to effective elastic moduli. This latter work, referred to as the CLM result, inspired a number of follow-up studies in the context of planar elasticity [7–15]. The present paper extends these earlier results to the linear plane elasticity with eigenstrains. The CLM concept was also found applicable to planar Cosserat materials [16], planar piezoelectric materials [17], planar electromagnetic thermoelastic materials [18, 19], and was explored for three-dimensional elasticity [20, 21].

First, we briefly present the main definitions of three-dimensional elasticity with eigenstrains, following the notation of MURA [1]. The total strain ε_{ij} is the sum of the elastic strain e_{ij} and the eigenstrain ε^*_{ij}

$$\varepsilon_{ij} = e_{ij} + \varepsilon^*_{ij}, \qquad i,j = 1,2,3. \tag{1.1}$$

The total strain ε_{ij}, for infinitesimal deformations, is related to displacements as $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$ and is compatible. For linear elastic materials the elastic strain components are related to stress σ_{ij} by Hooke's law as

$$\sigma_{ij} = C_{ijkl}e_{kl} = C_{ijkl}\left(\varepsilon_{kl} - \varepsilon^*_{kl}\right) \tag{1.2}$$

where C_{ijkl} is the elastic stiffness tensor.

The inverse of expression (1.2) is

$$e_{ij} = S_{ijkl}\sigma_{kl} \tag{1.3}$$

where $S_{ijkl} = \left(C_{ijkl}\right)^{-1}$ is the elastic compliance tensor. Using Eq. (1.1), Eq. (1.3) can be written in the form

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} + \varepsilon^*_{ij}. \tag{1.4}$$

Note that all quantities may depend on the spacial position $\mathbf{x}$. This dependence representation is omitted for simplicity of notation.

In elasticity with eigenstrains the material is assumed to be free from any external forces and surface constraints. If these conditions of free surface are not satisfied, the stress field can be obtained by a superposition of the stress of a free body and the stress obtained from the solution of a given boundary value problem with nonzero external forces or boundary conditions.

The stresses must satisfy the equations of equilibrium

$$\sigma_{ij,j} = 0 \qquad i,j = 1,2,3 \tag{1.5}$$

and traction free-boundary condition

$$\sigma_{ij}n_j = 0. \tag{1.6}$$

Following MURA [1], by substituting Eq. (1.2) into Eq. (1.5) and assuming homogeneous material, we have

$$C_{ijkl}\varepsilon_{kl,j} = C_{ijkl}\varepsilon^*_{kl,j} \tag{1.7}$$

and by substituting Eq. (1.2) into Eq. (1.6) we obtain

$$C_{ijkl}\varepsilon_{kl}n_j = C_{ijkl}\varepsilon^*_{kl}n_j. \tag{1.8}$$

Note that in the absence of eigenstrain ($\varepsilon^*_{kl} = 0$), the left-hand side of Eq. (1.7) corresponds to $\sigma_{ij,j}$ and the left-hand side of Eq. (1.8) to $\sigma_{ij}n_j$. Thus, Eq. (1.7) is in the form $\sigma_{ij,j} = -X_i$ where $X_i = -C_{ijkl}\varepsilon^*_{kl,j}$ and Eq. (1.8) is in the form $\sigma_{ij}n_j = t_i$ where $t_i = C_{ijkl}\varepsilon^*_{kl}n_j$. Therefore, the contribution of eigenstrain ε^*_{ij} to the equations of equilibrium (1.7) is mathematically equivalent to a body force, and contribution to the boundary conditions (1.8) is similar to a surface force.

In the next sections we focus on the planar elasticity with eigenstrains, assuming isotropy in elastic properties. In addition we relax the boundary condition (1.6) and admit nonzero tractions to make the formulation more general. This will not change our conclusions on the reduced parameter dependence.

Note that a special case of elasticity with eigenstrains is the uncoupled thermoelasticity when the eigenstrain ε^*_{ij} is defined as

$$\varepsilon^*_{ij} = \alpha_{ij}\Delta T \quad \alpha_{ij} = 0 \text{ if } i \neq j \quad i,j = 1,2,3, \tag{1.9}$$

where α_{ij} is thermal expansion coefficient and ΔT is temperature change. We will refer to this special case in examples.

2. Governing equations of plane elasticity with eigenstrains

The governing equations of linear plane elasticity with eigenstrains in the absence of body forces in a domain D are as follows:

i) The equations of equilibrium in terms of stresses

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0 \quad \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} = 0. \tag{2.1}$$

ii) Constitutive equations (1.4), assuming isotropy, accommodating both plane strain and plane stress, and including eigenstrains

$$4\varepsilon_{xx} = 2S\sigma_{xx} + (A - S)(\sigma_{xx} + \sigma_{yy}) + 4\varepsilon^*_{xx} + 4\eta\varepsilon^*_{zz},$$

$$4\varepsilon_{xy} = 2S\sigma_{xy} + 4\varepsilon^*_{xy}, \tag{2.2}$$

$$4\varepsilon_{yy} = 2S\sigma_{yy} + (A - S)(\sigma_{xx} + \sigma_{yy}) + 4\varepsilon^*_{yy} + 4\eta\varepsilon^*_{zz},$$

where A and S denote the bulk and shear compliances respectively, defined by

$$A = \frac{\kappa - 1}{2G}, \qquad S = \frac{1}{G}. \tag{2.3}$$

Here G denotes the shear modulus and κ is the Kolosov constant defined in terms of the Poisson's ratio as

$$\begin{aligned} \kappa &= 3 - 4\nu, \qquad \eta = \nu \text{ (plane strain)}, \\ \kappa &= \frac{3-\nu}{1+\nu}, \qquad \eta = 0 \text{ (plane stress)}. \end{aligned} \tag{2.4}$$

iii) Pointwise (local) compatibility in terms of total strains

$$\frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = 0 \qquad \text{in } D. \tag{2.5}$$

These equations are subject to boundary conditions on S, the boundary of domain D. In this paper we focus on the boundary value problems involving applied tractions

$$t_x = \sigma_{xx} n_x + \sigma_{xy} n_y \qquad t_y = \sigma_{xy} n_x + \sigma_{yy} n_y \qquad \text{on } S. \tag{2.6}$$

3. Inhomogeneous materials

First, we consider an inhomogeneous, isotropic, simply connected body D subjected to spacially varying eigenstrains ε^*_{ij} and traction boundary conditions (2.6). Using Eqs. (2.1) and (2.2), and assuming that both compliances and eigenstrains are smoothly varying functions of position, the compatibility condition (2.5) can be expressed in terms of stresses as

$$\begin{aligned} \nabla^2\left[(A+S)(\sigma_{xx}+\sigma_{yy})\right] - 2\frac{\partial^2 S}{\partial x^2}\sigma_{xx} - 2\frac{\partial^2 S}{\partial y^2}\sigma_{yy} - 4\frac{\partial^2 S}{\partial x \partial y}\sigma_{xy} = -4\frac{\partial^2 \varepsilon^*_{yy}}{\partial x^2} \\ - 4\frac{\partial^2 \varepsilon^*_{xx}}{\partial y^2} + 8\frac{\partial^2 \varepsilon^*_{xy}}{\partial x \partial y} - 4\nabla^2 \eta\varepsilon^*_{zz} - 8\frac{\partial \eta}{\partial x}\frac{\partial \varepsilon^*_{zz}}{\partial x} - 8\frac{\partial \eta}{\partial y}\frac{\partial \varepsilon^*_{zz}}{\partial y} - 4\eta\nabla^2\varepsilon^*_{zz}, \end{aligned} \tag{3.1}$$

where $\eta = \nu = \frac{1}{2}(1 - A/S)$ for plane strain and $\eta = 0$ for plane stress. Thus, the unknown planar stress components σ_{xx}, σ_{xy}, and σ_{yy} can be determined from the equilibrium equations (2.1), the compatibility condition (3.1) and boundary conditions (2.6). From these governing equations only Eq. (3.1) contains elastic compliances. Thus, in our investigation on reduced parameter dependence we focus on Eq. (3.1).

Following CHERKAEV, LURIE and MILTON [6] we seek the conditions for the invariance of the planar stresses with respect to the shift in shear and bulk compliances in this class of boundary value problems. In particular, the general form of the shift in elastic compliances, introduced by DUNDURS and MARKENSCOFF [8] is considered

$$\overline{A} = mA + a + bx + cy, \qquad \overline{S} = mS - a - bx - cy, \tag{3.2}$$

where a, b, c, and m are arbitrary constants provided that the compliances remain non-negative. In this analysis, in addition to the plane stress and plane strain cases, which lead to different results, the distinction is made between the cases when $m = 1$ and $m \neq 1$.

For the plane stress case and $m = 1$, Eq. (3.1) remains unchanged under the linear shift (3.2), i.e. the planar stress components remain unchanged, for any ε^*_{ij}, and thus there is a reduced parameter dependence.

For the plane strain case and $m = 1$, Eq. (3.1) remains invariant under the linear shift (3.2) when

$$\nabla^2\eta\varepsilon^*_{zz} + 2\frac{\partial\eta}{\partial x}\frac{\partial\varepsilon^*_{zz}}{\partial x} + 2\frac{\partial\eta}{\partial y}\frac{\partial\varepsilon^*_{zz}}{\partial y} + \eta\nabla^2\varepsilon^*_{zz} = 0. \tag{3.3}$$

For the special case of uniform eigenstrains, the condition (3.3) is satisfied provided that

$$\varepsilon^*_{zz} = 0 \qquad \text{or} \qquad \nabla^2\eta = 0, \tag{3.4}$$

while for the case of a homogeneous material the condition (3.3) is satisfied if

$$\nabla^2\varepsilon^*_{zz} = 0 \qquad \text{or} \qquad \eta = 0. \tag{3.5}$$

Recall that $\eta = \nu$ for plane strain case.

For the case of plane stress and $m \neq 1$, Eq. (3.1) is invariant under the shift (3.2) if

$$\frac{\partial^2\varepsilon^*_{yy}}{\partial x^2} - 2\frac{\partial^2\varepsilon^*_{xy}}{\partial x\partial y} + \frac{\partial^2\varepsilon^*_{xx}}{\partial y^2} = 0, \tag{3.6}$$

while for the plane strain case and $m \neq 1$, the linear shift is only possible when

$$\frac{\partial^2 \varepsilon^*_{yy}}{\partial x^2} - 2\frac{\partial^2 \varepsilon^*_{xy}}{\partial x \partial y} + \frac{\partial^2 \varepsilon^*_{xx}}{\partial y^2} + \nabla^2 \eta \varepsilon^*_{zz} + 2\frac{\partial \eta}{\partial x}\frac{\partial \varepsilon^*_{zz}}{\partial x} + 2\frac{\partial \eta}{\partial y}\frac{\partial \varepsilon^*_{zz}}{\partial y} + \eta \nabla^2 \varepsilon^*_{zz} = 0. \tag{3.7}$$

For multiply connected materials the compatibility condition (3.1) is a necessary but not a sufficient condition for the existence of continuous displacements. Thus, for such materials in addition to (3.1), the global compatibility conditions in the form of line integrals (called Cesaro integrals) need to be included. More details on this procedure are given in [8, 15]; the analysis presented there can be extended to multiply connected materials with eigenstrains. This class of problems is not addressed in this paper.

Next, two special cases are considered.

CASE 1. Assume a homogeneous material with uniform eigenstrains subject to zero traction boundary conditions. In this case the eigenstrains may represent thermal strains defined by Eq. (1.9) and the problem reduces to the case of uncoupled linear thermoelasticity for homogeneous materials. The governing equation (3.1) becomes

$$\nabla^2 (\sigma_{xx} + \sigma_{yy}) = 0$$

subject to zero tractions in the plane. This boundary value problem is satisfied identically by zero stresses, as expected. Thus, the concept of the reduced parameter dependence has no relevance for this boundary value problem.

CASE 2. Assume a homogeneous material with non-uniform eigenstrains subject to traction boundary conditions. Eq. (3.1) takes the form

$$(A + S)\nabla^2 (\sigma_{xx} + \sigma_{yy}) = -4\frac{\partial^2 \varepsilon^*_{yy}}{\partial x^2} + 8\frac{\partial^2 \varepsilon^*_{xy}}{\partial x \partial y} - 4\frac{\partial^2 \varepsilon^*_{xx}}{\partial y^2} - 4\eta \nabla^2 \varepsilon^*_{zz}. \tag{3.8}$$

In general, for this case, the stresses are non-zero, and the concept of reduced parameter dependence applies, subject to conditions on eigenstrains given in this section.

A related work addressing residual stresses in isotropic materials is due to HOGER [22]. It that reference, residual stress is defined as the stress present in a body in an unloaded reference configuration. In this definition there are no body forces and no surface tractions acting on material. For such class of problems in an isotropic material "the residual stresses must commute with all proper orthogonal tensors", and therefore have a restricted form of a hydrostatic pressure. For zero tractions on the boundary this stress must be identically zero. Thus, it is concluded in [22] that an isotropic body can support no residual stress.

In our case the stress is caused by external eigenstrains which, as mentioned in Sec. 1, are mathematically equivalent to body forces and surface tractions. Thus, the stress fields in our formulation differ from those in [22]. More specifically, the stresses have the unrestricted form, which is related to the unrestricted form of applied eigenstrains. The form of eigenstrains is determined solely by the physical problem considered. Thus, for nonuniform applied eigenstrains, the stresses are in general nonuniform and nonzero. This agrees with the statement in [22] that nonzero residual stress must be nonuniform.

4. Two-phase materials with perfectly bonded interfaces

Next, we consider the case of a domain consisting of two discontinuous phases 1 and 2, subjected to eigenstrains. We assume perfect bonding boundary conditions on the boundary between the two phases, and we assume traction boundary conditions applied on the outer boundary. Using the stress formulation, the governing equations are Eq. (3.1) for phase 1

$$\begin{aligned}(4.1)\quad \nabla^2\left[(A_1+S_1)\left(\sigma_{xx}^{(1)}+\sigma_{yy}^{(1)}\right)\right]-2\frac{\partial^2 S_1}{\partial x^2}\sigma_{xx}^{(1)}-2\frac{\partial^2 S_1}{\partial y^2}\sigma_{yy}^{(1)}-4\frac{\partial^2 S_1}{\partial x\partial y}\sigma_{xy}^{(1)}\\ =-4\frac{\partial^2\left(\varepsilon_{yy}^*\right)_1}{\partial x^2}-4\frac{\partial^2\left(\varepsilon_{xx}^*\right)_1}{\partial y^2}+8\frac{\partial^2\left(\varepsilon_{xy}^*\right)_1}{\partial x\partial y}-4\nabla^2\eta_1\left(\varepsilon_{zz}^*\right)_1\\ -8\frac{\partial\eta_1}{\partial x}\frac{\partial\left(\varepsilon_{zz}^*\right)_1}{\partial x}-8\frac{\partial\eta_1}{\partial y}\frac{\partial\left(\varepsilon_{zz}^*\right)_1}{\partial y}-4\eta_1\nabla^2\left(\varepsilon_{zz}^*\right)_1,\end{aligned}$$

and the analogous equation for phase 2, the equilibrium equations (2.1) applied for each phase, and boundary conditions (2.6).

Following [8, 23], for a domain consisting of two discontinuous phases (denoted by 1 and 2), the perfect bonding boundary conditions on the boundary S_{12} between these phases involve

a) the continuity of normal tractions

$$(4.2)\qquad \sigma_{nn}^{(1)}=\sigma_{nn}^{(2)}\qquad \text{on } S_{12},$$

b) the continuity of tangential tractions

$$(4.3)\qquad \sigma_{ns}^{(1)}=\sigma_{ns}^{(2)}\qquad \text{on } S_{12},$$

c) the continuity of change in curvature $\Delta K_1=\Delta K_2$, which in terms of stresses has the form

$$\begin{aligned}(4.4)\quad & \frac{\partial}{\partial n}\left[(A_2+S_2)\,\sigma_{ss}^{(2)}\right]-\frac{\partial}{\partial n}\left[(A_1+S_1)\,\sigma_{ss}^{(1)}\right] \\ & \qquad -\left[(A_2-A_1)+3\,(S_2-S_1)\right]\frac{\partial\sigma_{ns}}{\partial s} \\ & -4\frac{\partial}{\partial s}(S_2-S_1)\,\sigma_{ns}+\left\{\frac{\partial}{\partial n}\left[(A_2-A_1)-(S_2-S_1)\right]+2K\,(A_2-A_1)\right\}\sigma_{nn} \\ & -8\frac{\partial}{\partial s}\left[(\varepsilon^*_{sn})_2-(\varepsilon^*_{sn})_1\right]+\frac{\partial}{\partial n}\left\{4\left[(\varepsilon^*_{ss})_2-(\varepsilon^*_{ss})_1\right]+4\left[\eta_2\,(\varepsilon^*_{zz})_2-\eta_1\,(\varepsilon^*_{zz})_1\right]\right\} \\ & \qquad +K\left\{4\left[(\varepsilon^*_{nn})_2-(\varepsilon^*_{nn})_1\right]+4\left[\eta_2\,(\varepsilon^*_{zz})_2-\eta_1\,(\varepsilon^*_{zz})_1\right]\right\}=0 \quad \text{on} \quad S_{12}\end{aligned}$$

where K is curvature,

d) the continuity of stretch strains $\varepsilon_{ss}^{(1)}=\varepsilon_{ss}^{(2)}$, which expressed in terms of stresses gives

$$\begin{aligned}(4.5)\quad & (A_2+S_2)\,\sigma_{ss}^{(2)}-(A_1+S_1)\,\sigma_{ss}^{(1)}+\left[(A_2-A_1)-(S_2-S_1)\right]\sigma_{nn} \\ & \qquad +4\left[(\varepsilon^*_{ss})_2-(\varepsilon^*_{ss})_1\right]+4\left[\eta_2\,(\varepsilon^*_{zz})_2-\eta_1\,(\varepsilon^*_{zz})_1\right]=0 \quad \text{on} \quad S_{12},\end{aligned}$$

where subscripts and superscripts 1 and 2 denote quantities in phases 1 and 2, respectively, and n and s are normal and tangential directions in plane as defined by DUNDURS [23]. Note that the last two boundary conditions (4.4) and (4.5) replace the conventional conditions involving the continuity of normal and tangential displacements

$$(4.6)\qquad u_n^{(1)}=u_n^{(2)} \qquad u_s^{(1)}=u_s^{(2)}.$$

Next we explore the conditions for stress invariance. For two phase materials the linear shift (3.2) takes on the form

$$(4.7)\qquad \begin{aligned}\overline{A}_1&=mA_1+a+bx+cy \qquad & \overline{S}_1&=mS_1-a-bx-cy,\\ \overline{A}_2&=mA_2+a+bx+cy \qquad & \overline{S}_2&=mS_2-a-bx-cy.\end{aligned}$$

Note that only the boundary conditions (4.4) – (4.5) depend on compliances. Thus, we explore the invariance under a shift by focusing on these two conditions.

For the case of plane stress and $m=1$, Eqs. (4.2) – (4.5) are invariant under the linear shift (4.7) for any eigenstrain ε^*_{ij} in phase 1 and 2.

For the case of plane strain and $m=1$ the linear shift is possible if

$$(4.8)\qquad \eta_1\,(\varepsilon^*_{zz})_1=\eta_2\,(\varepsilon^*_{zz})_2\,.$$

For the plane stress case and $m \neq 1$ the boundary condition (4.4) does not change under a linear shift if

$$(4.9)\quad -2\frac{\partial}{\partial s}\left[(\varepsilon^*_{sn})_2 - (\varepsilon^*_{sn})_1\right] + \frac{\partial}{\partial n}\left[(\varepsilon^*_{ss})_2 - (\varepsilon^*_{ss})_1\right] + K\left[(\varepsilon^*_{nn})_2 - (\varepsilon^*_{nn})_1\right] = 0$$

and the boundary condition (4.5) remains unchanged under a shift if

$$(4.10)\qquad (\varepsilon^*_{ss})_1 = (\varepsilon^*_{ss})_2\,.$$

For plane strain case and $m \neq 1$ Eq. (4.4) is invariant if

$$(4.11)\quad -2\frac{\partial}{\partial s}\left[(\varepsilon^*_{sn})_2 - (\varepsilon^*_{sn})_1\right] + \frac{\partial}{\partial n}\left\{\left[(\varepsilon^*_{ss})_2 - (\varepsilon^*_{ss})_1\right] + \left[\eta_2\,(\varepsilon^*_{zz})_2 - \eta_1\,(\varepsilon^*_{zz})_1\right]\right\}$$
$$+ K\left\{\left[(\varepsilon^*_{nn})_2 - (\varepsilon^*_{nn})_1\right] + \left[\eta_2\,(\varepsilon^*_{zz})_2 - \eta_1\,(\varepsilon^*_{zz})_1\right]\right\} = 0 \quad \text{on} \quad S_{12}$$

and Eq. (4.5) is invariant if

$$(4.12)\qquad \left[(\varepsilon^*_{ss})_2 - (\varepsilon^*_{ss})_1\right] + \left[\eta_2\,(\varepsilon^*_{zz})_2 - \eta_1\,(\varepsilon^*_{zz})_1\right] = 0 \quad \text{on} \quad S_{12},$$

Thus, if (4.12) is used, the condition (4.11) reduces to

$$(4.13)\quad -2\frac{\partial}{\partial s}\left[(\varepsilon^*_{sn})_2 - (\varepsilon^*_{sn})_1\right] + K\left\{\left[(\varepsilon^*_{nn})_2 - (\varepsilon^*_{nn})_1\right]\right.$$
$$\left. + \left[\eta_2\,(\varepsilon^*_{zz})_2 - \eta_1\,(\varepsilon^*_{zz})_1\right]\right\} = 0.$$

In addition, the conditions on the invariance of Eq. (4.1) and on its counterpart for phase 2 must be satisfied for all the four cases discussed. These conditions are analogous to those in Sec. 3.

Note that the results presented in this section are applicable for both simply and multiply connected two phase materials. It has been shown by MARKENSCOFF [24] that for multiply connected materials with perfectly bonded interfaces, the Cesaro integrals do not need to be considered. Also, the results for two phase materials, discussed in this section, can be extended to multiphase materials in a straightforward way.

To illustrate the concepts presented in this section we include in the Appendix an example involving a single elastic circular inclusion embedded in the elastic matrix subjected to uniform eigenstrains, and more specifically – to a uniform temperature change.

5. Conclusions

We showed the reduced parameter dependence in the in-plane stress fields in the problems governed by plane elasticity with eigenstrains, if the eigenstrains satisfy the given conditions. Note that there are no conditions needed for the plane stress case for the form of shift with $m = 1$. These results can be applied for two-phase materials to linear plane uncoupled thermoelasticity, where eigenstrains are uniform and represent the product of the thermal expansion coefficient and temperature change. The analysis can also be extended to multi-phase materials with perfectly bonded interfaces, two-phase (or multi-phase) materials with slipping interfaces [25], and inhomogeneous multiply connected materials.

The reduced parameter dependence is of importance in parametric studies, both experimental and theoretical. It can be used as a check for numerical and analytical calculations, it reduces the number of output parameters, and facilitates the presentation of results. It results in savings in time, space and resources.

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Appendix

The analytical linear elastic solution for radial stresses due to a small circular homogeneous inclusion made of phase 2 of radius a embedded in a homogeneous matrix of phase 1 and subjected to a uniform temperature change ΔT is given by

$$\sigma_{rr}^{(1)} = \frac{2\left[\alpha_1 (1+\eta_1) - \alpha_2 (1+\eta_2)\right] \Delta T a^2}{\dfrac{\kappa_2 - 1}{2G_2} + \dfrac{1}{G_1}} \frac{1}{r^2} \tag{A.1}$$

and

$$\sigma_{rr}^{(2)} = \frac{\left[\alpha_1 (1+\eta_1) - \alpha_2 (1+\eta_2)\right] \Delta T}{\dfrac{\kappa_2 - 1}{2G_2} + \dfrac{1}{G_1}}, \tag{A.2}$$

where subscripts and superscripts 1 and 2 denote the matrix and inclusion respectively, and α is the coefficient of thermal expansion. κ is the Kolosov constant defined in Eq. (2.4) and G is a shear modulus. If we express this solution in a contracted form using the definitions (2.3), we have

$$\sigma_{rr}^{(1)} = \frac{2\left[\alpha_1 (1+\eta_1) - \alpha_2 (1+\eta_2)\right] \Delta T a^2}{A_2 + S_1} \frac{1}{r^2} \tag{A.3}$$

and

$$\sigma_{rr}^{(2)} = \frac{\left[\alpha_1 (1+\eta_1) - \alpha_2 (1+\eta_2)\right] \Delta T}{A_2 + S_1} \tag{A.4}$$

Note that the eigenstrains, defined in Eq. (1.9), are

$$(\varepsilon^*_{rr})_1 = (\varepsilon^*_{\theta\theta})_1 = (\varepsilon^*_{zz})_1 = \alpha_1 \Delta T \tag{A.5}$$

$$(\varepsilon^*_{rr})_2 = (\varepsilon^*_{\theta\theta})_2 = (\varepsilon^*_{zz})_2 = \alpha_2 \Delta T \tag{A.6}$$

and

$$\varepsilon^*_{rr} = \varepsilon^*_{nn} \qquad \varepsilon^*_{\theta\theta} = \varepsilon^*_{ss}. \tag{A.7}$$

Note that the stresses are invariant under the transformation (4.7) subject to conditions discussed in Sec. 4.

Thus, for plane stress case and $m = 1$, the in-plane stresses are invariant under the shift (4.7) and there is a reduced parameter dependence.

For plane stress case and $m \neq 1$, the condition (3.6) applied to $\left(\varepsilon^*_{ij}\right)_1$ and $\left(\varepsilon^*_{ij}\right)_2$ is satisfied automatically, but the conditions (4.9) and (4.10) impose

$$(\varepsilon^*_{rr})_1 = (\varepsilon^*_{rr})_2 \qquad (\varepsilon^*_{\theta\theta})_1 = (\varepsilon^*_{\theta\theta})_2. \tag{A.8}$$

The condition (5.8) implies $\alpha_1 = \alpha_2$, which gives a zero stress field. Thus, the reduced parameter dependence does not hold for this case. This conclusion can easily be verified by analyzing Eqs. (5.3) – (5.4).

For plane strain case and $m = 1$, the conditions (3.4) and (4.8) are satisfied if

$$(\varepsilon^*_{zz})_1 = (\varepsilon^*_{zz})_2 = 0 \tag{A.9}$$

Thus, there is parameter dependence subject to the condition (5.9).

For the plane strain case and $m \neq 1$, the condition (3.7) is satisfied for any eigenstrain value, while the conditions (4.12) and (4.13) are satisfied only if

$$(\varepsilon^*_{rr})_1 = (\varepsilon^*_{rr})_2 \qquad (\varepsilon^*_{\theta\theta})_1 = (\varepsilon^*_{\theta\theta})_2 \qquad (\varepsilon^*_{zz})_1 = (\varepsilon^*_{zz})_2 \tag{A.10}$$

which implies $\alpha_1 = \alpha_2$, and thus the zero stress field. Therefore, the reduced parameter dependence does not hold for this case.

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