

Linear stability of a 1D flow in porous media under transversal disturbance with adsorption

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*Dedicated to Professor Piotr Perzyna
on the occasion of his 70th birthday*

IN THIS PAPER we investigate the linear stability behavior of a flow within an adsorption/diffusion model for porous materials summarized in Section 2. We consider a 1D stationary flow through a poroelastic medium which is perturbed by transversal (2D) disturbances with mass exchange. The eigenvalue problem for the first step field equations is solved using a finite-difference scheme. We present the instability regions in dependence on the three most important model parameters, namely the bulk and surface permeability coefficients, and the mass density of the adsorbate on the inner surface.

1. Introduction

AT LEAST TWO kinds of instabilities may appear in porous media: structural instabilities, and flow instabilities. Examples of structural instabilities are piping (the breakdown of a granular medium) or shear band formation. The appearance of Bénard cells or the change from laminar to turbulent flows belong to the class of flow instabilities. All kinds of instability have in common that they appear due to competition of at least two mechanisms. In the case of structural instabilities, e.g. a threshold nonlinearity competes with the permeability of the material. A flow instability arises if a kinematic nonlinearity acts against, for example, viscosity.

This paper is devoted to the investigation of one special type of flow instability. We consider a steady state 1D flow in a porous medium. This defines the base flow on which we superpose a small disturbance with adsorption. The disturbances satisfy equations of the model for multicomponent systems with adsorption (summarized in Section 2 and introduced in [2]). It is considered that a fluid/adsorbate mixture flows through channels of a skeleton. In this case a kinematic nonlinearity acts against the permeability (diffusion) of the medium.

Adsorption processes contribute in a nonlinear way to the field equations, and essentially influence the stability properties.

Small disturbances are inevitably present in any real system but their effect on stable systems is mostly ephemeral. Consider, for example, the influence of wavelike disturbances on the circulation of the atmosphere. As we have shown in earlier works also the superposition of the same flow as this considered in this paper with disturbances without mass exchange led to completely stable situations. But the disturbance with adsorption yields the existence of an unstable region which means that adsorption slows down the relaxation process tremendously or even controls the loss of stability of the base flow (see: [3]) for a certain region of permeability coefficients. If the base flow is unstable, the disturbances will grow in amplitude with time and space. However, the values of the permeability coefficient for which the base flow is unstable are much smaller than those used in earlier works.

The region of instability for a transversal disturbance of the base flow with adsorption will be shown in dependence on three important model parameters.

First there appear two different permeability coefficients, namely the bulk permeability coefficient π , and the surface permeability α . While the first one enters the field equations and describes the effective resistance of the skeleton to the flow of the fluid, the latter enters the model through the boundary conditions of the third type, and it accounts for properties of the surface. It is one of the material parameters which determine the fluid velocity. The third parameter, the mass density of the adsorbate on the internal surface ρ_{ad}^A , enters the model through the mass source. It is proportional to the size of the internal surface which plays an enormous role for the global rate and amount of adsorption. For different soils it may vary by some orders of magnitude.

2. Adsorption/diffusion model

We investigate the stability behavior of a 1D base flow under perturbations with mass exchange. The disturbance satisfies the following equations of the adsorption/diffusion model introduced e.g. in B. ALBERS [2] Coupling. This model is based on the model with a balance equation for porosity by K. WILMAŃSKI (see e.g. [7, 8]). However, we assume the porosity to be constant so that the model in this work is simpler than the version introduced in the article.

We consider a process of physical adsorption in a three-component porous medium. A fluid-adsorbate mixture flows through the channels of the skeleton. The model takes into account three components: the skeleton, the fluid and an adsorbate which either flows with the same velocity as the fluid through the channels of the skeleton or it settles down on the inner surface of the porous body. Under the assumption of a small concentration of the adsorbate we can

neglect the influence of mass changes of the skeleton. Furthermore we neglect dynamical disturbances of the skeleton.

Mass balances

The mass balance equation for the liquid (fluid and adsorbate phases together) and the concentration balance have the following form:

$$(2.1) \quad \begin{aligned} \frac{\partial \rho^L}{\partial t} + \operatorname{div}(\rho^L \mathbf{v}^F) &= \hat{\rho}^A, \\ \rho^L \left(\frac{\partial c}{\partial t} + \mathbf{v}^F \cdot \operatorname{grad} c \right) &= (1 - c) \hat{\rho}^A, \end{aligned}$$

where ρ^L is the mass density of the liquid phases, i.e. the sum of the mass densities of the fluid and the adsorbate, and c denotes the concentration of the adsorbate in the fluid component. The common velocity of fluid and adsorbate is \mathbf{v}^F . The mass balance for the adsorbate is replaced by the concentration balance¹⁾.

Mass source

According to the model (see [2]) the mass source is given by the relation

$$(2.2) \quad \hat{\rho}^A = -\frac{m^A}{V} \frac{d(\xi f_{int})}{dt} = -\frac{m^A}{V} \left(f_{int} \frac{d\xi}{dt} + \xi \frac{df_{int}}{dt} \right),$$

whose derivation is based on the classical LANGMUIR adsorption theory about *occupied* (ξ) and *bare* ($1 - \xi$) sites (see [6]) on the internal surface f_{int} of the solid. V is the representative elementary volume *REV* and m_A denotes the reference mass of adsorbate per unit of the internal surface area.

¹⁾With

$$\frac{\partial}{\partial t} \left[\rho^L \frac{\rho^A}{\rho^L} \right] = \frac{\partial \rho^L}{\partial t} \frac{\rho^A}{\rho^L} + \rho^L \frac{\partial \rho^A}{\partial t}, \quad \text{and} \quad \operatorname{div} \left[\rho^L \frac{\rho^A}{\rho^L} \right] \mathbf{v}^F = \frac{\rho^A}{\rho^L} \operatorname{div} \rho^L \mathbf{v}^F + \rho^L \mathbf{v}^F \cdot \operatorname{grad} \frac{\rho^A}{\rho^L},$$

we get from the mass balance for the adsorbate, i.e. from

$$\frac{\partial \rho^A}{\partial t} + \operatorname{div}(\rho^A \mathbf{v}^F) = \hat{\rho}^A,$$

the relation:

$$c \underbrace{\left[\frac{\partial \rho^L}{\partial t} + \operatorname{div} \rho^L \mathbf{v}^F \right]}_{\hat{\rho}^A} + \rho^L \left[\frac{\partial c}{\partial t} + \mathbf{v}^F \cdot \operatorname{grad} c \right] = \hat{\rho}^A,$$

From this follows

$$\rho^L \left[\frac{\partial c}{\partial t} + \mathbf{v}^F \cdot \operatorname{grad} c \right] = (1 - c) \hat{\rho}^A,$$

which is the concentration balance.

The first contribution on the right-hand side of (2.2) describes changes in time of the fraction of occupied sites. It is specified by the Langmuir evolution equation which can be written in the form

$$(2.3) \quad \frac{\partial \xi}{\partial t} = \left[\frac{cp^L}{p_0} (1 - \xi) - \xi \right] \frac{1}{\tau_{ad}},$$

where p^L denotes the partial pressure in the liquid (fluid and adsorbate together), p_0 is a Langmuir reference pressure and τ_{ad} is the characteristic time of adsorption. In the case of time changes of ξ equal to zero the well-known *Langmuir isotherm of occupied sites* follows

$$(2.4) \quad \xi_L = \frac{\frac{cp^L}{p_0}}{1 + \frac{cp^L}{p_0}},$$

where according to Dalton's law for small concentrations of the adsorbate it is assumed that the partial pressure of the adsorbate $p^A \cong cp^L$.

The other part of (2.2) describes the change of the internal surface. We assumed this change to be coupled with the relaxation of porosity. However, under the assumption of constant porosity it drops out in calculations of this work. This does not influence essentially the results because, as we have shown in earlier works, the Langmuir part of the mass source dominates the part connected with the porosity. Consequently, we obtain the following form of the mass source

$$(2.5) \quad \hat{\rho}^A = -\rho_{ad}^A \left\{ \left[\frac{cp^L}{p_0} (1 - \xi) - \xi \right] \frac{1}{\tau_{ad}} \right\},$$

where $\rho_{ad}^A := \frac{m^A f_{int}}{V}$ is the mass density of the adsorbate on the internal surface. This is a very important parameter as we know, for example, from BEAR [4] that the internal surface may change by several orders of magnitude. Therefore ρ_{ad}^A is one of those parameters which we alter to see changes in the stability behavior for various porous materials.

Momentum balance

Due to the same velocity of fluid and adsorbate we need only one momentum balance jointly for these both components. We have

$$(2.6) \quad \frac{\partial \rho^L \mathbf{v}^F}{\partial t} + \text{div} (\rho^L \mathbf{v}^F \otimes \mathbf{v}^F + p^L \mathbf{1}) = (-\pi + \hat{\rho}^A) \mathbf{v}^F \approx -\pi \mathbf{v}^F.$$

The order of magnitude of π is much higher than that of $\hat{\rho}^A$ so that on the right-hand side of (3.5) $\hat{\rho}^A$ is negligible. Furthermore the permeability coefficient π

and the compressibility parameter κ are assumed to be constant. We assume the flow to be isochoric which leads to the following linear constitutive relation for the pressure in the liquid phase p^L

$$(2.7) \quad p^L = p^0 + \kappa \rho^1.$$

Governing equations in 2D

Summing up, the 2D model is governed by the following set of equations

$$(2.8) \quad \begin{aligned} \frac{\partial \rho^L}{\partial t} + \rho^L \left(\frac{\partial v_x^F}{\partial x} + \frac{\partial v_z^F}{\partial z} \right) + v_x^F \frac{\partial \rho^L}{\partial x} + v_z^F \frac{\partial \rho^L}{\partial z} &= -\frac{\rho_{ad}^A}{\tau_{ad}} \left[\frac{cp^L}{p_0} (1 - \xi) - \xi \right], \\ \rho^L \left[\frac{\partial c}{\partial t} + v_x^F \frac{\partial c}{\partial x} + v_z^F \frac{\partial c}{\partial z} \right] &= -(1 - c) \frac{\rho_{ad}^A}{\tau_{ad}} \left[\frac{cp^L}{p_0} (1 - \xi) - \xi \right], \\ \frac{\partial \xi}{\partial t} &= \left[\frac{cp^L}{p_0} (1 - \xi) - \xi \right] \frac{1}{\tau_{ad}}, \\ \rho^L \left[\frac{\partial v_x^F}{\partial t} + v_x^F \frac{\partial v_x^F}{\partial x} + v_z^F \frac{\partial v_x^F}{\partial z} \right] &= -\frac{\partial p^L}{\partial x} - \pi v_x^F, \\ \rho^L \left[\frac{\partial v_z^F}{\partial t} + v_x^F \frac{\partial v_z^F}{\partial x} + v_z^F \frac{\partial v_z^F}{\partial z} \right] &= -\frac{\partial p^L}{\partial z} - \pi v_z^F. \end{aligned}$$

3. Regular perturbation

The stability behavior of the 1D base flow is investigated with respect to transversal (2D) perturbations with mass exchange. For the analysis we use a regular perturbation method restricted to the zeroth and first order contributions. This means that we expect the fields to be a superposition of the base solution (indicated by 0) and a small perturbation (indicated by 1). The solutions have the following structure:

$$(3.1) \quad \begin{aligned} p^F &= p^0(x) + \kappa \rho^1(x, z, t), \quad c = c^0 + c^1(x, z, t), \quad \xi = \xi^0(x) + \xi^1(x, z, t), \\ v_x^F &= v^0 + v_x^1(x, z, t), \quad v_z^F = v_z^1(x, z, t). \end{aligned}$$

3.1. Base solution/zeroth step of perturbation

The base flow satisfies the following set of field equations. It is a steady state one-dimensional flow process in a porous medium,

$$(3.2) \quad \frac{\partial \rho^F}{\partial t} + \frac{\partial \rho^F v_x^F}{\partial x} = 0, \quad 0 < x < l$$

$$\rho^F \left(\frac{\partial v_x^F}{\partial t} + v_x^F \frac{\partial v_x^F}{\partial x} \right) = - \frac{\partial p^F}{\partial x} - \pi v_x^F.$$

Equations (3.2)_{1/2} are the mass and momentum balances of the fluid. Here, ρ^F is the mass density of the fluid component, v_x^F is the fluid velocity in x -direction and π is the bulk permeability coefficient. The partial pressure in the fluid is denoted by p^F . We investigate solely the steady state processes. Consequently, the base flow does not contain an influence of mass exchange.

It is assumed that a deformed skeleton does not contribute to a dynamical disturbance, and therefore relations for the skeleton are not quoted.

Boundary conditions of this problem have the following form:

$$(3.3) \quad - \rho^F v_x^F \Big|_{x=0} = \alpha [p^F \Big|_{x=0} - n_{EP} p_l],$$

$$\rho^F v_x^F \Big|_{x=l} = \alpha [p^F \Big|_{x=l} - n_{EP} p_r].$$

They are of the third type and express the fact that the flow through the boundary of the body depends on the pressure difference of the partial pressure in the fluid (p^F) and the external pressure which works on the fluid (on the left-hand side p_l and on the right-hand side p_r). The permeability of the surface denoted by α is the material property of the system.

For simplicity we assume the base flow to be isochoric. This means that in the zeroth step of perturbation the mass density is constant

$$(3.4) \quad \overset{0}{\rho} = \text{const} = \rho_0^F.$$

This simplification is supported by calculations which we performed in earlier papers on porous media. Bearing (3.2)₁ for the zeroth step in mind we obtain that also the fluid velocity in x -direction is constant (denoted by $\overset{0}{v}$ without subscribed x)

$$(3.5) \quad \overset{0}{v} = \text{const}.$$

The stationary form of (3.2)₂ allows us to calculate the solution for the partial pressure in the fluid in the zeroth step, $\overset{0}{p}$,

$$(3.6) \quad \frac{\partial \overset{0}{p}}{\partial x} = -\pi \overset{0}{v} \implies \overset{0}{p} = -\pi \overset{0}{v} x + C,$$

where C is an integration constant which can be determined by use of the boundary condition on the left-hand side of the system ($x = 0$)

$$(3.7) \quad -\rho_0^{F0} v = \alpha [p^F]_{x=0} - n_E p_l,$$

so that we obtain

$$(3.8) \quad \overset{0}{p} = -\pi \overset{0}{v} x + n_E p_l - \frac{\rho_0^{F0} v}{\alpha}.$$

As the second boundary condition for $x = l$ we have

$$(3.9) \quad \rho_0^{F0} v = \alpha [p^F]_{x=l} - n_E p_r,$$

from which the constant velocity of the zeroth step follows in the form

$$(3.10) \quad \overset{0}{v} = \frac{\alpha p_d}{2\rho_0^{F0} + \alpha \pi l}, \quad p_d := n_E (p_l - p_r).$$

The base solution forms the zeroth approximation of the above described perturbation. Obviously it depends solely on the x -variable. In addition to the above fields we need in the zeroth step with mass exchange the fields of concentration $\overset{0}{c}$ and of the number of occupied sites $\overset{0}{\xi}$ given by Eq. (2.8)_{2/3}.

In the stationary case they reduce to

- concentration balance

$$(3.11) \quad \rho_0^{F0} v \frac{\partial \overset{0}{c}}{\partial x} = - \left(1 - \overset{0}{c}\right) \frac{\rho_{ad}^A}{\tau_{ad}} \left[\overset{0}{c} \frac{\overset{0}{p}}{p_0} \left(1 - \overset{0}{\xi}\right) - \overset{0}{\xi} \right],$$

- evolution equation for fraction of occupied sites

$$(3.12) \quad 0 = \overset{0}{c} \frac{\overset{0}{p}}{p_0} \left(1 - \overset{0}{\xi}\right) - \overset{0}{\xi}.$$

From (3.12) follows the Langmuir adsorption isotherm which depends, of course, on x

$$(3.13) \quad \overset{0}{\xi} = \frac{\overset{0}{c} \frac{\overset{0}{p}}{p_0}}{1 + \overset{0}{c} \frac{\overset{0}{p}}{p_0}}.$$

On the other hand (3.11) yields $\overset{0}{c} = \text{const.} = c_0$, where c_0 denotes the initial concentration of the adsorbate in the fluid.

3.2. First step of perturbation

For the first step of perturbation (2D) we obtain the following set of equations:

$$(x, z) \in \mathcal{B} := (0, l) \times (-b, b)$$

$$(3.14) \quad \begin{aligned} \frac{\partial \rho^1}{\partial t} + \rho_0^F \left(\frac{\partial v_x^1}{\partial x} + \frac{\partial v_z^1}{\partial z} \right) + v^0 \frac{\partial \rho^1}{\partial x} &= \hat{\rho}^A, \\ \rho_0^F \left(\frac{\partial c^1}{\partial t} + v^0 \frac{\partial c^1}{\partial x} \right) &= (1 - c_0) \hat{\rho}^A - \frac{c^0 \rho_{ad}^A}{\tau_{ad}} \left[c_0 \frac{p^0}{p_0} \left(1 - \xi^0 \right) - \xi^0 \right], \\ \frac{\partial \xi^1}{\partial t} &= \left[\left(1 - \xi^0 \right) \left(\frac{c_0 \kappa \rho^1}{p_0} + \frac{c^0 p^0}{p_0} \right) - \left(1 + c_0 \frac{p^0}{p_0} \right) \xi^1 \right] \frac{1}{\tau_{ad}}, \\ \rho_0^F \left[\frac{\partial v_x^1}{\partial t} + v^0 \frac{\partial v_x^1}{\partial x} \right] &= -\kappa \frac{\partial \rho^1}{\partial x} - \pi v_x^1, \\ \rho_0^F \left[\frac{\partial v_z^1}{\partial t} + v^0 \frac{\partial v_z^1}{\partial z} \right] &= -\kappa \frac{\partial \rho^1}{\partial z} - \pi v_z^1, \end{aligned}$$

with

$$(3.15) \quad \hat{\rho}^A = -\frac{\rho_{ad}^A}{\tau_{ad}} \left[\left(1 - \xi^0 \right) \left(\frac{c_0 \kappa \rho^1}{p_0} + \frac{c^0 p^0}{p_0} \right) - \xi^1 \left(1 + c_0 \frac{p^0}{p_0} \right) \right].$$

The following boundary conditions are considered:

$$(3.16) \quad \begin{aligned} -\rho^F v_x^F \Big|_{x=0} &= \alpha [p^F \Big|_{x=0} - n_{EP} l], & v_z^F \Big|_{z=\pm b} &= 0, \\ \rho^F v_x^F \Big|_{x=l} &= \alpha [p^F \Big|_{x=l} - n_{EP} r], \end{aligned}$$

The two conditions on the left-hand side corresponding to the momentum balance in x -direction are already introduced in (3.3). They describe in the above explained way the in- and outflow in x -direction of the liquid to the porous body of length l . But we consider also the flow in z -direction. We assume the fields to be periodic in this direction with period $2b$. Application of a Fourier ansatz yields that the two boundary conditions on the right-hand side of (3.16) corresponding to the momentum balance in z -direction are automatically fulfilled.

In the first step of perturbation boundary conditions (3.16) have the form

$$(3.17) \quad \begin{aligned} -\rho_0^F v_x^1 \Big|_{x=0} - v^0 \rho^1 \Big|_{x=0} &= \alpha \kappa \rho^1 \Big|_{x=0}, & v_z^1 \Big|_{z=\pm b} &= 0, \\ \rho_0^F v_x^1 \Big|_{x=l} + v^0 \rho^1 \Big|_{x=l} &= \alpha \kappa \rho^1 \Big|_{x=l}, \end{aligned}$$

4. Wave ansatz for the disturbance

The perturbations in the first step are expressed in terms of the following simple wave ansatz:

$$(4.1) \quad \begin{aligned} \overset{1}{\rho} &= e^{\omega t} \bar{\rho}(x) \cos kz, & \overset{1}{v}_x &= e^{\omega t} \bar{v}_x(x) \cos kz, & \overset{1}{v}_z &= e^{\omega t} \bar{v}_z(x) \sin kz, \\ \overset{1}{c} &= e^{\omega t} \bar{c}(x) \cos kz, & \overset{1}{\xi} &= e^{\omega t} \bar{\xi}(x) \cos kz, \end{aligned}$$

where $\bar{\rho}(x), \bar{v}_x(x), \bar{v}_z(x), \bar{c}(x)$ and $\bar{\xi}(x)$ are the amplitudes of the disturbances, ω is the frequency, possibly complex, and k the real wave number defined as

$$(4.2) \quad k := \frac{\Pi m}{b}, \quad \Pi = 3.1415\dots$$

The Fourier ansatz in z -direction follows from the structure of the boundary conditions.

After insertion of this ansatz into the field equations (2.8) we have

$$(4.3) \quad \begin{aligned} \omega \bar{\rho} + \rho_0^F \left(\frac{\partial \bar{v}_x}{\partial x} + k \bar{v}_z \right) + \overset{0}{v} \frac{\partial \bar{\rho}}{\partial x} &= - \frac{\rho_{ad}^A}{\tau_{ad}} \left[\left(\frac{c_0 \kappa \bar{\rho}}{p_0} + \bar{c} \frac{\overset{0}{p}}{p_0} \right) \left(1 - \overset{0}{\xi} \right) - \bar{\xi} \left(1 + c_0 \frac{\overset{0}{p}}{p_0} \right) \right], \\ \rho_0^F \left(\omega \bar{c} + \overset{0}{v} \frac{\partial \bar{c}}{\partial x} \right) &= - (1 - c_0) \frac{\rho_{ad}^A}{\tau_{ad}} \left[\left(\frac{c_0 \kappa \bar{\rho}}{p_0} + \bar{c} \frac{\overset{0}{p}}{p_0} \right) \left(1 - \overset{0}{\xi} \right) \right. \\ &\quad \left. - \bar{\xi} \left(1 + c_0 \frac{\overset{0}{p}}{p_0} \right) \right] - \frac{\rho_{ad}^A}{\tau_{ad}} \bar{c} \left[c_0 \frac{\overset{0}{p}}{p_0} \left(1 - \overset{0}{\xi} \right) - \overset{0}{\xi} \right], \\ \omega \bar{\xi} &= \left[\left(\bar{\rho} \frac{c_0 \kappa}{p_0} + \bar{c} \frac{\overset{0}{p}}{p_0} \right) \left(1 - \overset{0}{\xi} \right) - \bar{\xi} \left(1 + c_0 \frac{\overset{0}{p}}{p_0} \right) \right] \frac{1}{\tau_{ad}}, \\ \rho_0^F \left(\omega \bar{v}_x + \overset{0}{v} \frac{\partial \bar{v}_x}{\partial x} \right) &= -\kappa \frac{\partial \bar{\rho}}{\partial x} - \pi \bar{v}_x, \\ \rho_0^F \left(\omega \bar{v}_z + \overset{0}{v} \frac{\partial \bar{v}_z}{\partial x} \right) &= k \kappa \bar{\rho} - \pi \bar{v}_z. \end{aligned}$$

Dispersion relation

We have to analyse the following dispersion relation:

$$(4.29) \quad (\omega I + \mathfrak{A}) \mathbf{u} + \mathfrak{B} \mathbf{u}' = 0,$$

with

$$\mathbf{u} = (\bar{\rho}, \bar{c}, \bar{\xi}, \bar{v}_x, \bar{v}_z)^T, \quad \mathbf{u}' = \left(\frac{\partial \bar{\rho}}{\partial x}, \frac{\partial \bar{c}}{\partial x}, \frac{\partial \bar{\xi}}{\partial x}, \frac{\partial \bar{v}_x}{\partial x}, \frac{\partial \bar{v}_z}{\partial x} \right)^T,$$

$$(4.30) \quad \mathbf{l} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathfrak{B} := \begin{pmatrix} 0 & 0 & 0 & \rho_0^F & 0 \\ \bar{v} & 0 & 0 & 0 & 0 \\ 0 & \bar{v} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\kappa}{\rho_0^F} & 0 & 0 & \bar{v} & 0 \\ 0 & 0 & 0 & 0 & \bar{v} \end{pmatrix},$$

$$\mathfrak{A} := \begin{pmatrix} \frac{\rho_{ad}^A c_0 \kappa}{\tau_{ad} p_0} \left(1 - \frac{0}{\xi}\right) & \frac{\rho_{ad}^A \bar{p}}{\tau_{ad} p_0} \left(1 - \frac{0}{\xi}\right) \\ \frac{1 - c_0}{\rho_0^F} \frac{\rho_{ad}^A c_0 \kappa}{\tau_{ad} p_0} \left(1 - \frac{0}{\xi}\right) & \frac{1 - c_0}{\rho_0^F} \frac{\rho_{ad}^A \bar{p}}{\tau_{ad} p_0} \left(1 - \frac{0}{\xi}\right) + \frac{1}{\rho_0^F} \frac{\rho_{ad}^A}{\tau_{ad}} \left[c_0 \frac{\bar{p}}{p_0} \left(1 - \frac{0}{\xi}\right) - \frac{0}{\xi} \right] \\ -\frac{c_0 \kappa}{p_0} \left(1 - \frac{0}{\xi}\right) \frac{1}{\tau_{ad}} & -\frac{\bar{p}}{p_0} \left(1 - \frac{0}{\xi}\right) \frac{1}{\tau_{ad}} \\ 0 & 0 \\ -k \frac{\kappa}{\rho_0^F} & 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\rho_{ad}^A}{\tau_{ad}} \left(1 + c_0 \frac{\bar{p}}{p_0}\right) & 0 & \rho_0^F k \\ -\frac{1 - c_0}{\rho_0^F} \frac{\rho_{ad}^A}{\tau_{ad}} \left(1 + c_0 \frac{\bar{p}}{p_0}\right) & 0 & 0 \\ \left(1 + c_0 \frac{\bar{p}}{p_0}\right) \frac{1}{\tau_{ad}} & 0 & 0 \\ 0 & \frac{\pi}{\rho_0^F} & 0 \\ 0 & 0 & \frac{\pi}{\rho_0^F} \end{pmatrix}$$

and boundary conditions

$$(4.31) \quad \bar{v}_x|_{x=0} = -\frac{\alpha \kappa + \bar{v}}{\rho_0^F} \bar{\rho}|_{x=0}, \quad \bar{v}_z|_{z=\pm b} = 0,$$

$$\bar{v}_x|_{x=l} = \frac{\alpha \kappa - \bar{v}}{\rho_0^F} \bar{\rho}|_{x=l},$$

Conditions in z -direction are fulfilled automatically due to ansatz (4.1).

5. Numerical investigation

We solve the eigenvalue problem for ω , using a second order finite difference scheme in a equidistant mesh (length of the body l divided into n parts of length h). The derivatives of disturbances are written as central differences $\left(\frac{\partial u}{\partial x} = \frac{u(x+h) - u(x-h)}{2h}\right)$ for inner mesh points, or as asymmetric ones for the first $\left(\frac{\partial u}{\partial x} = \frac{u(x+h) - u(x)}{h}\right)$ and the last point $\left(\frac{\partial u}{\partial x} = \frac{u(x) - u(x-h)}{h}\right)$. For this linear eigenvalue problem we obtain $5n + 3$ eigenvalues ω_i (number of linear equations: $5(n + 1) - 2$). The exponential form of the ansatz (4.1) yields that the base flow is stable if all real parts of ω_i are negative and unstable if at least one of the $5n + 3$ real parts is positive.

For a chosen pair (α, ρ_{ad}^A) we calculate $\max(Re(\omega))$ in dependence on π . In order to consider the precision of numerical calculations only those real parts of eigenvalues are considered whose absolute value is greater than 10^{-5} . We have determined this value by comparison of results for the same parameters running once a program without mass exchange and once a program with mass exchange where mass exchange was switched off. Many authors who solve eigenvalue problems numerically using a finite-difference scheme complain about slowly converging procedures (e.g. [5]) from which it follows that calculations for a large value of elements are necessary. In our case we do not observe such problems: Comparison of analytical and numerical results (without mass exchange) shows that the small number of 10 elements is sufficient to obtain a good agreement of results in the scope of the just mentioned precision.

In the calculations we use the data of the following two tables. The data are typical for geotechnical applications. In Table 1 parameters for a flow process without adsorption are given, in Table 2 these quantities are given which enter the model due to the adsorption process:

Table 1. Typical model parameters for flow processes in soils.

Length, width of the body l, b	1m	equilibrium porosity n_E	0.23
Compressibility κ	$2.25 \cdot 10^6 \frac{m^2}{s^2}$	Initial mass density ρ_0^L	$2.3 \cdot 10^2 \frac{kg}{m^3}$
Pressure left-h.s. p_l	110kPa	Pressure right-h.s. p_r	100kPa
Pressure difference working on the fluid		$p_d = n_E (p_l - p_r)$	2.3kPa

Table 2. Additional model parameters for adsorption processes in soils.

Initial concentration c_0	10^{-3}
Langmuir pressure p_0	10 kPa
Charact. time of adsorp. τ_{ad}	1s

In the following Fig. 1 we show the results of the stability analysis. The domain of instability is shown in 3D in dependence on the bulk permeability π , the surface permeability α , and the coefficient ρ_{ad}^A characterizing adsorption properties of the internal surface. The last coefficient may change by a few orders of magnitude because it is proportional to the internal surface. Below we see the 2D projections of the instability region.

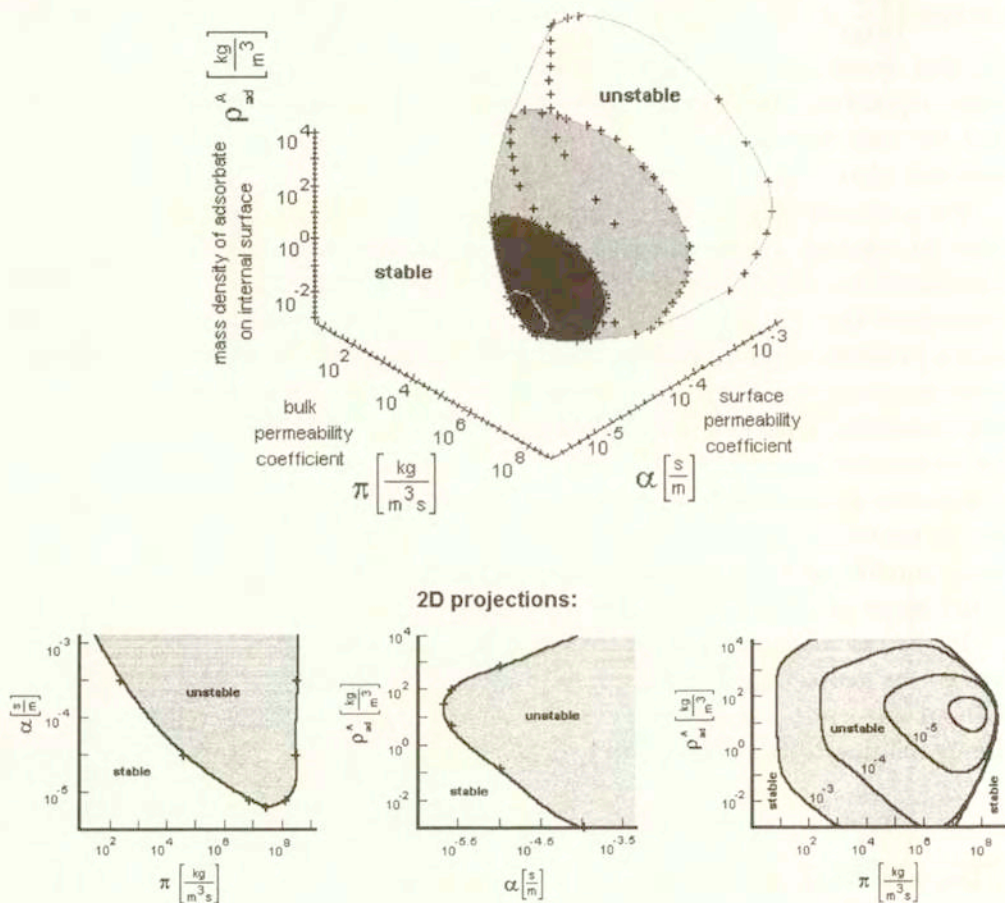


FIG. 1. Numerical result for transversal disturbance with mass exchange.

Transversal disturbance with mass exchange yields an instability of the steady state flow for some ranges of parameters. While for small values of π the border of this region strongly varies with the surface permeability parameter α (see the third 2D projection), on the side of large π the unstable region is limited by a value of π app. $10^{8.548}$. For greater values of π the base flow is stable for any α . In

the range of extremally large π not shown any more in the figure there appears again an instability for all values of α . However, the model is not applicable in this range any more. In addition, the numerical procedure is not stable any more for such very large values of π .

With decreasing α also the region of π decreases for which the flow is unstable until the value $\alpha \approx 10^{-5.690}$ is reached where the base flow is stable in the whole range of π (see the first 2D projection). A similar behavior exists in dependence on ρ_{ad}^A (see second 2D projection): the range of ρ_{ad}^A for which the flow is unstable decreases with decreasing α . For smaller values than $\alpha \approx 10^{-5.690}$ the flow is stable for any ρ_{ad}^A . It is worth mentioning that parameters which have been used for the works quoted earlier lie all in the stable region.

6. Properties of some other disturbances of the base flow

As in many other known cases of flow instabilities, a 1D disturbance either without or with mass exchange does not produce any instability of our base flow. The same is valid for a 2D disturbance without mass exchange. Results for these disturbances are shown in the paper [1]. Although there appears no region of instability for the three mentioned disturbances results show important features of the relaxation behavior. For the 1D and 2D disturbances without mass exchange even an analytical investigation was feasible which gave hints for the numerical approach also of this work.

From the stable cases we know that the relaxation properties do not change monotonously with the permeability π . They possess rather two different ranges. In the range of smaller values of π the relaxation is determined by a real part of the complex root, while for larger π this root does not possess the imaginary part. It means that the perturbation causes vibrations in the range of smaller π , whose frequencies cover the whole discrete spectrum. For values of π greater than this of the turning point, the disturbance is only damped but the damping is smaller than that predicted by the resistance to the diffusion (i.e. $< \frac{\pi}{\rho_0^F}$). The position of the turning point is determined by the compressibility coefficient of the fluid. We refer to [3] where this property is discussed. In general the damping is the smallest (i.e. the relaxation is the slowest) for large and small values of π .

Calculations for different disturbances made it evident that the mass exchange slows down the relaxation even by a few orders of magnitude. This effect is related to the characteristic time of adsorption τ_{ad} . Little can be said about its experimental values for porous materials because most experiments are conducted in quasistatic conditions. In [3] we show the influence of τ_{ad} on the relaxation properties. As expected, the disturbances relax faster for smaller characteristic times τ_{ad} but even for a very short time of adsorption $\tau_{ad} = 10^{-5}$ s

this relaxation is considerably longer than in the case of no mass exchange. For the current work we used a value of $\tau_{ad} = 1s$ as in all earlier works on adsorption.

Results for the 2D disturbance without mass exchange reveal the same behavior as these for the 1D perturbation. Admittedly there appear additional solutions but they do neither have influence on the stability behavior nor on the relaxation properties. In general the relaxation in the 2D case without mass exchange is faster than in the 1D case. Anyway it is interesting that the 2D disturbance with mass exchange yields an instability while the 1D disturbance with adsorption does not.

7. Conclusions

In this work we have shown that for transversal disturbances with mass exchange of the 1D steady state flow through the porous material there appears a region of parameters π , α , and ρ_{ad}^A in which the base flow is unstable. This is not the case with respect to a linear longitudinal disturbance without and with mass exchange and to a linear transversal disturbance without mass exchange. For the three latter disturbances the base flow is stable in the whole range of control permeability parameters π , and α .

This shows that a disturbance with adsorption has a great influence on the stability behavior of the base flow. While adsorption already in the case of 1D disturbances decreases the maximum values of real parts by some orders of magnitude, for the 2D disturbances it even decides whether the base flow is stable or unstable.

The existing unstable region for the disturbance investigated in this paper appears in the region of π where in the case of the other disturbances (1D without and with mass exchange, 2D without mass exchange) appear complex roots and thus vibrations. The range of instability corresponds to higher velocities for the same pressure difference due to a lower resistance of the boundary. Simultaneously it corresponds to a lower "internal friction" πv_x^F in momentum balance equation due to lower values of permeability π .

The work has shown that the order of magnitude of parameters which were used in earlier works on this model were determined correctly and lie in stable regions of the model.

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